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QUASILINEARIZATION METHOD FOR FIRST-ORDER IMPULSIVE INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this article we study first-order impulsive integro-differential equations with integral boundary conditions, employing the method of quasilinearization with reversed ordering upper and lower solutions. We obtain two monotone sequences of iterates converging uniformly and quadratically to the unique solution of the problem. Two examples are given to illustrate the applications of the established results.

1. Introduction

Integral differential equations arise in several engineering and scientific disciplines as the mathematical modelling of systems and processes, such as physics, mechanics, biology, economics and engineering [2, 7]. In consequence, the qualitative theory of integral differential equations creates an important branch of nonlinear analysis. Over the last twenty years, there are some results on the existence, uniqueness, continuation and other properties of solutions and extremal solutions for various boundary value problem involving integral boundary conditions, such as the monographs [5, 15, 17], the papers for differential equations [1, 2, 6, 8, 10, 12, 22, 27], for functional integro-differential equations [11, 25], for impulsive integro-differential equations [3, 9, 11, 13, 19, 20, 21, 24], for integro-differential equations of fractional order [3, 4], for integral boundary value problems with causal operators [26], and references given therein. However, we noticed that the previous studies mainly focused on the existence and uniformly convergence results for extremal solutions via the method of upper and lower solutions coupled with the monotone iterative technique, which gives a constructive procedure for approximation solutions, and offers monotone sequences uniformly converging to extremal solutions (see the monograph [17]). In terms of applications, it is important to pay attention to the high-order convergence of sequences of approximate solutions. Quasilinearization combined with the technique of upper and lower solutions is an effective and fruitful technique for obtaining approximate solutions to a wide variety of nonlinear problems. The main advantages of the method are the practicality of finding successive approximations of the unknown solution as well as the quadratic convergence rate. A systematic development of the quasilinearization method to ordinary differential equations has been provided by Lakshmikantham and Vatsala [18], and

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there are some generalized results for various types of differential systems, see the monographs [15, 16, 17].

The goal of this paper is to investigate the convergence of solutions for a class of first-order impulsive integro-differential equations with integral boundary conditions,

$$\begin{aligned} u'(t) &= f(t, u(t), (Su)(t)), \quad t \neq t_k,\ t \in J, \\ \Delta u(t_k) &= I_k(u(t_k)), \quad k = 1, 2, \dots, m, \\ u(0) + \mu \int_0^T u(s)ds &= \theta u(T), \end{aligned} \tag{1.1}$$

where $f \in C(J' \times R^2, R)$, $J = [0, T]$, $J' = J \backslash \{t_1, t_2, \dots, t_m\}$, $0 < t_1 < t_2 < \dots < t_m = T$. S is a Volterra operator defined by $(Su)(t) = \int_0^t r(t,s)u(s)ds$, $r \in C(D, R_+)$, $r_0 = \max_{(t,s)\in J\times J} r(t,s)$, $D = \{(t,s) \in J \times J : t \geq s\}$, $R_+ = [0, \infty)$. $I_k \in C(R, R)$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ denotes the jump of u at $t = t_k$, $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$ respectively, $k = 1, 2, \dots, m$. $\mu \leq 0$, $\theta = 1$ or -1 are constants.

By employing the method of quasilinearization with reversed ordering upper and lower solutions, we obtain the two monotone sequences of iterates converging uniformly and quadratically to the unique solution of the problem. Two examples are given to illustrate the applications of the established results. The impulsive integro-differential equation (1.1) has a lot of special types. For example, if $\mu = 0$, $\theta = 1$, problem (1.1) reduces to a periodic boundary value problem. If $\mu = 0$, $\theta = -1$, problem (1.1) reduces to an anti-periodic boundary value problem.

This article is organized as follows. In Section 2, we give the new definitions of reversed ordering upper and lower solutions and establish comparison theorems for the case of $\theta = 1$ and $\theta = -1$ in order to discuss the existence and uniqueness of the solutions for first-order impulsive integral boundary value problem. Then, we obtain its accelerated rate of convergence by using the technique of quasilinearization in section 3. Finally, we give two examples to illustrate the applications of the established results in Section 4.

2. Preliminaries

Firstly, we introduce the notation, definitions and a lemma. Let

$$\begin{aligned} PC(J) = \big\{&u : J \to R, u \text{ is continuous for } t \in J' \text{ and } u(t_k^+), u(t_k^-) \text{ exist with} \\ &u(t_k^-) = u(t_k),\ \text{for } k = 1, 2, \dots, m\big\}; \\ PC^1(J) = \big\{&u \in PC(J) : u \text{ is continuously differentiable for } t \in J',\ u'(t_k^+), u'(t_k^-) \\ &\text{exist and } u' \text{is left continuous at } t = t_k \text{ for } k = 1, 2, \dots, m\big\}. \end{aligned}$$

Note that $PC(J)$ and $PC^1(J)$ are Banach spaces with the norms

$$\|u\|_{PC(J)} = \sup\{|u(t)| : t \in J\}, \quad \|u\|_{PC^1(J)} = \max\{\|u(t)\|_{PC(J)},]; \|u'(t)\|_{PC(J)}\}.$$

A function $u \in PC^1(J)$ is called a solution of the integral boundary value problem (1.1) if it satisfies (1.1).

Definition 2.1. A function $\alpha \in PC^1(J)$ is called a lower solution of (1.1), if the following inequalities hold:

$$\begin{gathered}
\alpha'(t) \le f(t, \alpha(t), (S\alpha)(t)), \quad t \ne t_k,\ t \in J, \\
\Delta\alpha(t_k) \le I_k(\alpha(t_k)), \quad k = 1, 2, \dots, m, \\
\alpha(0) + \mu \int_0^T \alpha(s) ds \le \theta\alpha(T).
\end{gathered} \tag{2.1}$$

Definition 2.2. A function $\beta \in PC^1(J)$ is called an upper solution of the integral boundary value problem (1.1), if the following inequalities hold:

$$\begin{gathered}
\beta'(t) \ge f(t, \beta(t), (S\beta)(t)), \quad t \ne t_k,\ t \in J, \\
\Delta\beta(t_k) \ge I_k(\beta(t_k)),\ k = 1, 2, \dots, m, \\
\beta(0) + \mu \int_0^T \beta(s) ds \ge \theta\beta(T).
\end{gathered} \tag{2.2}$$

For the next lemma we use the following assumptions:

(A1) the sequence $\{t_k\}$ satisfies $0 < t_1 < t_2 < \dots < t_k < \dots < t_m = T$ with $\lim_{k\to\infty} t_k = +\infty$;

(A2) $m \in PC^1(R_+, R)$ is left continuous at t_k for $k = 1, 2 \dots$, and

$$\begin{gathered}
m'(t) \ge p(t)m(t) + q(t), \quad t \ne t_k,\ t \in J, \\
m(t_k^+) \ge d_k m(t_k) + b_k, \quad k = 1, 2, \dots, m,
\end{gathered}$$

where $p, q \in C(R_+, R)$, $d_k \ge 0$ and b_k are constant.

Lemma 2.3 (See [14])**.** *Assume* (A1), (A2)*. Then*

$$\begin{aligned}
m(t) \ge m(t_0) \prod_{t_0<t_k<t} d_k \exp\Big(\int_{t_0}^t p(s) ds\Big) + \int_{t_0}^t \prod_{s<t_k<t} d_k \exp\Big(\int_s^t p(\sigma) d\sigma\Big) q(s) ds \\
+ \sum_{t_0<t_k<t,} \ \prod_{t_k<t_j<t} d_j \exp\Big(\int_{t_k}^t p(s) ds\Big) b_k\,.
\end{aligned}$$

To study problem (1.1), we need to establish a comparison theorem and obtain its solution for the associated linear impulsive integral boundary value problem.

Lemma 2.4. *Assume that $m \in PC^1(J)$ satisfies the following inequalities*

$$\begin{gathered}
m'(t) \ge M_1 m(t) + M_2 (Sm)(t), \quad t \in J', \\
\Delta m(t_k) \ge L_k m(t_k), \quad k = 1, 2, \dots, m, \\
m(0) \ge m(T)\,.
\end{gathered} \tag{2.3}$$

If

$$\prod_{k=1}^m (1 + L_k)^{-1} \ge M_2 \int_0^t \Big[\int_0^s r(s,\sigma) e^{-M_1(s-\sigma)} d\sigma\Big] ds, \tag{2.4}$$

then $m(t) \le 0$ on J, where $M_1 > 0$, $M_2 > 0$ and $L_k \ge 0$ are constants, $k = 1, 2, \dots, m$.

Proof. Setting $u(t) = m(t)e^{-M_1 t}$, we have $u \in PC^1(J)$, and

$$\begin{gathered} u'(t) \geq M_1(\bar{S}u)(t), \quad t \in J', \\ \Delta u(t_k) \geq L_k u(t_k), \quad k = 1, 2, \dots, m, \\ u(0) \geq u(T)e^{M_1 T}, \end{gathered} \tag{2.5}$$

where $(\bar{S}u)(t) = \int_0^t r(t,s)e^{-M_1(t-s)}u(s)ds$.

We now prove that $u(t) \leq 0$ for any $t \in J$. Suppose on the contrary, that $u(t) > 0$ for some $t \in J$. Then there are two cases:

Case 1: There exists a $t_1^* \in J$ such that $u(t_1^*) > 0$ and $u(t) \geq 0$ for $t \in J$. Then (2.5) implies that

$$u'(t) \geq 0,\ t \in J';\ \Delta u(t_k) \geq 0,\ k = 1, 2, \dots, m.$$

This means that $u(t)$ is nondecreasing in J. Therefore, $u(T) \geq u(t_1^*) > 0$ and $u(T) \geq u(0) \geq u(T)e^{M_1 T}$, which is a contradiction.

Case 2: There exist $t_1^*, t_2^* \in J$ such that $u(t_1^*) > 0$ and $u(t_2^*) < 0$. Let $\inf_{t\in J} u(t) = -\lambda$, then $\lambda > 0$, and there exists a $t_i < t_0^* \leq t_{i+1}$ for some i such that $u(t_0^+) = -\lambda$ or $u(t_i^+) = -\lambda$. We may assume that $u(t_0^+) = -\lambda$. The case of $u(t_i^+) = -\lambda$ can be proved similarly.

Consider the inequalities:

$$\begin{gathered} u'(t) \geq -\lambda M_2 \int_0^t r(t,s)e^{-M_1(t-s)}ds, \quad t \in J', \\ u(t_k^+) \geq (1 + L_k)u(t_k), \quad k = 1, 2, \dots, m. \end{gathered} \tag{2.6}$$

By Lemma 2.3, we have

$$u(t) \geq u(0) \prod_{0<t_k<t}(1 + L_k) + \int_0^t \prod_{s<t_k<t}(1 + L_k)\Big[-\lambda M_2 \int_0^s r(s,\sigma)e^{-M_1(t-\sigma)}d\sigma\Big]ds.$$

Letting $t = t_0^*$, we have

$$u(0) \leq -\lambda \prod_{0<t_k<t_0^*}(1+L_k)^{-1} + \lambda M_2 \int_0^{t_0^*} \prod_{0<t_k<s}(1+L_k)^{-1}\Big[\int_0^s r(s,\sigma)e^{-M_1(s-\sigma)}d\sigma\Big]ds.$$

If $u(0) > 0$, then

$$\begin{aligned} \prod_{0<t_k<t_0^*}(1 + L_k)^{-1} &< M_2 \int_0^{t_0^*} \prod_{0<t_k<s}(1 + L_k)^{-1}\Big[\int_0^s r(s,\sigma)e^{-M_1(s-\sigma)}d\sigma\Big]ds \\ &< M_2 \int_0^{t_0^*} \Big[\int_0^s r(s,\sigma)e^{-M_1(s-\sigma)}d\sigma\Big]ds; \end{aligned}$$

that is,

$$\prod_{0<t_k<t_0^*}(1 + L_k)^{-1} < M_2 \int_0^T \Big[\int_0^s r(s,\sigma)e^{-M_1(s-\sigma)}d\sigma\Big]ds,$$

which contradicts with (2.4). Thus $u(0) \leq 0$. Furthermore, by (2.5), we can obtain $u(T) \leq u(0)e^{-M_1 T} < 0$, then $0 < t_1^* < T$.

Let $t_j < t_1^* \le t_{j+1}$ for some j. We first assume that $t_1^* < t_0^*$. Consider the inequalities

$$\begin{gathered} u'(t) \ge -\lambda M_2 \int_0^t r(t,s)e^{-M_1(t-s)}ds, \quad t \in J', \\ u(t_k^+) \ge (1+L_k)u(t_k), \ k = 1,2,\dots,m. \end{gathered}$$

Similar to the process above, using Lemma 2.3, we can also find a contradiction with (2.4). Similarly, we can prove the case of $t_0^* < t_1^*$. The proof is complete. □

Lemma 2.5. *Assume that $x \in PC^1(J)$, $\sigma \in PC(J)$. If*

$$M_1^{-1}M_2r_0T + \frac{e^{M_1T}}{e^{M_1T}-1}\sum_{k=1}^m L_k < 1. \tag{2.7}$$

Then the linear impulsive boundary value problem

$$\begin{gathered} x'(t) = M_1x(t) + M_2(Sx)(t) + \sigma(t), \quad t \in J', \\ \Delta x(t_k) = L_kx(t_k) + d_k, \quad k = 1,2,\dots,m, \\ x(0) + d = x(T), \quad d \in R \end{gathered} \tag{2.8}$$

has a unique solution, where $M_1 > 0$, $M_2 > 0$ and $L_k \ge 0$ are constants.

Proof. We define a map $A : PC(J) \to PC(J)$ by

$$(Ax)(t) = -\frac{e^{M_1T}}{e^{M_1T}-1}d + \int_0^T G(t,s)\big[\sigma(s) + M_2(Sx)(s)\big]ds + \sum_{k=1}^m G(t,t_k)\big[L_kx(t_k) + d_k\big],$$

where

$$G(t,s) = \begin{cases} \frac{e^{M_1(t-s)}}{e^{M_1T}-1}, & 0 \le s \le t \le T, \\ \frac{e^{M_1(T+t-s)}}{e^{M_1T}-1}, & 0 \le t \le s \le T. \end{cases}$$

It is easy to verify that $x(t)$ is a solution of (2.8), if and only if $x(t)$ is a fixed point of A. For any $u, v \in PC^1(J)$, we have

$$\begin{aligned} &|(Au)(t) - (Av)(t)| \\ &\le \int_0^T G(t,s)\big|M_2\big[(Su)(s) - (Sv)(s)\big]\big|ds + \sum_{k=1}^m G(t,t_k)|L_k(u(t_k) - v(t_k))| \\ &\le M_2^{-1}M_2r_0T + \frac{e^{M_1T}}{e^{M_1T}-1}\sum_{k=1}^m L_k\|u-v\|\,. \end{aligned}$$

Then

$$\|Au - Av\| \le \Big(M_1^{-1}M_2r_0T + \frac{e^{M_1T}}{e^{M_1T}-1}\sum_{k=1}^m L_k\Big)\|u-v\|,$$

which means that (2.7) implies that A is a contradiction mapping. Consequently, employing the Banach's Fixed Point Theorem, the map A has an unique fixed point. Thus (2.8) has an unique solution. The proof is complete. □

Similar to the proof Lemma 2.4 and Lemma 2.5, for the case of $\theta = -1$, we have the following Lemmas.

Lemma 2.6. *Assume that $m \in PC^1(J)$ satisfies the inequalities*

$$\begin{gathered} m'(t) \geq M_1 m(t) + M_2 (Sm)(t), \quad t \in J', \\ \Delta m(t_k) \geq L_k m(t_k), \quad k = 1, 2, \dots m, \\ m(0) \geq -m(T). \end{gathered} \tag{2.9}$$

If

$$\prod_{k=1}^{m} (1 + L_k)^{-1} \geq M_2 \int_0^t \Big[\int_0^s r(s, \sigma) e^{-M_1(s-\sigma)} d\sigma \Big] ds \,, \tag{2.10}$$

then $m(t) \leq 0$ on J, where $M_1 > 0$, $M_2 > 0$ and $L_k \geq 0$ are constants, $k = 1, 2, \dots, m$.

Lemma 2.7. *Assume that $u \in PC^1(J)$ and $\sigma \in PC(J)$. If*

$$M_1^{-1} M_2 r_0 T + \frac{e^{M_1 T}}{e^{M_1 T} - 1} \sum_{k=1}^{m} L_k < 1 \,, \tag{2.11}$$

then the impulsive differential equation

$$\begin{gathered} x'(t) = M_1 x(t) + M_2 (Sx)(t) + \sigma(t), \quad t \in J', \\ \Delta x(t_k) = L_k x(t_k) + d_k, \quad k = 1, 2, \dots, m, \\ x(0) + d = -x(T), \quad d \in R, \end{gathered} \tag{2.12}$$

has a unique solution, where $M_1 > 0$, $M_2 > 0$, $L_k \geq 0$ are constants.

3. Main results

In this section, we give the results which converge uniformly and quadratically to the unique solution of the integral boundary value problem (1.1). Consider the sets:

$$\begin{gathered} \Omega = \{(t, x) : \beta(t) \leq x(t) \leq \alpha(t), \ t \in J\}, \\ D_k = \{x \in R : \beta(t_k) \leq x(t_k) \leq \alpha(t_k), \ 1 \leq k \leq m\}. \end{gathered}$$

For the next theorem we the following assumptions:

(A3) There exist constants $M_1 > 0$ and $M_2 > 0$ such that

$$f(t, u, v) - f(t, \bar{u}, \bar{v}) \leq M_1 (u - \bar{u}) + M_2 (v - \bar{v}),$$

for $\beta \leq \bar{u} \leq u \leq \alpha$ and $T\beta \leq \bar{v} \leq v \leq T\alpha$;

(A4) there exist constants $L_k \geq 0, k = 1, 2 \dots m$, such that

$$I_k(x) - I_k(y) \leq L_k (x - y), \quad \text{for } \beta \leq y \leq x \leq \alpha.$$

Theorem 3.1. *Let α, β be lower and upper solutions respectively for problem* (1.1) *with $\beta \leq \alpha$ on J. Assume that* (A3), (A4), (2.4) *and* (2.7) *hold. Then there exist two monotone sequences $\{\alpha_n\}, \{\beta_n\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ such that $\lim_{n\to\infty} \alpha_n = \rho(t)$, $\lim_{n\to\infty} \beta_n = \gamma(t)$ uniformly on J, where $\rho(t)$, $\gamma(t)$ are the maximal and minimal solutions of* (1.1) *respectively, satisfying*

$$\beta_0 \leq \beta_1 \leq \beta_2 \leq \dots \beta_n \leq \gamma(t) \leq u(t) \leq \rho(t) \leq \alpha_n \leq \dots \leq \alpha_2 \leq \alpha_1 \leq \alpha_0$$

in which $u(t)$ is any solution of (1.1) *such that $\beta(t) \leq u(t) \leq \alpha(t)$ on J.*

Proof. For any $\eta \in [\beta, \alpha]$, consider the linear impulsive integral boundary value problem

$$\begin{gathered} u'(t) = f(t, \eta, S\eta) + M_1(u - \eta) + M_2(Su - S\eta), \quad t \in J', \\ \Delta u(t_k) = I_k(\eta(t_k)) + L_k(u(t_k) - \eta(t_k)), \quad k = 1, 2, \dots, m, \\ u(0) + \mu \int_0^T \eta(s) ds = u(T). \end{gathered} \tag{3.1}$$

By Lemma 2.5, problem (3.1) has a unique solution $u \in PC^1(J)$. Now we define an operator A by $u = A\eta$, then the operator has the following properties:

(i) $\beta \le A\beta,\ A\alpha \le \alpha$;

(ii) A is a monotone nondecreasing on $[\beta, \alpha]$, i.e., for any $\eta_1, \eta_2 \in C[\beta, \alpha],\ \eta_1 \le \eta_2$, implies $A\eta_1 \le A\eta_2$.

To prove (i), setting $m = \beta_0 - \beta_1$, where $\beta_1 = A\beta_0$. Then from $\beta(t) \le \alpha(t)$ and (2.7), we have

$$\begin{aligned} m'(t) &\ge f(t, \beta_0, S\beta_0) - f(t, \beta_0, S\beta_0) - M_1(\beta_1 - \beta_0) - M_2(S\beta_1 - S\beta_0) \\ &\ge M_1 m(t) + M_2(Sm)(t), \end{aligned}$$

and

$$\Delta m(t_k) \ge I_k(\beta_0(t_k)) - I_k(\beta_0(t_k)) - L_k(\beta_1(t_k) - \beta_0(t_k)) = L_k m(t_k).$$

Thus, by Lemma 2.4, we have $m(t) \le 0$ on J, that is $\beta_0 \le \beta_1 = A\beta_0$. Similarly, we can prove that $A\alpha_0 = \alpha_1 \le \alpha_0$.

To prove (ii), setting $u_1 = A\eta_1,\ u_2 = A\eta_2$, where $\eta_1 \le \eta_2$ with $\eta_1,\ \eta_2 \in [\beta, \alpha]$. Let $m(t) = u_1 - u_2$, then

$$\begin{aligned} m'(t) &= f(t, \eta_1, S\eta_1) + M_1(A\eta_1 - \eta_1) + M_2(S(A\eta_1) - S\eta_1) - f(t, \eta_2, S\eta_2) \\ &\quad - M_1(A\eta_2 - \eta_2) - M_2(S(A\eta_2) - S\eta_2) \\ &\ge M_1(A\eta_1 - A\eta_2) + M_2(S(A\eta_1) - S(A\eta_2)) \\ &= M_1 m(t) + M_2 Tm(t), \end{aligned}$$

and

$$\begin{aligned} \Delta m(t_k) &= I_k(\eta_1(t_k)) + L_k((A\eta_1)(t_k) - \eta_1(t_k)) - I_k(\eta_2(t_k)) \\ &\quad - L_k((A\eta_2)(t_k) - \eta_2(t_k)) \\ &\ge L_k((A\eta_1)(t_k) - (A\eta_2)(t_k)) \\ &= L_k m(t_k). \end{aligned}$$

Furthermore,

$$\begin{aligned} m(T) &= (A\eta_1)(0) + \mu \int_0^T \eta_1(s) ds - (A\eta_2)(0) - \mu \int_0^T \eta_2(s) ds \\ &= m(0) + \mu \int_0^T (\eta_1(s) - \eta_2(s)) ds \\ &\le m(0). \end{aligned}$$

In view of Lemma 2.4, we have $m(t) \le 0$ on J. Consequently, it is easy to define the sequences $\{\alpha_n\},\ \{\beta_n\}$ with $\alpha_0 = \alpha,\ \beta_0 = \beta$ such that $\alpha_{n+1} = A\alpha_n,\ \beta_{n+1} = A\beta_n$. From (i) and (ii), the sequences $\{\alpha_n\},\ \{\beta_n\}$ satisfying

$$\beta_0 \le \beta_1 \le \beta_2 \le \dots \le \beta_n \le \alpha_n \le \dots \le \alpha_2 \le \alpha_1 \le \alpha_0 \text{ on } J,$$

and there exist ρ, γ such that $\lim_{n\to\infty}\alpha_n = \rho(t)$, $\lim_{n\to\infty}\beta_n = \gamma(t)$ uniformly on J. Clearly, ρ, γ satisfy the integral boundary value problem (1.1) such that $u \in [\beta, \alpha]$ and that there exists a positive integer n such that $\beta_n \le \alpha_n$.

Then setting $m = \beta_{n+1} - u$, we have

$$\begin{aligned} m'(t) &= f(t, \beta_n, S\beta_n) + M_1(A\beta_{n+1} - \beta_n) + M_2(S(A\beta_{n+1}) - S\beta_n) - f(t, u, Su) \\ &\ge M_1 m(t) + M_2 Sm(t), \end{aligned}$$

and

$$\Delta m(t_k) = I_k(\beta_n(t_k)) + L_k(\beta_{n+1}(t_k) - \beta_n(t_k)) - I_k(u(t_k)) \ge L_k m(t_k).$$

Furthermore,

$$\begin{aligned} m(T) &= \beta_{n+1}(0) + \mu \int_0^T \beta_n(s)ds - u(0) - \mu \int_0^T u(s)ds \\ &= m(0) + \mu \int_0^T (\beta_n(s) - u(s))ds \\ &\le m(0). \end{aligned}$$

By Lemma 2.4, $m(t) \le 0$ on J, i.e., $\beta_{n+1} \le u$ on J. Similarly, we get $u(t) \le \alpha_{n+1}(t)$ on J. Noticing that $\beta_0(t) \le u(t) \le \alpha_0(t)$ on J, by induction, we can obtain $\beta_n(t) \le u(t) \le \alpha_n(t)$ on J for every n. Therefore, $\gamma(t) \le u(t) \le \rho(t)$ on J by taking limit as $n \to \infty$. The proof is complete. □

For the next theorem we use the following assumptions:

- (A5) $f_x, f_y, f_{xx}, f_{yy} \in C[\Omega, R]$, and $f_x \le 0$, $f_y \le 0$, $f_{xx} \ge 0$, $f_{yy} \ge 0$;
- (A6) $I_k \in C^2[D_k, R]$, $I_k' \ge 0$, $k = 1, 2, \dots, m$. If

$$1 - \prod_{k=1}^{m}(1 + \mu_k)\exp\Big(\int_0^T \lambda(s)ds\Big) > 0,$$

$$1 + \mu T\Big(1 - \prod_{k=1}^{m}(1 + \mu_k)\exp\{\int_0^T \lambda(s)ds\}\Big) > 0,$$

where $\mu_k = \sup_{x\in D_k} I_k'$, $\lambda(t) = \sup_{x\in D_t}\{f_x + f_y T r_0\}$, $D_t = [\beta(t), \alpha(t)]$, $t \in J$.

Theorem 3.2. *Assume* (A5), (A6) *and the conditions of Theorem 3.1. Then there are two monotone sequences* $\{\alpha_n\}, \{\beta_n\}$ *satisfying:*

1. $\beta_0 \le \beta_1 \le \beta_2 \le \dots \le \beta_n \le \dots \le \alpha_n \le \dots \le \alpha_2 \le \alpha_1 \le \alpha_0$ *on* J*;*
2. $\{\alpha_n\}, \{\beta_n\}$ *converging uniformly and quadratically to the unique solution of* (1.1).

Proof. Since $f_{xx} \ge 0$, $f_{yy} \ge 0$, then for any $(t, x_1, y_1), (t, x_2, y_2) \in \Omega$, we have

$$f(t, x_2, y_2) \ge f(t, x_1, y_1) + f_x(t, x_1, y_1)(x_2 - x_1) + f_y(t, x_1, y_1)(y_2 - y_1).$$

Similarly, for any $x, y \in D_k$, we have

$$I_k(y) \ge I_k(x) + I_k'(x)(y - x), \quad 1 \le k \le m.$$

Setting $\alpha_0 = \alpha$, $\beta_0 = \beta$, we consider the integral boundary value problem

$$\begin{aligned}
u'(t) &= f(t, \alpha(t), (S\alpha)(t)) + f_x(t, \alpha, S\alpha)(u - \alpha) + f_y(t, \alpha, S\alpha)(Su - S\alpha) \\
&= H_0(t, u, Su), \\
\Delta u(t_k) &= I_k(\alpha(t_k)) + I'(\alpha(t_k))(u(t_k) - \alpha(t_k)) = \Gamma_0(u(t_k)), \\
u(0) &+ \mu \int_0^T u(s)ds = u(T).
\end{aligned} \tag{3.2}$$

Since α and β are the lower and upper solutions of (1.1), we have

$$\begin{gathered}
\alpha'(t) \leq f(t, \alpha(t), (S\alpha)(t)) = H_0(t, \alpha, S\alpha), \\
\Delta\alpha(t_k) \leq I_k(\alpha(t_k)) = \Gamma_0(\alpha(t_k)), \\
\alpha(0) + \mu \int_0^T \alpha(s)ds \leq \alpha(T),
\end{gathered}$$

and

$$\begin{gathered}
\beta'(t) \geq f(t, \beta(t), (S\beta)(t)) \geq H_0(t, \beta, S\beta), \\
\Delta\beta(t_k) \geq I_k(\beta(t_k)) \geq \Gamma_0(\beta(t_k)), \\
\beta(0) + \mu \int_0^T \beta(s)ds \geq \beta(T).
\end{gathered}$$

Hence, α, β are the lower and upper solutions of (3.2) respectively.

By Theorem 3.1, there exists a solution $\alpha_1(t)$ of the integral boundary value problem (3.2) such that $\beta(t) \leq \alpha_1(t) \leq \alpha(t)$. Similarly, consider the integral boundary value problem:

$$\begin{aligned}
u'(t) &= f(t, \beta(t), (S\beta)(t)) + f_x(t, \beta, S\beta)(u - \beta) + f_y(t, \beta, S\beta)(Su - S\beta) \\
&= \overline{H}_0(t, u, Su), \\
\Delta u(t_k) &= I_k(\beta(t_k)) + I'(\beta(t_k))(u(t_k) - \beta(t_k)) = \overline{\Gamma}_0(u(t_k)), \\
u(0) &+ \mu \int_0^T u(s)ds = u(T),
\end{aligned} \tag{3.3}$$

we can also get a solution $\beta_1(t)$ of (3.3) such that $\beta(t) \leq \beta_1(t) \leq \alpha(t)$.

Next, we prove that $\beta_1(t) \leq \alpha_1(t)$. Since

$$\begin{gathered}
\alpha_1'(t) = H_0(t, \alpha_1, S\alpha_1) \leq f(t, \alpha_1(t), (S\alpha_1)(t)), \\
\Delta\alpha_1(t_k) = \Gamma_0(\alpha_1(t_k)) \leq I_k(\alpha_1(t_k)), \\
\alpha_1(0) + \mu \int_0^T \alpha_1(s)ds = \alpha_1(T),
\end{gathered} \tag{3.4}$$

thus, α_1 is a lower solution of (1.1). In the same manner, we can also prove that β_1 is an upper solution of (1.1). Therefore, it follows that $\beta_1(t) \leq \alpha_1(t)$ on J. Consequently, we have $\{\alpha_n\}$, $\{\beta_n\}$ such that

$$\beta_0 \leq \beta_1 \leq \beta_2 \leq \dots \beta_n \dots \leq \alpha_n \leq \dots \leq \alpha_2 \leq \alpha_1 \leq \alpha_0 \text{ on } J,$$

where

$$\begin{aligned}
\alpha_n'(t) = &f(t, \alpha_{n-1}(t), (S\alpha_{n-1})(t)) + f_x(t, \alpha_{n-1}(t), (S\alpha_{n-1})(t))(\alpha_n - \alpha_{n-1}) \\
&+ f_y(t, \alpha_{n-1}(t), (S\alpha_{n-1})(t))(S\alpha_n - S\alpha_{n-1}) \\
= &H_{n-1}(t, \alpha_n, S\alpha_n),
\end{aligned}$$

$$\begin{aligned}\Delta\alpha_n(t_k) &= I_k(\alpha_{n-1}(t_k)) + I'(\alpha_{n-1}(t_k))(\alpha_n(t_k) - \alpha_{n-1}(t_k))\\ &= \Gamma_{n-1}(\alpha_n(t_k)),\end{aligned}$$
$$\alpha_n(0) + \mu\int_0^T \alpha_n(s)ds = \alpha_n(T),$$

and

$$\begin{aligned}\beta_n'(t) &= f(t,\beta_{n-1}(t),(S\beta_{n-1})(t)) + f_x(t,\beta_{n-1}(t),(S\beta_{n-1})(t))(\beta_n - \beta_{n-1})\\ &\quad + f_y(t,\beta_{n-1}(t),(S\beta_{n-1})(t))(S\beta_n - S\beta_{n-1})\\ &= \overline{H}_{n-1}(t,\beta_n,S\beta_n),\end{aligned}$$
$$\begin{aligned}\Delta\beta_n(t_k) &= I_k(\beta_{n-1}(t_k)) + I'(\beta_{n-1}(t_k))(\beta_n(t_k) - \beta_{n-1}(t_k))\\ &= \overline{\Gamma}_{n-1}(\beta_n(t_k)),\end{aligned}$$
$$\beta_n(0) + \mu\int_0^T \beta_n(s)ds = \beta_n(T).$$

Since the sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ are monotonically bounded on $[0,T]$, then, it is easy to conclude that the sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ converge uniformly and monotonically to $\rho(t)$ and $\gamma(t)$, respectively, where

$$\rho'(t) = f(t,\rho(t),S\rho(t)),\quad \Delta\rho(t_k) = I_k(\rho(t_k)),\quad \rho(0) + \mu\int_0^T \rho(s)ds = \rho(T);$$
$$\gamma'(t) = f(t,\gamma(t),S\gamma(t)),\quad \Delta\gamma(t_k) = I_k(\gamma(t_k)),\quad \gamma(0) + \mu\int_0^T \gamma(s)ds = \gamma(T).$$

Thus, we get $\rho(t) = u(t) = \gamma(t)$ by Lemma 2.5, where $u(t)$ is the unique solution of (1.1). This proves that the sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ converge uniformly and monotonically to the unique solution $u(t)$ of (1.1).

Finally, we have to prove the quadratic convergence. Set

$$p_{n+1}(t) = u(t) - \beta_{n+1}(t) \geq 0,\ q_{n+1}(t) = \alpha_{n+1}(t) - u(t) \geq 0.$$

Now, using the mean value theorem, we have

$$\begin{aligned}p_{n+1}'(t) &= f(t,u(t),(Su)(t)) - H_0(t,\beta_{n+1}(t),(S\beta_{n+1})(t))\\ &= f_x(t,\xi_1(t),(S\xi_1)(t))p_n(t) + f_y(t,\xi_2(t),(S\xi_2)(t))(Sp_n)(t)\\ &\quad - f_x(t,\beta_n(t),(S\beta_n)(t))p_n(t) - f_y(t,\beta_n(t),(S\beta_n)(t))(Sp_n)(t)\\ &\quad + f_x(t,\beta_n(t),S\beta_n(t))p_{n+1}(t) + f_y(t,\beta_n(t),S\beta_n(t))Sp_{n+1}(t)\\ &\leq f_{xx}(t,\tau_1(t),(S\tau_1)(t))p_n^2(t) + f_{yy}(t,\tau_2(t),(S\tau_2)(t))[(Sp_n)(t)]^2\\ &\quad + f_x(t,\beta_n(t),(S\beta_n)(t))p_{n+1}(t) + f_y(t,\beta_n(t),(S\beta_n)(t))(Sp_{n+1})(t)\\ &\leq \lambda(t)p_{n+1}(t) + (A + BT^2r_0^2)\|p_n\|^2,\end{aligned} \tag{3.5}$$

where $A = \sup f_{xx}$, $B = \sup f_{yy}$. In the same way, we can obtain

$$\begin{aligned}\Delta p_{n+1}(t_k) &= I_k(u(t_k)) - I_k(\beta_n(t_k)) - I_k'(\beta_n(t_k))(\beta_{n+1}(t_k) - \beta_n(t_k))\\ &= I_k'(\eta(t_k))p_n(t_k) + I_k'(\beta_n(t_k))p_{n+1}(t_k) - I_k'(\beta_n(t_k))p_n(t_k)\\ &\leq I_k''(\theta(t_k))\|p_n\|^2 + \mu_k p_{n+1}(t_k)\\ &\leq c_k\|p_n\|^2 + \mu_k p_{n+1}(t_k),\end{aligned} \tag{3.6}$$

where $c_k = \sup I_k''$. By Lemma 2.3, we have

$$\begin{aligned} p_{n+1}(t) &\leq p_{n+1}(0)\prod_{k=1}^{m}(1+\mu_k)\exp\Big(\int_0^t \lambda(s)ds\Big) + \prod_{k=1}^{m}(1+\mu_k)\sum_{k=1}^{m}c_k\|p_n\|^2 \\ &\quad + \prod_{k=1}^{m}(1+\mu_k)T(A+BT^2r_0^2)\|p_n\|^2. \end{aligned} \tag{3.7}$$

Applying the boundary conditions of (1.1) and $p_{n+1}(0)+\mu\int_0^T p_{n+1}(s)ds = p_{n+1}(T)$, we obtain

$$\begin{aligned} p_{n+1}(T) &\leq p_{n+1}(0)\prod_{k=1}^{m}(1+\mu_k)\exp\{\int_0^t \lambda(s)ds\}\prod_{k=1}^{m}(1+\mu_k)\sum_{k=1}^{m}c_k\|p_n\|^2 \\ &\quad + \prod_{k=1}^{m}(1+\mu_k)T(A+BT^2r_0^2)\|p_n\|^2, \end{aligned}$$

and

$$\begin{aligned} p_{n+1}(0) &\leq \Big(1-\prod_{k=1}^{m}(1+\mu_k)\exp\{\int_0^T \lambda(s)ds\}\Big)^{-1}\Big[\prod_{k=1}^{m}(1+\mu_k)\sum_{k=1}^{m}c_k\|p_n\|^2 \\ &\quad + \prod_{k=1}^{m}(1+\mu_k)T(A+BT^2r_0^2)\|p_n\|^2 - \mu\int_0^T p_{n+1}(s)ds\Big]. \end{aligned} \tag{3.8}$$

Furthermore,

$$\begin{aligned} p_{n+1}(t) &\leq \Big(1-\prod_{k=1}^{m}(1+\mu_k)\exp\{\int_0^T \lambda(s)ds\}\Big)^{-1}\Big[\prod_{k=1}^{m}(1+\mu_k)\sum_{k=1}^{m}c_k\|p_n\|^2 \\ &\quad + \prod_{k=1}^{m}(1+\mu_k)T(A+BT^2r_0^2)\|p_n\|^2 - \mu\int_0^T p_{n+1}(s)ds\Big] \\ &\quad \times \prod_{k=1}^{m}(1+\mu_k)\exp\Big(\int_0^t \lambda(s)ds\Big) + \prod_{k=1}^{m}(1+\mu_k)\sum_{k=1}^{m}c_k\|p_n\|^2 \\ &\quad + \prod_{k=1}^{m}(1+\mu_k)T(A+BT^2r_0^2)\|p_n\|^2. \end{aligned}$$

Thus

$$\begin{aligned} \|p_{n+1}\| &\leq \Big[1+\mu T\big(1-\prod_{k=1}^{m}(1+\mu_k)e^{\int_0^T \lambda(s)ds}\big)^{-1}\Big]^{-1} \\ &\quad \times \Big\{\Big(1-\prod_{k=1}^{m}(1+\mu_k)\exp\{\int_0^T \lambda(s)ds\}\Big)^{-1} \\ &\quad \times \Big[\prod_{k=1}^{m}(1+\mu_k)\sum_{k=1}^{m}c_k\|p_n\|^2 + \prod_{k=1}^{m}(1+\mu_k)T(A+BT^2r_0^2)\|p_n\|^2\Big] \\ &\quad \times \prod_{k=1}^{m}(1+\mu_k)\exp\{\int_0^t \lambda(s)ds\} + \prod_{k=1}^{m}(1+\mu_k)\sum_{k=1}^{m}c_k\|p_n\|^2 \\ &\quad + \prod_{k=1}^{m}(1+\mu_k)T(A+BT^2r_0^2)\|p_n\|^2\Big\}; \end{aligned} \tag{3.9}$$

that is,

$$\|p_{n+1}\| \le Q_1 \|p_n\|^2,$$

where $Q_1 \ge 0$. Similarly, there exists a $Q_2 \ge 0$ such that

$$\|q_{n+1}\| \le Q_2 \|q_n\|^2.$$

This proves the quadratic convergence. □

Similar results can be obtained for $\theta = -1$, we omit their proof.

Theorem 3.3. *Assume that the conditions of Theorem 3.1 hold. Then there exist two monotone sequences $\{\alpha_n\}, \{\beta_n\}$ with $\alpha_0 = \alpha,\ \beta_0 = \beta$ such that $\lim_{n\to\infty}\alpha_n = \rho(t)$, $\lim_{n\to\infty}\beta_n = \gamma(t)$ uniformly on J, where $\rho(t),\ \gamma(t)$ are the maximal and minimal solutions of integral boundary value problem* (1.1) *respectively, satisfying*

$$\beta_0 \le \beta_1 \le \beta_2 \le \dots \beta_n \le \gamma(t) \le u(t) \le \rho(t) \le \alpha_n \le \dots \le \alpha_2 \le \alpha_1 \le \alpha_0$$

in which $u(t)$ is any solution of (1.1) *such that $\beta(t) \le u(t) \le \alpha(t)$ on J.*

Theorem 3.4. *Assume that the conditions of Theorem 3.2 hold. Then there exist two monotone sequences $\{\alpha_n\}$, $\{\beta_n\}$ satisfying:*

(1) $\beta_0 \le \beta_1 \le \beta_2 \le \dots \le \beta_n \le \dots \le \alpha_n \le \dots \le \alpha_2 \le \alpha_1 \le \alpha_0$ *on J;*
(2) $\{\alpha_n\}$, $\{\beta_n\}$ *converging uniformly and quadratically to the unique solution of* (1.1).

4. Examples

In this section, we give two examples to illustrate the results established in the previous section.

Example 4.1. Consider the impulsive integro-differential equation

$$\begin{gathered} u'(t) = -t\big(\cos u(t) + \sin u(t)\big) - \int_0^t \frac{u(s)}{(s+1)^2-1}ds, \quad t \in J = [0, \frac{\pi}{4}],\ t \ne \frac{\pi}{8}, \\ \Delta u(\frac{\pi}{8}) = \frac{1}{6}(u(\frac{\pi}{8})), \\ u(0) - \frac{1}{4}\int_0^{\frac{\pi}{4}} u(s)ds = u(\frac{\pi}{4}). \end{gathered} \tag{4.1}$$

It is easy to check that $\alpha_0 = 3\pi/4$ and $\beta_0 = 0$ are lower and upper solutions of (4.1) respectively, satisfy $\alpha_0 > \beta_0$, and $f_x \le 0$, $f_y < 0$, $f_{xx} \ge 0$, $f_{yy} = 0$. Problem (4.1) satisfies all the conditions of Theorem 3.2. Then there exist two monotone sequences $\{\alpha_n\}, \{\beta_n\}$ converging uniformly to the unique solution of (4.1).

Example 4.2. Consider the impulsive integro-differential equation

$$\begin{gathered} u'(t) = -t\cos u(t) - u(t) - \int_0^t \frac{u(s)}{(s+1)^2-1}ds, \quad t \in [0,1],\ t \ne \frac{1}{2}, \\ \Delta u(\frac{1}{2}) = \frac{1}{6}(u(\frac{1}{2})), \\ u(0) - 2\int_0^1 u(s)ds = -u(1). \end{gathered} \tag{4.2}$$

It is easy to check that $\alpha_0 = 1 - t$ and $\beta_0 = 0$ are lower and upper solutions of (4.2) respectively, and satisfying $\alpha_0 > \beta_0$. Meanwhile, problem (4.2) satisfies all the conditions of Theorem 3.4. Thus, we can apply the quasilinesrization method to find two monotone sequences $\{\alpha_n\}, \{\beta_n\}$ converging uniformly to the unique solution of (4.2).

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