INTERPOLATION INEQUALITIES BETWEEN LORENTZ SPACE AND BMO: THE ENDPOINT CASE \((L^{1,\infty}, \text{BMO})\)

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Abstract. We prove interpolation inequalities by means of the Lorentz norm, BMO norm, and the fractional Sobolev norm. In particular, we obtain an interpolation inequality for \((L^{1,\infty}, \text{BMO})\), that we call the endpoint case.

1. Introduction and statement of main results

The main purpose of this article is to study the interpolation inequalities between the Lorentz space \(L^{p,\alpha}(\mathbb{R}^n)\) and the BMO(\(\mathbb{R}^n\)) space, where \(n \geq 1\). It is known that the interpolation inequalities play a crucial role in studying the boundedness of operators and in studying PDEs, see, e.g. [12, 5, 6, 7, 8]. Thus, such an extension of the inequalities of this type is involved many purposes, for instance: the theory of Marcinkiewicz interpolation; the boundedness of the operators acting on Lorentz spaces (the Hardy-Littlewood maximal function, the Hilbert transform, and the Riesz transform); and the estimates in PDEs.

In this article, we want to prove an interpolation inequality between the Lorentz space \(L^{q,\alpha}(\mathbb{R}^n)\) and BMO(\(\mathbb{R}^n\)), for \(q \geq 1\), and \(\alpha > 0\). And we call the endpoint case when \(q = 1\). Our result is as follows.

**Theorem 1.1.** Let \(1 \leq q < p\), and \(0 < \alpha < \infty\). Let \(f \in L^{q,\infty}(\mathbb{R}^n) \cap \text{BMO}(\mathbb{R}^n)\). Then

\[
\|f\|_{L^{p,\alpha}(\mathbb{R}^n)} \lesssim \|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{q/p} \|f\|_{\text{BMO}(\mathbb{R}^n)}^{1 - \frac{q}{p}}.
\]  

(1.1)

This result extends the recent results in [2, 3]. As a consequence of Theorem 1.1 we obtain an interpolation inequality between \(L^{q,\infty}\) and the critical Sobolev space \(\dot{W}^{s,n/s}(\mathbb{R}^n)\) for \(s \in (0, 1)\).

**Corollary 1.2.** Let \(1 \leq q < p\), and \(\alpha > 0\). For any \(0 < s < 1\), we have

\[
\|f\|_{L^{p,\alpha}(\mathbb{R}^n)} \lesssim \|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{q/p} \|f\|_{\dot{W}^{s,n/s}(\mathbb{R}^n)}^{1 - \frac{q}{p}}.
\]  

(1.2)
Theorem 1.3. Let \( p > 1 \). Then
\[
\|f\|_{L^p(\mathbb{R}^n)} \lesssim \|f^t\|_{L^p(\mathbb{R}^n)},
\]
whenever the right hand side is well-defined.

After that, we denote by
\[
BMO(\mathbb{R}^n) = \{ f \in L^1_{loc}(\mathbb{R}^n) : \|f\|_{BMO(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} f^t(x) < \infty \}.
\]

Finally, we denote the homogeneous fractional Sobolev space by
\[
\dot{W}^{s,p}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{W}^{s,p}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{1/p} < \infty \right\},
\]
where \( \mathcal{S}'(\mathbb{R}^n) \) is the dual space of \( \mathcal{S}(\mathbb{R}^n) \) (the Schwartz space). To end this part, we denote \( A \lesssim B \) if \( A \leq cB \), where \( c > 0 \) is a constant.

2. Proof of Theorem 1.1

It suffices to show that (1.1) holds for \( q = 1 \). To start, we prove the following result.

Lemma 2.1. Let \( 0 < q < p < r \leq \infty \) and \( \alpha > 0 \). If \( f \in L^{q,\infty}(\mathbb{R}^n) \cap L^{r,\infty}(\mathbb{R}^n) \), then \( f \in L^{p,\alpha}(\mathbb{R}^n) \), and
\[
\|f\|_{L^{p,\alpha}(\mathbb{R}^n)} \lesssim \|f\|_{L^{q,\infty}(\mathbb{R}^n)} \|f\|_{L^{r,\infty}(\mathbb{R}^n)}^{1-\theta},
\]
with \( \frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{r} \).

Proof. We rewrite
\[
\|f\|_{L^{p,\alpha}(\mathbb{R}^n)} = p \int_0^{\lambda_0} \lambda^\alpha \left\{ |\{ f > \lambda\}|^{\alpha/p} \frac{d\lambda}{\lambda} + p \int_{\lambda_0}^{\infty} \lambda^\alpha \left\{ |\{ f > \lambda\}|^{\alpha/p} \frac{d\lambda}{\lambda} \right\} \right\},
\]
and
\[
\int_0^{\lambda_0} \lambda^\alpha |\{ f > \lambda\}|^{\alpha/p} \frac{d\lambda}{\lambda} \leq \int_0^{\lambda_0} \lambda^\alpha \left( \frac{\|f\|_{L^{q,\infty}(\mathbb{R}^n)}^q}{\lambda^q} \right)^{\alpha/p} \frac{d\lambda}{\lambda},
\]
and
\[
\int_0^{\lambda_0} \lambda^\alpha |\{ f > \lambda\}|^{\alpha/p} \frac{d\lambda}{\lambda} = \frac{\|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{\alpha q/p}}{\alpha (1-q/p)} \lambda_0^{\alpha(1-q/p)},
\]
which completes the proof.
and
\[
\int_{\lambda_0}^{\infty} \lambda^\alpha \left\{ |f| > \lambda \right\} d\lambda \leq \int_{\lambda_0}^{\infty} \lambda^\alpha \left( \frac{\|f\|_{L^{r,\infty}(\mathbb{R}^n)}^{\alpha r/p}}{\lambda^r} \right) \frac{\alpha p}{\lambda^r} d\lambda \\
= \|f\|_{L^{r,\infty}(\mathbb{R}^n)}^{\alpha r/p} \lambda_0^{\alpha(r/p - 1)}.
\]
\[(2.4)\]

By \((2.2), (2.3)\) and \((2.4)\), we obtain
\[
\|f\|_{L^{p,\infty}(\mathbb{R}^n)}^\alpha \leq p \left( \frac{\|f\|_{L^{r,\infty}(\mathbb{R}^n)}^{\alpha q/p}}{\alpha(1 - q/p)} \lambda_0^{\alpha(1 - q/p)} + \frac{\|f\|_{L^{r,\infty}(\mathbb{R}^n)}^{\alpha r/p}}{\alpha(r/p - 1)} \lambda_0^{\alpha(1 - r/p)} \right).
\]

Now, we take
\[
\lambda_0^{\theta - q} = \frac{\|f\|_{L^{r,\infty}(\mathbb{R}^n)}^\alpha}{\|f\|_{L^{p,\infty}(\mathbb{R}^n)}},
\]
so the proof is complete. \(\square\)

Thanks to Lemma \(2.1\), we have for any \(r > p\)
\[
\|f\|_{L^{p,\infty}(\mathbb{R}^n)} \lesssim \|f\|_{L^{1,\infty}(\mathbb{R}^n)} \|f\|_{L^{r,\infty}(\mathbb{R}^n)}^{1 - \theta},
\]
\[(2.5)\]
where \(\frac{1}{p} = \theta + \frac{1 - \theta}{r} \).

Since \(r > p > 1\), and by \((1.3)\), we obtain
\[
\|f\|_{L^{r,\infty}(\mathbb{R}^n)} \leq \|f\|_{L^{r,\infty}(\mathbb{R}^n)}^{\theta} \|f\|_{L^{r,\infty}(\mathbb{R}^n)}^{1 - \theta} \lesssim \|f\|_{BMO(\mathbb{R}^n)} \|f\|_{L^{p,\infty}(\mathbb{R}^n)}^{\theta} \|f\|_{L^{p,\infty}(\mathbb{R}^n)}^{1 - \theta}.
\]
\[(2.6)\]

Combining \((2.5)\) and \((2.6)\) yields
\[
\|f\|_{L^{p,\infty}(\mathbb{R}^n)} \lesssim \|f\|_{L^{1,\infty}(\mathbb{R}^n)} \left( \|f\|_{BMO(\mathbb{R}^n)} \|f\|_{L^{p,\infty}(\mathbb{R}^n)} \right)^{1 - \theta} \lesssim \|f\|_{L^{1,\infty}(\mathbb{R}^n)} \left( \|f\|_{BMO(\mathbb{R}^n)} \|f\|_{L^{p,\infty}(\mathbb{R}^n)} \right)^{1 - \theta}.
\]

Then
\[
\|f\|_{L^{p,\infty}(\mathbb{R}^n)}^{1 - \frac{\theta}{p}(1 - \theta)} \lesssim \|f\|_{L^{1,\infty}(\mathbb{R}^n)} \|f\|_{BMO(\mathbb{R}^n)}^{(1 - \frac{\theta}{p})(1 - \theta)},
\]
\[
\|f\|_{L^{p,\infty}(\mathbb{R}^n)} \lesssim \|f\|_{L^{1,\infty}(\mathbb{R}^n)} \|f\|_{BMO(\mathbb{R}^n)}^{1 - \frac{1}{p}}.
\]

Thus, the proof is complete.

References


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