Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 57, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

NONLINEAR DIRICHLET PROBLEMS WITH THE COMBINED EFFECTS OF SINGULAR AND CONVECTION TERMS

YUNRU BAI, LESZEK GASIŃSKI, NIKOLAOS S. PAPAGEORGIOU

ABSTRACT. We consider a nonlinear Dirichlet elliptic problem driven by the p-Laplacian. In the reaction term of the equation we have the combined effects of a singular term and a convection term. Using a topological approach based on the fixed point theory (the Leray-Schauder alternative principle), we prove the existence of a positive smooth solution.

1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this article we study the nonlinear Dirichlet problem

$$\begin{gathered} -\Delta_p u(z) = u(z)^{-\gamma} + f(z, u(z), Du(z)) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u > 0, \end{gathered} \tag{1.1}$$

where $1 < p < +\infty$ and $0 < \gamma < 1$. In this problem Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2} Du) \quad \forall u \in W_0^{1,p}(\Omega).$$

In the right-hand side of (1.1) (the reaction of the problem), we have the combined effects of a singular term $u^{-\gamma}$ $(0 < \gamma < 1)$ and of a convection term $f(z, u, Du)$. The convection term f is a Carathéodory function, that is, for all $(x, y) \in \mathbb{R} \times \mathbb{R}^N$, $z \mapsto f(z, x, y)$ is measurable and for a.a. $z \in \Omega$, $(x, y) \mapsto f(z, x, y)$ is continuous. We assume that $f(z, \cdot, y)$ exhibits $(p-1)$-linear growth near $+\infty$ and we have nonuniform non-resonance with respect to the principal eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. We look for positive solutions. The dependence of the gradient Du of the perturbation f, removes from consideration a variational approach directly on the equation. Instead our method of proof is topological based on fixed point theory. More precisely, we employ the Leray-Schauder alternative principle. This leads to the existence of a positive smooth solution for problem (1.1).

In the past, singular problems and problems with convection, were investigated mostly separately. For singular problems, we mention the following works: Bai-Gasiński-Papageorgiou [2], Gasiński-Papageorgiou [9], Giacomoni-Schindler-Takáč [13], Hirano-Saccon-Shioji [17], Papageorgiou-Rădulescu [24], Papageorgiou-Rădulescu-Repovš [25], Papageorgiou-Smyrlis [27, 28], Perera-Zhang [29], Sun-WuLong

2010 *Mathematics Subject Classification.* 35J92, 35P30.
Key words and phrases. Singular term; convection term; strong comparison; Leray-Schauder alternative principle.

Submitted November 12, 2018. Published May 3, 2019.

[32]. For problems with convection, we mention the following works Bai-Gasiński-Papageorgiou [1], Faraci-Motreanu-Puglisi [3], de Figueiredo-Girardi-Matzeu [4], Gasiński-Papageorgiou [12], Girardi-Matzeu [14], Huy-Quan-Khanh [19], Papageorgiou-Rădulescu-Repovš [26], Ruiz [30].

2. Preliminaries and hypotheses

If V and W are two Banach spaces, a map $h\colon V \to W$ is said to be "compact" if it is continuous and maps bounded sets in V onto relatively compact sets in W. As we already mentioned in the Introduction, we will use the Leray-Schauder alternative principle which we recall below (see e.g., Gasiński-Papageorgiou [7, p. 827]).

Theorem 2.1. *If X is a Banach space and $h\colon X \to X$ is compact, then exactly one of the following holds:*

(a) *h has a fixed point;*
(b) *the set $K = \{x \in X : \ x = th(x),\ 0 < t < 1\}$ is unbounded.*

In the analysis of problem (1.1) we will use the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}.$$

By $\|\cdot\|$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. On account of Poincaré's inequality, we can have

$$\|u\| = \|Du\|_p \quad \forall u \in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : \ u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \big\{u \in C_+ : \ u(z) > 0 \ \forall z \in \Omega,\ \frac{\partial u}{\partial n}|_{\partial\Omega} < 0\big\}.$$

Here $\frac{\partial u}{\partial n}$ denotes the normal derivative of u, that is $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$ with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

We know that $W_0^{1,p}(\Omega)^* = W^{-1,p'}(\Omega)$ (where $\frac{1}{p} + \frac{1}{p'} = 1$). Let $A\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ be the nonlinear operator defined by

$$\langle A(u), h\rangle = \int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \forall u, h \in W_0^{1,p}(\Omega).$$

This operator has the following properties (see Gasiński-Papageorgiou [11, Problem 2.192, p.279] or [8, Lemma 3.2]).

Proposition 2.2. *The map $A\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$; that is,*

"if $u_n \to u$ weakly in $W_0^{1,p}(\Omega)$ and $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leqslant 0$, then $u_n \to u$ in $W_0^{1,p}(\Omega)$."

Consider the nonlinear eigenvalue problem

$$\begin{gathered} -\Delta_p u(z) = \widehat{\lambda}|u(z)|^{p-2}u(z) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{gathered} \tag{2.1}$$

This problem has a smallest eigenvalue $\widehat{\lambda}_1$, which has the following properties:

- $\widehat{\lambda}_1 > 0$ and is isolated (that is, if $\widehat{\sigma}(p)$ is the spectrum of (2.1), we can find $\varepsilon > 0$ such that $(\widehat{\lambda}_1, \widehat{\lambda}_1 + \varepsilon) \cap \widehat{\sigma}(p) = \emptyset$).
- $\widehat{\lambda}_1$ is simple (that is, if $\widehat{u}, \widehat{v} \in W_0^{1,p}(\Omega)$ are eigenfunctions corresponding to $\widehat{\lambda}_1$, then $\widehat{u} = \xi\widehat{v}$ for some $\xi \in \mathbb{R} \setminus \{0\}$).
- We have

$$\widehat{\lambda}_1 = \inf\Big\{\frac{\|Du\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega),\ u \neq 0\Big\}. \tag{2.2}$$

  The infimum in (2.2) is realized on the corresponding one-dimensional eigenspace.

The nonlinear regularity theory of Lieberman [21], implies that if $\widehat{u}$ is an eigenvalue of (2.1), then $\widehat{u} \in C_0^1(\overline{\Omega})$. The above properties of $\widehat{\lambda}_1$ imply that the eigenfunctions corresponding to $\widehat{\lambda}_1$ do not change sign.

By $\widehat{u}_1$ we denote the positive, L^p-normalized (that is, $\|\widehat{u}_1\|_p = 1$) eigenfunction corresponding to $\widehat{\lambda}_1 > 0$. From the nonlinear maximum principle (see e.g., Gasiński-Papageorgiou [7, p. 738]), we have that $\widehat{u}_1 \in \operatorname{int} C_+$. Using these properties, we can easily prove the following result (see Filippakis-Gasiński-Papageorgiou [5, Lemma 3.2] or Motreanu-Motreanu-Papageorgiou [23, p. 305]).

Lemma 2.3. *Let $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \leqslant \widehat{\lambda}_1$ for a.a. $z \in \Omega$ and the inequality is strict on a set of positive measure, then there exists $c_0 > 0$ such that*

$$\|Du\|_p^p - \int_\Omega \vartheta(z)|u|^p\,dz \geqslant c_0\|u\|^p \quad \forall u \in W_0^{1,p}(\Omega).$$

For $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then given $u \in W_0^{1,p}(\Omega)$, we set $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W_0^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

The hypotheses on the perturbation term f are the following:

(H1) $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that $f(z,0,y) = 0$ for a.a. $z \in \Omega$, all $y \in \mathbb{R}^N$, $f(z,x,y) = f_0(z,y)$ for a.a. $z \in \Omega$, all $x \leqslant 0$, all $y \in \mathbb{R}^N$ with f_0 being a Carathéodory function such that $f_0 \geqslant 0$ and

(i) we have

$$f(z,x,y) \leqslant a(z) + \vartheta(z)x^{p-1} + c|y|^{p-1} \quad \text{for a.a. } z \in \Omega,\ \text{all } x \geqslant 0,\ y \in \mathbb{R}^N,$$

with $a, \vartheta \in L^\infty(\Omega)$, $0 < c < \widehat{\lambda}_1^{1/p}$, $\vartheta(z) \leqslant (1 - \frac{c}{\widehat{\lambda}_1^{1/p}})\widehat{\lambda}_1$ a.e. on Ω and the last inequality is strict on a set of positive measure;

(ii) there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ there exists $c_\delta > 0$ such that

$$0 < c_\delta \leqslant f(z,x,y) \quad \text{for a.a. } z \in \Omega,\ \text{all } 0 < \delta \leqslant x \leqslant \delta_0,\ y \in \mathbb{R}^N;$$

(iii) for every $\varrho > 0$, there exists $\widehat{\xi}_\varrho > 0$ such that for a.a. $z \in \Omega$, all $|y| \leqslant \varrho$, the map $x \mapsto f(z,x,y) + \widehat{\xi}_p x^{p-1}$ is nondecreasing on $[0, \varrho]$;

(iv) for a.a. $z \in \Omega$, all $x \geqslant 0$, $y \in \mathbb{R}^N$ and $t \in (0,1)$, we have

$$f\big(z, \tfrac{1}{t}x, y\big) \leqslant \tfrac{1}{t^{p-1}} f(z,x,y).$$

Remark 2.4. Hypothesis (H1)(i) implies that asymptotically at $+\infty$ we may have nonuniform non-resonance with respect to the principal eigenvalue $\widehat{\lambda}_1 > 0$. Hypothesis $H(f)(iv)$ is satisfied if for a.a. $z \in \Omega$, all $y \in \mathbb{R}^N$, the function

$$x \mapsto \frac{f(z,x,y)}{x^{p-1}}$$

is non-increasing on $(0,+\infty)$.

Example 2.5. The following function satisfies hypotheses (H1). For the sake of simplicity we drop the z-dependence.

$$f(x,y) = \begin{cases} 0 & \text{if } x < 0, \\ \widehat{\vartheta}(x^{p-1} - x^{\tau-1}) + \eta|y|^{p-1} & \text{if } 0 \leqslant x \leqslant 1, \quad \forall y \in \mathbb{R}^N, \\ \vartheta[x^{p-1} - x^{q-1}] + \eta|y|^{p-1} & \text{if } 1 < x, \end{cases}$$

with $0 < \eta < \widehat{\lambda}_1^{1/p}$, $0 < \vartheta < (1 - \frac{\eta}{\widehat{\lambda}_1^{1/p}})\widehat{\lambda}_1$, $\widehat{\vartheta} > 0$, $1 < q < p < \tau < +\infty$.

3. Positive solutions

We start by considering the purely singular problem

$$\begin{gathered} -\Delta_p u(z) = u(z)^{-\gamma} \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u > 0. \end{gathered} \tag{3.1}$$

From Papageorgiou-Smyrlis [28, Proposition 5], we have the following result.

Proposition 3.1. *Problem* (3.1) *admits a unique positive solution* $\overline{u} \in \operatorname{int} C_+$.

Let $\delta_0 > 0$ be as postulated by hypothesis (H1)(ii). We choose $t \in (0,1)$ small such that

$$\widetilde{u} = t\overline{u} \leqslant \delta_0. \tag{3.2}$$

For every $y \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned} -\Delta_p \widetilde{u}(z) &= t^{p-1}[-\Delta_p \overline{u}(z)] = t^{p-1}\overline{u}(z)^{-\gamma} = t^{p-1+\gamma}\widetilde{u}(z)^{-\gamma} \\ &< \widetilde{u}(z)^{-\gamma} + f(z, \widetilde{u}(z), Dy(z)) \quad \text{for a.a. } z \in \Omega, \end{aligned} \tag{3.3}$$

(see (3.2) and hypothesis (H1)(ii)).

Given $v \in C_0^1(\overline{\Omega})$, we consider the nonlinear Dirichlet problem

$$\begin{gathered} -\Delta_p u(z) = u(z)^{-\gamma} + f(z, u(z), Dv(z)) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \ u > 0, \end{gathered} \tag{3.4}$$

Proposition 3.2. *If hypotheses* (H1) *hold and* $v \in C_0^1(\overline{\Omega})$*, then problem* (3.4) *admits a positive solution* $u_v \in \operatorname{int} C_+$ *and* $\widetilde{u} \leqslant u_v$.

Proof. We consider the following truncation of the reaction in problem (1.1),

$$\widehat{f}_v(z,x) = \begin{cases} \widetilde{u}(z)^{-\gamma} + f(z, \widetilde{u}(z), Dv(z)) & \text{if } x \leqslant \widetilde{u}(z), \\ x^{-\gamma} + f(z,x,Dv(z)) & \text{if } \widetilde{u}(z) < x. \end{cases} \tag{3.5}$$

Evidently this is a Carathéodory function.

Since $\widetilde{u}, \widehat{u}_1 \in \operatorname{int} C_+$, on account of [22, Proposition 2.1], we can find $c_1 > 0$ such that $\widehat{u}_1 \leqslant c_1 \widetilde{u}^{p'}$, so

$$\widehat{u}_1^{1/p'} \leqslant c_1^{1/p'} \widetilde{u},$$

thus

$$\widetilde{u}^{-\gamma} \leqslant c_2 \widehat{u}_1^{-\gamma/p'},$$

for some $c_2 > 0$.

Using a Lemma in Lazer-McKenna [20], we have that $\widehat{u}_1^{-\gamma/p'} \in L^{p'}(\Omega)$. Therefore

$$\widetilde{u}^{-\gamma} \in L^{p'}(\Omega). \tag{3.6}$$

We set

$$\widehat{F}_v(z,x) = \int_0^x \widehat{f}_v(z,s)\,ds$$

and consider the functional $\widehat{\varphi}_v \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_v(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega \widehat{F}_v(z,u)\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

From hypothesis (H1)(i) and (3.6), we infer that $\widehat{\varphi}_v \in C^1(W_0^{1,p}(\Omega))$ (see also Papageorgiou-Smyrlis [28, Proposition 3]).

Claim. $\widehat{\varphi}_v$ is coercive.

Clearly it suffices to check when $u(z) \geqslant \widetilde{u}(z)$. We have

$$\begin{aligned}
\widehat{F}_v(z,u(z)) &= \int_0^{u(z)} \widehat{f}_v(z,x)\,dx \\
&= \int_0^{\widetilde{u}(z)} \widehat{f}_v(z,x)\,dz + \int_{\widetilde{u}(z)}^{u(z)} \widehat{f}_v(z,x)\,dx \\
&\leqslant (\widetilde{u}(z)^{-\gamma} + f(z,\widetilde{u}(z),Dv(z)))\widetilde{u}(z) \\
&\quad + \int_{\widetilde{u}(z)}^{u(z)} (\widetilde{u}(z)^{-\gamma} + \widehat{a}(z) + \vartheta(z)x^{p-1})\,dx \\
&\leqslant \widehat{a}_0(z) + \frac{1}{p}\vartheta(z)|u(x)|^p
\end{aligned}$$

with $\widehat{a} \in L^\infty(\Omega)$, $\widehat{a}_0 \in L^{p'}(\Omega)$. Therefore

$$\begin{aligned}
\widehat{\varphi}_v(u) &= \frac{1}{p}\|Du\|_p^p - \int_\Omega \widehat{F}_v(z,u(z))\,dz \\
&\geqslant \frac{1}{p}\Big(\|Du\|_p^p - \int_\Omega \vartheta(z)|u|^p\,dz\Big) - \widehat{c}_1 \\
&\geqslant \widehat{c}_2\|Du\|_p^p - \widehat{c}_1,
\end{aligned}$$

for some $\widehat{c}_1, \widehat{c}_2 > 0$ (see Lemma 2.3). Thus $\widehat{\varphi}_v$ is coercive and so the Claim is proved.

From (3.6) and the Sobolev embedding theorem, we see that $\widehat{\varphi}_v$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_v \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_v(u_v) = \inf_{u \in W_0^{1,p}(\Omega)} \widehat{\varphi}_v(u),$$

so $\widehat{\varphi}_v'(u_v) = 0$ and thus

$$\langle A(u_v), h\rangle = \int_\Omega \widehat{f}_v(z, u_v) h\, dz \quad \forall h \in W_0^{1,p}(\Omega). \tag{3.7}$$

In (3.7) we choose $h = (\widetilde{u} - u_v)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}\langle A(u_v), (\widetilde{u} - u_v)^+\rangle &= \int_\Omega (\widetilde{u}^{-\gamma} + f(z, \widetilde{u}, Dv))(\widetilde{u} - u_v)^+\, dz \\ &\geqslant \langle A(\widetilde{u}), (\widetilde{u} - u_v)^+\rangle\end{aligned}$$

(see (3.5) and (3.3) with $y = v$), so

$$\langle A(\widetilde{u}) - A(u_v), (\widetilde{u} - u_v)^+\rangle \leqslant 0,$$

and

$$\widetilde{u} \leqslant u_v. \tag{3.8}$$

From (3.8), (3.5) and (3.7), we infer that

$$\begin{gathered}-\Delta_p u_v(z) = u_v(z)^{-\gamma} + f(z, u_v(z), Dv(z)) \quad \text{in } \Omega, \\ u_v|_{\partial\Omega} = 0.\end{gathered} \tag{3.9}$$

Then from (3.7) and Giacomoni-Schindler-Takáč [13, Theorem B.1] we get that $u_v \in \operatorname{int} C_+$ (see (3.8)). $\square$

Given $v \in C_0^1(\overline{\Omega})$, let

$$S_v = \{u \in W_0^{1,p}(\Omega) : u \text{ is a solution of (3.4)}, \widetilde{u} \leqslant u\}.$$

From Proposition 3.2 we know that

$$\emptyset \neq S_v \subseteq \operatorname{int} C_+.$$

In the next proposition we prove a useful property of the elements of S_v.

Proposition 3.3. *If hypotheses* (H1) *hold,* $v \in C_0^1(\overline{\Omega})$ *and* $u \in S_v$*, then* $u - \widetilde{u} \in \operatorname{int} C_+$.

Proof. We know that $u \in \operatorname{int} C_+$. Let $\varrho = \|u\|_{C_0^1(\overline{\Omega})}$ and let $\widetilde{\xi}_\varrho > 0$ be as postulated by hypothesis (H1)(iii). We have

$$\begin{aligned}&-\Delta_p \widetilde{u}(z) + \widehat{\xi}_p \widetilde{u}(z)^{p-1} - \widetilde{u}(z)^{-\gamma} \\ &< f(z, \widetilde{u}(z), Dv(z)) + \widehat{\xi}_p \widetilde{u}(z)^{p-1} \\ &\leqslant f(z, u(z), Dv(z)) + \widehat{\xi}_p u(z)^{p-1} \\ &= -\Delta_p u(z) + \widehat{\xi}_p u(z)^{p-1} - u(z)^{-\gamma} \quad \text{for a.a. } z \in \Omega\end{aligned} \tag{3.10}$$

(see (3.3) with $y = v$, hypothesis (H1)(iii), recall that $\widetilde{u} \leqslant u$ and see (3.9)).

We know that

$$\begin{aligned}-\Delta_p \widetilde{u}(z) + \widehat{\xi}_p \widetilde{u}(z)^{p-1} &= t^{p-1}(-\Delta_p \overline{u}(z) + \widehat{\xi}_p \overline{u}(z)^{p-1}) \\ &= t^{p-1}(\overline{u}(z)^{-\gamma} + \widehat{\xi}_p \overline{u}(z)^{p-1}) \\ &= t^{p-1+\gamma}(t\overline{u}(z))^{-\gamma}(1 + \widehat{\xi}_p \overline{u}(z)^{p-1+\gamma}) \\ &< \widetilde{u}(z)^{-\gamma} \quad \text{for a.a. } z \in \Omega\end{aligned}$$

for $t \in (0,1)$ sufficiently small (as $\overline{u} \in L^\infty(\Omega)$ and see Proposition 3.1), so

$$-\Delta_p \widetilde{u}(z) + \widehat{\xi}_p \widetilde{u}(z)^{p-1} - \widetilde{u}(z)^{-\gamma} < 0 \quad \text{for a.a. } z \in \Omega. \tag{3.11}$$

Since $\widetilde{u} \in \operatorname{int} C_+$, for $K \subseteq \Omega$ compact, we have

$$0 < \delta_K \leqslant \widetilde{u}(z) \quad \forall z \in K.$$

Then hypothesis (H1)(ii) implies that there exists $c_K = c_{\delta_K} > 0$ such that

$$0 < c_K \leqslant f(z, \widetilde{u}(z), Dv(z)) \quad \text{for a.a. } z \in K. \tag{3.12}$$

From (3.10), (3.11), (3.12) and Papageorgiou-Smyrlis [28, Proposition 4] (the strong comparison principle), we have that $u - \widetilde{u} \in \operatorname{int} C_+$. □

Next we show that the set S_v has a smallest element, that is there exists $\widehat{u}_v \in S_v$ such that $\widehat{u}_v \leqslant u$ for all $u \in S_v$.

Proposition 3.4. *If hypotheses* (H1) *hold and* $v \in C_0^1(\overline{\Omega})$, *then there exists* $\widehat{u}_v \in S_v$ *such that* $\widehat{u}_v \leqslant u$ *for all* $u \in S_v$.

Proof. From Filippakis-Papageorgiou [6] we know that S_v is downward directed (that is, if $u, \widehat{u} \in S_v$, then there exists $y \in S_v$ such that $y \leqslant u$, $y \leqslant \widehat{u}$). Invoking Hu-Papageorgiou [18, Lemma 3.10, p. 178], we can find a decreasing sequence $\{u_n\}_{n \geqslant 1} \subseteq S_v$ such that

$$\inf S_v = \inf_{n \geqslant 1} u_n.$$

We have

$$\langle A(u_n), h \rangle = \int_\Omega (u_n^{-\gamma} + f(z, u_n, Dv)) h \, dz \quad \forall h \in W_0^{1,p}(\Omega),\ n \geqslant 1. \tag{3.13}$$

Let $h = u_n \in W_0^{1,p}(\Omega)$ in (3.13). Then

$$\|Du_n\|_p^p = \int_\Omega (u_n^{1-\gamma} + f(z, u_n, Dv) u_n) \, dz,$$

so

$$\|Du_n\|_p^p \leqslant c_3 \quad \forall n \geqslant 1,$$

for some $c_3 > 0$. Here we used that $0 \leqslant u_n \leqslant u_1 \in \operatorname{int} C_+$ for all $n \geqslant 1$ and Hewitt-Stromberg [16, Theorem 13.17, p. 196] and hypothesis (H1)(i). It follows that the sequence $\{u_n\}_{n \geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, passing to a subsequence if necessary, we may assume that

$$u_n \to \widehat{u}_v \text{ weakly in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to \widehat{u}_v \text{ in } L^p(\Omega). \tag{3.14}$$

In (3.13) we choose $h = u_n - \widehat{u}_v \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (3.14) and (3.6). Then

$$\lim_{n \to +\infty} \langle A(u_n), u_n - \widehat{u}_v \rangle = 0,$$

so

$$u_n \to \widehat{u}_v \quad \text{in } W_0^{1,p}(\Omega) \tag{3.15}$$

(see Proposition 2.2).

If in (3.13) we pass to the limit as $n \to +\infty$ and use (3.15), then we obtain

$$\langle A(\widehat{u}_v), h \rangle = \int_\Omega (\widehat{u}_v^{-\gamma} + f(z, \widehat{u}_v, Dv)) h \, dz \quad \forall h \in W_0^{1,p}(\Omega),$$

so $\widehat{u}_v \in S_v \subseteq \operatorname{int} C_+$ and $\widehat{u}_v = \inf S_v$. □

We define a map $g\colon C_0^1(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ by setting

$$g(v) = \widehat{u}_v.$$

This map is well-defined and clearly a fixed point of g is a solution of (1.1). To produce a fixed point of g, we will use the Leray-Schauder alternative principle (see Theorem 2.1). To this end, we need to show that the minimal solution map g is compact (that is, g is continuous and maps bounded sets to relatively compact sets). The next lemma will be useful in this respect.

Lemma 3.5. *If hypotheses* (H1) *hold,* $\{v_n\}_{n\geqslant 1} \subseteq C_0^1(\overline{\Omega})$*,* $v_n \to v$ *in* $C_0^1(\overline{\Omega})$ *and* $u \in S_v$*, then we can find* $u_n \in S_{v_n}$ *for* $n \geqslant 1$ *such that* $u_n \to u$ *in* $C_0^1(\overline{\Omega})$*.*

Proof. We start by considering the nonlinear Dirichlet problem

$$\begin{gathered} -\Delta_p y(z) = u(z)^{-\gamma} + f(z, u(z), Dv_n(z)) \quad \text{in } \Omega, \\ y|_{\partial\Omega} = 0, \end{gathered} \tag{3.16}$$

for $n \geqslant 1$. As in the proof of Proposition 3.2, using Marano-Papageorgiou [22, Proposition 2.1] and a Lemma by Lazer-McKenna [20], we have that $u^{-\gamma} \in L^q(\Omega)$ with $q > N$. We set

$$k_n(z) = u(z)^{-\gamma} + f(z, u(z), Dv_n(z)).$$

Then hypothesis (H1)(i) implies that

$$k_n \in L^q(\Omega), \quad k_n \geqslant 0, \quad k_n \not\equiv 0, \quad \|k_n\|_q \leqslant c_4 \quad \forall n \geqslant 1,$$

for some $c_4 > 0$. Hence problem (3.16) has a unique solution $y_n^0 \in W_0^{1,p}(\Omega)$, $y_n^0 \geqslant 0$, $y_n^0 \not\equiv 0$ and using Guedda-Véron [15, Proposition 1.3], we have

$$y_n^0 \in L^\infty(\Omega), \quad \|y_n^0\|_\infty \leqslant c_5 \quad \forall n \geqslant 1, \tag{3.17}$$

for some $c_5 > 0$. Consider the linear Dirichlet problem

$$\begin{gathered} -\Delta w(z) = k_n(z) \quad \text{in } \Omega, \\ w|_{\partial\Omega} = 0 \end{gathered}$$

for all $n \geqslant 1$. Standard regularity theory (see e.g., Struwe [31, p. 218]), implies that this problem has a unique solution w_n such that

$$w_n \in W_0^{2,q}(\Omega) \subseteq C_0^{1,\alpha}(\overline{\Omega}) = C^{1,\alpha}(\overline{\Omega}) \cap C_0^1(\overline{\Omega}), \quad \|w_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \leqslant c_6 \quad \forall n \geqslant 1,$$

with $\alpha = q - \frac{N}{q} > 0$ and for some $c_6 > 0$. We put $\sigma_n(z) = \nabla w_n(z)$ for all $z \in \overline{\Omega}$ and all $n \geqslant 1$. Evidently $\sigma_n \in C^\alpha(\overline{\Omega})$ for all $n \geqslant 1$. Then from (3.16) we see that y_n^0 satisfies

$$\begin{gathered} -\operatorname{div}\big(|\nabla y_n^0(z)|^{p-2}\nabla y_n^0(z) - \sigma_n(z)\big) = 0 \quad \text{in } \Omega, \\ y_n^0|_{\partial\Omega} = 0, \end{gathered}$$

for $n \geqslant 1$. Invoking Lieberman [21, Theorem 1] (see also Guedda-Véron [15, Corollary 1.1]) and using (3.17), we infer that there exists $\beta \in (0,1)$ and $c_7 > 0$ such that

$$y_n^0 \in C_0^{1,\beta}(\overline{\Omega}) \cap \operatorname{int} C_+, \quad \|y_n^0\|_{C_0^{1,\beta}(\overline{\Omega})} \leqslant c_7 \quad \forall n \geqslant 1. \tag{3.18}$$

Recall that $C_0^{1,\beta}(\overline{\Omega})$ is embedded compactly in $C_0^1(\overline{\Omega})$. So, from (3.18) it follows that there exists a subsequence $\{y_{n_k}^0\}_{k\geqslant 1}$ of $\{y_n^0\}_{n\geqslant 1}$ such that

$$y_{n_k}^0 \to y^0 \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{as } k \to +\infty, \tag{3.19}$$

with $y^0 \geqslant 0$. Note that

$$k_n \to k \quad \text{in } L^q(\Omega), \tag{3.20}$$

with $k(z) = u(z)^{-\gamma} + f(z, u(z), Dv(z))$. From (3.16), (3.19), (3.20), in the limit as $n \to +\infty$, we have

$$\begin{gathered} -\Delta_p y^0(z) = k(z) \quad \text{in } \Omega, \\ y^0|_{\partial\Omega} = 0. \end{gathered} \tag{3.21}$$

This problem has a unique solution $y^0 \in C_0^1(\overline{\Omega})$. On the other hand, since $u \in S_v$, from (3.20) it follows that u also solves (3.21). Hence $y^0 = u$. It follows that for the original sequence we have

$$y_n^0 \to u \quad \text{in } C_0^1(\overline{\Omega}). \tag{3.22}$$

Next we consider the nonlinear Dirichlet problem

$$\begin{gathered} -\Delta_p y(z) = y_n^0(z)^{-\gamma} + f(z, y_n^0(z), Dv_n(z)) \quad \text{in } \Omega, \\ y_n^0|_{\partial\Omega} = 0, \end{gathered}$$

for $n \geqslant 1$. Again this problem has a unique solution $y_n^1 \in \operatorname{int} C_+$ for $n \geqslant 1$ and as above (see (3.22)), we have

$$y_n^1 \to u \quad \text{in } C_0^1(\overline{\Omega}).$$

Continuing this way, we generate a sequence $\{y_n^k\}_{n\geqslant 1} \subseteq \operatorname{int} C_+$ for all $k \geqslant 1$ such that

$$\begin{gathered} -\Delta_p y_n^k(z) = y_n^{k-1}(z)^{-\gamma} + f(z, y_n^{k-1}(z), Dv_n(z)) \quad \text{in } \Omega, \\ y_n^k|_{\partial\Omega} = 0, \end{gathered} \tag{3.23}$$

for $k, n \geqslant 1$ and

$$y_n^k \to u \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{as } n \to +\infty \quad \forall k \geqslant 1. \tag{3.24}$$

As before from (3.23) and Lieberman [21, Theorem 1], we know that $\{y_n^k\}_{k\geqslant 1} \subseteq C_0^1(\overline{\Omega})$ is relatively compact.

So, we can find a subsequence $\{y_n^{k_m}\}_{m\geqslant 1}$ of $\{y_n^k\}_{k\geqslant 1}$ such that

$$y_n^{k_m} \to \widehat{y}_n \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{as } m \to +\infty \ \forall n \geqslant 1.$$

From (3.23) in the limit as $m \to +\infty$, we obtain

$$\begin{gathered} -\Delta_p \widehat{y}_n(z) = \widehat{y}_n(z)^{-\gamma} + f(z, \widehat{y}_n(z), Dv_n(z)) \quad \text{in } \Omega, \\ \widehat{y}_n|_{\partial\Omega} = 0, \end{gathered} \tag{3.25}$$

for $n \geqslant 1$.

From (3.25) we have

$$\|D\widehat{y}_n\|_p^p = \int_\Omega \widehat{y}_n^{1-\gamma}\, dz + \int_\Omega f(z, \widehat{y}_n, Dv_n)\widehat{y}_n\, dz \leqslant \widehat{c}_3 + \int_\Omega \vartheta(z)\widehat{y}_n^p\, dz$$

for some $\widehat{c}_3 > 0$, so

$$\|D\widehat{y}_n\|_p^p - \int_\Omega \vartheta(z)\widehat{y}_n^p\, dz \leqslant \widehat{c}_3$$

and hence the sequence $\{\widehat{y}_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded (by Lemma 2.3).

From this and Lieberman [21, Theorem 1], it follows that the sequence $\{\widehat{y}_n\}_{n\geqslant 1} \subseteq C_0^1(\overline{\Omega})$ is relatively compact. Passing to a subsequence if necessary, we may assume that

$$\widehat{y}_n \to \widehat{u} \quad \text{in } C_0^1(\overline{\Omega}).$$

By the double limit lemma (see e.g., Gasiński-Papageorgiou [10, Problem 1.175, p. 61]), we have

$$y_n^{k_m(n)} \to \widehat{u} \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{as } n \to +\infty.$$

If $\widehat{u} \neq u$, then $0 < \varepsilon_0 \leqslant \|u - \widehat{u}\|_{C_0^1(\overline{\Omega})}$, so

$$0 < \frac{\varepsilon_0}{2} \leqslant \|u - y_n^{k_m(n)}\|_{C_0^1(\overline{\Omega})} \quad \forall n \geqslant n_0,$$

a contradiction (see (3.24)). So, we have

$$\widehat{y}_n \to u \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{as } n \to +\infty.$$

Recall that $u - \widetilde{u} \in \operatorname{int} C_+$ (see Proposition 3.3). So, it follows that

$$\widehat{y}_n - \widetilde{u} \in \operatorname{int} C_+ \quad \forall n \geqslant n_0,$$

and $\widehat{y}_n \in S_{v_n} \quad \forall n \geqslant n_0$ (see (3.25)). □

Using this lemma, we can show that the minimal solution map is compact.

Proposition 3.6. *If hypotheses* (H1) *hold, then the minimal solution map* $g\colon C_0^1(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ *defined by* $g(v) = \widehat{u}_v$ *is compact.*

Proof. First we show that g is continuous. To this end let $v_n \to v$ in $C_0^1(\overline{\Omega})$. We set $\widehat{u}_n = \widehat{u}_{v_n} = g(v_n)$ for all $n \geqslant 1$. We have

$$\begin{gathered} -\Delta_p \widehat{u}_n(z) = \widehat{u}_n(z)^{-\gamma} + f(z, \widehat{u}_n(z), Dv_n(z)) \quad \text{in } \Omega, \\ \widehat{u}_n|_{\partial\Omega} = 0, \end{gathered} \tag{3.26}$$

for $n \geqslant 1$.

As in the proof of Lemma 3.5, using Guedda-Véron [15, Proposition 1.3] and Lieberman [21, Theorem 1], we have that the sequence $\{\widehat{u}_n\}_{n\geqslant 1} \subseteq C_0^1(\overline{\Omega})$ is relatively compact (see also Giacomoni-Schindler-Takáč [13, Theorem B.1]). So, we may assume that

$$\widehat{u}_n \to \widehat{u}_0 \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{as } n \to +\infty. \tag{3.27}$$

Passing to the limit as $n \to +\infty$ in (3.26) and using (3.27), we obtain that

$$\widehat{u}_0 \in S_v. \tag{3.28}$$

From Lemma 3.5, we know that we can find $u_n \in S_{v_n}$ for $n \geqslant 1$ such that

$$u_n \to \widehat{u} = \widehat{u}_v = g(v) \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{as } n \to +\infty. \tag{3.29}$$

We have $\widehat{u}_n \leqslant u_n \quad \forall n \geqslant 1$, so

$$\widehat{u}_0 \leqslant \widehat{u} = g(v)$$

(see (3.27) and (3.29)). Since $\widehat{u}_0 \in S_v$ (see (3.28)), we conclude that

$$\widehat{u}_0 = g(v) = \widehat{u}.$$

Therefore for the original sequence we have $\widehat{u}_n \to \widehat{u}$ in $C_0^1(\overline{\Omega})$; thus g is continuous.

Also, if $B \subseteq C_0^1(\overline{\Omega})$ is bounded, then as before via the results by Guedda-Véron [15] and Lieberman [21], we obtain that $g(B) \subseteq C_0^1(\overline{\Omega})$ is relatively compact and thus g is compact. □

Now we can employ the Leray-Schauder alternative principle (see Theorem 2.1) to produce a positive solution to problem (1.1).

Theorem 3.7. *If hypotheses* (H1) *hold, then problem* (1.1) *admits a positive solution* $\widehat{u}_0 \in \operatorname{int} C_+$.

Proof. From Proposition 3.6 we know that the minimal solution map $g\colon C_0^1(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ is compact. Let $K \subseteq C_0^1(\overline{\Omega})$ be the set

$$K = \{u \in C_0^1(\overline{\Omega}) : \ u = tg(u),\ 0 < t < 1\}.$$

If $u \in K$, then $\frac{1}{t}u = g(u)$, so

$$-\Delta_p u(z) = t^{p-1}\Big(\frac{t^\gamma}{u(z)^\gamma} + f(z, \frac{1}{t}u(z), Du(z))\Big) \quad \text{a.e. in } \Omega. \tag{3.30}$$

Hypothesis (H1)(iv) implies that

$$f(z, \tfrac{1}{t}u(z), Du(z)) \leqslant \frac{1}{t^{p-1}} f(z, u(z), Du(z)) \quad \text{for a.a. } z \in \Omega. \tag{3.31}$$

Returning to (3.30) and using (3.31) and hypothesis (H1)(i), we have

$$\begin{aligned}
-\Delta_p u(z) &\leqslant \frac{t^{p+\gamma-1}}{u(z)^\gamma} + f(z, u(z), Du(z)) \\
&\leqslant \frac{1}{\widetilde{u}(z)^\gamma} + a(z) + \vartheta(z)u(z)^{p-1} + c|Du(z)|^{p-1},
\end{aligned} \tag{3.32}$$

for a.a. $z \in \Omega$, so

$$\begin{aligned}
\|Du\|_p^p &\leqslant \widehat{c}_4 + \int_\Omega \vartheta(z)u^p\,dz + c\int_\Omega |Du|^{p-1}u\,dz \\
&\leqslant \widehat{c}_4 + \int_\Omega \vartheta(z)u^p\,dz + c\|Du\|_p^{p-1}\|u\|_p \\
&\leqslant \widehat{c}_4 + \int_\Omega \vartheta(z)u^p\,dz + \frac{c}{\widehat{\lambda}_1^{1/p}}\|Du\|_p^p,
\end{aligned}$$

for some $\widehat{c}_4 > 0$ (by Hölder's inequality and using (2.2)), thus

$$\Big(1 - \frac{c}{\widehat{\lambda}_1^{1/p}}\Big)\|Du\|_p^p - \int_\Omega \vartheta(z)u^p\,dz \le \widehat{c}_4,$$

hence, by Lemma 2.3, we have

$$\widehat{c}_5\|Du\|_p^p \leqslant \widehat{c}_4,$$

for some $\widehat{c}_5 > 0$. This proves the boundedness of $K \subseteq W_0^{1,p}(\Omega)$.

Invoking Theorem 2.1 (the Leray-Schauder alternative principle), we can find $\widehat{u}_0 \in C_0^1(\overline{\Omega})$ such that

$$\widehat{u}_0 = g(\widehat{u}_0) \in S_{\widehat{u}_0} \subseteq \operatorname{int} C_+.$$

This is a positive solution of (1.1). $\square$

Remark 3.8. It will be interesting to know if we can have multiplicity of positive solutions (for example a pair of positive solutions). For purely singular elliptic problem such a result was proved by Papageorgiou-Rădulescu-Repovš [25]. Also another interesting open problem is whether we can treat resonant equations.

Acknowledgments. L. Gasiński was supported by the National Science Center of Poland under Project No. 2015/19/B/ST1/01169, and the H2020-MSCA-RISE-2018 Research and Innovation Staff Exchange Scheme Fellowship within the Project no. 823731CONMECH.

References

[1] Bai Y.; Gasiński, L.; Papageorgiou, N. S.; *Nonlinear nonhomogeneous Robin problems with dependence on the gradient*, Bound. Value Probl., 2018, Paper No. 17, 24.

[2] Bai Y.; Gasiński, L.; Papageorgiou, N. S.; *Positive solutions for nonlinear singular superlinear elliptic equations*, Positivity, published online, doi: 10.1007/s11117-018-0636-8.

[3] Faraci, F.; Motreanu, D.; Puglisi, D.; *Positive solutions of quasi-linear elliptic equations with dependence on the gradient*, Calc. Var. Partial Differential Equations, 54:1 (2015), 525–538.

[4] de Figueiredo, D.; Girardi, M.; Matzeu, M.; *Semilinear elliptic equations with dependence on the gradient via mountain-pass techniques*, Differential Integral Equations, 17:1–2 (2004), 119–126.

[5] Filippakis, M. E.; Gasiński L.; Papageorgiou, N. S.; *On the existence of positive solutions for hemivariational inequalities driven by the p-Laplacian*, J. Global Optim., 31:1 (2005), 173–189.

[6] Filippakis, M. E.; Papageorgiou, N. S.; *Multiple constant sign and nodal solutions for nonlinear elliptic equations with the p-Laplacian*, J. Differential Equations, 245:7 (2008), 1883–1922.

[7] Gasiński, L.; Papageorgiou, N. S.; *Nonlinear Analysis.* Chapman & Hall/CRC, Boca Raton, FL, 2006.

[8] Gasiński, L.; Papageorgiou, N. S.; *Nodal and multiple constant sign solutions for resonant p-Laplacian equations with a nonsmooth potential*, Nonlinear Anal., 71:11 (2009), 5747–5772.

[9] Gasiński, L.; Papageorgiou, N. S.; *Nonlinear elliptic equations with singular terms and combined nonlinearities*, Ann. Henri Poincaré, 13:3 (2012), 481–512.

[10] Gasiński, L.; Papageorgiou, N. S.; *Exercises in Analysis. Part 1*, Problem Books in Mathematics, Springer, Cham, 2014.

[11] Gasiński L.; Papageorgiou N. S.; *Exercises in Analysis. Part 2. Nonlinear Analysis*, Springer, Cham (2016).

[12] Gasiński, L.; Papageorgiou, N. S.; *Positive solutions for nonlinear elliptic problems with dependence on the gradient*, J. Differential Equations, 263:2 (2017), 1451–1476.

[13] Giacomoni, J.; Schindler, I.; Takáč, P.; *Sobolev versus Hölder local minimizers and existence of multiple solutions for a singular quasilinear equation*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 6:1 (2007), 117–158.

[14] Girardi, M.; Matzeu, M.; *Positive and negative solutions of a quasi-linear elliptic equation by a mountain pass method and truncature techniques*, Nonlinear Anal., 59:1–2 (2004), 199–210.

[15] Guedda, M.; Véron, L.; *Quasilinear elliptic equations involving critical Sobolev exponents*, Nonlinear Anal., 13:8 (1989), 879–902.

[16] Hewitt, E.; Stromberg, K.; *Real and Abstract Analysis*, Springer-Verlag, New York-Heidelberg, 1975.

[17] Hirano, N.; Saccon, C.; Shioji, N.; *Brezis-Nirenberg type theorems and multiplicity of positive solutions for a singular elliptic problem*, J. Differential Equations, 245:8 (2008), 1997–2037.

[18] Hu, S.; Papageorgiou, N. S.; *Handbook of Multivalued Analysis. Vol. I.* Theory. Mathematics and its Applications, 419. Kluwer Academic Publishers, Dordrecht, 1997.

[19] Huy, N. B.; Quan, B. T.; Khanh, N. H.; *Existence and multiplicity results for generalized logistic equations*, Nonlinear Anal., 144 (2016), 77–92.

[20] Lazer, A. C.; McKenna, P. J.; *On a singular nonlinear elliptic boundary-value problem*, Proc. Amer. Math. Soc., 111:3 (1991), 721–730.

[21] Lieberman, G. M.; *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal., 12:11 (1988), 1203–1219.

[22] Marano, S. A.; Papageorgiou, N. S.; *Positive solutions to a Dirichlet problem with p-Laplacian and concave-convex nonlinearity depending on a parameter*, Commun. Pure Appl. Anal., 12:2 (2013), 815–829.

[23] Motreanu, D.; Motreanu, V. V.; Papageorgiou, N. S.; *Topological and Variational Methods with Applications to Nonlinear Boundary Value Problems*, Springer, New York, 2014.

[24] Papageorgiou, N. S.; Rădulescu, V. D.; *Combined effects of singular and sublinear nonlinearities in some elliptic problems*, Nonlinear Anal., 109 (2014), 236–244.

[25] Papageorgiou, N. S.; Rădulescu, V. D.; Repovš, D. D.; *Pairs of positive solutions for resonant singular equations with the p-Laplacian*, Electron. J. Differential Equations, 2017, Paper No. 249, 1-13.

[26] Papageorgiou, N. S.; Rădulescu, V. D.; Repovš, D. D.; *Nonlinear elliptic inclusions with unilateral constraint and dependence on the gradient*, Appl. Math. Optim., 79 (2018), 1–23.
[27] Papageorgiou, N. S.; Smyrlis, G.; *Nonlinear elliptic equations with singular reaction*, Osaka J. Math., 53:2 (2016), 489–514.
[28] Papageorgiou, N. S.; Smyrlis, G.; *A bifurcation-type theorem for singular nonlinear elliptic equations*, Methods Appl. Anal., 22:2 (2015), 147–170.
[29] Perera, K.; Zhang, Z.; *Multiple positive solutions of singular p-Laplacian problems by variational methods*, Bound. Value Probl., 2005:3 (2005) 377–382.
[30] Ruiz, D.; *A priori estimates and existence of positive solutions for strongly nonlinear problems*, J. Differential Equations, 199:1 (2004), 96–114.
[31] Struwe, M.; *Variational Methods*, Springer-Verlag, Berlin, 2008.
[32] Sun, Y.; Wu, S.; Long, Y.; *Combined effects of singular and superlinear nonlinearities in some singular boundary value problems*, J. Differential Equations, 176:2, (2001), 511–531.

Yunru Bai
Jagiellonian University, Faculty of Mathematics and Computer Science, ul. Łojasiewicza 6, 30-348 Kraków, Poland
Email address: yunrubai@163.com

Leszek Gasiński
Department of Mathematics, Pedagogical University of Cracow, Podchorazych 2, 30-084 Cracow, Poland
Email address: leszek.gasinski@up.krakow.pl

Nikolaos S. Papageorgiou
National Technical University, Department of Mathematics, Zografou Campus, Athens 15780, Greece
Email address: npapg@math.ntua.gr