EXISTENCE AND MULTICIPLICITY OF POSITIVE SOLUTIONS TO SYSTEMS OF NONLINEAR HAMMERSTEIN INTEGRAL EQUATIONS

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Abstract. This article studies the existence and multiplicity of component-wise positive solutions for systems of nonlinear Hammerstein integral equations. In this system one nonlinear term is uniformly superlinear or uniformly sublinear, and the other is locally uniformly superlinear or locally uniformly sublinear. Discussions are undertaken by means of the fixed point index theory in cones. As applications, we show the existence and multiplicity of component-wise positive solutions for systems of second-order ordinary differential equations with the Dirichlet boundary value conditions and mixed boundary value conditions, respectively.

1. Introduction

We consider the nonlinear Hammerstein integral equations:

\[ u(x) = \int_{\Omega} k_1(x, y) f_1(y, u(y), v(y)) \, dy, \quad x \in \overline{\Omega}, \]
\[ v(x) = \int_{\Omega} k_2(x, y) f_2(y, u(y), v(y)) \, dy, \quad x \in \overline{\Omega}, \] (1.1)

where \( \Omega \subset \mathbb{R}^n \) is a bounded domain, \( k_i \in C(\overline{\Omega} \times \overline{\Omega}, \mathbb{R}^+) \), \( f_i \in C(\overline{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+) \) \( (i = 1, 2) \) and \( \mathbb{R}^+ = [0, +\infty) \). In the past decades, considerable attention has been drawn to the study of the existence of nontrivial solutions of nonlinear Hammerstein integral equations \[2, 3, 4, 12, 13, 14, 16\], especially in component-wise positive solutions for system (1.1) \[11, 15, 18, 19, 20, 22, 23, 24\]. A survey of the existing results in the literature was presented in \[18\]. The existence of nontrivial solutions for systems of perturbed Hammerstein integral equations and Hammerstein integral equations with singularities was established in \[1, 12, 11, 15, 18, 19, 23\]. For the systems of second-order ordinary differential equations, some results on the existence and multiplicity of component-wise positive solutions were derived by applying the fixed point index theory in cones \[5, 7, 6, 8, 9\].

In this article, we are concerned with the existence and multiplicity of component-wise positive solutions for system (1.1), in which one nonlinear term is uniformly superlinear or uniformly sublinear and the other is locally uniformly superlinear or...
Then we say that \( f(x, u, v) \) is one positive solution to system (1.1), if \((u, v) \in [C^+(\overline{\Omega}) \times C^+(\overline{\Omega})] \setminus \{(0, 0)\} \). We say that \((u, v)\) is one component-wise positive solution to system (1.1), if \((u, v) \in [C^+(\overline{\Omega}) \setminus \{0\}] \times [C^+(\overline{\Omega}) \setminus \{0\}] \) satisfies system (1.1).

In this study, we suppose that the kernel functions \( k_i(x, y) (i = 1, 2) \) satisfy the following three conditions:

(i) \( k_i(x, y) = k_i(y, x) \), for all \( x, y \in \overline{\Omega} \);

(ii) there exist \( p_i \in C(\overline{\Omega}) \) and \( 0 \leq p_i(x) \leq 1 \) such that \( k_i(x, y) \geq p_i(x)k_i(z, y) \), for all \( x, y, z \in \overline{\Omega} \); and

(iii) \( \max_{x \in \Omega} \int_{\Omega} k_i(x, y)p_i(y) \, dy \) is positive.

**Proposition 1.2.** Let \( B_i : C(\overline{\Omega}) \rightarrow C(\overline{\Omega}) \) be defined by

\[
B_iu(x) = \int_{\Omega} k_i(x, y)u(y) \, dy \quad (i = 1, 2).
\]

Then the spectral radius of \( B_i \), denoted by \( r(B_i) \), is positive.

**Proof.** In view of definition of \( B_i \) and the given conditions (i)–(iii) about \( k_i \), we have

\[
B_i p_i(x) = \int_{\Omega} k_i(x, y)p_i(y) \, dy \geq p_i(x) \int_{\Omega} k_i(z, y)p_i(y) \, dy, \quad x \in \Omega,
\]

so

\[
B_i p_i(x) \geq p_i(x)\|B_i p_i\|, \quad x \in \Omega,
\]

where \( \|B_i p_i\| = \max_{x \in \Omega} \int_{\Omega} k_i(x, y)p_i(y) \, dy > 0 \).

By mathematical induction, we see that \( B_i^n p_i(x) \geq p_i(x)\|B_i p_i\|^n \) for \( x \in \overline{\Omega} \) and \( n \in \mathbb{N} \), thus \( \|B_i^n\| \geq \|B_i p_i\|^n \). Using the formula of spectral radius, one deduces that

\[
r(B_i) = \lim_{n \to \infty} \|B_i^n\|^{1/n} \geq \|B_i p_i\| > 0.
\]

**Definition 1.3.** Suppose that \( f_1 \) and \( f_2 \) in system (1.1) satisfy the following two hypotheses, respectively:

\[\text{(H1)} \quad \lim_{u \to 0^+} \sup_{x \in \Omega} \frac{f_1(x, u, v)}{u} < \frac{1}{r(B_1)} < \lim_{u \to +\infty} \inf_{x \in \Omega} \frac{f_1(x, u, v)}{u} \quad \text{uniformly w.r.t.} \ v \in \mathbb{R}^+;\]

\[\text{(H2)} \quad \lim_{v \to 0^+} \inf_{x \in \Omega} \frac{f_2(x, u, v)}{v} > \frac{1}{r(B_2)} > \lim_{v \to +\infty} \sup_{x \in \Omega} \frac{f_2(x, u, v)}{v} \quad \text{uniformly w.r.t.} \ u \in [0, M], \text{ where } M \in \mathbb{R}^+ \text{ is arbitrary.}\]

Then we say that \( f_1 \) is uniformly superlinear at both ends (i.e., \( u = 0, +\infty \)) with respect to \( v \), and that \( f_2 \) is locally uniformly sublinear at both ends (i.e., \( v = 0, +\infty \)) with respect to \( u \).

**Definition 1.4.** Suppose that \( f_1 \) and \( f_2 \) in system (1.1) satisfy the following two hypotheses, respectively:
(H3) \[
\liminf_{u \to 0^+} \min_{x \in \Omega} \frac{f_1(x, u, v)}{u} > \frac{1}{r(B_1)} > \limsup_{u \to +\infty} \max_{x \in \Omega} \frac{f_1(x, u, v)}{u}
\]
uniformly w.r.t. \(v \in \mathbb{R}^+\);

(H4) \[
\limsup_{v \to 0^+} \max_{x \in \Omega} \frac{f_2(x, u, v)}{v} < \frac{1}{r(B_2)} < \liminf_{v \to +\infty} \min_{x \in \Omega} \frac{f_2(x, u, v)}{v}
\]
uniformly w.r.t. \(u \in [0, M]\), where \(M \in \mathbb{R}^+\) is arbitrary.

Then we say that \(f_1\) is uniformly sublinear at both ends (i.e., \(u = 0, +\infty\)) with respect to \(v\), and that \(f_2\) is locally uniformly superlinear at both ends (i.e., \(v = 0, +\infty\)) with respect to \(u\).

We now summarize our main results regarding the case that the nonlinear term is so-called “super-sublinear”.

**Theorem 1.5.** Assume that \(f_1\) is uniformly superlinear at \(u = 0\) and \(u = +\infty\) with respect to \(v\), and that \(f_2\) is locally uniformly sublinear at \(v = 0\) and \(v = +\infty\) with respect to \(u\). Then system (1.1) has at least one component-wise positive solution.

In particular, when \(f_1\) and \(f_2\) are independent of \(v\) and \(u\), respectively, Theorem 1.5 incorporates into the well-known results on the nonlinear Hammerstein integral equations.

**Corollary 1.6** ([21]). If \(h_1, h_2 \in C(\Omega \times \mathbb{R}^+, \mathbb{R}^+)\), and satisfy the following two conditions:

(H1\*) \[
\limsup_{u \to 0^+} \max_{x \in \Omega} \frac{h_1(x, u)}{u} < \frac{1}{r(B_1)} < \liminf_{u \to +\infty} \min_{x \in \Omega} \frac{h_1(x, u)}{u}
\]
(i.e., superlinear case);

(H2\*) \[
\liminf_{u \to 0^+} \min_{x \in \Omega} \frac{h_2(x, u)}{u} > \frac{1}{r(B_2)} > \limsup_{u \to +\infty} \max_{x \in \Omega} \frac{h_2(x, u)}{u}
\]
(i.e., sublinear case),

then the integral equation
\[
u(x) = \int_{\Omega} k_i(x, y) h_i(y, u(y))\,dy, \quad x \in \Omega \text{ for } i = 1, 2,
\]
has at least one positive solution.

The “sub-superlinear” case is different from the “super-sublinear” case, since the uniformly sublinear term \(f_1\) needs to be controlled at infinity for a priori estimates of the solution component \(u\). For this purpose, we impose the condition (H5) in the following theorem.

**Theorem 1.7.** Assume that \(f_2\) is locally uniformly superlinear at \(v = 0\) and \(v = +\infty\) with respect to \(u\), and \(f_1\) is uniformly sublinear at \(u = 0\) and \(u = +\infty\) with respect to \(v\) and satisfies

(H5) \[
\limsup_{u \to +\infty} \max_{x \in \Omega} f_1(x, u, v) = g(u) \text{ uniformly with respect to } u \in [0, M]
\]
(for all \(M > 0\)), where \(g\) is a locally bounded function.

Then system (1.1) has at least one component-wise positive solution.
From the proofs of Theorems 1.5 and 1.7 presented in Section 3, it is not difficult to derive the following results immediately on the mixed case of “super-superlinear” and “sub-sublinear”.

**Theorem 1.8.** Assume that \( f_1 \) is uniformly superlinear at \( u = 0 \) and \( u = +\infty \) with respect to \( v \), and \( f_2 \) is locally uniformly superlinear at \( v = 0 \) and \( v = +\infty \) with respect to \( u \). Then system \((1.1)\) has at least one component-wise positive solution.

**Theorem 1.9.** Assume that \( f_1 \) is uniformly sublinear at \( u = 0 \) and \( u = +\infty \) with respect to \( v \) and satisfies condition \((H5)\), and \( f_2 \) is locally uniformly sublinear at \( v = 0 \) and \( v = +\infty \) with respect to \( u \). Then system \((1.1)\) has at least one component-wise positive solution.

**Definition 1.10.** For \( u^*, v^* \in C(\Omega) \), \((u^*, v^*)\) is said to be a strict upper solution of system \((1.1)\), provided it satisfies
\[
\begin{align*}
    u^*(x) &> \int_\Omega k_1(x,y)f_1(y,u^*(y),v^*(y))\,dy, \quad x \in \Omega, \\
v^*(x) &> \int_\Omega k_2(x,y)f_2(y,u^*(y),v^*(y))\,dy, \quad x \in \Omega.
\end{align*}
\]
(1.2)

Otherwise, \((u_*, v_*) \in C(\Omega) \times C(\Omega)\) is called a strict lower solution of system \((1.1)\), provided it satisfies the reverse of the above inequalities.

**Theorem 1.11.** Suppose that \( f_1 \) and \( f_2 \) satisfy the following three conditions:
\((H6)\) \( f_i(x, u, v) \leq f_i(x, u, \bar{v}) \) as \( x \in \Omega \), and \( u \leq \bar{u} \) and \( v \leq \bar{v} \), \( i = 1,2; \)
\((H7)\) \( \liminf_{x \to 0^+} \min_{x \in \Omega} \frac{f_i(x,u,0)}{u} > \frac{1}{r(B_1)} \) and \( \liminf_{x \to 0^+} \min_{x \in \Omega} \frac{f_i(x,0,v)}{v} > \frac{1}{r(B_2)} \);
\((H8)\) \( \liminf_{x \to \infty} \min_{x \in \Omega} \frac{f_i(x,u,0)}{u} > \frac{1}{r(B_1)} \) and \( \liminf_{x \to \infty} \min_{x \in \Omega} \frac{f_i(x,0,v)}{v} > \frac{1}{r(B_2)} \).

If system \((1.1)\) has a strict upper solution \((u^*, v^*)\), then system \((1.1)\) has at least two component-wise positive solutions.

As mentioned in [12, 13, 16], it is usually difficult to find a strict upper solution for such kind of systems. For the sake of applications to the associated systems of ordinary differential equations, we give the following useful lemma.

**Lemma 1.12.** If there exist constants \( M_1, M_2 > 0 \) such that \( f_1(x,u,v) < M_1 \) and \( f_2(x,u,v) < M_2 \) for all \((x,u,v) \in \Omega \times [0,\|\int_\Omega k_1(\cdot,y)\,dy\|][M_1] \times [0,\|\int_\Omega k_2(\cdot,y)\,dy\|][M_2], \) then system \((1.1)\) has a strict upper solution.

Proof. Let
\[
\begin{align*}
u(x) &= \int_\Omega M_1 k_1(x,y)\,dy, \quad x \in \Omega, \\
v(x) &= \int_\Omega M_2 k_2(x,y)\,dy, \quad x \in \Omega.
\end{align*}
\]
(1.3)

It follows that \( \|u\| \leq \|\int_\Omega k_1(\cdot,y)\,dy\|M_1 \) and \( \|v\| \leq \|\int_\Omega k_2(\cdot,y)\,dy\|M_2 \). Hence, \( f_1(x,u(x),v(x)) < M_1 \) and \( f_2(x,u(x),v(x)) < M_2 \) for \( x \in \Omega \).

We choose \((u^*, v^*) = (u + \varepsilon, v + \varepsilon), \) where \( \varepsilon > 0 \) is sufficiently small such that \( f_1(x,u(x) + \varepsilon, v(x) + \varepsilon) < M_1 \) and \( f_2(x,u(x) + \varepsilon, v(x) + \varepsilon) < M_2 \) for \( x \in \Omega \). Thus, it is easy to see that \((u^*, v^*)\) is a strict upper solution of system \((1.1)\). \(\square\)
This article is organized as follows. In Section 2, we present some preliminary results on the fixed point index. In Section 3, we show the proofs of our main results. Section 4 is dedicated to the existence and multiplicity of component-wise solutions to systems of second-order ordinary differential equations with the Dirichlet boundary conditions or mixed boundary conditions.

2. Preliminaries

Let \( C(\overline{\Omega}) \) be a Banach space with the maximum norm \( \|u\| = \max_{x \in \overline{\Omega}} |u(x)| \), and \( C^+(\overline{\Omega}) \) be a total cone of \( C(\overline{\Omega}) \). Choose bounded domains \( \Omega_i \subset \Omega \) \((i = 1, 2)\) such that

\[
\delta_i := \min_{x \in \overline{\Omega}_i} p_i(x) > 0,
\]

which is feasible by the hypotheses of \( p_i \). We now construct sub-cones and subsets as follows:

\[
K_i = \{ u \in C^+(\overline{\Omega}) \mid u(x) \geq \delta_i \|u\|, \forall x \in \overline{\Omega}_i \} \quad (i = 1, 2),
\]

\[
K_{r_i} = \{ u \in K_i \mid \|u\| < r_i \}, \quad \partial K_{r_i} = \{ u \in K_i \mid \|u\| = r_i \}, \quad \forall r_i > 0.
\]

From Proposition 1.2 and the Krein-Rutman theorem (see [17]), we know that \( r(B_i) \) is one of eigenvalues for \( B_i \) and there exist positive eigenfunctions corresponding to \( r(B_i) \).

**Lemma 2.1.** Let \( \psi_i(x) \) be the positive eigenfunctions of \( B_i \) corresponding to \( r(B_i) \) with \( \int_{\Omega} \psi_i(x) \, dx = 1 \). Then the following three statements are true.

(a) \( \int_{\Omega} \psi_i(x) u(x) \, dx \leq \|u\| \), for all \( u \in K_i \).

(b) \( \psi_i(x) \geq p_i(x) \|\psi_i\| \), for all \( x \in \overline{\Omega} \).

(c) There exist constants \( c_i > 0 \) such that \( \int_{\Omega} \psi_i(x) u(x) \, dx \geq c_i \|u\| \), for all \( u \in K_i \).

**Proof.** (a) Obviously,

\[
\int_{\Omega} \psi_i(x) u(x) \, dx \leq \int_{\Omega} \psi_i(x) \, dx \cdot \|u\| = \|u\|.
\]

(b) Notice that \( k_i(x, y) \geq p_i(x)k_i(z, y) \), for all \( x, y, z \in \overline{\Omega} \). So it is straightforward to obtain

\[
\int_{\Omega} k_i(x, y)\psi_i(y) \, dy \geq \int_{\Omega} p_i(x)k_i(z, y)\psi_i(y) \, dy, \quad \forall x, z \in \overline{\Omega},
\]

\[
r(B_i)\psi_i(x) \geq r(B_i)p_i(x)\psi_i(z), \quad \forall x, z \in \overline{\Omega}.
\]

This implies that \( \psi_i(x) \geq p_i(x)\|\psi_i\|, \forall x \in \overline{\Omega} \).

(c) It follows from (b) and the definition of \( K_i \) immediately. \( \square \)

For \( \tau \in I := [0, 1] \) and \( u, v \in C^+(\overline{\Omega}) \), we define the mappings \( T^\tau_1(\cdot, \cdot), T^\tau_2(\cdot, \cdot) : C^+(\overline{\Omega}) \times C^+(\overline{\Omega}) \to C^+(\overline{\Omega}) \) and \( T^+ : C^+(\overline{\Omega}) \times C^+(\overline{\Omega}) \to C^+(\overline{\Omega}) \times C^+(\overline{\Omega}) \) by

\[
T^\tau_1(u, v)(x) = \int_{\Omega} k_1(x, y)[\tau f_1(y, u(y), v(y)) + (1 - \tau)f_1(y, u(y), 0)] \, dy,
\]

\[
T^\tau_2(u, v)(x) = \int_{\Omega} k_2(x, y)[\tau f_2(y, u(y), v(y)) + (1 - \tau)f_2(y, 0, v(y))] \, dy,
\]

\[
T^+(u, v)(x) = (T^\tau_1(u, v)(x), T^\tau_2(u, v)(x)).
\]

**Lemma 2.2.** \( T^+ : K_1 \times K_2 \to K_1 \times K_2 \) is completely continuous.
Lemma 2.3 \([1, 12, 25]\) \(T_1(u, v)(x) = \int \Omega k_1(x, y)[\tau f_1(y, u(y), v(y)) + (1 - \tau)f_1(y, u(y), 0)] dy \geq p_1(x)\int \Omega k_1(z, y)[\tau f_1(y, u(y), v(y)) + (1 - \tau)f_1(y, u(y), 0)] dy = p_1(x)T_1^1(u, v)(z), \ \forall x, z \in \Omega,\)

which implies that \(T_1^1(u, v)(x) \geq \delta_1\|T_1^1(u, v)\|\) for \(x \in \overline{\Omega}_1\). Similarly, one can obtain \(T_2^2(u, v)(x) \geq \delta_2\|T_2^2(u, v)\|, \ x \in \overline{\Omega}_2\).

Hence, \(T^\tau(K_1 \times K_2) \subset K_1 \times K_2\). By Arzelà-Ascoli theorem, we know that \(T^\tau : K_1 \times K_2 \rightarrow K_1 \times K_2\) is completely continuous.

Lemma 2.3 \([1, 12, 25]\). Let \(E\) be a Banach space and \(P \subset E\) be a closed convex cone in \(E\). For \(r > 0\), let \(P_r = \{u \in P | \|u\| < r\}\) and \(\partial P_r = \{u \in P | \|u\| = r\}\). Suppose that \(A : P \rightarrow P\) is completely continuous. Then the following two statements are true.

(I) If \(\mu Au \neq u\) for every \(u \in \partial P_r\) and \(\mu \in (0, 1]\), then \(i(A, P_r, P) = 1\).

(II) Suppose that the mapping \(A\) satisfies the following two conditions:

(a) \(\inf_{u \in \partial P_r} \|Au\| > 0\); and

(b) \(\mu Au \neq u\) for every \(u \in \partial P_r\) and \(\mu > 1\).

Then \(i(A, P_r, P) = 0\).

Lemma 2.4 \([8, 9]\). Let \(X\) be a real Banach space, \(P_i \subset X\) be a closed convex cone, \(W_i\) be a bounded open subset of \(X\) with the boundary \(\partial W_i\) \((i = 1, 2)\), and \(P = P_1 \times P_2\) and \(W = W_1 \times W_2\). Assume that \(T : P \cap W \rightarrow P\) is completely continuous, and that there exist compactly continuous mappings \(A_i : P_i \cap W_i \rightarrow P_i\) and \(H : (P \cap W) \times (0, 1] \rightarrow P\) such that

(a) \(H(\cdot, 1) = T\) and \(H(\cdot, 0) = \mathbb{0}\), where \(A(u, v) := (A_1 u, A_2 v)\) and \((u, v) \in P \cap W;\)

(b) \(A_i u_i \neq u_i\), for all \(u_i \in P_i \cap \partial W_i\); and

(c) \(H(w, \tau) \neq w\), for all \((w, \tau) \in (P \cap \partial W) \times (0, 1]\).

Then we have \(i(T, P \cap W, P) = i(A_1, P_1 \cap W_1, P_1) \cdot i(A_2, P_2 \cap W_2, P_2)\).

3. PROOFS OF MAIN RESULTS

Proof of Theorem 1.3. Choose a bounded open set \(D = (K_{r_1} \setminus \overline{K_{r_1}}) \times (K_{r_2} \setminus \overline{K_{r_2}})\) in the product cone \(K_1 \times K_2\), where \(R_i > r_j > 0\) \((j = 1, 2)\) are to be determined such that the family of operators \(\{T^\tau\}_{\tau \in I}\) satisfies the sufficient condition of the homotopy invariance of the fixed point index on \(\partial D\). We will in turn determine \(r_1, R_1, r_2\) and \(R_2\) in the following process.

(A) From the uniformly superlinear assumption on \(f_1\) at \(u = 0\), there are \(\varepsilon \in (0, 1/r(B_1))\) and \(r_1 > 0\) such that

\[\tau f_1(x, u, v) + (1 - \tau)f_1(x, u, 0) \leq (1/r(B_1) - \varepsilon)u,\]  

for all \(x \in \overline{\Omega}, (u, v) \in [0, r_1] \times \mathbb{R}^+.\) We can obtain

\[\mu T_1^\tau(u, v) \neq u, \ \forall \mu \in (0, 1] \text{ and } (u, v) \in \partial K_{r_1} \times K_2.\]  

(3.2)
Otherwise, there exist \( \mu_0 \in (0, 1] \) and \( (u_0, v_0) \in \partial K_r \times K_2 \) such that \( \mu_0 T^\tau_1(u_0, v_0) = u_0 \). In combination with (3.1), it follows that

\[
u_0(x) \leq T^\tau_1(u_0, v_0)(x) \leq \int_{\Omega} k_1(x, y)(1/r(B_1) - \varepsilon)u_0(y) \, dy.
\]

Multiplying both sides of this inequality by \( \psi_1(x) \) and integrating it on \( \overline{\Omega} \) yields

\[
\int_{\Omega} \nu_0(x)\psi_1(x) \, dx \leq [1 - r(B_1)]\varepsilon \int_{\Omega} u_0(y)\psi_1(y) \, dy.
\]  

(3.3)

Since \( \int_{\Omega} \nu_0(x)\psi_1(x) dx > 0 \) and \( r(B_1) > 0 \), by (3.3) it gives \( 1 \leq 1 - r(B_1)\varepsilon \). This apparently leads to a contradiction.

(B) Because of the uniformly superlinear hypothesis on \( f_1 \) at \( u = +\infty \), there exist \( \varepsilon > 0 \) and \( m > 0 \) such that

\[
\tau f_1(x, u, v) + (1 - \tau)f_1(x, u, 0) \geq (1/r(B_1) + \varepsilon)u,
\]  

(3.4)

for all \( x \in \overline{\Omega} \), \( (u, v) \in [m, +\infty) \times \mathbb{R}^+ \). So we have

\[
\tau f_1(x, u, v) + (1 - \tau)f_1(x, u, 0) \geq (1/r(B_1) + \varepsilon)u - C_1,
\]  

(3.5)

for all \( x \in \overline{\Omega} \), \( (u, v) \in \mathbb{R}^+ \times \mathbb{R}^+ \), where \( C_1 = (1/r(B_1) + \varepsilon)m \).

Then there exists an \( R_1 > r_1 \) such that

\[
\mu T^\tau_1(u, v) \neq u \quad \text{and} \quad \inf_{u \in \partial K_{R_1}} \| T^\tau_1(u, v) \| > 0,
\]  

(3.6)

for all \( \mu \geq 1 \), \( (u, v) \in \partial K_{R_1} \times K_2 \).

If there exist \( (u_0, v_0) \in K_1 \times K_2 \) and \( \mu_0 \geq 1 \) such that \( u_0 = \mu_0 T^\tau_1(u_0, v_0) \), from (3.5) we deduce that

\[
u_0(x) \geq T^\tau_1(u_0, v_0)(x) \geq \int_{\Omega} k_1(x, y)(1/r(B_1) + \varepsilon)u_0(y) \, dy - C.
\]

It follows that

\[
\int_{\Omega} \nu_0(x)\psi_1(x) \, dx \geq (1 + r(B_1))\varepsilon \int_{\Omega} \nu_0(y)\psi_1(y) \, dy - C,
\]

which yields

\[
\int_{\Omega} \nu_0(x)\psi_1(x) \, dx \leq \frac{C}{r(B_1)\varepsilon}.
\]

In view of Lemma 2.1 (c), we obtain

\[
\| u_0 \| \leq \frac{C}{c_1 r(B_1)\varepsilon} =: R^*.
\]  

(3.7)

Thus, when \( R > R^* \), \( u \neq \mu T^\tau_1(u, v) \) holds for all \( (u, v) \in \partial K_R \times K_2 \) and \( \mu \geq 1 \). In addition, if \( R > m/\delta_1 \), by (3.4) we know that for all \( (u, v) \in \partial K_R \times K_2 \), it holds

\[
\| T^\tau_1(u, v) \| \geq \int_{\Omega} T^\tau_1(u, v)(x)\psi_1(x) \, dx
\]

\[
\geq \int_{\Omega} \int_{\Omega} k_1(y, x)(1/r(B_1) + \varepsilon)u(y) \, dy \psi_1(x) \, dx
\]

\[
\geq (1 + r(B_1))\varepsilon \int_{\Omega} u(y)\psi_1(y) \, dy
\]

\[
\geq (1 + r(B_1))\varepsilon \text{mes}(\Omega_1)\delta_1^2\| \psi_1 \| R.
\]
That is,
\[ \inf_{u \in \partial K_R} \| T^+_{11} (u, v) \| > 0. \]

Hence, we choose \( R_1 > \max \{ r_1, R^*, m/\delta \} \).

(C) Based on the locally uniformly sublinear assumption on \( f_2 \) at \( v = 0 \), there exist \( \varepsilon > 0 \) and \( r_2 > 0 \) such that
\[ \tau f_2(x, u, v) + (1 - \tau)f_2(x, 0, v) \geq (1/r(B_2) + \varepsilon)v, \]
for all \( x \in \bar{\Omega}, (u, v) \in [0, R_1] \times [0, r_2] \). Then we have
\[ \mu T^+_{22} (u, v) \neq v \quad \text{and} \quad \inf_{v \in \partial K_{r_2}} \| T^+_{22} (u, v) \| > 0, \]
for all \( \mu \geq 1, (u, v) \in \bar{K}_{R_1} \times \partial K_{r_2} \).

(D) Given the locally uniformly sublinear hypothesis on \( f_2 \) at \( v = +\infty \), there exist \( \varepsilon \in (0, 1/r(B_2)) \), \( n > 0 \) and \( C > 0 \) such that
\[ \tau f_2(x, u, v) + (1 - \tau)f_2(x, 0, v) \leq (1/r(B_2) - \varepsilon)v, \quad \forall x \in \bar{\Omega}, (u, v) \in [0, R_1] \times [n, +\infty) \]
and
\[ \tau f_2(x, u, v) + (1 - \tau)f_2(x, 0, v) \leq (1/r(B_2) - \varepsilon)v + C, \]
for all \( x \in \bar{\Omega}, (u, v) \in [0, R_1] \times \mathbb{R}^+ \).

As we did in the discussion of \((3.7)\), one can prove that if \( v_0 = \mu_0 T^+_{22} (u_0, v_0) \) for \( (u_0, v_0) \in \bar{K}_{R_1} \times K_2 \) and \( \mu_0 \in (0, 1) \), then
\[ \| v_0 \| \leq R' \overset{\text{def}}{=} \frac{C}{c_2 r(B_2) \varepsilon}. \]

Hence, we take \( R_2 > \max\{r_2, R'\} \), and have
\[ \mu T^+_{22} (u, v) \neq v, \quad \forall \mu \in (0, 1] \quad \text{and} \quad (u, v) \in \bar{K}_{R_1} \times \partial K_{R_2}. \]

Now, we choose an open set \( D = (K_{R_1} \setminus K_{r_j}) \times (K_{R_2} \setminus K_{r_j}) \). By using \((3.2), (3.6), (3.9)\) as well as \((3.13)\), we see that \( \{ T^+ \}_{\tau \in I} \) satisfies the sufficient conditions for the homotopy invariance of the fixed point index on \( \partial D \). By virtue of Lemmas \( 2.3 \) and \( 2.4 \), we have
\[ i(T^1, D, K_1 \times K_2) = \prod_{j=1}^{2} i(T^0_j, K_{R_j} \setminus K_{r_j}, K_j) \]
\[ = \prod_{j=1}^{2} [i(T^0_j, K_{R_j}, K_j) - i(T^0_j, K_{r_j}, K_j)] \]
\[ = (0 - 1) \times (1 - 0) = -1. \]

Consequently, system \((1.1)\) has at least one component-wise positive solution. \( \square \)

Proof of Theorem \( 1.7 \) Similar to the arguments described in the proof of Theorem \( 1.5 \), we separate our discussions into four steps and determine \( r_1, R_1, r_2, R_2 \) one by one.

Step 1. Given the uniformly sublinear assumption on \( f_1 \) at \( u = 0 \), there are \( \varepsilon > 0 \) and \( r_1 > 0 \) such that
\[ \tau f_1(x, u, v) + (1 - \tau)f_1(x, 0, v) \geq (1/r(B_1) + \varepsilon)u, \]
for all $x \in \overline{\Omega}$, $(u, v) \in [0, r_1] \times \mathbb{R}^+$. We can obtain
\[
\mu T_1^\tau(u, v) \neq u, \quad \min_{u \in \partial K_{r_1}}\|T_1^\tau(u, v)\| > 0, \quad \forall \mu \geq 1, (u, v) \in \partial K_{r_1} \times K_2. \tag{3.15}
\]
Otherwise, suppose that there exist $\mu_0 \geq 1$ and $(u_0, v_0) \in \partial K_{r_1} \times K_2$ such that
\[
\mu_0 T_1^\tau(u_0, v_0) = u_0.
\]
It follows from $(3.14)$ that
\[
u_0(x) \geq T_1^\tau(u_0, v_0)(x) \geq \int_{\Omega} k_1(x, y)(1/r(B_1) + \varepsilon) u_0(y) \, dy.
\]
Multiplying both sides of this inequality by $\psi_1(x)$ and integrating it on $\overline{\Omega}$, we obtain
\[
\int_{\Omega} \nu_0(x) \psi_1(x) \, dx \geq (1 + r(B_1)\varepsilon) \int_{\Omega} u_0(y) \psi_1(y) \, dy. \tag{3.16}
\]
Since $\int_{\Omega} \nu_0(x) \psi_1(x) \, dx > 0$ and $r(B_1) > 0$, by $(3.16)$ it gives $1 \geq 1 + r(B_1)\varepsilon$, which is obviously a contradiction.

By $(3.14)$ we know that for all $(u, v) \in \partial K_{r_1} \times K_2$, it holds
\[
\|T_1^\tau(u, v)\| \geq \int_{\Omega} T_1^\tau(u, v)(x) \psi_1(x) \, dx \geq \int_{\Omega} \int_{\Omega} k_1(y, x)(1/r(B_1) + \varepsilon) u(y) \, dy \psi_1(x) \, dx \geq (1 + r(B_1)\varepsilon) \int_{\Omega} u(y) \psi_1(y) \, dy \geq c_1[1 + r(B_1)\varepsilon] r_1.
\]
Clearly, this implies
\[
\min_{u \in \partial K_{r_1}}\|T_1^\tau(u, v)\| > 0.
\]

**Step 2.** Under the uniformly sublinear hypothesis on $f_1$ at $u = +\infty$ and condition (H5), there exist $\varepsilon \in (0, 1/r(B_1))$, $m > 0$ and $C > 0$ such that
\[
\tau f_1(x, u, v) + (1 - \tau) f_1(x, u, 0) \leq (1/r(B_1) - \varepsilon) u, \tag{3.17}
\]
for all $x \in \overline{\Omega}$, $(u, v) \in [m, +\infty) \times \mathbb{R}^+$, and
\[
\tau f_1(x, u, v) + (1 - \tau) f_1(x, u, 0) \leq (1/r(B_1) - \varepsilon) u + C, \tag{3.18}
\]
for all $x \in \overline{\Omega}$, $(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+$.

Using $(3.18)$ and processing in like manner as we did in the discussion of $(3.7)$, one can see that if $u_0 = \mu_0 T_1^\tau(u_0, v_0)$ for $(u_0, v_0) \in K_2 \times K_2$ and $\mu_0 \in (0, 1]$, then
\[
\|u_0\| \leq R^* = \frac{C}{c_1 r(B_1)\varepsilon}. \tag{3.19}
\]
Choosing $R_1 > \max\{r_1, R^*\}$, we then obtain
\[
\mu T_1^\tau(u, v) \neq u, \quad \forall \mu \in (0, 1] \text{ and } (u, v) \in \partial K_{r_1} \times K_2. \tag{3.20}
\]

**Step 3.** By the locally uniformly superlinear assumption on $f_2$ at $v = 0$, there are $\varepsilon \in (0, 1/r(B_2))$ and $r_2 > 0$ such that
\[
\tau f_2(x, u, v) + (1 - \tau) f_2(x, 0, v) \leq (1/r(B_2) - \varepsilon) v, \tag{3.21}
\]
for all $x \in \overline{\Omega}$, $(u, v) \in [0, R_1] \times [0, r_2]$. 

We claim that
\[ \mu T_2^+ (u, v) \neq v, \quad \forall \mu \in (0, 1] \text{ and } (u, v) \in K_{R_1} \times \partial K_{R_2}. \] (3.22)

Otherwise, there exist \( \mu_0 \in (0, 1] \) and \((u_0, v_0) \in K_{R_1} \times \partial K_{R_2} \) so that \( \mu_0 T_2^+ (u_0, v_0) = v_0 \). In combination with (3.21), it follows that
\[ v_0(x) \leq T_2^+(u_0, v_0)(x) \leq \int_{\Omega} k_2(x, y)(1/r(B_2) - \varepsilon)v_0(y) \, dy. \]

It follows that
\[ \int_{\Omega} v_0(x)\psi_2(x) \, dx \leq (1 - r(B_2)\varepsilon)\int_{\Omega} v_0(y)\psi_2(y) \, dy. \] (3.23)

Since \( \int_{\Omega} v_0(x)\psi_2(x) \, dx > 0 \) and \( r(B_2) > 0 \), it follows from (3.23) that \( 1 \leq 1 - r(B_2)\varepsilon \), which yields a contradiction.

**Step 4.** By utilizing the locally uniformly superlinear hypothesis on \( f_2 \) at \( v = +\infty \), there exist \( \varepsilon > 0 \) and \( n > 0 \) such that
\[ \tau f_2(x, u, v) + (1 - \tau) f_2(x, 0, v) \geq (1/r(B_2) + \varepsilon)v, \] (3.24)
for all \( x \in \Omega, (u, v) \in [0, R_1] \times [n, +\infty) \). It gives
\[ \tau f_2(x, u, v) + (1 - \tau) f_2(x, 0, v) \geq (1/r(B_2) + \varepsilon)v - C_2, \] (3.25)
for all \( x \in \Omega, (u, v) \in [0, R_1] \times \mathbb{R}^+ \), where \( C_2 = (1/r(B_2) + \varepsilon)n \).

We now prove that there exists an \( R_2 > r_2 \) such that
\[ \mu T_2^+ (u, v) \neq v \quad \text{and} \quad \inf_{v \in \partial K_{R_2}} \| T_2^+ (u, v) \| > 0, \] (3.26)
for all \( \mu \geq 1 \), \((u, v) \in K_{R_1} \times \partial K_{R_2} \). If there are \((u_0, v_0) \in K_{R_1} \times K_2 \) and \( \mu_0 \geq 1 \) such that \( v_0 = \mu_0 T_2^+(u_0, v_0) \), then it follows from (3.25) that
\[ v_0(x) \geq T_2^+(u_0, v_0)(x) \geq \int_{\Omega} k_2(x, y)(1/r(B_2) + \varepsilon)v_0(y) \, dy - C. \] (3.27)

Moreover,
\[ \int_{\Omega} v_0(x)\psi_2(x) \, dx \geq (1 + r(B_2)\varepsilon)\int_{\Omega} v_0(y)\psi_2(y) \, dy - C. \]

It further leads to
\[ \int_{\Omega} v_0(x)\psi_2(x) \, dx \leq \frac{C}{r(B_2)\varepsilon}. \]

In view of Lemma 2.1 (c), we know that
\[ \| v_0 \| \leq R' = \frac{C}{c_2 r(B_2)\varepsilon}. \] (3.28)

When \( R > R' \), \( v \neq \mu T_2^+ (u, v) \) holds for all \((u, v) \in K_{R_1} \times \partial K_{R_2} \) and \( \mu \geq 1 \). In addition, if \( R > n/\delta_2 \), then by (3.24) we know that for all \((u, v) \in K_{R_1} \times \partial K_{R_2} \), it holds
\[ \| T_2^+(u, v) \| \geq \int_{\Omega} T_2^+(u, v)(x)\psi_2(x) \, dx \]
\[ \geq \int_{\Omega} \int_{\Omega} k_2(y, x)(1/r(B_2) + \varepsilon)v(y) \, dy \psi_2(x) \, dx \]
\[ \geq (1 + r(B_2)\varepsilon) \int_{\Omega} v(y)\psi_2(y) \, dy \]
\[ \geq (1 + r(B_2)\varepsilon)\text{mes}(\overline{D}_2)\delta_2^2\|\psi_2\|R. \]

This indicates that
\[ \inf_{v \in \partial K_R} \|T^*_2(u, v)\| > 0. \]

Hence, we choose \( R > \max\{r_2, R', n/\delta_2\} \).

Based on \( 3.15 \), \( 3.20 \), \( 3.22 \) and \( 3.26 \), one can easily verify that \( \{T^*_\tau\}_{\tau \in I} \) satisfies the sufficient conditions for the homotopy invariance of the fixed point index on \( \partial D \), where \( D = (K_{R_1} \setminus \overline{K_{r_1}}) \times (K_{R_2} \setminus \overline{K_{r_2}}) \). It then follows Lemmas \( 2.3 \) and \( 2.4 \) that
\[
i(T^1, D, K_1 \times K_2) = \prod_{j=1}^{2} i(T^0_j, K_{R_j} \setminus \overline{K_{r_j}}, K_j)
= \prod_{j=1}^{2} [i(T^0_j, K_{R_j}, K_j) - i(T^0_j, K_{r_j}, K_j)]
= (1 - 0) \times (0 - 1) = -1.
\]

Hence, system \( 1.1 \) has at least one component-wise positive solution.

**Proof of Theorem 1.11** Let \( \varepsilon > 0 \), and
\[
W_1 = \{u \in K_1 : -\varepsilon < u(x) < u^*(x), x \in \overline{\Omega}\},
W_2 = \{v \in K_2 : -\varepsilon < v(x) < v^*(x), x \in \overline{\Omega}\},
W = W_1 \times W_2.
\]

Then, \( W \) is an open subset of \( K_1 \times K_2 \) and \( (0, 0) \in W \).

We first show that
\[ T^\tau(u, v) \neq (u, v), \forall (\tau, u, v) \in [0, 1] \times \partial W. \quad (3.29) \]

In fact, if there exists some \( (\tau_0, u_0, v_0) \in [0, 1] \times \partial W \) such that \( T^{\tau_0}(u_0, v_0) = (u_0, v_0) \), then \( (u_0, v_0) \in \partial W_1 \times W_2 \) or \( (u_0, v_0) \in W_1 \times \partial W_2 \). Without loss of generality, we assume that \( (u_0, v_0) \in \partial W_1 \times W_2 \), then \( v_0 \leq v^* \), \( u_0 \leq u^* \), and \( u_0(x_0) = u^*(x_0) \) for some \( x_0 \in \Omega \). Hence, by \( (F_1) \), in view of the definitions of \( T^1 \) and the strict upper solution, it yields
\[
u^*(x_0) = u_0(x_0) = T^{\tau_0}(u_0, v_0)(x_0) \leq T^0_1(u^*, v^*)(x_0) \leq T^1_1(u^*, v^*)(x_0) < u^*(x_0).
\]

This is obviously a contradiction.

To prove that \( T^0_1(u, v) \neq \mu u \), for all \( \mu \geq 1 \) and all \( u \in \partial W_1 \), we let \( \mu \geq 1 \) and \( u \in \partial W_1 \). Then \( u \leq u^* \), and there is an \( x_0 \in \Omega \) such that \( u(x_0) = u^*(x_0) \). Thus it follows that
\[
T^0_1(u, v)(x_0) = T^0_1(u, 0)(x_0) \leq T^0_1(u^*, 0)(x_0) < u^*(x_0) = u(x_0) \leq \mu u(x_0).
\]

Similarly, we find that \( T^0_2(u, v) \neq \mu v \) for all \( \mu \geq 1 \) and all \( v \in \partial W_2 \). It then follows from Lemma \( 2.3 \) that
\[ i(T^0_1, W_1, K_1) = i(T^0_2, W_2, K_2) = 1. \]

In view of \( 3.6 \), \( 3.9 \), \( 3.15 \) and \( 3.26 \), we choose \( r_1, r_2 \) small enough and \( R_1, R_2 \) large enough such that
\[
0 < r_1 < \min_{x \in \overline{\Omega}} u^*(x) \leq \|u^*\| < R_1,
0 < r_2 < \min_{x \in \overline{\Omega}} v^*(x) \leq \|v^*\| < R_2.
\]
Combining (3.29) and (3.30), by Lemmas 2.3 and 2.4, we obtain
\[
i(T^1, (W_1 \backslash K_r_1) \times (W_2 \backslash K_r_2), K_1 \times K_2)
\]
\[
= \prod_{j=1}^{2} i(T^0_j, W_j \backslash K_r_j, K_j)
\]
\[
= \prod_{j=1}^{2} [i(T^0_j, W_j, K_j) - i(T^0_j, K_r_j, K_j)]
\]
\[
= (1 - 0) \times (1 - 0) = 1, \quad (3.31)
\]
and further deduce that
\[
i(T^1, D, (W_1 \backslash K_r_1) \times (W_2 \backslash K_r_2), K_1 \times K_2)
\]
\[
= i(T^1, D, K_1 \times K_2) - i(T^1, (W_1 \backslash K_r_1) \times (W_2 \backslash K_r_2), K_1 \times K_2)
\]
\[
= \prod_{j=1}^{2} [i(T^0_j, K_{R_j}, K_j) - i(T^0_j, K_r_j, K_j)] - 1
\]
\[
= 0 - 1 = -1. \quad (3.32)
\]
Thus, system (1.1) has at least two component-wise positive solutions. \(\square\)

4. Applications

Consider the existence and multiplicity of component-wise positive solutions for the following system of second-order ordinary differential equations
\[
-u''(x) = f_1(x, u(x), v(x)), \quad x \in (0, 1),
\]
\[
-v''(x) = f_2(x, u(x), v(x)), \quad x \in (0, 1), \quad (4.1)
\]
subject to the Dirichlet boundary conditions
\[
u(0) = u(1) = v(0) = v(1) = 0, \quad (4.2)
\]
or the mixed boundary conditions
\[
u(0) = u(1) = v(0) = v'(1) = 0, \quad (4.3)
\]
where \(f_1, f_2 \in C([0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+).\)

4.1. Dirichlet boundary value problem.

**Theorem 4.1.** Assume that \(f_1\) and \(f_2\) satisfy the following two conditions:

(H9)
\[
\limsup_{u \to 0^+} \max_{x \in [0, 1]} \frac{f_1(x, u, v)}{u} < \pi^2 < \liminf_{u \to +\infty} \min_{x \in [0, 1]} \frac{f_1(x, u, v)}{u}
\]
uniformly w.r.t. \(v \in \mathbb{R}^+;\)

(H10)
\[
\liminf_{v \to 0^+} \min_{x \in [0, 1]} \frac{f_2(x, u, v)}{v} > \pi^2 > \limsup_{v \to +\infty} \max_{x \in [0, 1]} \frac{f_2(x, u, v)}{v}
\]
uniformly w.r.t. \(u \in [0, M],\) where \(M \in \mathbb{R}^+\) is arbitrary.

Then problem (4.1)–(4.2) has at least one component-wise positive solution.
Proof. It is clear that system (4.1) subject to (4.2) is equivalent to the system of nonlinear Hammerstein integral equations

\[ u(x) = \int_0^1 k_1(x, y)f_1(y, u(y), v(y)) \, dy, \quad x \in [0, 1], \]
\[ v(x) = \int_0^1 k_2(x, y)f_2(y, u(y), v(y)) \, dy, \quad x \in [0, 1], \]

where

\[ k_1(x, y) = k_2(x, y) = \begin{cases} x(1 - y), & x \leq y, \\ y(1 - x), & y \leq x. \end{cases} \]

It is easy to verify that the kernel functions \( k_1 \) and \( k_2 \) satisfy all conditions (i)–(iii).

According to Theorem 1.5, we only need to prove that \( r(B_1) = r(B_2) = \pi^{-2} \).

To this end, it suffices to observe that the following linear eigenvalue problem

\[ -u''(x) = \lambda u(x), \quad x \in (0, 1), \]
\[ u(0) = u'(1) = 0, \]

has the minimal eigenvalue \( \lambda_1 = \pi^2 \).


**Theorem 4.2.** Assume that \( f_1 \) satisfies condition (H9), and that \( f_2 \) satisfies (H11)

\[ \liminf_{v \to 0^+} \min_{x \in [0, 1]} \frac{f_2(x, u, v)}{v} > \frac{\pi^2}{4} > \limsup_{v \to +\infty} \max_{x \in [0, 1]} \frac{f_2(x, u, v)}{v} \]

uniformly w.r.t. \( u \in [0, M] \), where \( M \in \mathbb{R}^+ \) is arbitrary.

Then system (4.1) with the mixed boundary condition (4.3) has at least one component-wise positive solution.

Proof. We know that system (4.1) subject to (4.3) is equivalent to the system of nonlinear Hammerstein integral equations

\[ u(x) = \int_0^1 k_1(x, y)f_1(y, u(y), v(y)) \, dy, \quad x \in [0, 1], \]
\[ v(x) = \int_0^1 k_2(x, y)f_2(y, u(y), v(y)) \, dy, \quad x \in [0, 1], \]

where

\[ k_1(x, y) = \begin{cases} x(1 - y), & x \leq y, \\ y(1 - x), & y \leq x, \end{cases} \]
\[ k_2(x, y) = \begin{cases} x, & x \leq y, \\ y, & y \leq x. \end{cases} \]

It is easy to verify that two kernel functions \( k_1 \) and \( k_2 \) satisfy conditions (i)–(iii).

From the proof of Theorem 4.1 we find that \( r(B_1) = \pi^{-2} \). Since \( \mu_1 = \pi^2/4 \) is the minimal eigenvalue of the linear eigenvalue problem

\[ -v''(x) = \mu v(x), \quad x \in (0, 1), \]
\[ v(0) = v'(1) = 0, \]

one can derive that \( r(B_2) = 4\pi^{-2} \). Hence, we obtain the desired result according to Theorem 1.5. □
Remark 4.3. There are many types of functions $f_1$ and $f_2$ satisfying conditions given in Theorems 4.1 and 4.2. For instance, let $f_1(x,u,v) = \max\{u, u^2\}(1 + \tan^{-1}v)$ and $f_2(x,u,v) = \sqrt{ve^v}$. It is easy to see that $f_1$ satisfies condition (H9), and $f_2$ satisfies conditions (H10) and (H11). Thus system (4.1) subject to (4.2) or (4.3) has at least one component-wise positive solution.

Theorem 4.4. Suppose that $f_1$ and $f_2$ satisfy the following three conditions:

(H12) $f_i(x, u, v) \leq f_i(x, \bar{u}, \bar{v})$ as $x \in [0, 1]$, $u \leq \bar{u}$ and $v \leq \bar{v}$, for $i = 1, 2$;

(H13) $\lim_{u \to 0^+} \min_{x \in [0,1]} \frac{f_1(x, u, 0)}{u} > \pi^2$ and $\lim_{v \to 0^+} \min_{x \in [0,1]} \frac{f_2(x, 0, v)}{v} > \frac{\pi^2}{4}$;

(H14) $\lim_{u \to +\infty} \min_{x \in [0,1]} \frac{f_1(x, u, 0)}{u} > \pi^2$ and $\lim_{v \to +\infty} \min_{x \in [0,1]} \frac{f_2(x, 0, v)}{v} > \frac{\pi^2}{4}$.

In addition, if there exist constants $M_1, M_2 > 0$ such that $f_1(x, u, v) < M_1$ and $f_2(x, u, v) < M_2$ for all $(x, u, v) \in [0, 1] \times [0, M_1] \times [0, M_2]$, then system (4.1) subject to (4.3) has at least two component-wise positive solutions.

Proof. Similar to the arguments in the proof of Theorem 4.2 one can derive that $r(B_1) = \pi^{-2}$ and $r(B_2) = 4\pi^{-2}$. In addition, it is straightforward to calculate that

$$\max_{x \in [0,1]} \int_{\Omega} k_1(x, y) dy = 1/8 \quad \text{and} \quad \max_{x \in [0,1]} \int_{\Omega} k_2(x, y) dy = 1/2.$$ 

The desired result follows Theorem 1.11 and Lemma 1.12. \qed

Remark 4.5. As an illustration of Theorem 4.4 we consider

$$f_1(x, u, v) = \begin{cases} \frac{\sqrt{2}}{2} \tan^{-1}(v + 1), & u \in [0, 1], \\ \left(\frac{2\pi}{2} - 3\right) \tan^{-1}(v + 1), & u \in (1, 2), \\ u^2 \tan^{-1}(v + 1), & u \geq 2, \end{cases}$$

$$f_2(x, u, v) = \begin{cases} \frac{\sqrt{2}}{4} e^u, & v \in [0, 1], \\ \left(1 + \frac{\pi}{2} - \frac{7}{2}\right) e^u, & v \in (1, 2), \\ u^2 e^u, & v \geq 2. \end{cases}$$

It is easy to see that $f_1$ and $f_2$ satisfy all the conditions given in Theorem 4.4 by choosing $M_1 = M_2 = 1$. Thus system (4.1) subject to (4.3) has at least two component-wise positive solutions.

Remark 4.6. Note that Theorems 1.7 and 1.9 can also be applied to system (4.1) subject to conditions (4.2) or (4.3) in a similar manner. Here, we omit the details for the corresponding results on the existence and multiplicity of component-wise positive solutions for such kind of systems. It is worth mentioning that the ideas described herein are also applicable to nonlinear systems of Hammerstein integral equations with weighted functions. We will consider this problem in a subsequent work.

Remark 4.7. It is notable that assumptions (H12), (H13) and (H14) are also satisfied for coupling functions $f_i(x,u,v)$ which are negative for small values of $u$ or/and $v$. Such type of couplings often arises in Chemical Engineering, see [10] and...
references therein, and leads to solutions such that may be zero in a subset of the interval \((0, 1)\), or at least may have zero derivative in some of the boundary points (the so called “flat solutions”).

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