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EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS TO SYSTEMS OF NONLINEAR HAMMERSTEIN INTEGRAL EQUATIONS

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Abstract. This article studies the existence and multiplicity of component-wise positive solutions for systems of nonlinear Hammerstein integral equations. In this system one nonlinear term is uniformly superlinear or uniformly sublinear, and the other is locally uniformly superlinear or locally uniformly sublinear. Discussions are undertaken by means of the fixed point index theory in cones. As applications, we show the existence and multiplicity of component-wise positive solutions for systems of second-order ordinary differential equations with the Dirichlet boundary value conditions and mixed boundary value conditions, respectively.

1. Introduction

We consider the nonlinear Hammerstein integral equations:

$$
\begin{aligned}
u(x) &= \int_{\overline{\Omega}} k_1(x,y) f_1(y,u(y),v(y))\,\mathrm{d}y, \quad x \in \overline{\Omega},\\
v(x) &= \int_{\overline{\Omega}} k_2(x,y) f_2(y,u(y),v(y))\,\mathrm{d}y, \quad x \in \overline{\Omega},
\end{aligned}
\tag{1.1}
$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $k_i \in C(\overline{\Omega}\times\overline{\Omega}, \mathbb{R}^+)$, $f_i \in C(\overline{\Omega}\times\mathbb{R}^+\times\mathbb{R}^+, \mathbb{R}^+)$ $(i = 1, 2)$ and $\mathbb{R}^+ = [0, +\infty)$. In the past decades, considerable attention has been drawn to the study of the existence of nontrivial solutions of nonlinear Hammerstein integral equations [2, 3, 4, 12, 13, 14, 16], especially in component-wise positive solutions for system (1.1) [11, 15, 18, 19, 20, 22, 23, 24]. A survey of the existing results in the literature was presented in [18]. The existence of nontrivial solutions for systems of perturbed Hammerstein integral equations and Hammerstein integral equations with singularities was established in [1, 12, 11, 15, 17, 18, 19, 23]. For the systems of second-order ordinary differential equations, some results on the existence and multiplicity of component-wise positive solutions were derived by applying the fixed point index theory in cones [5, 7, 6, 8, 9].

In this article, we are concerned with the existence and multiplicity of component-wise positive solutions for system (1.1), in which one nonlinear term is uniformly superlinear or uniformly sublinear and the other is locally uniformly superlinear or

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locally uniformly sublinear. In order to make the paper sufficiently self-contained and present our discussions in a straightforward manner, let us recall some basic definitions related to system (1.1).

Definition 1.1. Let $C^+(\overline{\Omega}) = \{u \in C(\overline{\Omega}) | \, u(x) \geqslant 0, \ \forall x \in \overline{\Omega}\}$. We say that (u, v) is one positive solution to system (1.1), if $(u, v) \in [C^+(\overline{\Omega}) \times C^+(\overline{\Omega})] \backslash \{(0, 0)\}$ satisfies system (1.1). We say that (u, v) is one component-wise positive solution to system (1.1), if $(u, v) \in [C^+(\overline{\Omega}) \backslash \{0\}] \times [C^+(\overline{\Omega}) \backslash \{0\}]$ satisfies system (1.1).

In this study, we suppose that the kernel functions $k_i(x, y)$ $(i = 1, 2)$ satisfy the following three conditions:

(i) $k_i(x, y) = k_i(y, x)$, for all $x, y \in \overline{\Omega}$;
(ii) there exist $p_i \in C(\overline{\Omega})$ and $0 \leqslant p_i(x) \leqslant 1$ such that $k_i(x, y) \geqslant p_i(x)k_i(z, y)$, for all $x, y, z \in \overline{\Omega}$; and
(iii) $\max_{x\in\overline{\Omega}} \int_{\overline{\Omega}} k_i(x, y)p_i(y)\,\mathrm{d}y$ is positive.

Proposition 1.2. *Let $B_i : C(\overline{\Omega}) \to C(\overline{\Omega})$ be defined by*
$$B_i u(x) = \int_{\overline{\Omega}} k_i(x, y)u(y)\,\mathrm{d}y \quad (i = 1, 2).$$
Then the spectral radius of B_i, denoted by $r(B_i)$, is positive.

Proof. In view of definition of B_i and the given conditions (i)–(iii) about k_i, we have
$$B_i p_i(x) = \int_{\overline{\Omega}} k_i(x, y)p_i(y)\,\mathrm{d}y \geqslant p_i(x) \int_{\overline{\Omega}} k_i(z, y)p_i(y)\,\mathrm{d}y, \quad x \in \overline{\Omega},$$
$$B_i p_i(x) \geqslant p_i(x)\|B_i p_i\|, \ x \in \overline{\Omega},$$
where $\|B_i p_i\| = \max_{x\in\overline{\Omega}} \int_{\overline{\Omega}} k_i(x, y)p_i(y)\,\mathrm{d}y > 0$.

By mathematical induction, we see that $B_i^n p_i(x) \geqslant p_i(x)\|B_i p_i\|^n$ for $x \in \overline{\Omega}$ and $n \in \mathbb{N}$, thus $\|B_i^n\| \geqslant \|B_i p_i\|^n$. Using the formula of spectral radius, one deduces that
$$r(B_i) = \lim_{n\to\infty} \|B_i^n\|^{1/n} \geqslant \|B_i p_i\| > 0.$$
□

Definition 1.3. Suppose that f_1 and f_2 in system (1.1) satisfy the following two hypotheses, respectively:

(H1)
$$\limsup_{u\to0^+} \max_{x\in\overline{\Omega}} \frac{f_1(x, u, v)}{u} < \frac{1}{r(B_1)} < \liminf_{u\to+\infty} \min_{x\in\overline{\Omega}} \frac{f_1(x, u, v)}{u}$$
uniformly w.r.t. $v \in \mathbb{R}^+$;

(H2)
$$\liminf_{v\to0^+} \min_{x\in\overline{\Omega}} \frac{f_2(x, u, v)}{v} > \frac{1}{r(B_2)} > \limsup_{v\to+\infty} \max_{x\in\overline{\Omega}} \frac{f_2(x, u, v)}{v}$$
uniformly w.r.t. $u \in [0, M]$, where $M \in \mathbb{R}^+$ is arbitrary.

Then we say that f_1 is uniformly superlinear at both ends (i.e., $u = 0, +\infty$) with respect to v, and that f_2 is locally uniformly sublinear at both ends (i.e., $v = 0, +\infty$) with respect to u.

Definition 1.4. Suppose that f_1 and f_2 in system (1.1) satisfy the following two hypotheses, respectively:

(H3)
$$\liminf_{u\to 0^+}\min_{x\in\overline{\Omega}}\frac{f_1(x,u,v)}{u} > \frac{1}{r(B_1)} > \limsup_{u\to+\infty}\max_{x\in\overline{\Omega}}\frac{f_1(x,u,v)}{u}$$
uniformly w.r.t. $v\in\mathbb{R}^+$;

(H4)
$$\limsup_{v\to 0^+}\max_{x\in\overline{\Omega}}\frac{f_2(x,u,v)}{v} < \frac{1}{r(B_2)} < \liminf_{v\to+\infty}\min_{x\in\overline{\Omega}}\frac{f_2(x,u,v)}{v}$$
uniformly w.r.t. $u\in[0,M]$, where $M\in\mathbb{R}^+$ is arbitrary.

Then we say that f_1 is uniformly sublinear at both ends (i.e., $u=0,+\infty$) with respect to v, and that f_2 is locally uniformly superlinear at both ends (i.e., $v=0,+\infty$) with respect to u.

We now summarize our main results regarding the case that the nonlinear term is so-called “super-sublinear”.

Theorem 1.5. *Assume that f_1 is uniformly superlinear at $u=0$ and $u=+\infty$ with respect to v, and that f_2 is locally uniformly sublinear at $v=0$ and $v=+\infty$ with respect to u. Then system* (1.1) *has at least one component-wise positive solution.*

In particular, when f_1 and f_2 are independent of v and u, respectively, Theorem 1.5 incorporates into the well-known results on the nonlinear Hammerstein integral equations.

Corollary 1.6 ([21])**.** *If $h_1,h_2\in C(\overline{\Omega}\times\mathbb{R}^+,\mathbb{R}^+)$, and satisfy the following two conditions:*

(H1*)
$$\limsup_{u\to 0^+}\max_{x\in\overline{\Omega}}\frac{h_1(x,u)}{u} < \frac{1}{r(B_1)} < \liminf_{u\to+\infty}\min_{x\in\overline{\Omega}}\frac{h_1(x,u)}{u}$$
(i.e., superlinear case);

(H2*)
$$\liminf_{u\to 0^+}\min_{x\in\overline{\Omega}}\frac{h_2(x,u)}{u} > \frac{1}{r(B_2)} > \limsup_{u\to+\infty}\max_{x\in\overline{\Omega}}\frac{h_2(x,u)}{u}$$
(i.e., sublinear case),

then the integral equation
$$u(x)=\int_{\overline{\Omega}}k_i(x,y)h_i(y,u(y))\,\mathrm{d}y,\quad x\in\overline{\Omega}\ \text{for}\ i=1,2,$$
has at least one positive solution.

The “sub-superlinear” case is different from the “super-sublinear” case, since the uniformly sublinear term f_1 needs to be controlled at infinity for a priori estimates of the solution component u. For this purpose, we impose the condition (H5) in the following theorem.

Theorem 1.7. *Assume that f_2 is locally uniformly superlinear at $v=0$ and $v=+\infty$ with respect to u, and f_1 is uniformly sublinear at $u=0$ and $u=+\infty$ with respect to v and satisfies*

(H5) $\limsup_{v\to+\infty}\max_{x\in\overline{\Omega}}f_1(x,u,v)=g(u)$ *uniformly with respect to $u\in[0,M]$ (for all $M>0$), where g is a locally bounded function.*

Then system (1.1) *has at least one component-wise positive solution.*

From the proofs of Theorems 1.5 and 1.7 presented in Section 3, it is not difficult to derive the following results immediately on the mixed case of "super-superlinear" and "sub-sublinear".

Theorem 1.8. *Assume that f_1 is uniformly superlinear at $u = 0$ and $u = +\infty$ with respect to v, and f_2 is locally uniformly superlinear at $v = 0$ and $v = +\infty$ with respect to u. Then system* (1.1) *has at least one component-wise positive solution.*

Theorem 1.9. *Assume that f_1 is uniformly sublinear at $u = 0$ and $u = +\infty$ with respect to v and satisfies condition (H5), and f_2 is locally uniformly sublinear at $v = 0$ and $v = +\infty$ with respect to u. Then system* (1.1) *has at least one component-wise positive solution.*

Definition 1.10. For $u^*, v^* \in C(\overline{\Omega})$, (u^*, v^*) is said to be a strict upper solution of system (1.1), provided it satisfies

$$\begin{aligned} u^*(x) &> \int_{\overline{\Omega}} k_1(x,y) f_1(y, u^*(y), v^*(y))\,\mathrm{d}y, \quad x \in \overline{\Omega}, \\ v^*(x) &> \int_{\overline{\Omega}} k_2(x,y) f_2(y, u^*(y), v^*(y))\,\mathrm{d}y, \quad x \in \overline{\Omega}. \end{aligned} \tag{1.2}$$

Otherwise, $(u_*, v_*) \in C(\overline{\Omega}) \times C(\overline{\Omega})$ is called a strict lower solution of system (1.1), provided it satisfies the reverse of the above inequalities.

Theorem 1.11. *Suppose that f_1 and f_2 satisfy the following three conditions:*

- (H6) *$f_i(x, \underline{u}, \underline{v}) \leqslant f_i(x, \overline{u}, \overline{v})$ as $x \in \overline{\Omega}$, and $\underline{u} \leqslant \overline{u}$ and $\underline{v} \leqslant \overline{v}$, $i = 1, 2$;*
- (H7) $\liminf_{u\to 0^+} \min_{x\in\overline{\Omega}} \frac{f_1(x,u,0)}{u} > \frac{1}{r(B_1)}$, *and* $\liminf_{v\to 0^+} \min_{x\in\overline{\Omega}} \frac{f_2(x,0,v)}{v} > \frac{1}{r(B_2)}$;
- (H8) $\liminf_{u\to +\infty} \min_{x\in\overline{\Omega}} \frac{f_1(x,u,0)}{u} > \frac{1}{r(B_1)}$ *and* $\liminf_{v\to +\infty} \min_{x\in\overline{\Omega}} \frac{f_2(x,0,v)}{v} > \frac{1}{r(B_2)}$.

If system (1.1) *has a strict upper solution $(u^*,\ v^*)$, then system* (1.1) *has at least two component-wise positive solutions.*

As mentioned in [12, 13, 16], it is usually difficult to find a strict upper solution for such kind of systems. For the sake of applications to the associated systems of ordinary differential equations, we give the following useful lemma.

Lemma 1.12. *If there exist constants $M_1, M_2 > 0$ such that $f_1(x,u,v) < M_1$ and $f_2(x,u,v) < M_2$ for all $(x,u,v) \in \overline{\Omega} \times [0, \|\int_\Omega k_1(\cdot,y)\,\mathrm{d}y\| M_1] \times [0, \|\int_\Omega k_2(\cdot,y)\,\mathrm{d}y\| \cdot M_2]$, then system* (1.1) *has a strict upper solution.*

Proof. Let

$$\begin{aligned} u(x) &= \int_{\overline{\Omega}} M_1 k_1(x,y)\,\mathrm{d}y, \quad x \in \overline{\Omega}, \\ v(x) &= \int_{\overline{\Omega}} M_2 k_2(x,y)\,\mathrm{d}y, \quad x \in \overline{\Omega}. \end{aligned} \tag{1.3}$$

It follows that $\|u\| \leqslant \|\int_\Omega k_1(\cdot,y)\,\mathrm{d}y\| M_1$ and $\|v\| \leqslant \|\int_\Omega k_2(\cdot,y)\,\mathrm{d}y\| M_2$. Hence, $f_1(x,u(x),v(x)) < M_1$ and $f_2(x,u(x),v(x)) < M_2$ for $x \in \overline{\Omega}$.

We choose $(u^*, v^*) = (u + \varepsilon,\ v + \varepsilon)$, where $\varepsilon > 0$ is sufficiently small such that $f_1(x, u(x) + \varepsilon,\ v(x) + \varepsilon) < M_1$ and $f_2(x, u(x) + \varepsilon,\ v(x) + \varepsilon) < M_2$ for $x \in \overline{\Omega}$. Thus, it is easy to see that (u^*, v^*) is a strict upper solution of system (1.1). $\square$

This article is organized as follows. In Section 2, we present some preliminary results on the fixed point index. In Section 3, we show the proofs of our main results. Section 4 is dedicated to the existence and multiplicity of component-wise positive solutions to systems of second-order ordinary differential equations with the Dirichlet boundary conditions or mixed boundary conditions.

2. Preliminaries

Let $C(\overline{\Omega})$ be a Banach space with the maximum norm $\|u\| = \max_{x\in\overline{\Omega}} |u(x)|$, and $C^+(\overline{\Omega})$ be a total cone of $C(\overline{\Omega})$. Choose bounded domains $\Omega_i \subset \Omega$ $(i = 1, 2)$ such that

$$\delta_i := \min_{x\in\overline{\Omega}_i} p_i(x) > 0,$$

which is feasible by the hypotheses of p_i. We now construct sub-cones and subsets as follows:

$$K_i = \big\{u \in C^+(\overline{\Omega}) |\, u(x) \geqslant \delta_i\|u\|,\ \forall\ x \in \overline{\Omega}_i\big\} \quad (i = 1, 2),$$
$$K_{r_i} = \{u \in K_i |\, \|u\| < r_i\}\,, \quad \partial K_{r_i} = \{u \in K_i |\, \|u\| = r_i\}, \quad \forall r_i > 0\,.$$

From Proposition 1.2 and the Krein-Rutman theorem (see [17]), we know that $r(B_i)$ is one of eigenvalues for B_i and there exist positive eigenfunctions corresponding to $r(B_i)$.

Lemma 2.1. *Let $\psi_i(x)$ be the positive eigenfunctions of B_i corresponding to $r(B_i)$ with $\int_{\overline{\Omega}} \psi_i(x)\,\mathrm{d}x = 1$. Then the following three statements are true.*

(a) $\int_{\overline{\Omega}} \psi_i(x)u(x)\,\mathrm{d}x \leqslant \|u\|$, *for all $u \in K_i$.*
(b) $\psi_i(x) \geqslant p_i(x)\|\psi_i\|$, *for all $x \in \overline{\Omega}$.*
(c) *There exist constants $c_i > 0$ such that $\int_{\overline{\Omega}} \psi_i(x)u(x)\,\mathrm{d}x \geqslant c_i\|u\|$, for all $u \in K_i$.*

Proof. (a) Obviously,

$$\int_{\overline{\Omega}} \psi_i(x)u(x)\,\mathrm{d}x \leqslant \int_{\overline{\Omega}} \psi_i(x)\,\mathrm{d}x \cdot \|u\| = \|u\|.$$

(b) Notice that $k_i(x, y) \geqslant p_i(x)k_i(z, y)$, for all $x, y, z \in \overline{\Omega}$. So it is straightforward to obtain

$$\int_{\overline{\Omega}} k_i(x, y)\psi_i(y)\,\mathrm{d}y \geqslant \int_{\overline{\Omega}} p_i(x)k_i(z, y)\psi_i(y)\,\mathrm{d}y, \quad \forall x, z \in \overline{\Omega}\,,$$
$$r(B_i)\psi_i(x) \geqslant r(B_i)p_i(x)\psi_i(z),\ \forall x, z \in \overline{\Omega}.$$

This implies that $\psi_i(x) \geqslant p_i(x)\|\psi_i\|$, $\forall x \in \overline{\Omega}$.

(c) It follows from (b) and the definition of K_i immediately. □

For $\tau \in I := [0, 1]$ and $u, v \in C^+(\overline{\Omega})$, we define the mappings $T_1^\tau(\cdot,\cdot)$, $T_2^\tau(\cdot,\cdot)$: $C^+(\overline{\Omega}) \times C^+(\overline{\Omega}) \to C^+(\overline{\Omega})$ and $T^\tau(\cdot,\cdot) : C^+(\overline{\Omega}) \times C^+(\overline{\Omega}) \to C^+(\overline{\Omega}) \times C^+(\overline{\Omega})$ by

$$T_1^\tau(u, v)(x) = \int_{\overline{\Omega}} k_1(x, y)[\tau f_1(y, u(y), v(y)) + (1 - \tau)f_1(y, u(y), 0)]\,\mathrm{d}y,$$
$$T_2^\tau(u, v)(x) = \int_{\overline{\Omega}} k_2(x, y)[\tau f_2(y, u(y), v(y)) + (1 - \tau)f_2(y, 0, v(y))]\,\mathrm{d}y,$$
$$T^\tau(u, v)(x) = (T_1^\tau(u, v)(x), T_2^\tau(u, v)(x)).$$

Lemma 2.2. *$T^\tau : K_1 \times K_2 \to K_1 \times K_2$ is completely continuous.*

Proof. For $(u,v)\in K_1\times K_2$, we show that $T^\tau(u,v)\in K_1\times K_2$, i.e., $T_1^\tau(u,v)\in K_1$ and $T_2^\tau(u,v)\in K_2$. It follows from the definitions of $k_i(x,y)$ that

$$\begin{aligned}
T_1^\tau(u,v)(x) &= \int_{\overline{\Omega}} k_1(x,y)[\tau f_1(y,u(y),v(y))+(1-\tau)f_1(y,u(y),0)]\,\mathrm{d}y\\
&\geqslant p_1(x)\int_{\overline{\Omega}} k_1(z,y)[\tau f_1(y,u(y),v(y))+(1-\tau)f_1(y,u(y),0)]\,\mathrm{d}y\\
&= p_1(x)T_1^\tau(u,v)(z), \quad \forall x,z\in\overline{\Omega},
\end{aligned}$$

which implies that $T_1^\tau(u,v)(x)\geqslant \delta_1\|T_1^\tau(u,v)\|$ for $x\in\overline{\Omega}_1$. Similarly, one can obtain

$$T_2^\tau(u,v)(x)\geqslant \delta_2\|T_2^\tau(u,v)\|,\ x\in\overline{\Omega}_2.$$

Hence, $T^\tau(K_1\times K_2)\subset K_1\times K_2$. By Arzelà-Ascoli theorem, we know that $T^\tau: K_1\times K_2\to K_1\times K_2$ is completely continuous. □

Lemma 2.3 ([1, 12, 25])**.** *Let E be a Banach space and $P\subset E$ be a closed convex cone in E. For $r>0$, let $P_r=\{u\in P|\,\|u\|<r\}$ and $\partial P_r=\{u\in P|\,\|u\|=r\}$. Suppose that $A:P\to P$ is completely continuous. Then the following two statements are true.*

(I) *If $\mu Au\neq u$ for every $u\in\partial P_r$ and $\mu\in(0,1]$, then $i(A,P_r,P)=1$.*
(II) *Suppose that the mapping A satisfies the following two conditions:*
  (a) $\inf_{u\in\partial P_r}\|Au\|>0$*; and*
  (b) *$\mu Au\neq u$ for every $u\in\partial P_r$ and $\mu\geqslant 1$.*

Then $i(A,P_r,P)=0$.

Lemma 2.4 ([6, 8])**.** *Let X be a real Banach space, $P_i\subset X$ be a closed convex cone, W_i be a bounded open subset of X with the boundary ∂W_i $(i=1,2)$, and $P=P_1\times P_2$ and $W=W_1\times W_2$. Assume that $T:P\cap\overline{W}\to P$ is completely continuous, and that there exist compactly continuous mappings $A_i:P_i\cap\overline{W_i}\to P_i$ and $H:(P\cap\overline{W})\times[0,1]\to P$ such that*

(a) *$H(\cdot,1)=T$ and $H(\cdot,0)=A$, where $A(u,v):=(A_1u,A_2v)$ and $(u,v)\in P\cap\overline{W}$;*
(b) *$A_iu_i\neq u_i$, for all $u_i\in P_i\cap\partial W_i$; and*
(c) *$H(w,\tau)\neq w$, for all $(w,\tau)\in(P\cap\partial W)\times(0,1]$.*

Then we have

$$i(T,P\cap W,P)=i(A_1,P_1\cap W_1,P_1)\cdot i(A_2,P_2\cap W_2,P_2).$$

3. Proofs of main results

Proof of Theorem 1.5. Choose a bounded open set $D=(K_{R_1}\backslash\overline{K_{r_1}})\times(K_{R_2}\backslash\overline{K_{r_2}})$ in the product cone $K_1\times K_2$, where $R_j>r_j>0$ $(j=1,2)$ are to be determined such that the family of operators $\{T^\tau\}_{\tau\in I}$ satisfies the sufficient condition of the homotopy invariance of the fixed point index on ∂D. We will in turn determine r_1,R_1,r_2 and R_2 in the following process.

(A) From the uniformly superlinear assumption on f_1 at $u=0$, there are $\varepsilon\in(0,1/r(B_1))$ and $r_1>0$ such that

$$\tau f_1(x,u,v)+(1-\tau)f_1(x,u,0)\leqslant(1/r(B_1)-\varepsilon)u, \tag{3.1}$$

for all $x\in\overline{\Omega},(u,v)\in[0,r_1]\times\mathbb{R}^+$. We can obtain

$$\mu T_1^\tau(u,v)\neq u,\quad \forall\mu\in(0,1] \text{ and } (u,v)\in\partial K_{r_1}\times K_2. \tag{3.2}$$

Otherwise, there exist $\mu_0 \in (0,1]$ and $(u_0, v_0) \in \partial K_{r_1} \times K_2$ such that $\mu_0 T_1^\tau(u_0, v_0) = u_0$. In combination with (3.1), it follows that

$$u_0(x) \leqslant T_1^\tau(u_0, v_0)(x) \leqslant \int_{\overline{\Omega}} k_1(x,y)(1/r(B_1) - \varepsilon)u_0(y)\,\mathrm{d}y.$$

Multiplying both sides of this inequality by $\psi_1(x)$ and integrating it on $\overline{\Omega}$ yields

$$\int_{\overline{\Omega}} u_0(x)\psi_1(x)\,\mathrm{d}x \leqslant [1 - r(B_1)\varepsilon] \int_{\overline{\Omega}} u_0(y)\psi_1(y)\,\mathrm{d}y. \tag{3.3}$$

Since $\int_{\overline{\Omega}} u_0(x)\psi_1(x)\mathrm{d}x > 0$ and $r(B_1) > 0$, by (3.3) it gives $1 \leqslant 1 - r(B_1)\varepsilon$. This apparently leads to a contradiction.

(B) Because of the uniformly superlinear hypothesis on f_1 at $u = +\infty$, there exist $\varepsilon > 0$ and $m > 0$ such that

$$\tau f_1(x,u,v) + (1-\tau)f_1(x,u,0) \geqslant (1/r(B_1) + \varepsilon)u, \tag{3.4}$$

for all $x \in \overline{\Omega}$, $(u,v) \in [m, +\infty) \times \mathbb{R}^+$. So we have

$$\tau f_1(x,u,v) + (1-\tau)f_1(x,u,0) \geqslant (1/r(B_1) + \varepsilon)u - C_1, \tag{3.5}$$

for all $x \in \overline{\Omega}$, $(u,v) \in \mathbb{R}^+ \times \mathbb{R}^+$, where $C_1 = (1/r(B_1) + \varepsilon)m$.

Then there exists an $R_1 > r_1$ such that

$$\mu T_1^\tau(u,v) \neq u \quad \text{and} \quad \inf_{u \in \partial K_{R_1}} \|T_1^\tau(u,v)\| > 0, \tag{3.6}$$

for all $\mu \geqslant 1$, $(u,v) \in \partial K_{R_1} \times K_2$.

If there exist $(u_0, v_0) \in K_1 \times K_2$ and $\mu_0 \geqslant 1$ such that $u_0 = \mu_0 T_1^\tau(u_0, v_0)$, from (3.5) we deduce that

$$u_0(x) \geqslant T_1^\tau(u_0, v_0)(x) \geqslant \int_{\overline{\Omega}} k_1(x,y)(1/r(B_1) + \varepsilon)u_0(y)\,\mathrm{d}y - C.$$

It follows that

$$\int_{\overline{\Omega}} u_0(x)\psi_1(x)\,\mathrm{d}x \geqslant (1 + r(B_1)\varepsilon) \int_{\overline{\Omega}} u_0(y)\psi_1(y)\,\mathrm{d}y - C,$$

which yields

$$\int_{\overline{\Omega}} u_0(x)\psi_1(x)\,\mathrm{d}x \leqslant \frac{C}{r(B_1)\varepsilon}.$$

In view of Lemma 2.1 (c), we obtain

$$\|u_0\| \leqslant \frac{C}{c_1 r(B_1)\varepsilon} =: R^*. \tag{3.7}$$

Thus, when $R > R^*$, $u \neq \mu T_1^\tau(u,v)$ holds for all $(u,v) \in \partial K_R \times K_2$ and $\mu \geqslant 1$. In addition, if $R > m/\delta_1$, by (3.4) we know that for all $(u,v) \in \partial K_R \times K_2$, it holds

$$\begin{aligned}
\|T_1^\tau(u,v)\| &\geqslant \int_{\overline{\Omega}} T_1^\tau(u,v)(x)\psi_1(x)\,\mathrm{d}x \\
&\geqslant \int_{\overline{\Omega}} \int_{\overline{\Omega}_1} k_1(y,x)(1/r(B_1) + \varepsilon)u(y)\,\mathrm{d}y\,\psi_1(x)\mathrm{d}x \\
&\geqslant (1 + r(B_1)\varepsilon) \int_{\overline{\Omega}_1} u(y)\psi_1(y)\,\mathrm{d}y \\
&\geqslant (1 + r(B_1)\varepsilon)\mathrm{mes}(\overline{\Omega}_1)\delta_1^2\|\psi_1\|R.
\end{aligned}$$

That is,

$$\inf_{u\in\partial K_R} \|T_1^\tau(u,v)\| > 0.$$

Hence, we choose $R_1 > \max\{r_1, R^*, m/\delta_1\}$.

(C) Based on the locally uniformly sublinear assumption on f_2 at $v = 0$, there exist $\varepsilon > 0$ and $r_2 > 0$ such that

$$\tau f_2(x,u,v) + (1-\tau) f_2(x,0,v) \geqslant (1/r(B_2) + \varepsilon)v, \tag{3.8}$$

for all $x \in \overline{\Omega}$, $(u,v) \in [0,R_1] \times [0,r_2]$. Then we have

$$\mu T_2^\tau(u,v) \neq v \quad \text{and} \quad \inf_{v\in\partial K_{r_2}} \|T_2^\tau(u,v)\| > 0, \tag{3.9}$$

for all $\mu \geqslant 1$, $(u,v) \in \overline{K_{R_1}} \times \partial K_{r_2}$.

(D) Given the locally uniformly sublinear hypothesis on f_2 at $v = +\infty$, there exist $\varepsilon \in (0, 1/r(B_2))$, $n > 0$ and $C > 0$ such that

$$\tau f_2(x,u,v) + (1-\tau) f_2(x,0,v) \leqslant (1/r(B_2) - \varepsilon)v, \ \forall\, x \in \overline{\Omega},\ (u,v) \in [0,R_1] \times [n,+\infty) \tag{3.10}$$

and

$$\tau f_2(x,u,v) + (1-\tau) f_2(x,0,v) \leqslant (1/r(B_2) - \varepsilon)v + C, \tag{3.11}$$

for all $x \in \overline{\Omega}$, $(u,v) \in [0,R_1] \times \mathbb{R}^+$.

As we did in the discussion of (3.7), one can prove that if $v_0 = \mu_0 T_2^\tau(u_0,v_0)$ for $(u_0,v_0) \in \overline{K_{R_1}} \times K_2$ and $\mu_0 \in (0,1]$, then

$$\|v_0\| \leqslant R' \stackrel{\text{def}}{=} \frac{C}{c_2 r(B_2)\varepsilon}. \tag{3.12}$$

Hence, we take $R_2 > \max\{r_2, R'\}$, and have

$$\mu T_2^\tau(u,v) \neq v, \quad \forall \mu \in (0,1] \text{ and } (u,v) \in \overline{K_{R_1}} \times \partial K_{R_2}. \tag{3.13}$$

Now, we choose an open set $D = (K_{R_1} \backslash \overline{K_{r_1}}) \times (K_{R_2} \backslash \overline{K_{r_2}})$. By using (3.2), (3.6), (3.9) as well as (3.13), we see that $\{T^\tau\}_{\tau\in I}$ satisfies the sufficient conditions for the homotopy invariance of the fixed point index on ∂D. By virtue of Lemmas 2.3 and 2.4, we have

$$\begin{aligned} i(T^1, D, K_1 \times K_2) &= \prod_{j=1}^{2} i(T_j^0, K_{R_j} \backslash \overline{K_{r_j}}, K_j) \\ &= \prod_{j=1}^{2} [i(T_j^0, K_{R_j}, K_j) - i(T_j^0, K_{r_j}, K_j)] \\ &= (0-1) \times (1-0) = -1. \end{aligned}$$

Consequently, system (1.1) has at least one component-wise positive solution. $\square$

Proof of Theorem 1.7. Similar to the arguments described in the proof of Theorem 1.5, we separate our discussions into four steps and determine r_1, R_1, r_2, R_2 one by one.

Step 1. Given the uniformly sublinear assumption on f_1 at $u = 0$, there are $\varepsilon > 0$ and $r_1 > 0$ such that

$$\tau f_1(x,u,v) + (1-\tau) f_1(x,u,0) \geqslant (1/r(B_1) + \varepsilon)u, \tag{3.14}$$

for all $x \in \overline{\Omega}$, $(u,v) \in [0,r_1] \times \mathbb{R}^+$. We can obtain

$$\mu T_1^\tau(u,v) \neq u, \quad \inf_{u\in\partial K_{r_1}} \|T_1^\tau(u,v)\| > 0, \quad \forall \mu \geqslant 1,\ (u,v) \in \partial K_{r_1} \times K_2. \tag{3.15}$$

Otherwise, suppose that there exist $\mu_0 \geqslant 1$ and $(u_0,v_0) \in \partial K_{r_1} \times K_2$ such that

$$\mu_0 T_1^\tau(u_0,v_0) = u_0.$$

It follows from (3.14) that

$$u_0(x) \geqslant T_1^\tau(u_0,v_0)(x) \geqslant \int_{\overline{\Omega}} k_1(x,y)(1/r(B_1)+\varepsilon)u_0(y)\,\mathrm{d}y.$$

Multiplying both sides of this inequality by $\psi_1(x)$ and integrating it on $\overline{\Omega}$, we obtain

$$\int_{\overline{\Omega}} u_0(x)\psi_1(x)\,\mathrm{d}x \geqslant [1+r(B_1)\varepsilon] \int_{\overline{\Omega}} u_0(y)\psi_1(y)\,\mathrm{d}y. \tag{3.16}$$

Since $\int_{\overline{\Omega}} u_0(x)\psi_1(x)\mathrm{d}x > 0$ and $r(B_1) > 0$, by (3.16) it gives $1 \geqslant 1 + r(B_1)\varepsilon$, which is obviously a contradiction.

By (3.14) we know that for all $(u,v) \in \partial K_{r_1} \times K_2$, it holds

$$\begin{aligned}
\|T_1^\tau(u,v)\| &\geqslant \int_{\overline{\Omega}} T_1^\tau(u,v)(x)\psi_1(x)\,\mathrm{d}x \\
&\geqslant \int_{\overline{\Omega}} \int_{\overline{\Omega}} k_1(y,x)(1/r(B_1)+\varepsilon)u(y)\,\mathrm{d}y\,\psi_1(x)\mathrm{d}x \\
&\geqslant (1+r(B_1)\varepsilon) \int_{\overline{\Omega}} u(y)\psi_1(y)\,\mathrm{d}y \\
&\geqslant c_1[1+r(B_1)\varepsilon]r_1.
\end{aligned}$$

Clearly, this implies

$$\inf_{u\in\partial K_{r_1}} \|T_1^\tau(u,v)\| > 0.$$

Step 2. Under the uniformly sublinear hypothesis on f_1 at $u = +\infty$ and condition (H5), there exist $\varepsilon \in (0, 1/r(B_1))$, $m > 0$ and $C > 0$ such that

$$\tau f_1(x,u,v) + (1-\tau) f_1(x,u,0) \leqslant (1/r(B_1) - \varepsilon)u, \tag{3.17}$$

for all $x \in \overline{\Omega}$, $(u,v) \in [m,+\infty) \times \mathbb{R}^+$, and

$$\tau f_1(x,u,v) + (1-\tau) f_1(x,u,0) \leqslant (1/r(B_1) - \varepsilon)u + C, \tag{3.18}$$

for all $x \in \overline{\Omega}$, $(u,v) \in \mathbb{R}^+ \times \mathbb{R}^+$.

Using (3.18) and processing in like manner as we did in the discussion of (3.7), one can see that if $u_0 = \mu_0 T_1^\tau(u_0,v_0)$ for $(u_0,v_0) \in K_1 \times K_2$ and $\mu_0 \in (0,1]$, then

$$\|u_0\| \leqslant R^* = \frac{C}{c_1 r(B_1)\varepsilon}. \tag{3.19}$$

Choosing $R_1 > \max\{r_1, R^*\}$, we then obtain

$$\mu T_1^\tau(u,v) \neq u, \quad \forall \mu \in (0,1] \text{ and } (u,v) \in \partial K_{R_1} \times K_2. \tag{3.20}$$

Step 3. By the locally uniformly superlinear assumption on f_2 at $v = 0$, there are $\varepsilon \in (0, 1/r(B_2))$ and $r_2 > 0$ such that

$$\tau f_2(x,u,v) + (1-\tau) f_2(x,0,v) \leqslant (1/r(B_2) - \varepsilon)v, \tag{3.21}$$

for all $x \in \overline{\Omega}$, $(u,v) \in [0,R_1] \times [0,r_2]$.

We claim that

$$\mu T_2^\tau(u,v) \neq v, \quad \forall \mu \in (0,1] \text{ and } (u,v) \in \overline{K_{R_1}} \times \partial K_{r_2}. \tag{3.22}$$

Otherwise, there exist $\mu_0 \in (0,1]$ and $(u_0,v_0) \in \overline{K_{R_1}} \times \partial K_{r_2}$ so that $\mu_0 T_2^\tau(u_0,v_0) = v_0$. In combination with (3.21), it follows that

$$v_0(x) \leqslant T_2^\tau(u_0,v_0)(x) \leqslant \int_{\overline{\Omega}} k_2(x,y)(1/r(B_2)-\varepsilon)v_0(y)\,\mathrm{d}y.$$

It follows that

$$\int_{\overline{\Omega}} v_0(x)\psi_2(x)\,\mathrm{d}x \leqslant (1-r(B_2)\varepsilon)\int_{\overline{\Omega}} v_0(y)\psi_2(y)\,\mathrm{d}y. \tag{3.23}$$

Since $\int_{\overline{\Omega}} v_0(x)\psi_2(x)\mathrm{d}x > 0$ and $r(B_2) > 0$, it follows from (3.23) that $1 \leqslant 1-r(B_2)\varepsilon$, which yields a contradiction.

Step 4. By utilizing the locally uniformly superlinear hypothesis on f_2 at $v = +\infty$, there exist $\varepsilon > 0$ and $n > 0$ such that

$$\tau f_2(x,u,v) + (1-\tau)f_2(x,0,v) \geqslant (1/r(B_2)+\varepsilon)v, \tag{3.24}$$

for all $x \in \overline{\Omega}$, $(u,v) \in [0,R_1] \times [n,+\infty)$. It gives

$$\tau f_2(x,u,v) + (1-\tau)f_2(x,0,v) \geqslant (1/r(B_2)+\varepsilon)v - C_2, \tag{3.25}$$

for all $x \in \overline{\Omega}$, $(u,v) \in [0,R_1] \times \mathbb{R}^+$, where $C_2 = (1/r(B_2)+\varepsilon)n$.

We now prove that there exists an $R_2 > r_2$ such that

$$\mu T_2^\tau(u,v) \neq v \quad \text{and} \quad \inf_{v \in \partial K_{R_2}} \|T_2^\tau(u,v)\| > 0, \tag{3.26}$$

for all $\mu \geqslant 1$, $(u,v) \in \overline{K_{R_1}} \times \partial K_{R_2}$. If there are $(u_0,v_0) \in \overline{K_{R_1}} \times K_2$ and $\mu_0 \geqslant 1$ such that $v_0 = \mu_0 T_2^\tau(u_0,v_0)$, then it follows from (3.25) that

$$v_0(x) \geqslant T_2^\tau(u_0,v_0)(x) \geqslant \int_{\overline{\Omega}} k_2(x,y)(1/r(B_2)+\varepsilon)v_0(y)\,\mathrm{d}y - C. \tag{3.27}$$

Moreover,

$$\int_{\overline{\Omega}} v_0(x)\psi_2(x)\,\mathrm{d}x \geqslant (1+r(B_2)\varepsilon)\int_{\overline{\Omega}} v_0(y)\psi_2(y)\,\mathrm{d}y - C.$$

It further leads to

$$\int_{\overline{\Omega}} v_0(x)\psi_2(x)\,\mathrm{d}x \leqslant \frac{C}{r(B_2)\varepsilon}.$$

In view of Lemma 2.1 (c), we know that

$$\|v_0\| \leqslant R' = \frac{C}{c_2 r(B_2)\varepsilon}\,. \tag{3.28}$$

When $R > R'$, $v \neq \mu T_2^\tau(u,v)$ holds for all $(u,v) \in \overline{K_{R_1}} \times \partial K_R$ and $\mu \geqslant 1$. In addition, if $R > n/\delta_2$, then by (3.24) we know that for all $(u,v) \in \overline{K_{R_1}} \times \partial K_R$, it holds

$$\begin{aligned}
\|T_2^\tau(u,v)\| &\geqslant \int_{\overline{\Omega}} T_2^\tau(u,v)(x)\psi_2(x)\,\mathrm{d}x \\
&\geqslant \int_{\overline{\Omega}}\int_{\overline{\Omega}_2} k_2(y,x)(1/r(B_2)+\varepsilon)v(y)\,\mathrm{d}y\,\psi_2(x)\mathrm{d}x \\
&\geqslant (1+r(B_2)\varepsilon)\int_{\overline{\Omega}_2} v(y)\psi_2(y)\,\mathrm{d}y
\end{aligned}$$

$$\geqslant (1 + r(B_2)\varepsilon)\mathrm{mes}(\overline{\Omega}_2)\delta_2^2\|\psi_2\|R.$$

This indicates that

$$\inf_{v\in\partial K_R} \|T_2^\tau(u,v)\| > 0.$$

Hence, we choose $R_2 > \max\{r_2,\, R',\, n/\delta_2\}$.

Based on (3.15), (3.20), (3.22) and (3.26), one can easily verify that $\{T^\tau\}_{\tau\in I}$ satisfies the sufficient conditions for the homotopy invariance of the fixed point index on ∂D, where $D = (K_{R_1}\backslash\overline{K_{r_1}}) \times (K_{R_2}\backslash\overline{K_{r_2}})$. It then follows Lemmas 2.3 and 2.4 that

$$\begin{aligned}
i(T^1, D, K_1 \times K_2) &= \prod_{j=1}^{2} i(T_j^0, K_{R_j}\backslash\overline{K_{r_j}}\,, K_j) \\
&= \prod_{j=1}^{2} [i(T_j^0, K_{R_j}, K_j) - i(T_j^0, K_{r_j}, K_j)] \\
&= (1-0) \times (0-1) = -1.
\end{aligned}$$

Hence, system (1.1) has at least one component-wise positive solution. □

Proof of Theorem 1.11. Let $\varepsilon > 0$, and

$$\begin{gathered}
W_1 = \{u \in K_1 : -\varepsilon < u(x) < u^*(x),\ x \in \overline{\Omega}\}, \\
W_2 = \{v \in K_2 : -\varepsilon < v(x) < v^*(x),\ x \in \overline{\Omega}\}, \\
W = W_1 \times W_2.
\end{gathered}$$

Then, W is an open subset of $K_1 \times K_2$ and $(0, 0) \in W$.

We first show that

$$T^\tau(u,v) \neq (u,v),\ \forall(\tau,u,v) \in [0,1] \times \partial W. \tag{3.29}$$

In fact, if there exists some $(\tau_0, u_0, v_0) \in [0,1]\times\partial W$ such that $T^{\tau_0}(u_0, v_0) = (u_0, v_0)$, then $(u_0, v_0) \in \partial W_1 \times W_2$ or $(u_0, v_0) \in W_1 \times \partial W_2$. Without loss of generality, we assume that $(u_0, v_0) \in \partial W_1 \times W_2$, then $v_0 \leqslant v^*$, $u_0 \leqslant u^*$ and $u_0(x_0) = u^*(x_0)$ for some $x_0 \in \overline{\Omega}$. Hence, by (F_1), in view of the definitions of T_1^τ and the strict upper solution, it yields

$$u^*(x_0) = u_0(x_0) = T_1^{\tau_0}(u_0, v_0)(x_0) \leqslant T_1^{\tau_0}(u^*, v^*)(x_0) \leqslant T_1^1(u^*, v^*)(x_0) < u^*(x_0).$$

This is obviously a contradiction.

To prove that $T_1^0(u,v) \neq \mu u$, for all $\mu \geqslant 1$ and all $u \in \partial W_1$, we let $\mu \geqslant 1$ and $u \in \partial W_1$. Then $u \leqslant u^*$, and there is an $x_0 \in \overline{\Omega}$ such that $u(x_0) = u^*(x_0)$. Thus it follows that

$$T_1^0(u,v)(x_0) = T_1^0(u,0)(x_0) \leqslant T_1^0(u^*,0)(x_0) < u^*(x_0) = u(x_0) \leqslant \mu u(x_0).$$

Similarly, we find that $T_2^0(u,v) \neq \mu v$ for all $\mu \geqslant 1$ and all $v \in \partial W_2$. It then follows from Lemma 2.3 that

$$i(T_1^0, W_1, K_1) = i(T_2^0, W_2, K_2) = 1. \tag{3.30}$$

In view of (3.6), (3.9), (3.15) and (3.26), we choose r_1, r_2 small enough and R_1, R_2 large enough such that

$$\begin{gathered}
0 < r_1 < \min_{x\in\overline{\Omega}} u^*(x) \leqslant \|u^*\| < R_1, \\
0 < r_2 < \min_{x\in\overline{\Omega}} v^*(x) \leqslant \|v^*\| < R_2.
\end{gathered}$$

Combining (3.29) and (3.30), by Lemmas 2.3 and 2.4, we obtain

$$\begin{aligned} &i(T^1, (W_1\backslash\overline{K_{r_1}}) \times (W_2\backslash\overline{K_{r_2}}), K_1 \times K_2) \\ &= \prod_{j=1}^{2} i(T_j^0, W_j\backslash\overline{K_{r_j}}, K_j) \\ &= \prod_{j=1}^{2} [i(T_j^0, W_j, K_j) - i(T_j^0, K_{r_j}, K_j)] \\ &= (1-0) \times (1-0) = 1, \end{aligned} \tag{3.31}$$

and further deduce that

$$\begin{aligned} &i(T^1, D\backslash\overline{(W_1\backslash\overline{K_{r_1}}) \times (W_2\backslash\overline{K_{r_2}})}, K_1 \times K_2) \\ &= i(T^1, D, K_1 \times K_2) - i(T^1, (W_1\backslash\overline{K_{r_1}}) \times (W_2\backslash\overline{K_{r_2}}), K_1 \times K_2) \\ &= \prod_{j=1}^{2} [i(T_j^0, K_{R_j}, K_j) - i(T_j^0, K_{r_j}, K_j)] - 1 \\ &= 0 - 1 = -1. \end{aligned} \tag{3.32}$$

Thus, system (1.1) has at least two component-wise positive solutions. □

4. Applications

Consider the existence and multiplicity of component-wise positive solutions for the following system of second-order ordinary differential equations

$$\begin{aligned} -u''(x) &= f_1(x, u(x), v(x)), \quad x \in (0,1), \\ -v''(x) &= f_2(x, u(x), v(x)), \quad x \in (0,1), \end{aligned} \tag{4.1}$$

subject to the Dirichlet boundary conditions

$$u(0) = u(1) = v(0) = v(1) = 0, \tag{4.2}$$

or the mixed boundary conditions

$$u(0) = u(1) = v(0) = v'(1) = 0, \tag{4.3}$$

where $f_1, f_2 \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$.

4.1. **Dirichlet boundary value problem.**

Theorem 4.1. *Assume that f_1 and f_2 satisfy the following two conditions:*

(H9)
$$\limsup_{u\to 0^+} \max_{x\in[0,1]} \frac{f_1(x,u,v)}{u} < \pi^2 < \liminf_{u\to +\infty} \min_{x\in[0,1]} \frac{f_1(x,u,v)}{u}$$
uniformly w.r.t. $v \in \mathbb{R}^+$;

(H10)
$$\liminf_{v\to 0^+} \min_{x\in[0,1]} \frac{f_2(x,u,v)}{v} > \pi^2 > \limsup_{v\to +\infty} \max_{x\in[0,1]} \frac{f_2(x,u,v)}{v}$$
uniformly w.r.t. $u \in [0, M]$, where $M \in \mathbb{R}^+$ is arbitrary.

Then problem (4.1)-(4.2) *has at least one component-wise positive solution.*

Proof. It is clear that system (4.1) subject to (4.2) is equivalent to the system of nonlinear Hammerstein integral equations

$$\begin{aligned} u(x) &= \int_0^1 k_1(x,y) f_1(y,u(y),v(y))\,\mathrm{d}y, \quad x \in [0,1], \\ v(x) &= \int_0^1 k_2(x,y) f_2(y,u(y),v(y))\,\mathrm{d}y, \quad x \in [0,1], \end{aligned}$$

where

$$k_1(x,y) = k_2(x,y) = \begin{cases} x(1-y), & \text{if } x \leqslant y, \\ y(1-x), & \text{if } y \leqslant x. \end{cases}$$

It is easy to verify that the kernel functions k_1 and k_2 satisfy all conditions (i)–(iii).

According to Theorem 1.5, we only need to prove that $r(B_1) = r(B_2) = \pi^{-2}$. To this end, it suffices to observe that the following linear eigenvalue problem

$$\begin{gathered} -u''(x) = \lambda u(x), \quad x \in (0,1), \\ u(0) = u(1) = 0, \end{gathered}$$

has the minimal eigenvalue $\lambda_1 = \pi^2$. □

4.2. **Mixed boundary value problem.**

Theorem 4.2. *Assume that f_1 satisfies condition (H9), and that f_2 satisfies*

(H11)
$$\liminf_{v\to 0^+} \min_{x\in[0,1]} \frac{f_2(x,u,v)}{v} > \frac{\pi^2}{4} > \limsup_{v\to+\infty} \max_{x\in[0,1]} \frac{f_2(x,u,v)}{v}$$
uniformly w.r.t. $u \in [0,M]$, where $M \in \mathbb{R}^+$ is arbitrary.

Then system (4.1) with the mixed boundary condition (4.3) has at least one component-wise positive solution.

Proof. We know that system (4.1) subject to (4.3) is equivalent to the system of nonlinear Hammerstein integral equations

$$\begin{aligned} u(x) &= \int_0^1 k_1(x,y) f_1(y,u(y),v(y))\,\mathrm{d}y, \quad x \in [0,1], \\ v(x) &= \int_0^1 k_2(x,y) f_2(y,u(y),v(y))\,\mathrm{d}y, \quad x \in [0,1], \end{aligned}$$

where

$$k_1(x,y) = \begin{cases} x(1-y), & x \leqslant y, \\ y(1-x), & y \leqslant x, \end{cases} \quad \text{and} \quad k_2(x,y) = \begin{cases} x, & x \leqslant y, \\ y, & y \leqslant x. \end{cases}$$

It is easy to verify that two kernel functions k_1 and k_2 satisfy conditions (i)–(iii).

From the proof of Theorem 4.1, we find that $r(B_1) = \pi^{-2}$. Since $\mu_1 = \pi^2/4$ is the minimal eigenvalue of the linear eigenvalue problem

$$\begin{gathered} -v''(x) = \mu v(x), \quad x \in (0,1), \\ v(0) = v'(1) = 0, \end{gathered}$$

one can derive that $r(B_2) = 4\pi^{-2}$. Hence, we obtain the desired result according to Theorem 1.5. □

Remark 4.3. There are many types of functions f_1 and f_2 satisfying conditions given in Theorems 4.1 and 4.2. For instance, let $f_1(x,u,v) = \max\{u,u^2\}(1+\tan^{-1} v)$ and $f_2(x,u,v) = \sqrt{v}\, e^u$. It is easy to see that f_1 satisfies condition (H9), and f_2 satisfies conditions (H10) and (H11). Thus system (4.1) subject to (4.2) or (4.3) has at least one component-wise positive solution.

Theorem 4.4. *Suppose that f_1 and f_2 satisfy the following three conditions:*

(H12) $f_i(x,\underline{u},\underline{v}) \leqslant f_i(x,\overline{u},\overline{v})$ *as* $x \in [0,1]$, $\underline{u} \leqslant \overline{u}$ *and* $\underline{v} \leqslant \overline{v}$, *for* $i = 1,2$;

(H13)
$$\liminf_{u\to 0^+} \min_{x\in[0,1]} \frac{f_1(x,u,0)}{u} > \pi^2 \quad \textit{and} \quad \liminf_{v\to 0^+} \min_{x\in[0,1]} \frac{f_2(x,0,v)}{v} > \frac{\pi^2}{4};$$

(H14)
$$\liminf_{u\to +\infty} \min_{x\in[0,1]} \frac{f_1(x,u,0)}{u} > \pi^2 \quad \textit{and} \quad \liminf_{v\to +\infty} \min_{x\in[0,1]} \frac{f_2(x,0,v)}{v} > \frac{\pi^2}{4}.$$

In addition, if there exist constants M_1, $M_2 > 0$ such that $f_1(x,u,v) < M_1$ and $f_2(x,u,v) < M_2$ for all $(x,u,v) \in [0,1]\times[0,M_1/8]\times[0,M_2/2]$, then system (4.1) *subject to* (4.3) *has at least two component-wise positive solutions.*

Proof. Similar to the arguments in the proof of Theorem 4.2, one can derive that $r(B_1) = \pi^{-2}$ and $r(B_2) = 4\pi^{-2}$. In addition, it is straightforward to calculate that
$$\max_{x\in[0,1]} \int_\Omega k_1(x,y)\,\mathrm{d}y = 1/8 \quad \text{and} \quad \max_{x\in[0,1]} \int_\Omega k_2(x,y)\,\mathrm{d}y = 1/2.$$
The desired result follows Theorem 1.11 and Lemma 1.12. □

Remark 4.5. As an illustration of Theorem 4.4, we consider
$$f_1(x,u,v) = \begin{cases} \frac{\sqrt{u}}{2}\tan^{-1}(v+1), & u\in[0,1],\\ \big(\frac{7u}{2}-3\big)\tan^{-1}(v+1), & u\in(1,2),\\ u^2\tan^{-1}(v+1), & u\geqslant 2,\end{cases}$$
$$f_2(x,u,v) = \begin{cases} \frac{\sqrt{v}}{4}e^u, & v\in[0,1],\\ \big(\frac{15v}{4}-\frac{7}{2}\big)e^u, & v\in(1,2),\\ v^2e^u, & v\geqslant 2.\end{cases}$$
It is easy to see that f_1 and f_2 satisfy all the conditions given in Theorem 4.4 by choosing $M_1 = M_2 = 1$. Thus system (4.1) subject to (4.3) has at least two component-wise positive solutions.

Remark 4.6. Note that Theorems 1.7-1.9 can also be applied to system (4.1) subject to conditions (4.2) or (4.3) in a similar manner. Here, we omit the details for the corresponding results on the existence and multiplicity of component-wise positive solutions for such kind of systems. It is worth mentioning that the ideas described herein are also applicable to nonlinear systems of Hammerstein integral equations with weighted functions. We will consider this problem in a subsequent work.

Remark 4.7. It is notable that assumptions (H12), (H13) and (H14) are also satisfied for coupling functions $f_i(x,u,v)$ which are negative for small values of u or/and v. Such type of couplings often arises in Chemical Engineering, see [10] and references therein, and leads to solutions such that may be zero in a subset of the interval $(0,1)$, or at least may have zero derivative in some of the boundary points (the so called "flat solutions").

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References

[1] H. Amann; *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Review, 18 (1976), 620–709.

[2] H. Brezis, F. E. Browder; *Some new results about Hammerstein equations*, Bull. Amer. Math. Soc., 80 (1974), 567–572.

[3] H. Brezis, F.E. Browder; *Nonlinear integral equations and systems of Hammerstein type*, Advances in Math., 18 (1975), 115–147.

[4] H. Brezis, F. E. Browder; *Maximal monotone operators in nonreflexive Banach spaces and nonlinear integral equations of Hammerstein type*, Bull. Amer. Math. Soc., 81 (1975), 82–88.

[5] X. Cheng; *Existence of positive solutions for a class of second-order ordinary differential systems*, Nonlinear Anal., 69 (2008), 3042–3049.

[6] X. Cheng, H. Lü; *Multiplicity of positive solutions for a (p_1, p_2)-Laplacian system and its applications*, Nonlinear Anal. RWA, 13 (2012), 2375–2390.

[7] X. Cheng, Z. Zhang; *Existence of positive solutions to systems of nonlinear integral or differential equations*, Topol. Meth. Nonlinear Anal., 34 (2009), 267–277.

[8] X. Cheng, Z. Zhang; *Positive solutions for a multi-parameter system of second-order ordinary differential equations*, Sci. China Math., 54 (2011), 959–972.

[9] X. Cheng, C.Zhong; *Existence of positive solutions for a second-order ordinary differential system*, J. Math. Anal. Appl., 312 (2005), 14–23.

[10] J. I. Diaz, I. I. Vrabie; *Existence for reaction diffusion systems. A compactness method approach*, J. Math. Anal. Appl., 188 (1994), 521–540.

[11] D. Franco, G. Infante, D. O'Regan; *Nontrivial solutions in abstract cones for Hammerstein integral systems*, Dyn. Contin. Discrete Impuls. Syst. Ser. A, 14 (2007), 837–850.

[12] D. Guo, V. Lakshmikantham; *Nonlinear Problems in Abstract Cones*, Academic Press, New York, 1988.

[13] D. Guo, J. Sun; *Nonlinear Integral Equations* (in Chinese), Shandong Press of Science and Technology, Jinan, 1987.

[14] A. Hammerstein; *Nichtlineare Intergralgleichungen nebst Anwendungen*, Acta Math., 54 (1929), 117–176.

[15] G. Infante, P. Pietramala; *Existence and multiplicity of non-negative solutions for systems of perturbed Hammerstein integral equations*, Nonlinear Anal., 71 (2009), 1301–1310.

[16] M. A. Krasnoselskii; *Topological methods in the Theory of Nonlinear Integral Equations*, Pergamon, Oxford, 1964.

[17] M. G. Krein, M. A. Rutman; *Linear operators leaving invariant a cone in a Banach space*, Uspekhi Mat. Nauk, 3 (1948), 3–95.

[18] K. Q. Lan, W. Lin; *Multiple positive solutions of systems of Hammerstein integral equations with applications to fractional differential equations*, J. London Math. Soc., 83 (2011), 449–469.

[19] K. Q. Lan, W. Lin; *Positive solutions of systems of singular Hammerstein integral equations with applications to semilinear elliptic equations in annuli*, Nonlinear Anal., 74 (2011), 7184–7197.

[20] J. Sun, X. Liu; *Computation for topological degree and its applications*, J. Math. Anal. Appl., 202 (1996), 785–796.

[21] J. R. L. Webb, K. Q. Lan; *Eigenvalue criteria for existence of multiple positive solutions of nonlinear boundary value problems of local and nonlocal type*, Topol. Meth. Nonlinear Anal., 27 (2006), 91–116.

[22] Z. Yang, D. O'Regan; *Positive solvability of systems of nonlinear Hammerstein integral equations*, J. Math. Anal. Appl., 311 (2005), 600–614.

[23] Z. Yang, Z. Zhang; *Positive solutions for a system of nonlinear singular Hammerstein integral equations via nonnegative matrices and applications*, Positivity, 16 (2012), 783–800.

[24] Z. Zhang; *Existence of nontrivial solutions for superlinear systems of integral equations and applications*, Acta Math. Sinica, 15 (1999), 153-162.
[25] C. Zhong, X. Fan, W. Chen; *An Introduction to Nonlinear Functional Analysis* (in Chinese), Lanzhou University Press, Lanzhou, 1998.

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