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BIPOLYNOMIAL FRACTIONAL DIRICHLET-LAPLACE PROBLEM

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Abstract. In the article, we derive the existence of solutions for a nonlinear non-autonomous partial elliptic system on an open bounded domain with Dirichlet boundary conditions. This problem contains fractional powers of the weak Dirichlet-Laplace operator in the Stone-von Neumann operator calculus sense. We apply a direct variational method and some results based on the dual least action principle. Both methods give strong solutions of the problem under consideration.

1. Introduction

In this article, we study strong solutions to the problem

$$\sum_{i,j=0}^{k} \alpha_i \alpha_j [(-\Delta)_\omega]^{\beta_i+\beta_j} u(x) - au(x) = D_u F(x, u(x)), \quad x \in \Omega \text{ a.e.}, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, $a \in \mathbb{R}$, $\alpha_i > 0$ for $i = 0, \dots, k$ ($k \in \mathbb{N} \cup \{0\}$) and $0 \le \beta_0 < \beta_1 < \dots < \beta_k$, $[(-\Delta)_\omega]^\gamma$ with a real $\gamma \ge 0$ is a γ-power (in the sense of Stone-von Neumann operator calculus) of a self-adjoint extension $(-\Delta)_\omega : D((-\Delta)_\omega) \subset L^2 \to L^2$ of the Dirichlet-Laplace operator $(-\Delta) : C_c^\infty \subset L^2 \to L^2$. We name this extension the weak Dirichlet-Laplace operator. Moreover, $F : \Omega \times \mathbb{R} \to \mathbb{R}$, $D_u F$ is the partial derivative of F with respect to u, $C_c^\infty = C_c^\infty(\Omega, \mathbb{R})$, $L^2 = L^2(\Omega, \mathbb{R})$ are real spaces of smooth functions with compact supports and square integrable functions, respectively.

Particular cases of the above problem are: the classical Dirichlet-Laplace problem

$$[(-\Delta)_\omega] u(x) - au(x) = D_u F(x, u(x)), \quad x \in \Omega \text{ a.e.}; \tag{1.2}$$

the biharmonic equation (see [17])

$$[(-\Delta)_\omega]^2 u(x) = D_u F(x, u(x)), \quad x \in \Omega \text{ a.e.}; \tag{1.3}$$

and the standard fractional problem

$$[(-\Delta)_\omega]^\beta u(x) = D_u F(x, u(x)), \quad x \in \Omega \text{ a.e.}, \tag{1.4}$$

In recent years, fractional Laplacians (including biharmonic case) have been extensively studied because of their numerous applications. The authors use different approaches to such operators: via Riesz type potential [10, 11, 12, 13, 14, 24], via

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Fourrier transform [7, 10, 13], and a distributional approach [14]. Definition of the fractional Dirichlet-Laplacian adopted in our paper comes from the Stone-von Neumann operator calculus and is based on the spectral integral representation theorem for a self-adjoint operator in Hilbert space. It reduces to a series form which is taken by some authors as a starting point (see [5, 6, 9]). Our approach allows us to obtain useful properties of fractional operators in an effortless way. Let us point out that in all above mentioned papers one considers powers $\gamma \in (0,1)$. The Stone-von Neumann approach allows us to consider any nonnegative powers.

The aim of our paper is to obtain existence results for problem (1.1). First, in the case of $a = 0$, we apply a direct variational method. Such a method was used by other authors (see e.g. [5, 6]) but to problems containing only a single fractional Dirichlet-Laplacian. An important issue of our study is the equivalence of the solutions obtained with the aid of this variational method and the strong solutions, that, to the best of our knowledge, was not noticed up to now. Next, in the general case of any $a \in \mathbb{R}$ (including resonance equation) we apply some results due to Mawhin and Willem ([20, 25]; see also [21]) obtained with the aid of the dual least action principle.

This article consists of three parts. In the first part, we give some basics from the spectral theory of self-adjoint operators in real Hilbert space and Stone-von Neumann operator calculus. In the second part, we investigate selected properties of the powers of the weak Dirichlet-Laplace operator including a connection between weak and strong solutions of equation (1.1). In the third part, we derive existence results for problem (1.1).

2. Self-adjoint operators in real Hilbert space

This subsection contains the results from the theory of self-adjoint operators in real Hilbert space. Results presented in this section comes from [2, 22] where they are derived in the case of complex Hilbert space but their proofs can be moved without any or with small changes to the case of real Hilbert space (one can also consult the book [18]).

Let H be a real Hilbert space and $E : \mathcal{B} \to \Pi(H)$ where $\Pi(H)$ is the set of all projections of H on closed linear subspaces and $\mathcal{B}$ - the σ-algebra of Borel subsets of $\mathbb{R}$, a spectral measure. If $b : \mathbb{R} \to \mathbb{R}$ is a bounded Borel measurable function, defined E - a.e., then, for any $x \in H$ such that

$$\int_{-\infty}^{\infty} |b(\lambda)|^2 \|E(d\lambda)x\|^2 < \infty, \tag{2.1}$$

we define the value $\big(\int_{-\infty}^{\infty} b(\lambda)E(d\lambda)\big)x$ of the operator $\int_{-\infty}^{\infty} b(\lambda)E(d\lambda)$ by

$$\Big(\int_{-\infty}^{\infty} b(\lambda)E(d\lambda)\Big)x = \lim \int_{-\infty}^{\infty} b_n(\lambda)E(d\lambda)x$$

where

$$b_n : \mathbb{R} \ni \lambda \mapsto \begin{cases} b(\lambda) & \text{if } |b(\lambda)| \le n \\ 0 & \text{if } |b(\lambda)| > n \end{cases}$$

for $n \in \mathbb{N}$ and the integrals $\int_{-\infty}^{\infty} b_n(\lambda)E(d\lambda)x$ (with respect to the vector measure $\mathcal{B} \ni P \to E(P)x \in H$) are defined in a standard way, with the aid of the sequence of simple functions converging $E(d\lambda)x$ - a.e. to b (see [15]).

Let us point out that the set D of all points x with property (2.1) is dense linear subspace of H and the operator $\int_{-\infty}^{\infty} b(\lambda)E(d\lambda) : D \subset H \to H$ is self-adjoint.

Remark 2.1. To integrate a Borel measurable function $b : B \to \mathbb{R}$ where B is a Borel set containing the support of the measure E (the complement of the sum of all open subsets of $\mathbb{R}$ with zero spectral measure), it is sufficient to extend b on $\mathbb{R}$ to a whichever Borel measurable function (putting, for example, $b(\lambda) = 0$ for $\lambda \notin B$).

Remark 2.2. If $b : \mathbb{R} \to \mathbb{R}$ is Borel measurable and $\sigma \in \mathcal{B}$, then by the integral $\int_\sigma b(\lambda)E(d\lambda)$ we mean the integral $\int_{-\infty}^{\infty} \chi_\sigma(\lambda)b(\lambda)E(d\lambda)$ where χ_σ is the characteristic function of the set σ.

The next theorem plays the fundamental role in the spectral theory of self-adjoint operators (below, Λ is the support of a spectral measure E and $\sigma(A)$ denotes the spectrum of an operator $A : D(A) \subset H \to H$).

Theorem 2.3. *If $A : D(A) \subset H \to H$ is self-adjoint and the resolvent set $\rho(A)$ is non-empty, then there exists a unique spectral measure E with the closed support $\Lambda = \sigma(A)$, such that*

$$A = \int_{-\infty}^{\infty} \lambda E(d\lambda) = \int_{\sigma(A)} \lambda E(d\lambda).$$

The basic notion in the Stone-von Neumann operator calculus is a function of a self-adjoint operator. Namely, if $A : D(A) \subset H \to H$ is self-adjoint and E is the spectral measure determined according to the above theorem, then, for any Borel measurable function $b : \mathbb{R} \to \mathbb{R}$, one defines the operator $b(A)$ by

$$b(A) = \int_{-\infty}^{\infty} b(\lambda)E(d\lambda) = \int_{\sigma(A)} b(\lambda)E(d\lambda).$$

It is known that the spectrum $\sigma(b(A))$ of $b(A)$ is given by

$$\sigma(b(A)) = \overline{b(\sigma(A))} \tag{2.2}$$

provided that b is continuous (it is sufficient to assume that b is continuous on $\sigma(A)$). We have the following results.

Proposition 2.4. *If E is the spectral measure for a self-adjoint operator $A : D(A) \subset H \to H$ with non-empty resolvent set, then*

$$\alpha_k A^k + \dots + \alpha_1 A + \alpha_0 I = \int_{-\infty}^{\infty} (\alpha_k \lambda^k + \dots + \alpha_1 \lambda^1 + \alpha_0)E(d\lambda)$$

and, for any Borel measurable function $b : \mathbb{R} \to \mathbb{R}$,

$$(b(A))^n = b^n(A)$$

with any fixed positive integer $n \geq 2$.

Now, let $\beta > 0$ and $\sigma(A) \subset [0, \infty)$. According to the Remark 2.1 by A^β we mean the operator

$$A^\beta = \int_{-\infty}^{\infty} b(\lambda)E(d\lambda)$$

where

$$b : \mathbb{R} \ni \lambda \to \begin{cases} \lambda^\beta, & \lambda \geq 0 \\ 0, & \lambda < 0 \,. \end{cases}$$

Proposition 2.5. *If E is the spectral measure for a self-adjoint operator $A : D(A) \subset H \to H$ with $\sigma(A) \subset [0, \infty)$, then*

$$\alpha_k A^{\beta_k} + \dots + \alpha_1 A^{\beta_1} + \alpha_0 A^{\beta_0} = \int_{-\infty}^{\infty} w(\lambda) E(d\lambda).$$

where

$$w : \mathbb{R} \ni \lambda \to \begin{cases} \alpha_k \lambda^{\beta_k} + \dots + \alpha_1 \lambda^{\beta_1} + \alpha_0 \lambda^{\beta_0}, & \lambda \geq 0 \\ 0, & \lambda < 0, \end{cases} \tag{2.3}$$

and $0 \leq \beta_0 < \beta_1 < \dots < \beta_k$. Moreover,

$$A^{\beta_2} \circ A^{\beta_1} = A^{\beta_2 + \beta_1} \tag{2.4}$$

for β_2, $\beta_1 > 0$.

Proposition 2.6. *If E is the spectral measure for a self-adjoint operator $A : D(A) \subset H \to H$ and $b : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function such that $b(\lambda) \neq 0$ a.e. with respect to E, then there exists the inverse operator $[b(A)]^{-1}$ and*

$$[b(A)]^{-1} = \int_{-\infty}^{\infty} \frac{1}{b(\lambda)} E(d\lambda).$$

3. Weak Dirichlet-Laplace operator

In this section we shall present definition and selected properties of the weak Dirichlet-Laplace operator. The last three subsections contain, to the best of our knowledge, the original results not presented up to now.

3.1. **Friedrich's extension.** Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. We shall say that $u : \Omega \to \mathbb{R}$ has a (weak) Dirichlet-Laplacian (see [3]) if $u \in H_0^1$ and there exists a function $g \in L^2$ such that

$$\int_\Omega \nabla u(x) \nabla v(x) dx = \int_\Omega g(x) v(x) dx$$

for any $v \in H_0^1$. The function g will be called the weak Dirichlet-Laplacian and denoted by $(-\Delta)_\omega u$.

Applying the Fridrich's procedure of extension of a densely defined symmetric and positive-definite operator $T_0 : D(T_0) \subset H \to H$ to a self-adjoint one $S : D(S) \subset H \to H$ where H is a real Hilbert space (see [18]), in the case of the classical Dirichlet-Laplace operator

$$T_0 = -\Delta : D(T_0) = C_c^\infty \subset L^2 \to L^2,$$

we state that the domain $D(S)$ coincides with the set of all functions $u : \Omega \to \mathbb{R}$ possessing the weak Dirichlet-Laplacian $(-\Delta)_\omega u$ and

$$Su = (-\Delta)_\omega u$$

for $u \in D(S)$.

Clearly, $-\Delta \subset (-\Delta)_\omega$ where $-\Delta : H_0^1 \cap H^2 \subset L^2 \to L^2$ is the strong Dirichlet-Laplace operator, i.e.

$$H_0^1 \cap H^2 \subset D((-\Delta)_\omega)$$

and $(-\Delta)_\omega u = (-\Delta) u$ for $u \in H_0^1 \cap H^2$.

Finally, we obtain the following result.

Theorem 3.1. *The operator*
$$(-\Delta)_\omega : D((-\Delta)_\omega) \subset L^2 \to L^2$$
is bijective, self-adjoint and $T_0 \subset (-\Delta) \subset (-\Delta)_\omega$.

Remark 3.2. If $N = 1$ and $\Omega = (0, \pi)$, then $(-\Delta) = (-\Delta)_\omega$ because $(-\Delta)$ is self-adjoint and, consequently, no proper self-adjoint extension of it exists. Using the results from [16, 17] one can show that if $\Omega \subset \mathbb{R}^N$ is an open bounded set of class $C^{1,1}$ or $\Omega \subset \mathbb{R}^2$ is an open bounded convex polygon, then $(-\Delta)_\omega = (-\Delta)$. Such an equality in the case of $\Omega \subset \mathbb{R}^N$ being of class C^2 has been derived in [1].

3.2. **Spectrum of $(-\Delta)_\omega$.** Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. It is known (see [3]) that the spectrum of $(-\Delta)_\omega$ consists of denumerable number the eigenvalues λ_j such that $0 < \lambda_1 \le \lambda_2 \le \dots \le \lambda_j \to \infty$ (similarly, as in [3], we count the eigenvalues of $(-\Delta)_\omega$ according to their multiplicity, i.e. each λ_j is repeated k_j times where k_j is the multiplicity of λ_j) and there exists a system $\{e_j\}$ of eigenfunctions of the operator $(-\Delta)_\omega$, corresponding to λ_j, which is a Hilbertian basis in L^2. In [8] it is proved that one can choose $e_j \in H_0^1 \cap C^\infty$. Thus, for any $u \in L^2$ there exist real numbers a_j, $j \in \mathbb{N}$, such that
$$u(t) = \sum a_j e_j(t) \quad \text{in } L^2 \text{ and } \|u\|_{L^2}^2 = \sum |a_j|^2.$$

3.3. **Hilbert space $D([(-\Delta)_\omega]^\beta)$.** Let us fix a number $\beta > 0$, an open bounded set $\Omega \subset \mathbb{R}^N$ and consider the operator
$$[(-\Delta)_\omega]^\beta : D([(-\Delta)_\omega]^\beta) \subset L^2 \to L^2$$
given by
$$([(-\Delta)_\omega]^\beta u)(t) = \Big(\Big(\int_{\sigma((-\Delta)_\omega)} \lambda^\beta E(d\lambda)\Big)u\Big)(t) = \Big(\sum \lambda_j^\beta a_j e_j\Big)(t)$$
where
$$D([(-\Delta)_\omega]^\beta) = \{u(t) \in L^2;\ \int_{\sigma((-\Delta)_\omega)} |\lambda^\beta|^2 \|E(d\lambda)u\|^2 = \sum ((\lambda_j)^\beta)^2 a_j^2 < \infty,$$
where a_j's are such that
$$u(t) = \Big(\int_{\sigma((-\Delta)_\omega)} 1E(d\lambda)u)(t) = (\sum a_j e_j)(t)\},$$
E is the spectral measure given by $(-\Delta)_\omega$ and the convergence of the series is in L^2. Of course, $[(-\Delta)_\omega]^\beta$ is self-adjoint, the spectrum $\sigma([(-\Delta)_\omega]^\beta)$ consists of eigenvalues λ_j^β, $j \in \mathbb{N}$, and eigenspaces corresponding to λ_j^β's are the same as eigenspaces for $(-\Delta)_\omega$, corresponding to λ_j's.

It is clear that if $0 < \beta_1 < \beta_2$, then
$$D([(-\Delta)_\omega]^{\beta_2}) \subset D([(-\Delta)_\omega]^{\beta_1}). \tag{3.1}$$

In $D([(-\Delta)_\omega]^\beta)$, we define the scalar product
$$\langle u, v\rangle_\beta = \langle u, v\rangle_{L^2} + \langle [(-\Delta)_\omega]^\beta u, [(-\Delta)_\omega]^\beta v\rangle_{L^2},$$
and the corresponding norm
$$\|u\|_\beta = (\|u\|_{L^2}^2 + \|[(-\Delta)_\omega]^\beta u\|_{L^2}^2)^{1/2}.$$
Since $[(-\Delta)_\omega]^\beta$ is closed (being self-adjoint operator), therefore it is easy to see that $D([(-\Delta)_\omega]^\beta)$ with the scalar product $\langle \cdot, \cdot\rangle_\beta$ is Hilbert space.

Let us also observe that the scalar product

$$\langle u, v\rangle_{\sim\beta} = \langle [(-\Delta)_\omega]^\beta u, [(-\Delta)_\omega]^\beta v\rangle_{L^2}$$

determines the equivalent norm

$$\|u\|_{\sim\beta} = \|[(-\Delta)_\omega]^\beta u\|_{L^2}.$$

More precisely,

$$\|u\|_{L^2}^2 \le M_\beta \|u\|_{\sim\beta}^2 \tag{3.2}$$

where $M_\beta = 1$ when the set $\{\lambda_j;\ \lambda_j < 1\}$ is empty and $M_\beta = \frac{1}{\lambda_1{}^{2\beta}} > 1$ in the opposite case and, consequently,

$$\|u\|_{\sim\beta} \le \|u\|_\beta \le \sqrt{M_\beta + 1}\|u\|_{\sim\beta}.$$

3.4. **Equivalence of weak and strong solutions.** Let E be the spectral measure for a self-adjoint operator $A : D(A) \subset H \to H$ with non-empty resolvent set and $b : \mathbb{R} \to \mathbb{R}$ - a Borel measurable function, defined E - a.e. Fact that the operator $b(A)$ is self-adjoint means that its domain satisfies the equality

$$\begin{aligned} D(b(A)) = \Big\{ &u \in L^2 : \text{there exists } z \in L^2 \text{ such that} \\ &\int_\Omega u(t)b(A)v(t)dt = \int_\Omega z(t)v(t)dt \text{ for all } v \in D(b(A))\Big\} \end{aligned} \tag{3.3}$$

and

$$b(A)u = z \quad \text{for } u \in D(b(A))\,. \tag{3.4}$$

From Proposition 2.4 it follows that

$$b(A)(b(A)u) = b^2(A)u. \tag{3.5}$$

In particular, $u \in D(b^2(A))$ if and only if $u \in D(b(A))$ and $b(A)u \in D(b(A))$. Using this fact and (3.3), (3.4), we obtain the following result.

Theorem 3.3. *For $g \in L^2$, we have that $u \in D(b^2(A))$ and*

$$b^2(A)u = g \tag{3.6}$$

if and only if $u \in D(b(A))$ and

$$\int_\Omega b(A)u(t)b(A)v(t)dt = \int_\Omega g(t)v(t)dt \tag{3.7}$$

for any $v \in D(b(A))$.

Consequently, if $A : D(A) \subset H \to H$ is self-adjoint with $\sigma(A) \subset [0, \infty)$, w is given by (2.3), then we have the following corollary.

Corollary 3.4. *Assume $g \in L^2$. Then $u \in D(w^2(A))$ and $w^2(A)u = g$ if and only if $u \in D(w(A))$ and*

$$\int_\Omega w(A)u(t)w(A)v(t)dt = \int_\Omega g(t)v(t)dt$$

for any $v \in D(w(A))$.

Clearly $w^2(A) = \sum_{i,j=0}^k \alpha_i\alpha_j A^{\beta_i+\beta_j}$. Moreover if the numbers $\beta_0, \beta_1, \dots, \beta_k$ are non-negative integers, then we can omit the assumption $\sigma(A) \subset [0, \infty)$ and consider the function

$$w(\lambda) = \alpha_k\lambda^{\beta_k} + \dots + \alpha_1\lambda^{\beta_1} + \alpha_0\lambda^{\beta_0}, \quad \lambda \in \mathbb{R}\,.$$

Remark 3.5. The above theorem states that u is the strong solution to problem (3.6) if and only if it is the weak one (in a sense). Consequently, it can be obtained with the aid of a variational method (see Section 4). Let us observe that in the case of $A = (-\Delta)_\omega$, $w(\lambda) = \lambda^{1/2}$ and $\Omega \subset \mathbb{R}^N$ being an open bounded set, the unique strong solution of problem

$$(-\Delta)_\omega u = g$$

(in fact, weak solution to the equation $(-\Delta)u = g$) is a function $u \in H_0^1$ such that

$$\int_\Omega \nabla u(t)\nabla v(t)dt = \int_\Omega g(t)v(t)dt$$

for any $v \in H_0^1$. From the above theorem it follows that $u \in D([(-\Delta)_\omega]^{1/2})$ and

$$\int_\Omega [(-\Delta)_\omega]^{1/2} u(t)[(-\Delta)_\omega]^{\frac{1}{2}} v(t)dt = \int_\Omega g(t)v(t)dt$$

If additionally $\Omega \subset \mathbb{R}^N$ is of class $C^{1,1}$ or convex polygon in $\mathbb{R}^2$, then the unique function $u \in H_0^1$ such that

$$\int_\Omega \nabla u(t)\nabla v(t)dt = \int_\Omega g(t)v(t)dt$$

for any $v \in H_0^1$ belongs to $H_0^1 \cap H^2$ and

$$(-\Delta)u = g.$$

Let us point out that even in the case of $N = 1$ and $\Omega = (0, \pi)$ the operator $[(-\Delta)_\omega]^{1/2} = (-\Delta)^{1/2}$ differs from the operator $H_0^1 \subset L^2 \to L^2$, defined by

$$x \mapsto \nabla x = x'\,.$$

This operator is not self-adjoint. So, we have a new variational approach to the equation $(-\Delta)u = g$.

3.5. **Compactness of the inverse** $(w^2((-\Delta)_\omega))^{-1}$**.** Let us consider the operator $w((-\Delta)_\omega)$ assuming additionally that $\alpha_i > 0$ for $i = 0, \dots, k$. From (3.1) it follows that $D(w((-\Delta)_\omega)) = D([(-\Delta)_\omega]^{\beta_k})$. Introduce in $D(w((-\Delta)_\omega))$ a new scalar product

$$\langle u, v\rangle_{w((-\Delta)_\omega)} = \langle w((-\Delta)_\omega)u, w((-\Delta)_\omega)v\rangle_{L^2}.$$

Lemma 3.6. *The scalar products $\langle \cdot, \cdot\rangle_{\sim\beta_k}$ and $\langle \cdot, \cdot\rangle_{w((-\Delta)_\omega)}$ generate the equivalent norms*

$$\|u\|_{\sim\beta_k} = \|[(-\Delta)_\omega]^{\beta_k} u\|_{L^2}$$

and

$$\|u\|_{w((-\Delta)_\omega)} = \|w((-\Delta)_\omega)u\|_{L^2}$$

in $D(w((-\Delta)_\omega))$ and, consequently, $D(w((-\Delta)_\omega))$ is complete under the scalar product $\langle \cdot, \cdot\rangle_{w((-\Delta)_\omega)}$.

Proof. First, let us observe that if $\beta_i < \beta_j$, then (see (2.4))

$$\begin{aligned}
&\alpha_i\alpha_j\langle [(-\Delta)_\omega]^{\beta_i} u, [(-\Delta)_\omega]^{\beta_j} u\rangle_{L^2}\\
&= \alpha_i\alpha_j\langle [(-\Delta)_\omega]^{\beta_i} u, [(-\Delta)_\omega]^{\beta_j-\beta_i}([(-\Delta)_\omega]^{\beta_i} u)\rangle_{L^2}\\
&= \alpha_i\alpha_j\langle [(-\Delta)_\omega]^{\frac{\beta_j-\beta_i}{2}}([(-\Delta)_\omega]^{\beta_i} u), [(-\Delta)_\omega]^{\frac{\beta_j-\beta_i}{2}}([(-\Delta)_\omega]^{\beta_i} u)\rangle_{L^2}\\
&= \alpha_i\alpha_j\|[(-\Delta)_\omega]^{\frac{\beta_j-\beta_i}{2}+\beta_i} u\|_{L^2}^2 \ge 0.
\end{aligned}$$

Using this property we obtain

$$\begin{aligned}
\|u\|^2_{\sim\beta_k} &= \frac{1}{\alpha_k^2}\|\alpha_k[(-\Delta)_\omega]^{\beta_k}u\|^2_{L^2} \\
&\le \frac{1}{\alpha_k^2}\|w((-\Delta)_\omega)u\|^2_{L^2} \\
&\le \frac{C_1}{\alpha_k^2}\sum_{i=0}^{k}\|[(-\Delta)_\omega]^{\beta_i}u\|^2_{L^2} \\
&= \frac{C_1}{\alpha_k^2}\sum_{i=0}^{k}\Big(\sum_{j=1}^{\infty}((\lambda_j)^{\beta_i})^2a_j^2\Big) \\
&\le \frac{C_1}{\alpha_k^2}\Big(\sum_{j=1}^{\infty}((\lambda_j)^{\beta_k})^2a_j^2 + kC_2^2\sum_{j=1}^{\infty}((\lambda_j)^{\beta_k})^2a_j^2\Big) \\
&= \frac{C_1}{\alpha_k^2}(1+kC_2^2)\sum_{j=1}^{\infty}((\lambda_j)^{\beta_k})^2a_j^2 \\
&= \frac{C_1}{\alpha_k^2}(1+kC_2^2)\|u\|^2_{\sim\beta_k}
\end{aligned}$$

where $u(x) = \sum_{j=1}^{\infty} a_je_j(x)$, ($\{e_j : j \in \mathbb{N}\}$ is a Hilbertian basis in L^2 consisting of eigenfunctions corresponding to eigenvalues λ_j), $C_1 > 0$ is a constant that does not depend on u and C_2 is such that

$$(\lambda_j)^{\beta_i} \le C_2(\lambda_j)^{\beta_k}$$

for any $i = 0, \dots, k-1$ and $j \in \{j \in \mathbb{N},\ \lambda_j < 1\}$ (if the set $\{j \in \mathbb{N},\ \lambda_j < 1\}$ is empty, we put $C_2 = 1$). Completeness is obvious. $\square$

Now, let us fix $g \in L^2$ and consider the equation

$$w^2((-\Delta)_\omega)u = g$$

in $D(w^2((-\Delta)_\omega))$. According to Corollary 3.4, to show that there exists a unique solution to this equation it is equivalent to prove that there exists a unique function $u \in D(w((-\Delta)_\omega))$ such that

$$\int_\Omega w((-\Delta)_\omega)u(t)w((-\Delta)_\omega)v(t)dt = \int_\Omega g(t)v(t)dt$$

for any $v \in D(w((-\Delta)_\omega))$. Indeed, the functional

$$D(w((-\Delta)_\omega)) \ni u \mapsto \int_\Omega g(x)u(x)dx \in \mathbb{R}$$

is linear and continuous with respect to the norm $\|u\|_{w((-\Delta)_\omega)}$ (continuity follows from (3.2) and Lemma 3.6). So, from the Riesz theorem it follows that there exists a unique function $u_g \in D(w((-\Delta)_\omega))$ such that

$$\langle u_g, v\rangle_{w((-\Delta)_\omega)} = \int_\Omega g(x)v(x)dx$$

for any $v \in D(w((-\Delta)_\omega))$, i.e.

$$\int_\Omega w((-\Delta)_\omega)u_g(t)w((-\Delta)_\omega)v(t)dt = \int_\Omega g(t)v(t)dt$$

for any $v \in D(w((-\Delta)_\omega))$. Thus, we have proved the following theorem.

Theorem 3.7. *For any function $g \in L^2$, there exists a unique solution*
$$u_g \in D(w^2((-\Delta)_\omega))$$
to the equation
$$w^2((-\Delta)_\omega)u = g.$$

So, the operator $w^2((-\Delta)_\omega) : D(w^2((-\Delta)_\omega)) \subset L^2 \to L^2$ is bijective and, consequently, there exists an inverse operator
$$(w^2((-\Delta)_\omega))^{-1} : L^2 \to L^2$$
defined on the whole space L^2. Moreover, for every $g \in L^2$, we have
$$\begin{aligned}
\|(w^2((-\Delta)_\omega))^{-1}g\|_{L^2}^2 &= \|u_g\|_{L^2}^2 \le M_{2\beta_k}\|u_g\|_{\sim 2\beta_k}^2 \\
&= M_{2\beta_k}\|[(-\Delta)_\omega]^{2\beta_k}u_g\|_{L^2}^2 \\
&= M_{2\beta_k}\langle[(-\Delta)_\omega]^{2\beta_k}u_g, [(-\Delta)_\omega]^{2\beta_k}u_g\rangle_{L^2} \\
&= \frac{M_{2\beta_k}}{\alpha_k^4}\langle\alpha_k^2[(-\Delta)_\omega]^{2\beta_k}u_g, \alpha_k^2[(-\Delta)_\omega]^{2\beta_k}u_g\rangle_{L^2} \\
&\le \frac{M_{2\beta_k}}{\alpha_k^4}\langle w^2((-\Delta)_\omega)u_g, w^2((-\Delta)_\omega)u_g\rangle_{L^2} \\
&= \frac{M_{2\beta_k}}{\alpha_k^4}\|w^2((-\Delta)_\omega)u_g\|_{L^2}^2 = \frac{M_{2\beta_k}}{\alpha_k^4}\|g\|_{L^2}^2,
\end{aligned}$$
i.e. $(w^2((-\Delta)_\omega))^{-1}$ is bounded. Using Proposition 2.6 with the operator $A = (-\Delta)_\omega$ and the function $b(\lambda) = w^2(\lambda)$, we assert that
$$(w^2((-\Delta)_\omega))^{-1} = \int_{-\infty}^{\infty} \frac{1}{w^2(\lambda)} E(d\lambda).$$

Note that
$$\{\lambda \in \mathbb{R} : w^2(\lambda) = 0\} = \{\lambda \in \mathbb{R} : \lambda \le 0\}$$
and $E(\{\lambda \in \mathbb{R} : \lambda \le 0\}) = 0$ because
$$\sigma((-\Delta)_\omega) \subset [\lambda_1, \infty)$$
where $\lambda_1 > 0$ is the first eigenvalue of $(-\Delta)_\omega$.

Thus, the operator $(w^2((-\Delta)_\omega))^{-1}$ is self-adjoint and (see (2.2)) the spectrum $\sigma((w^2((-\Delta)_\omega))^{-1})$ consists of 0 and eigenvalues $\mu_j = \frac{1}{w^2(\lambda_j)}$ (λ_j-s are eigenvalues of $(-\Delta)_\omega$) such that $0 \leftarrow \mu_j < \dots < \mu_2 < \mu_1$ (we used here the fact that $\alpha_j > 0$ for $j = 0, \dots, k$ and, consequently, $w^2(\lambda)$ is increasing, $w^2(\lambda) \to \infty$ as $\lambda \to \infty$ and $w^2(\lambda) \ne 0$ for $\lambda > 0$). Since eigenspaces N_{μ_j}, $N_{w^2(\lambda_j)}$ and N_{λ_j} of operators $(w^2((-\Delta)_\omega))^{-1}$, $w^2((-\Delta)_\omega)$ and $(-\Delta)_\omega$, corresponding to μ_j, $w^2(\lambda_j)$ and λ_j, respectively, are the same, therefore multiplicity of each μ_j is the same as multiplicity of $w^2(\lambda_j)$ and λ_j.

Finally, we have the operator $(w^2((-\Delta)_\omega))^{-1}$ which is defined on L^2, bounded, self-adjoint with countable spectrum consisting of 0 and eigenvalues of finite multiplicity, tending to 0. So, (see [22, Part VI.6]) we obtain the following theorem.

Theorem 3.8. *The operator $(w^2((-\Delta)_\omega))^{-1}$ is compact, i.e. the image of any bounded set in L^2 is relatively compact in L^2.*

Remark 3.9. The case of $w(\lambda) = \lambda^{1/2}$ of the above theorem is proved in [3, Proposition 8.2.1] and that proof is based on the Rellich-Kondrakov theorem.

Using the above theorem we obtain the following property.

Proposition 3.10. *If $u_k \rightharpoonup u_0$ weakly in $D(w((-\Delta)_\omega))$, then $u_k \to u_0$ strongly in L^2 and $w((-\Delta)_\omega)u_k \rightharpoonup w((-\Delta)_\omega)u_0$ weakly in L^2.*

Proof. First, let us assume that $w(\lambda) = z^2(\lambda)$ where z is a polynomial of type (2.3) with positive coefficients α_i. From the continuity of the linear operators

$$D(z^2((-\Delta)_\omega)) \ni u \mapsto u \in L^2,$$
$$D(z^2((-\Delta)_\omega)) \ni u \mapsto z^2((-\Delta)_\omega)u \in L^2$$

it follows that $u_k \rightharpoonup u_0$ weakly in L^2 and $z^2((-\Delta)_\omega)u_k \rightharpoonup z^2((-\Delta)_\omega)u_0$ weakly in L^2. Theorem 3.8 implies that the sequence (u_k) contains a subsequence (u_{k_i}) converging strongly in L^2 to a limit. Of course, this limit is the function u_0, i.e. $u_{k_i} \to u_0$ strongly in L^2. Supposing contrary and repeating the above argumentation we assert that $u_k \to u_0$ strongly in L^2.

Now, let us consider any polynomial $w(\lambda)$ of type (2.3) with positive coefficients α_i. Clearly, weak convergence $u_k \rightharpoonup u_0$ in $D(w((-\Delta)_\omega))$ implies the weak convergence $w((-\Delta)_\omega)u_k \rightharpoonup w((-\Delta)_\omega)u_0$ in L^2. Moreover,

$$D(w((-\Delta)_\omega)) = D([(-\Delta)_\omega]^{\beta_k}) = D([(-\Delta)_\omega]^{2\frac{\beta_k}{2}}) = D(z^2((-\Delta)_\omega))$$

where $z(\lambda) = \lambda^{\frac{\beta_k}{2}}$. Applying the proved case of the proposition to the polynomial $z(\lambda)$ we assert that $u_k \to u_0$ strongly in L^2 (positivity of coefficients α_i guaranties equivalence of norms $\|u\|_{\sim\beta_k}$ and $\|u\|_{w((-\Delta)_\omega)}$). $\square$

3.6. **Closedness of the range of $w^2((-\Delta)_\omega) - aI$.** Now, let us consider the operator

$$L = w^2((-\Delta)_\omega) - aI : D(w^2((-\Delta)_\omega) - aI) \subset L^2 \to L^2$$

where $a \in \mathbb{R}$ and $I : L^2 \to L^2$ is the identity operator. As in the previous section we assume that $\alpha_i > 0$ for $i = 0, \dots, k$. It is clear that the spectrum $\sigma(L)$ of L consists of eigenvalues

$$w^2(\lambda_1) - a < w^2(\lambda_2) - a < \dots \tag{3.8}$$

where $0 < \lambda_1 < \lambda_2 < \dots$ are eigenvalues of the operator $(-\Delta)_\omega$, of finite multiplicity. Since eigenspaces N_{λ_i} and $N_{w^2(\lambda_i)-a}$ of operators $(-\Delta)_\omega$ and $w^2((-\Delta)_\omega) - aI$, corresponding to λ_i and $w^2(\lambda_i) - a$, respectively, are the same, therefore multiplicity of each eigenvalue $w^2(\lambda_i) - a$ is also finite.

Now, we shall show that the range $R(L)$ of the operator L is closed.

First, let us consider the case when $a \notin \{w^2(\lambda_i) : i \in \mathbb{N}\}$ (non-resonance case). So, 0 belongs to the resolvent set $\rho(L)$. It means that the operator L^{-1} exists, is bounded and $\overline{D(L^{-1})} = \overline{R(L)} = L^2$. Consequently (see [22, Part III, Lemma 7.1]), $(L^{-1})^* \in \mathcal{L}(L^2)$ (the set of linear bounded operators defined on L^2). Moreover, since the operators L^{-1}, L^*, $(L^{-1})^*$ exist, therefore $(L^*)^{-1}$ exists and $(L^*)^{-1} = (L^{-1})^*$ (see [22, Part III, Theorem 6.2]). Thus ($L = L^*$),

$$R(L) = D(L^{-1}) = D((L^*)^{-1}) = D((L^{-1})^*) = L^2.$$

Now, let us assume that $a = w^2(\lambda_1)$. Since $L^2 = N(L) \oplus \overline{R(L)}$, i.e. $\overline{R(L)} = N(L)^\perp$ (orthogonal subspace), therefore it is sufficient to show that

$$N(L)^\perp \subset R(L).$$

Indeed, let $v \in N(L)^{\perp} = N_{w^2(\lambda_1)-a}{}^{\perp}$. Since $L^2 = \bigoplus_{i\geq 1} N_{w^2(\lambda_i)-a}$ (orthogonal sum), therefore $N_{w^2(\lambda_1)-a}{}^{\perp} = \bigoplus_{i>1} N_{w^2(\lambda_i)-a}$. Consequently $v = \sum_{i>1} v_i$ where $v_i \in N_{w^2(\lambda_i)-a}$. Consider the point $u = \sum_{i>1} \frac{1}{w^2(\lambda_i)-a} v_i \in L^2$ and observe that (below, $E(d\lambda)$ is the spectral measure connected with $(-\Delta)_\omega$ according to Theorem 2.3)

$$\begin{aligned}\int_{\sigma((-\Delta)_\omega)} |w^2(\lambda) - a|^2 \|E(d\lambda)u\|^2 &= \sum_{i>1} |w^2(\lambda_i) - a|^2 \|E(\{\lambda_i\})u\|^2 \\ &= \sum_{i>1} |w^2(\lambda_i) - a|^2 \|\frac{1}{w^2(\lambda_i) - a} v_i\|^2 \\ &= \sum_{i>1} \|v_i\|^2 = \|v\|^2 < \infty,\end{aligned}$$

i.e. $u \in D(L)$. Moreover,

$$\begin{aligned}Lu &= \int_{\sigma((-\Delta)_\omega)} (w^2(\lambda) - a)E(d\lambda)u \\ &= \sum_{i>1} (w^2(\lambda_i) - a)E(\{\lambda_i\})u \\ &= \sum_{i>1} (w^2(\lambda_i) - a)\frac{1}{w^2(\lambda_i) - a} v_i \\ &= \sum_{i>1} v_i = v.\end{aligned}$$

So, $v \in R(L)$ and, finally, $R(L)$ is closed. In a similar way, one can prove that $R(L)$ is closed when $a = w^2(\lambda_i)$ for $i > 1$.

4. Boundary value problem

Now, we shall study existence of solutions to boundary value problem (1.1). By a solution to (1.1) we mean a function $u \in D(w^2((-\Delta)_\omega) - aI) = D(w^2((-\Delta)_\omega))$ satisfying (1.1) a.e. on Ω. We shall apply two approaches. First of them, applied in the case of $a = 0$, is based on a direct method of calculus of variations and the second one, applied in the non-resonance and resonance cases, is based on the results obtained with the aid of the dual least action principle (see [20, 25, 21]).

4.1. **Direct method.** Let us consider problem (1.1) with $a = 0$. According to Corollary 3.4, to derive existence of a solution to (1.1) it is equivalent to show that there exists $u \in D(w((-\Delta)_\omega))$ such that

$$\int_\Omega w((-\Delta)_\omega)u(x)w((-\Delta)_\omega)v(x)dx = \int_\Omega D_u F(x, u(x))v(x)dx \tag{4.1}$$

for any $v \in D(w((-\Delta)_\omega))$. In such a case, a solution to (4.1) is the solution to (1.1). Of course, such a point u is a critical point of the functional

$$f : D(w((-\Delta)_\omega)) \ni u \mapsto \int_\Omega (\frac{1}{2}|w((-\Delta)_\omega)u(x)|^2 - F(x, u(x)))dx \in \mathbb{R} \tag{4.2}$$

(clearly, under assumptions guaranteeing Gateaux differentiability of f).

4.1.1. *Gateaux differentiability of F.* Assume that function F is measurable in $x \in \Omega$, continuously differentiable in $u \in \mathbb{R}$ and

$$|F(x,u)| \le a|u|^2 + b(x), \tag{4.3}$$

$$|D_u F(x,u)| \le c|u| + d(x) \tag{4.4}$$

for $x \in \Omega$ a.e., $u \in \mathbb{R}$, where a, $c \ge 0$ and $b \in L^1$, $d \in L^2$.

Proposition 4.1. *Functional f is differentiable in Gateaux sense and the differential $f'(u) : D(w((-\Delta)_\omega)) \to \mathbb{R}$ of f at u is given by*

$$f'(u)v = \int_\Omega w((-\Delta)_\omega)u(x)w((-\Delta)_\omega)v(x) - D_u F(x,u(x))v(x)dx$$

for $v \in D(w((-\Delta)_\omega))$.

Proof. Of course, the first term of f, equal to $\frac{1}{2}\|u\|^2_{w((-\Delta)_\omega)}$, is Gateaux (even continuously Gateaux) differentiable and its Gateaux differential at u is of the form

$$D(w((-\Delta)_\omega)) \ni v \mapsto \langle u, v\rangle_{w((-\Delta)_\omega)}.$$

So, let us consider the mapping

$$g : D(w((-\Delta)_\omega)) \ni u \mapsto \int_\Omega F(x,u(x))dx \in \mathbb{R}.$$

In a standard way, using the Lebesgue dominated convergence theorem we state that

$$g'(u) : D(w((-\Delta)_\omega)) \ni v \mapsto \int_\Omega D_u F(x,u(x))v(x)dx \in \mathbb{R},$$

is Gateaux differential of g at u. $\square$

4.1.2. *Existence of a solution to* (1.1)*.* First, we shall prove the following two propositions.

Proposition 4.2. *If there exist constants $A < \frac{\alpha_k^2}{M_{\beta_k}}$, B, $C \in \mathbb{R}$ such that*

$$F(x,u) \le \frac{A}{2}|u|^2 + B|u| + C \tag{4.5}$$

for $x \in \Omega$ a.e., $u \in \mathbb{R}$, then the functional (4.2) *is coercive, i.e. $f(u) \to \infty$ as $\|u\|_{w((-\Delta)_\omega)} \to \infty$.*

Proof. Let us assume, without loss of the generality, that $A, B \ge 0$. For any $u \in D(w((-\Delta)_\omega))$, we have

$$\begin{aligned}
f(u) &= \int_\Omega (\frac{1}{2}|w((-\Delta)_\omega)u(x)|^2 - F(x,u(x)))dx \\
&\ge \frac{1}{2}\|u\|^2_{w((-\Delta)_\omega)} - \frac{A}{2}\|u\|^2_{L^2} - B\sqrt{|\Omega|}\|u\|_{L^2} - C|\Omega| \\
&\ge \frac{1}{2}\|u\|^2_{w((-\Delta)_\omega)} - \frac{A}{2}M_{\beta_k}\|u\|^2_{\sim\beta_k} - B\sqrt{|\Omega|M_{\beta_k}}\|u\|_{\sim\beta_k} - C|\Omega| \\
&\ge \frac{1}{2}(1 - \frac{AM_{\beta_k}}{\alpha_k^2})\|u\|^2_{w((-\Delta)_\omega)} - B\sqrt{|\Omega|M_{\beta_k}}\frac{1}{\alpha_k}\|u\|_{w((-\Delta)_\omega)} - C|\Omega|
\end{aligned}$$

where $|\Omega|$ is the Lebesgue measure of Ω. It means that f is coercive. $\square$

Proposition 4.3. *Functional* (4.2) *is weakly sequentially lower semicontinuous.*

Proof. Weak sequential lower semicontinuity of the second power of the norm in Banach space is a classical result. So, it sufficient to show that the functional

$$D(w((-\Delta)_\omega)) \ni u \mapsto \int_\Omega F(x, u(x)))dx \in \mathbb{R}$$

is weakly sequentially continuous. But this fact follows immediately from Lebesgue dominated convergence theorem. Indeed, the weak convergence of a sequence (u_n) to u_0 in $D(w((-\Delta)_\omega))$ implies (see Proposition 3.10) the convergence $u_n \to u_0$ in L^2. From [8, Theorem 4.9] it follows that one can choose a subsequence (u_{n_k}) converging a.e. on Ω to u_0 and pointwise bounded by a function belonging to L^2. Using growth condition (4.3) we assert that

$$\int_\Omega F(x, u_{n_k}(x)))dx \to \int_\Omega F(x, u_0(x)))dx.$$

Supposing that the convergence

$$\int_\Omega F(x, u_n(x)))dx \to \int_\Omega F(x, u_0(x)))dx.$$

does not hold and repeating the above reasoning we obtain a contradiction. $\square$

Now, let us recall the following classical result:

> If E is a reflexive Banach space and functional $f : E \to \mathbb{R}$ is weakly sequentially lower semicontinuous and coercive, then there exists a global minimum point of f.

Thus, the functional f given by (4.2) has a global minimum point $u \in D(w((-\Delta)_\omega))$. Differentiability of f means that u satisfies (4.1). Consequently, u is a solution to (1.1).

Example 4.4. Let us consider the Dirichlet problem for the equation

$$(\alpha_k[(-\Delta)_\omega]^{\beta_k} + \dots + \alpha_0[(-\Delta)_\omega]^{\beta_0})^2 u(x) = A\cos(x_1 + \dots + x_N)u(x) - b(x)\sin(u(x))$$

in a bounded open set $\Omega \subset \mathbb{R}^N$ with $\alpha_i > 0$ for $i = 0, \dots, k$ ($k \in \mathbb{N} \cup \{0\}$) and $0 \le \beta_0 < \beta_1 < \dots < \beta_k$, where $0 < A < \frac{\alpha_k^2}{M_{\beta_k}}$, $M_{\beta_k} = \max\{\frac{1}{((\lambda_j)^{\beta_k})^2},\ \lambda_j < 1\} = \frac{1}{(\lambda_1)^{2\beta_k}}$ (recall that if there is no $\lambda_j < 1$, then $M_{\beta_k} = 1$), $b \in L^\infty(\Omega, \mathbb{R})$. It is clear that the function

$$F(x, u) = \frac{A}{2}\cos(x_1 + \dots + x_N)u^2 + b(x)\cos u$$

satisfies growth conditions (4.3), (4.4), (4.5). Consequently, there exists a solution $u \in D(w^2((-\Delta)_\omega)) = D([(-\Delta)_\omega]^{2\beta_k})$ to the problem under consideration. As we know, in the case of the domain $\Omega \subset \mathbb{R}^N$ being of class $C^{1,1}$ or a bounded open convex polygon in $\mathbb{R}^2$, $(-\Delta)_\omega$ can be replaced by $(-\Delta)$. If $\Omega = (0, \pi) \times (0, \pi)$ then (see [3, Proposition 8.5.3]) the first eigenvalue λ_1 of the operator $(-\Delta)_\omega = (-\Delta)$ is equal to 2 and consequently $M_{\beta_k} = 1$.

4.2. **Dual approach.**

4.2.1. *Abstract results.* In [25] (see also [21]) the following abstract results have been derived. Let $L : D(L) \subset H \to H$ be self-adjoint with closed range $R(L)$ and let $g : H \to \mathbb{R}$ be convex continuous on H and Gateaux differentiable at any point $u \in D(L)$. By the gradient of g at u we mean a unique element $\nabla g(u) \in H$ such that

$$g'(u)h = \langle \nabla g(u), h\rangle$$

for any $h \in H$.

Theorem 4.5. *If there exist numbers b, c, d, $\alpha \in \mathbb{R}$ such that*

- $\sigma(L) \cap]0, \alpha[= \emptyset$
- $\sigma(L) \cap [\alpha, \infty[$ *consists of at most countable amount of isolated eigenvalues of L of finite multiplicity*
- $0 < b \leq c < \alpha$
- *for any $u \in H$,*

$$b\frac{\|u\|^2}{2} - d \leq g(u) \leq c\frac{\|u\|^2}{2} + d,$$

then there exists a solution u_0 to the equation

$$Lu = \nabla g(u)$$

such that $v_0 = Lu_0$ minimizes the dual functional

$$\widetilde{f} : R(L) \ni v \mapsto g^*(v) - \frac{1}{2}\langle Kv, v\rangle \in \mathbb{R} \cup \{+\infty\}$$

where

$$g^* : H \ni v \mapsto \sup\{\langle v, u\rangle - g(u);\ u \in H\} \in \mathbb{R} \cup \{+\infty\}$$

is the Fenchel transform of g and $K = (L\,|_{D(L)\cap R(L)})^{-1} : R(L) \to R(L)$.

If, additionally, we assume that $N(L) \neq \{0\}$ (resonance case), then one can weaken the assumption

$$b\frac{\|u\|^2}{2} - d \leq g(u),\ u \in H.$$

Clearly, such an assumption implies coercivity of g on H ($g(x) \to \infty$ as $\|x\| \to \infty$, $x \in H$). When $N(L) \neq \{0\}$, it is sufficient to assume coercivity of g only on $N(L)$. Namely, we have the following theorem.

Theorem 4.6. *If $N(L) \neq \{0\}$ and there exist numbers c, d, $\alpha \in \mathbb{R}$ such that*

- $\sigma(L) \cap]0, \alpha[= \emptyset$
- $\sigma(L) \cap [\alpha, \infty[$ *consists of at most countable amount of isolated eigenvalues of L of finite multiplicity*
- $0 < c < \alpha$
- *for any $u \in H$,*

$$g(u) \leq c\frac{\|u\|^2}{2} + d$$

- *g is coercive on $N(L)$, i.e.*

$$g(u) \to \infty \text{ as } \|u\| \to \infty,\ u \in N(L),$$

then there exists a solution u_0 to the equation

$$Lu = \nabla g(u)$$

such that $v_0 = Lu_0$ minimizes the dual functional $\widetilde{f}$.

4.2.2. *Existence of a solution to* (1.1). Assume that $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable in $x \in \Omega$, continuously differentiable in $u \in \mathbb{R}$ and satisfies (4.3), (4.4). Thus, the functional

$$g : L^2 \ni u \mapsto \int_\Omega F(x, u(x))dx \in \mathbb{R}$$

is continuous and differentiable in Gateaux sense on L^2 with differential $g'(u)$ given by

$$g'(u)v = \int_\Omega D_u F(x, u(x))v(x)dx$$

for $u,\ v \in L^2$. So, $\nabla g(u) = D_u F(\cdot, u(\cdot))$. Additionally, we assume that F is convex in $u \in \mathbb{R}$. From Theorem 4.5 and characterization (3.8) of the spectrum of $w^2((-\Delta)_\omega) - aI$ we obtain the following theorem.

Theorem 4.7. *If $a,\ b,\ c,\ d \in \mathbb{R}$ are such that*

- $0 < b \le c < w^2(\lambda_1) - a$ *or* $0 < b \le c < w^2(\lambda_{i_0+1}) - a$ *where i_0 is such that* $w^2(\lambda_{i_0}) - a < 0 < w^2(\lambda_{i_0+1}) - a$,
- *for $x \in \Omega$ a.e., $u \in \mathbb{R}$,*

$$b\frac{|u|^2}{2} - d \le F(x, u) \le c\frac{|u|^2}{2} + d$$

then there exists a solution u_0 to the equation (1.1) *such that* $v_0 = w^2((-\Delta)_\omega)u_0 - au_0$ *minimizes the dual functional $\widetilde{f}$.*

Theorem 4.6 implies the following result.

Theorem 4.8. *If $a = w^2(\lambda_{i_0})$ for some $i_0 \in \mathbb{N}$ and $c,\ d \in \mathbb{R}$ are such that*

- $0 < c < w^2(\lambda_{i_0+1}) - a$
- *for $x \in \Omega$ a.e., $u \in \mathbb{R}$,*

$$F(x, u) \le c\frac{|u|^2}{2} + d \tag{4.6}$$

- $\int_\Omega F(x, u(x))dx \to \infty$ *as* $\|u\| \to \infty,\ u \in N_{w^2(\lambda_{i_0})}$

then there exists a solution u_0 to the equation (1.1) *such that* $v_0 = w^2((-\Delta)_\omega)u_0 - au_0$ *minimizes the dual functional $\widetilde{f}$.*

Example 4.9. Let us consider the Dirichlet problem for the equation

$$[(-\Delta)_\omega]^{\frac{3}{4}+\frac{3}{4}}u(x_1, x_2) - 2^{\frac{3}{2}}u(x_1, x_2) = (x_1 + x_2 + 1)u(x_1, x_2) - \sin(u(x_1, x_2))$$

in the set $\Omega = (0, \pi) \times (0, \pi) \subset \mathbb{R}^2$. It is known (see [3, Proposition 8.5.3]) that the eigenspace corresponding to the first eigenvalue $\lambda_1 = 2$ of $(-\Delta)_\omega = (-\Delta)$ is the set $\{\eta \sin x_1 \sin x_2;\ \eta \in \mathbb{R}\}$ and the second eigenvalue λ_2 is equal to 5. Of course, the function

$$F(x_1, x_2, u) = \frac{1}{2}(x_1 + x_2 + 1)u^2 + \cos u,\ (x_1, x_2) \in \Omega,\ u \in \mathbb{R},$$

satisfies growth conditions (4.3), (4.4). Moreover, it satisfies (4.6) with $i_0 = 1$, $c = 2\pi + 1$ ($c < 5^{3/2} - 2^{3/2}$), $d = 1$ and

$$\begin{aligned} &\int_\Omega F(x_1, x_2, \eta \sin x_1 \sin x_2)dx \\ &= \int_\Omega (\frac{1}{2}(x_1 + x_2 + 1)\eta^2 \sin^2 x_1 \sin^2 x_2 + \cos(\eta \sin x_1 \sin x_2))dx \to \infty \end{aligned}$$

as $|\eta| \to \infty$. Convexity of F in $u \in \mathbb{R}$ is obvious because $F''_u(x_1, x_2, u) = x_1 + x_2 + 1 - \cos u \geq 0$ for $(x_1, x_2) \in \Omega$ and $u \in \mathbb{R}$. Consequently, there exists a solution

$$u_0 \in D((-\Delta)^{3/2} - 2^{3/2} I) = D((-\Delta)^{3/2})$$

to (1.1) such that $v_0 = [(-\Delta)_\omega]^{3/2} u_0 - 2^{3/2} u_0$ minimizes the dual functional

$$\widetilde{f} : R(L) \to \mathbb{R} \cup \{+\infty\}$$

where $L = [(-\Delta)_\omega]^{3/2} - 2^{3/2} I$, and

$$R(L) = \{v \in L^2 : \int_\Omega v(x_1, x_2) \sin x_1 \sin x_2) dx = 0\}.$$

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