Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 60, pp. 1–21.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

MULTIPLICITY OF SOLUTIONS TO AN ELLIPTIC PROBLEM WITH SINGULARITY AND MEASURE DATA

SEKHAR GHOSH, AKASMIKA PANDA, DEBAJYOTI CHOUDHURI

Abstract. In this article, we prove the existence of multiple nontrivial solutions to the equation

$$\begin{gathered} -\Delta_p u = \frac{\lambda}{u^\gamma} + g(u) + \mu \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ u > 0 \quad \text{in } \Omega, \end{gathered}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with $N \geq 3$, $1 < p-1 < q$, $\lambda > 0$, $\gamma > 0$, g satisfies certain conditions, $\mu \geq 0$ is a bounded Radon measure.

1. Introduction

Elliptic equations with singularity has gained a huge attention owing to its richness both from the theoretical and application point of view. Early traces of research pertaining to problems involving singularity can be found in [24], where the authors addressed the problem

$$\begin{gathered} -\Delta u = \frac{f(x)}{u^\gamma} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a strictly convex, bounded domain in $\mathbb{R}^N$ with C^2 boundary. The existence of a unique solution was guaranteed if and only if $0 < \gamma < 3$. The authors in [24], also showed the existence of a solution in $C^1(\bar{\Omega})$, for $0 < \gamma < 1$. Haitao [22] studied the perturbed singular problem

$$\begin{gathered} -\Delta u = \frac{\lambda}{u^\gamma} + u^p, \quad u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

and guaranteed the existence of two weak solutions for $\lambda < \Lambda$, no solution for $\lambda > \Lambda$ and at least one solution for $0 < \gamma < 1 < p \leq \frac{N+2}{N-2}$ and some $\Lambda > 0$. A further generalization to this problem can be found in [19], where the existence of two solutions were shown for some $0 < \gamma < 1 < p - 1 < q \leq p^* - 1$. An important problem involving singularity in the literature can be found in the work due to

2010 *Mathematics Subject Classification.* 35J60, 35J75, 35R06.
Key words and phrases. Elliptic PDEs; p-Laplacian; Radon measure.

Submitted April 16, 2018. Published May 6, 2019.

Crandall et al [11], where the authors addressed the problem

$$\begin{aligned} -\Delta u &= f(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

where f is a function with singularity near 0. The authors in [11], whoed the existence of a unique classical solution in $C^2(\Omega) \cap C(\bar{\Omega})$. Another noteworthy work is due to Giacomoni and Sreenadh [18], where the authors investigated the quasilinear and singular problem

$$\begin{aligned} -\Delta_p u &= \frac{\lambda}{u^\delta} + u^q \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \\ u &> 0 \quad \text{in } \Omega, \end{aligned} \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $1 < p - 1 < q$ and $\lambda, \delta > 0$. The authors have shown the existence of weak solutions for small $\lambda > 0$ in $W_0^{1,p}(\Omega) \cap C(\bar{\Omega})$ if and only if $\delta < 2 + \frac{1}{p-1}$. Further they have investigated the radial symmetry case, i.e. for $\Omega = B_R(0)$, where they have proved the global multiplicity of solutions in $C(\bar{\Omega})$ with $\delta > 0$, $1 < p - 1 < q$, by using shooting method. Readers interested in 'singularity involving problem' can refer to [26, 8, 9, 29] and of late Panda et al [27], who investigated a problem involving singularity and a measure. Motivated by the work in [6], which stemmed out from the work in[3], by generalizing their result for the p-Laplacian, we will study the problem

$$\begin{aligned} -\Delta_p u &= \frac{\lambda}{u^\gamma} + g(u) + \mu \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \\ u &> 0 \quad \text{in } \Omega, \end{aligned} \tag{1.5}$$

where Ω is a strictly convex, bounded domain in $\mathbb{R}^N$ with C^2 boundary, $N > 2$, $1 < p < N$, $\Delta_p u = \operatorname{div}\{|\nabla u|^{p-2} \nabla u\}$, $\lambda > 0$, $\gamma > 0$ and μ is a bounded Radon measure. The function g obeys certain growth conditions, i.e. there exists some constants $C > 0$ such that,

$$C^{-1} t^{1+q} \le t g(t) \le C t^{1+q},$$

where $p - 1 < q < \frac{N(p-1)}{N-p}$.

2. Definitions and notation

We will use the notations due to [15], to denote $W_0^{k,p}(\Omega)$ as the space obtained by considering the closure of $C_c^\infty(\Omega)$ in the Sobolev space $W^{k,p}(\Omega)$ and $W_{\text{loc}}^{k,p}(\Omega)$ to be the local Sobolev space, which consists of functions u such that for any compact $K \subset \Omega$, $u \in W^{k,p}(K)$. The Hölder Space is denoted by $C^{k,\beta}(\bar{\Omega})$ with $0 < \beta \le 1$ (again a notation borrowed from [15]), which consists of all functions $u \in C^k(\bar{\Omega})$ such that the norm

$$\sum_{|\alpha| \le k} \sup |D^\alpha u| + \sup_{x \ne y} \Big\{ \frac{|D^k u(x) - D^k u(y)|}{|x - y|^\beta} \Big\} < \infty.$$

We will use the truncation functions for fixed $k > 0$,

$$T_k(t) = \max\{-k, \min\{k, t\}\} \text{ and } G_k(t) = (|t| - k)^+ sign(t)$$

with $t \in \mathbb{R}$. Observe that $T_k(t) + G_k(t) = t$ for any $t \in \mathbb{R}$ and $k > 0$.

We denote by $\mathbb{M}(\Omega)$ the space of all finite Radon measures on Ω. For every $\mu \in \mathbb{M}(\Omega)$, we define

$$\|\mu\|_{\mathbb{M}(\Omega)} = \int_\Omega d|\mu|.$$

We will use the Marcinkiewicz space $\mathcal{M}^q(\Omega)$ (or weak $L^q(\Omega)$) defined for every $0 < q < \infty$, as the space of all measurable functions $f : \Omega \to \mathbb{R}$ such that the corresponding distribution functions satisfy an estimate of the form

$$m(\{x \in \Omega : |f(x)| > t\}) \leq \frac{C}{t^q} \quad t > 0,\ C < \infty.$$

Indeed, for bounded domain Ω we have $\mathcal{M}^q \subset \mathcal{M}^{\bar{q}}$ if $q \geq \bar{q}$, for some fixed positive $\bar{q}$. Further, the following continuous embeddings hold

$$L^q(\Omega) \hookrightarrow \mathcal{M}^q(\Omega) \hookrightarrow L^{q-\epsilon}(\Omega), \tag{2.1}$$

for every $1 < q < \infty$ and $0 < \epsilon < q - 1$. We will use this embedding result to show the existence of solutions. We now give the definition of convergence in the measure space.

Definition 2.1. Let (μ_n) be the sequence of measurable functions in $\mathbb{M}(\Omega)$. We say (μ_n) converges to $\mu \in \mathbb{M}(\Omega)$ in the sense of measure [17] i.e. $\mu_n \rightharpoonup \mu$ in $\mathbb{M}(\Omega)$, if

$$\int_\Omega f d\mu_n \to \int_\Omega f d\mu, \quad \forall f \in C_0(\Omega).$$

To show the existence of solutions to problem (1.5), we consider the following sequence of problems (P_n).

$$\begin{gathered} -\Delta_p u = \frac{\lambda}{(u + \frac{1}{n})^\gamma} + g(u) + \mu_n \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ u > 0 \quad \text{in } \Omega, \end{gathered} \tag{2.2}$$

whose solutions are denoted by u_n. The weak formulation of (2.2) is

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla\phi dx \\ &= \lambda \int_\Omega \frac{\phi}{(u_n + \frac{1}{n})^\gamma} + \int_\Omega g(u_n)\phi dx + \int_\Omega \mu_n \phi dx, \quad \forall \phi \in C_0^1(\bar{\Omega}), \end{aligned} \tag{2.3}$$

where (μ_n) is a sequence of smooth non-negative functions bounded in $L^1(\Omega)$ and converging weakly to μ in the sense of Definition 2.1. We now give the definition of weak solution to the problem (2.2) in (1.5).

Definition 2.2. We say a function $u \in W^{1,p}_{\rm loc}(\Omega) \cap L^\infty(\Omega)$ is a weak solution to (1.5) if $\frac{\phi}{u^\gamma} \in L^1(\Omega)$ and it satisfies

$$\int_\Omega |\nabla u|^{p-2} \cdot \nabla u \cdot \nabla\phi \, dx = \lambda \int_\Omega \frac{\phi}{u^\gamma} dx + \int_\Omega g(u)\phi dx + \int_\Omega \phi \, d\mu \tag{2.4}$$

for every $\phi \in W_0^{1,p}(\Omega')$ with $\Omega' \subset\subset \Omega$.

In the subsequent section, we will prove some lemmas required in the proof of our main result in Section 4. Note that the solution will be named as u_n in multiple places for different problems.

3. Auxiliary Lemmas

In this section we will prove important lemmas that are the main tools for proving the main result for existence of solution to problem (1.5).

Lemma 3.1. *The problem*

$$\begin{gathered}-\Delta_p u=\frac{\lambda}{(u+\frac{1}{n})^\gamma}\quad \text{in } \Omega,\\ u=0\quad \text{on } \partial\Omega,\end{gathered}\tag{3.1}$$

possesses a nonnegative weak solution in $W^{1,p}_{\rm loc}(\Omega)\cap L^\infty(\Omega)$ for each $n\in\mathbb{N}$.

Proof. The idea of the proof is to apply Schauder's fixed point argument. For a fixed $n\in\mathbb{N}$ and a fixed $v\in L^p(\Omega)$, we define the map $J_\lambda: W^{1,p}_0(\Omega)\to\mathbb{R}$, as follows,

$$J_\lambda(u)=\frac{1}{p}\int_\Omega|\nabla u|^p dx-\lambda\int_\Omega\frac{u}{(|v|+\frac{1}{n})^\gamma}dx.$$

It is easy to see that, J_λ is continuous, coercive and strictly convex in $W^{1,p}_0(\Omega)$. Therefore, the existence of a unique minimizer $w\in W^{1,p}_0(\Omega)$ corresponding to a $v\in L^p(\Omega)$ is certain.

We define $H: L^p(\Omega)\to L^p(\Omega)$ by

$$H(v)=(-\Delta_p)^{-1}\Big[\frac{\lambda}{(|v|+\frac{1}{n})^\gamma}\Big]:=w.$$

On choosing w as a test function from $W^{1,p}_0(\Omega)$ in the weak formulation (3.1), we have

$$\int_\Omega|\nabla w|^p=\int_\Omega|\nabla w|^{p-2}\nabla w\cdot\nabla w=\int_\Omega\frac{\lambda}{(|v|+\frac{1}{n})^\gamma}w\le\lambda n^\gamma\int_\Omega|w|.$$

Hence, by using the Poincaré inequality and the Hölder's inequality on the left and right hand side respectively, we obtain

$$\|w\|_p\le C(n,\gamma,\lambda).\tag{3.2}$$

Let us consider a sequence (v_k) that converges to v in $L^p(\Omega)$. By using the dominated convergence theorem, we have

$$\Big\|\frac{\lambda}{(|v_k|+\frac{1}{n})^\gamma}-\frac{\lambda}{(|v|+\frac{1}{n})^\gamma}\Big\|_{L^p(\Omega)}\to 0.$$

Thus, the convergence of $w_k=H(v_k)$ to $w=H(v)$ in $L^p(\Omega)$ can be followed from the uniqueness of the weak solution. Hence, the continuity of H over $L^p(\Omega)$ is followed. By the estimate (3.2) and by the Rellich-Kondrochov theorem, we obtain that $H(L^p(\Omega))$ is relatively compact in $L^p(\Omega)$. We now can apply the Schauder's fixed point theorem to guarantee the existence of a fixed point say w. By the regularity theorem of Lieberman [23], we have $u_n\in C^1(\bar\Omega)$ for all $n\in\mathbb{N}$. Using the strong maximum principle [21], we have $w>0$ in Ω and this concludes the proof. □

Lemma 3.2. *The sequence (u_n) is increasing wit respect to n and for every $K\subset\subset\Omega$, there exists C_K (only depends on K) such that $u_n\ge C_K>0$, a.e. in K with $\|u_n\|_\infty\le R\lambda^{\frac{1}{\gamma+p-1}}$ for all $n\in\mathbb{N}$, R is independent of n.*

Proof. Consider a sequence of problems

$$
\begin{gathered}
-\Delta_p u = \frac{\lambda}{(u+\frac{1}{n})^\gamma} \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{3.3}
$$

For each n, let u_n be the solution to the problem (3.3). Consider

$$
\begin{aligned}
&\int_\Omega (|\nabla u_n|^{p-2}\cdot\nabla u_n - |\nabla u_{n+1}|^{p-2}\cdot\nabla u_{n+1})\cdot\nabla\phi\,dx \\
&= \lambda\int_\Omega \Big((u_n+\frac{1}{n})^{-\gamma} - (u_{n+1}+\frac{1}{n+1})^{-\gamma}\Big)\phi\,dx.
\end{aligned}
$$

We choose, the test function $\phi = (u_n - u_{n+1})^+$ to obtain

$$
\begin{aligned}
&\int_\Omega (|\nabla u_n|^{p-2}\cdot\nabla u_n - |\nabla u_{n+1}|^{p-2}\cdot\nabla u_{n+1})\cdot\nabla(u_n-u_{n+1})^+dx \\
&\le \lambda\int_\Omega \Big((u_n+\frac{1}{n+1})^{-\gamma} - (u_{n+1}+\frac{1}{n+1})^{-\gamma}\Big)(u_n-u_{n+1})^+dx.
\end{aligned}
$$

Using the inequalities from [14], for $p \ge 2$, we obtain

$$
\begin{aligned}
&\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_{n+1}|^{p-2}\cdot\nabla u_{n+1})\cdot\nabla(u_n-u_{n+1})^+dx \\
&\ge C_p\|\nabla(u_n-u_{+n+1})^+\|^p \ge 0,
\end{aligned}
$$

and for $1<p<2$,

$$
\begin{aligned}
&\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_{n+1}|^{p-2}\nabla u_{n+1})\cdot\nabla(u_n-u_{n+1})^+dx \\
&\ge C_p\frac{\|u_n-u_{+n+1}\|^2}{(\|u_n\|+\|u_{+n+1}\|)^{2-p}} \ge 0.
\end{aligned}
$$

Therefore,

$$
0 \le \lambda\int_\Omega \Big\{\Big(u_n+\frac{1}{n+1}\Big)^{-\gamma} - \Big(u_{n+1}+\frac{1}{n+1}\Big)^{-\gamma}\Big\}(u_n-u_{n+1})^+dx \le 0.
$$

Hence, $\|(u_n-u_{n+1})^+\| = 0$. This implies u_n is monotonically increasing w.r.t n. Now, using the Strong Maximum principle [31], we obtain $u_1 > 0$ in Ω, where u_1 is the solution of (3.3) with $n=1$. Since, u_n is monotonically increasing with respect to n, we have $u_n > u_1$ in Ω and hence we conclude that $u_n > C_K > 0$, for every $K \subset\subset \Omega$ with C_K being independent of n.

Claim: (u_n) is uniformly bounded in Ω.

Case 1: When $\lambda = 1$. Define, $M(k) = \{x\in\Omega : u_n > k\}$ and

$$
S_k(u_n) = \begin{cases} u_n - k; & \text{if } u_n > k \\ 0; & \text{if } u_n \le k. \end{cases}
$$

We choose, $S_k(u_n)$ as the test function in the weak formulation (3.3) to obtain

$$
\begin{aligned}
\int_{M(k)} |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla u_n &= \int_{M(k)} |\nabla u_n|^p \\
&= \int_{M(k)} \frac{u_n-k}{(u_n+\frac{1}{n})^\gamma}
\end{aligned}
$$

$$
\begin{aligned}
&< \int_{M(k)} \frac{u_n - k}{u_n^\gamma} \\
&\leq \|u_n - k\|_{L^p(M(k))} |M(k)|^{1/p'} \\
&\leq C \|\nabla u_n\|_{L^p(M(k))} |M(k)|^{1/p'},
\end{aligned}
$$

by the Poincaré inequality. Using the Sobolev embedding theorem, we obtain

$$
\|u_n\|^{p-1}_{L^{p^*}(M(k))} < \frac{C}{S^{p-1}} |M(k)|^{1/p'},
$$

where $p^* = \frac{Np}{N-p}$ which is the Sobolev conjugate of p. It is easy to see that $M(l) \subset M(k)$ for $1 < k < l$. Hence,

$$
|M(l)| \leq \big\{ \frac{C}{S^{p-1}} \big\}^{\frac{p^*}{p-1}} \frac{1}{(l-k)^{p^*}} |M(k)|^{\frac{p^*}{p}}.
$$

By [28, Lemma 4.1], we can guarantee the existence of a $T > 0$ independent of n such that $|M(T)| = 0$. Therefore, $\|u_n\|_\infty \leq T$.

Case 2: Suppose v is such that

$$
\int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla \phi < \lambda \int_\Omega \frac{\phi}{v^\gamma} \quad \forall \lambda \in W_0^{1,p}(\Omega),\ \phi \geq 0. \tag{3.4}
$$

Let $\lambda > 0$. Choose $v = (\frac{1}{\lambda})^{\frac{1}{\gamma+p-1}} w$. We can see that v satisfies

$$
\int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla \phi < \int_\Omega \frac{\phi}{v^\gamma}, \quad \forall \phi \in W_0^{1,p}(\Omega),\ \phi > 0.
$$

Therefore, using the result from Case 1, for $\lambda = 1$, we have $\|v\|_\infty \leq T$, which implies that $\|u_n\|_\infty \leq R\lambda^{\frac{1}{\gamma+p-1}}$. Hence, (u_n) is uniformly bounded in Ω. Finally, on using a result due to Lieberman [23], we conclude that $u_n \in C^1(\Omega)$ for all $n \in \mathbb{N}$. $\square$

Lemma 3.3. *Every bounded nontrivial solution v of the problem $-\Delta_p u = g(u) + \mu_n$ in Ω, is uniformly bounded below in $L^\infty(\Omega)$, i.e. $\|v\|_\infty > \delta$, for some $\delta > 0$.*

Before proving the above lemma, we prove the following lemma.

Lemma 3.4. *Every bounded nontrivial solution u of the problem $-\Delta_p u = g(u)$ in Ω, is uniformly bounded below in $L^\infty(\Omega)$, i.e. $\|u\|_\infty > \delta$, for some $\delta > 0$.*

Proof. Let us consider a sequence of nontrivial solutions (u_m) such that $\|u_m\|_\infty \to 0$ as $m \to \infty$. Then we can define $w_m(x) = u_m(x)\|u_m\|_\infty^{-1}$. Clearly, $\|w_m\|_\infty = 1$. As u_m satisfies $-\Delta_p u = g(u)$, we have

$$
\begin{aligned}
\Delta_p w_m &= \Delta_p (u_m(x)\|u_m\|_\infty^{-1}) \\
&= \nabla(|\nabla(u_m(x)\|u_m\|_\infty^{-1})|^{p-2} \nabla(u_m(x)\|u_m\|_\infty^{-1})) \\
&= \Delta_p u_m \|u_m\|_\infty^{1-p} \\
&= g(u_m)\|u_m\|_\infty^{1-p} \\
&\leq C u_m^q \|u_m\|_\infty^{1-p} \\
&\leq C w_m^q \|u_m\|_\infty^{1-p+q} = f_m.
\end{aligned}
$$

Now for very large m, these f_m's are uniformly bounded in $L^\infty(\Omega)$. So, $\|w_m\|_{C^{1,\beta}(\bar{\Omega})} \leq M$ for some $\beta \in (0,1)$, by regularity results in [30], where M is independent of m. Hence, by the Ascoli-Arzela theorem, the sequence (w_m) converges uniformly to w

in $C_0^1(\Omega)$. This implies $w = 0$. But with the consideration of [4, Lemma 1.1], we have a unique solution w in $C_0^1(\Omega)$, which contradicts the fact that $\|w_m\|_\infty = 1$. Hence, there exists $\delta > 0$ such that $\|u\|_\infty > \delta$. □

Proof of Lemma 3.3. Since $\mu_n \geq 0$, then the solutions of the problem in Lemma 3.3 are supersolutions of the problem in Lemma 3.4. Therefore, if v and u are solutions of the problem in Lemma 3.3 and Lemma 3.4 respectively, then $\|v\|_\infty \geq \|u\|_\infty > \delta > 0$, for some $\delta > 0$. □

Lemma 3.5. *There exists a $\bar{\lambda} > 0$ such that the problem*

$$
\begin{gathered}
-\Delta_p u = \frac{\lambda}{(u + \frac{1}{n})^\gamma} + g(u) + \mu_n \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega, \\
u > 0 \quad \text{in } \Omega
\end{gathered}
\tag{3.5}
$$

does not have any weak solution $u \in W_0^{1,p}(\Omega)$ for $\lambda \geq \bar{\lambda}$.

Proof. Let λ_1 be the first eigenvalue of the operator $-\Delta_p$ and its corresponding eigenfunction $\phi_1 \geq 0$ be such that

$$
\begin{gathered}
-\Delta_p \phi_1 = \lambda_1 \phi_1^{p-1} \quad \text{in } \Omega, \\
\phi_1 = 0 \quad \text{on } \partial\Omega.
\end{gathered}
$$

Its weak formulation with the test function $\phi = \phi_1$ is given by

$$
\int_\Omega |\nabla \phi_1|^p = \lambda_1 \int_\Omega \phi_1^p.
$$

Let u_n be the weak solution of (2.2), then by the strong maximum principle [31], we obtain $\frac{\phi_1^p}{u_n^{p-1}} \in W_0^{1,p}(\Omega)$. On applying the Picone's Identity [6, Theorem 2.1], we have

$$
\begin{aligned}
&\int_\Omega |\nabla \phi_1|^p dx - \int_\Omega \nabla(\frac{\phi_1^p}{u_n^{p-1}}) |\nabla u_n|^{p-2} \nabla u_n dx \geq 0 \\
\Rightarrow &\int_\Omega \lambda_1 \phi_1^p - \frac{\phi_1^p}{u_n^{p-1}} \frac{\lambda}{(u_n + \frac{1}{n})^\gamma} - g(u_n) \frac{\phi_1^p}{u_n^{p-1}} - \mu_n \frac{\phi_1^p}{u_n^{p-1}} dx \geq 0 \\
\Rightarrow &\int_\Omega \Big(\lambda_1 u_n^{p-1} - \lambda (u_n + \frac{1}{n})^{-\gamma} - g(u_n) - \mu_n \Big) \phi_1^p dx \geq 0.
\end{aligned}
$$

Consider $\bar{\lambda}$ defined as $\bar{\lambda} = \max_{x \in \Omega} \frac{\lambda_1 u_n^{p-1} - g(u_n) - \mu_n}{(u_n + 1)^{-\gamma}}$. Now for every $\epsilon > 0$, there exists a $\delta > 0$ such that $v^q < \epsilon v^{p-1}$ for all $v \in [0, \delta]$. Therefore, $\bar{\lambda} > 0$ for some ϵ and for $\lambda \geq \bar{\lambda}$, we have

$$
\begin{aligned}
\lambda &\geq \max_{x \in \Omega} \frac{\lambda_1 u_n^{p-1} - g(u) - \mu_n}{(u + 1)^{-\gamma}} \\
&\geq \frac{\lambda_1 u_n^{p-1} - g(u) - \mu_n}{(u + \frac{1}{n})^{-\gamma}} \\
&\Rightarrow \Big(\lambda_1 u_n^{p-1} - \lambda \big(u + \frac{1}{n}\big)^{-\gamma} - g(u) - \mu_n \Big) < 0
\end{aligned}
\tag{3.6}
$$

which is a contradiction to our assumption. Hence, for $\lambda \geq \bar{\lambda}$, the problem (2.2) does not possess any solution $u \in W_0^{1,p}(\Omega)$. □

Lemma 3.6. *Let Ω be a strictly convex domain and u_n be a solution of problem (2.2). Then there exists $M > 0$, which does not depend on n, such that $\|u_n\|_\infty \leq M$.*

Proof. We divide the proof of this lemma into six steps.

Step 1 (Uniform Höpf Lemma). Our aim is to show that $\frac{\partial u_n}{\partial n}(x) < c < 0$ for any $n \in \mathbb{N}$, where c is some constant which is independent of n but depends on x. $\hat{n}$ is the unit outward normal to the boundary $\partial\Omega$ at the point x.

Now Ω satisfies the interior ball condition as it has a C^2 boundary, i.e., for some $x_0 \in \partial\Omega$, there exists a $B_r(y) \subset \Omega$ such that $\partial B_r(y) \cap \partial\Omega = \{x_0\}$. Let us define $v : B_r(y) \to \mathbb{R}$ given by

$$v(x) = [2^{\frac{N-p}{p-1}} - 1]^{-1} r^{\frac{N-p}{p-1}} |x-y|^{\frac{p-N}{p-1}} - [2^{\frac{N-p}{p-1}} - 1]^{-1}.$$

We observe that

(i) $v(x) = 1$ on $\partial B_{\frac{r}{2}}(y)$ and $v(x) = 0$ on $\partial B_r(y)$, and
(ii) if $x \in B_r(y) \setminus B_{\frac{r}{2}}(y)$ with $|\nabla v(x)| > c > 0$ for some constant c independent of n.

Therefore, we have $0 < v(x) < 1$. Let us define $m = \inf\{u_n(x) | x \in \partial B_{\frac{r}{2}}(y)\}$. By using the Lemma 3.2, we can conclude that $m > 0$ and is independent of n. on choosing $w = mv$, we see that w satisfies

$$\begin{aligned} -\Delta_p w &= 0 \quad \text{in } B_r(y) - \overline{B_{\frac{r}{2}}(y)}, \\ w &= m \quad \text{if } x \in \partial B_{\frac{r}{2}}(y), \\ w &= 0 \quad \text{if} x \in \partial B_r(y). \end{aligned}$$

We have $u_n \geq w$ on the boundary of $B_r(y) - \overline{B_{\frac{r}{2}}(y)}$ and $-\Delta_p w \leq -\Delta_p u_n$ in Ω. Hence, by the weak comparison principle, we have $u_n \geq w$ in $B_r(y) - \overline{B_{\frac{r}{2}}(y)}$. Since, $u_n(x_0) = w(x_0) = 0$, then from the properties of v in (i) and (ii) above, we obtain

$$\begin{aligned} \frac{\partial u_n}{\partial \hat{n}}(x_0) &= \lim_{t\to 0} \frac{u_n(x_0 - t\hat{n})}{t} \leq \lim_{t\to 0} \frac{w(x_0 - t\hat{n})}{t} \\ &= \frac{\partial w}{\partial \hat{n}}(x_0) = m\frac{\partial w}{\partial \hat{n}} < -c < 0, \end{aligned}$$

where $c > 0$ is independent of n.

Step 2 (Existence of a neighbourhood of the boundary which does not contain any critical points of u_n). Let us denote $C(u_n) = \{x \in \Omega : \nabla u_n(x) = 0\}$, as the set of critical points of u_n. From Step 1, we have $\frac{\partial u_n}{\partial \eta} < 0$ on the boundary. Hence, $\operatorname{dist}(\partial\Omega, C(u_n)) = b_n > 0$ for all $n \in \mathbb{N}$ as $\partial\Omega$ and $C(u_n)$ are compact subsets in Ω.

Claim: There exists $\epsilon > 0$, independent of n, such that $b_n > \epsilon > 0$. In other words there exists a neighbourhood $\Omega_\epsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \epsilon\}$, such that $C(u_n) \cap \Omega_\epsilon = \phi$.

We prove this claim by a contrapositive argument. Assume there is no $\epsilon > 0$ such that $C(u_n) \cap \Omega_\epsilon \neq \phi$. Then there exists $x_n \in C(u_n)$ such that $\operatorname{dist}(x_n, \partial\Omega) \to 0$ as $n \to \infty$. Therefore, up to a subsequence $x_{n_k} \to x_0$ and $x_0 \in \partial\Omega$. But from Step 1, we obtain $\frac{\partial u_n}{\partial \eta}(x_0) < c < 0$. Hence, there exists $l > 0$ such that $|\nabla u_n(x)| > \frac{c}{2}$ for $x \in B_l(x_0) \cap \Omega$, where c is independent of n. This implies that $B_l(x_0) \cap C(u_n) = \phi$. This is a contradiction, since we can find $x_{n_0} \in B_l(x_0) \cap \Omega$ such that $\nabla u_{n_0}(x_{n_0}) = 0$. Hence the claim follows.

Step 3 (Monotonicity of u_n). Let $e \in \mathbb{S}^{N-1}, \delta \in \mathbb{R}$, then for a fixed $n \in \mathbb{N}$, we define the following:

(i) The hyperplane $\mathbb{L}_{\delta,e} = \{x \in \mathbb{R}^N : x.e = \delta\}$ and $\sigma_{\delta,e} = \{x \in \mathbb{R}^N : x.e < \delta\}$.
(ii) $\hat{x}$ be the reflection of x with respect to the hyperplane $\mathbb{L}_{\delta,e}$ i.e. $\hat{x} = x + 2(\delta - x.e)e$.
(iii) $a(e) = \inf_{x\in\Omega}\{x.e\}$ and the reflected cap of $\sigma_{\delta,e}$ with respect to $\mathbb{L}_{\delta,e}$ for any $\delta > a(e)$ denoted as $\hat{\sigma}_{\delta,e}$.
(iv) $\hat{\sigma}_{\delta,e}$ is not internally tangent to $\partial\Omega$ at some point $p \notin \mathbb{L}_{\delta,e}$.
(v) $\hat{n}(x)$ be the unit inward normal to $\partial\Omega$ at x, then $\hat{n}(x).e \neq 0$ for all $x \in \partial\Omega \cap \mathbb{L}_{\delta,e}$.
(vi) $\xi(e) = \{\mu_0 > a(e) : \forall\delta \in (a(e), \mu_0), \text{(iv) and (v) hold}\}$. and $\bar{\xi}(e) = \sup\{\xi(e)\}$.

If Ω is strictly convex, then the map $e \mapsto \bar{\xi}(e)$ is continuous by Proposition 2 of [5]. Let us denote $v_n(x) = u_n(\hat{x})$. Considering the strict convexity of Ω and the property (iv), we see that $\hat{\sigma}_{\delta,e}$ is contained in Ω for any $\delta \leq \delta_1$ where δ_1 only depends on Ω. Since, Δ_p is invariant under reflection and both u_n and v_n satisfy equation (2.2) hence both the functions take the same value on the hyperplane $\mathbb{L}_{\delta,e}$. Let us define $\delta_0 = \min(\delta_1, \epsilon)$. Also for $x \in \partial\Omega \cap \partial\sigma_{\delta,e}$, we have $u_n(x) = 0$ and $v_n(x) = u_n(\hat{x}) > 0$ as $\hat{x} \in \Omega$. Therefore,

$$\begin{aligned}
&-\Delta_p u_n + \frac{\lambda}{(u_n + \frac{1}{n})^\gamma} + g(u_n) + \mu_n \\
&= -\Delta_p v_n + \frac{\lambda}{(v_n + \frac{1}{n})^\gamma} + g(v_n) + \mu_n \quad \text{in } \sigma_{\delta,e} \\
&\qquad u_n \leq v_n \quad \text{on } \partial\sigma_{\delta,e} \cap \partial\Omega.
\end{aligned}$$

Then $u_n \leq v_n$ in $\sigma_{\delta,e}$ for any $\delta \in (a(e), \delta_0)$, by the comparison principle [12]. Hence, u_n is nondecreasing for all $x \in \sigma_{\delta_0,e}$ along the e-direction.

Step 4 (Existence of a measurable proper subset of Ω of nonzero measure on which u is nondecreasing). For a fixed $x_0 \in \partial\Omega$, let $e = e(x_0)$ be the unit outward normal to $\partial\Omega$ at x_0. Then by the results in Step 3, we conclude that u_n is nondecreasing in the direction of e for all $x \in \sigma_{\delta,e}$ and $a(e) < \delta < \delta_0$. For any $\theta \in \mathbb{S}^{N-1}$ in a small neighbourhood of e, the reflection of $\sigma_{\delta,\theta}$ w.r.t. $\mathbb{L}_{\delta,\theta}$ is a member of Ω, since the domain is strictly convex and hence the sequence u_n will be nondecreasing in the θ direction. Fix $\delta = \delta_0/2$. Since Ω is strictly convex, there exists a neighbourhood $\Theta \in \mathbb{S}^{N-1}$ such that $\sigma_{\delta_0/2,e} \subset \sigma_{\delta_0,\theta}$ for all $\theta \in \Theta$. Thus, we can conclude that u_n is nondecreasing in every direction for $\theta \in \Theta$ and for any x with $x \cdot e < \delta_0/2$. Consider

$$\sigma_0 = \Big\{x \in \Omega : \frac{\delta_0}{8} < x \cdot e < \frac{3\delta_0}{8}\Big\}.$$

Obviously, $\sigma_0 \subset \sigma_{\delta_0/2,e}$ and u_n is nondecreasing in every direction $\theta \in \Theta$ and $x \in \sigma_0$. Choose $\epsilon = \delta_0/8$ and fix a point $x \in \Omega_\epsilon$. Let x_0 be the projection of the point x onto $\partial\Omega$. We define $\mathbb{I}_x \subset \sigma_0$ to be the truncated cone having vertex at $x_0 - \epsilon e$ and an opening angle $\theta/2$. Then $\mathbb{I}_x$ satisfies the following properties.

(i) $|\mathbb{I}_x| > k$ for some k, where k depends only on Ω and ϵ,
(ii) $u_n(x) \leq u_n(y)$ for all $y \in \mathbb{I}_x$ and $n \in \mathbb{N}$.

Then, we have $u_n(x) \leq u_n(x_0 - \epsilon e) \leq u_n(y)$, for all $y \in \mathbb{I}_x$.

Step 5 (A boundary estimate). Let us consider the first eigenfunction ϕ_1 of the p-Laplacian eigenvalue problem over Ω. Using the Picone's identity on ϕ_1, u_n and then applying the strong maximum principle [31], we have $\frac{\phi_1^p}{u_n^{p-1}} \in W_0^{1,p}(\Omega)$. Denote $f_n(u_n) = \frac{\lambda}{(u_n+\frac{1}{n})^\gamma} + \mu_n$. Then, we have

$$\begin{aligned}\int_\Omega \frac{[f_n(u_n)+g(u_n)]\phi_1^p}{u_n^{p-1}} &= \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\big(\frac{\phi_1^p}{u_n^{p-1}}\big)\\ &\le \int_\Omega |\nabla \phi_1|^p dx \le C(\Omega).\end{aligned} \tag{3.7}$$

Let $\phi_1(z) \ge \xi > 0$ for all $z \in \Omega - \Omega_{\frac{\epsilon}{2}}$. Hence, from (3.7), we have

$$\xi^p \int_{\Omega-\Omega_{\frac{\epsilon}{2}}} \frac{[f_n(u_n)+g(u_n)]}{u_n^{p-1}} \le C(\Omega).$$

This implies

$$\int_{\mathbb{I}_x} \frac{[f_n(u_n)+g(u_n)]}{u_n^{p-1}} \le \frac{C(\Omega)}{\xi^p}.$$

Now since,

$$\int_{\mathbb{I}_x} \frac{[f_n(u_n)+g(u_n)]}{u_n^{p-1}} \ge \int_{\mathbb{I}_x} g(u_n)u_n^{1-p}(z)dz \ge u_n^{q-p+1}(x)|\mathbb{I}_x| \tag{3.8}$$

we have

$$u_n^{q-p+1}(x) \le \frac{C_1(\Omega)}{\xi^p},$$

for some constant $C_1 > 0$, i.e. $u_n(x) \le C'$, for all $x \in \Omega_\epsilon$ and for all $n \in \mathbb{N}$.

Step 6 (Blow-up analysis). We will show that for every open set, $K \subset\subset \Omega$, there exists $C_K > 0$ such that $\|u_n\|_\infty < C_K$, for every solution u_n of (2.2). We will prove it by contrapositive argument. Suppose, there exist a sequence (u_n) of positive solutions of the problem (2.2) and a sequence of points $(Z_n) \subset \Omega$ such that $M_n = u_n(Z_n) = \max\{u_n(x) : x \in \bar{K}\} \to \infty$ as $n \to \infty$. Using the boundary estimates one can assume that $Z_n \to x_0$ as $n \to \infty$, where $x_0 \in \bar{K}$. Let $\operatorname{dist}(\bar{K}, \partial\Omega) = 2d$ and $\Omega_d = \{x \in \Omega : \operatorname{dist}(x,\Omega) < d\}$.

Let R_n be the sequence of positive real numbers with $R_n^{\frac{p}{q-p+1}} M_n = 1$. Observe that $M_n \to \infty$ if and only if $R_n \to 0$ as $n \to \infty$. Define, $w_n : B_{\frac{d}{R_n}(0) \to \mathbb{R}}$ such that

$$w_n(y) = R_n^{\frac{p}{q-p+1}} u_n(Z_n + R_n y).$$

Now u_n has a maximum at Z_n, hence we have $\|w_n\|_\infty = w_n(0) = 1$. Since $R_n \to 0$ there exists n_0 such that $B_R(0) \subset B_{\frac{d}{R_n}}(0)$ for fixed $R > 0$. Again, we have that w_n satisfies

$$\nabla w_n(y) = R_n^{\frac{p}{q-p+1}+1} \nabla u_n(Z_n + R_n y)$$

and

$$\begin{aligned}-\Delta_p w_n(y) = R_n^{\frac{pq}{q-p+1}} \Big[&\lambda f_n(u_n(Z_n+R_n y)) + R_n^{\frac{-pq}{q-p+1}} w_n^q(Z_n+R_n y)\\ &+ R_n^{\frac{-pq}{q-p+1}} \mu_n(Z_n+R_n y)\Big].\end{aligned}$$

From Lemmas 3.1 and 3.3, for any $y \in B_R(0)$, we have $Z_n + R_n y \in \bar{\Omega}_d \subset \Omega$ and

$$\begin{aligned} &R_n^{\frac{pq}{q-p+1}}\Big[\lambda f_n(u_n(Z_n+R_ny)) + R_n^{\frac{-pq}{q-p+1}} w_n^q(Z_n+R_ny) \\ &+ R_n^{\frac{-pq}{q-p+1}}\mu_n(Z_n+R_ny)\Big] \le C(\bar{\Omega}_d), \end{aligned} \tag{3.9}$$

for every $n \ge n_0$. Let us fix a ball B such that $\bar{B} \subset B_{\frac{d}{R_n}}(0)$ for all $n \ge n_0$. Then by the interior estimates of Lieberman [23] and Tolksdorf [30], we have the existence of a constant $C = C(N,p,B) > 0$ and $\beta = \beta(N,p,B) \in (0,1)$ such that

$$w_n \in C^{1,\beta}(\bar{B}) \quad \text{and} \quad \|w_n\|_{1,\beta} \le C.$$

Using the Arzela-Ascoli theorem, we guarantee the existence of a function $w \in C^1(\bar{B})$ such that there exists a convergent subsequence $w_n \to w$ in $C^1(\bar{B})$. On passing the limit $n \to \infty$, we have

$$\int_B |\nabla w|^{p-2}\nabla w \cdot \nabla\phi \ge C\int_B w^q\phi, \quad \forall \phi \in C_c^\infty(B),\ w \in C^1(\bar{B}),\ w \ge 0 \text{ on } \bar{B},$$

where the constant is obtained from the growth condition over g and the condition in (3.9). Also, we have $\|w\|_\infty = 1$. Hence, by using the strong maximum principle [31], we have $w(x) > 0$, $\forall\, x \in B$. Now for a sequence of balls with increasing radius, the Cantor diagonal subsequence converges to $w \in C^1(\mathbb{R}^N)$, on every compact subsets of $\mathbb{R}^N$ and satisfy

$$\int_{\mathbb{R}^N} |\nabla w|^{p-2}\nabla w \cdot \nabla\phi \ge C\int_{\mathbb{R}^N} w^q\phi,$$

for all $\phi \in C_c^\infty(\mathbb{R}^N)$, $w \in C^1(\mathbb{R}^N)$, $w > 0$ on $\mathbb{R}^N$. This contradicts Theorem 4.9. ◻

Lemma 3.7. *For a strictly convex domain Ω, there exists $\bar{\lambda} > 0$ such that for $0 < \lambda < \bar{\lambda}$ and $\gamma > 0$ at least two solutions (say u_n, v_n) exist for the problem* (2.2) *in $W^{1,p}_{\rm loc}(\Omega)$.*

Proof. We define $\bar{J}_\lambda : C(\bar{\Omega}) \to C(\bar{\Omega})$ by

$$\bar{J}_\lambda(u) = (-\Delta_p)^{-1}\Big(\frac{\lambda}{(u+\frac{1}{n})^\gamma} + g(u) + \mu_n\Big), \quad \lambda \ge 0.$$

Now equation (2.2) can be written as $u = \bar{J}_\lambda(u)$. The map $\bar{J}_\lambda$ is compact since, we know $(-\Delta_p)^{-1}$ is a compact operator on $C(\bar{\Omega})$. So, we assume the map $\bar{J}_\lambda$ is also compact. For $0 < \lambda < \bar{\lambda}$, we have (u_n) as solutions to the problem (2.2) and $\|u_n\|_\infty \le M$, using Lemmas 3.5 and 3.6. Let us define, $S_1 = \{u \in C(\bar{\Omega}) : u \ge 0 \text{ in } \Omega\}$, $\bar{J}_0 : S_1 \to S_1$ by $\bar{J}_0(u) = (-\Delta_p)^{-1}(g(u)+\mu_n)$ and $G : \bar{B}_R \times [0,\infty) \to S_1$ such that $G(u,\lambda) = \bar{J}_\lambda$.

Claim 1. There exists a supersolution to problem (2.2). Let us define, $N(r) = \frac{1}{3}\big((\frac{r}{R})^{\gamma+p-1} - Cr^{\gamma+q}\big)$, for $r \in [0,\infty)$ where R is the bound used in Lemma 3.2 and $C > 0$ is the constant used in the growth condition of g and $\eta = \max_{0 \le r \le \beta_0} N(r)$, where

$$\beta_0 = \frac{1}{2}(2q-2p+3)^{\frac{1}{p-q-1}} R^{\frac{\gamma+p-1}{p-q-1}}.$$

Observe that, $N(r) > 0$ for $r \in (0,\beta_1)$, where $\beta_1 \in (0, \min(\gamma,\beta_0))$. Now applying the intermediate value property of continuous functions, we obtain that there exists a $\beta_2 \in (0,\beta_1)$ such that $N(\beta_2) = \lambda_0$. Denote $\lambda^* = \big(\frac{\beta_2}{R}\big)^{\gamma+p-1}$. So

$$\lambda_0 = N(\beta_2) = \frac{1}{2}\left(\lambda^* - C\beta_2^{\gamma+q}\right),$$

$$\lambda^* > \lambda_0 + \beta_2^{\gamma+q} = \lambda_0 + C[R(\lambda^*)^{\frac{1}{\gamma+p-1}}]^{\gamma+q}.$$

Let u_{n,λ^*} satisfy (2.2). Then for $n \geq n_0$, we have

$$\begin{aligned}
\lambda^* &> \lambda_0 + C\Big(\|u_{n,\lambda^*}\|_\infty\Big)^q\Big(\|u_{n,\lambda^*}\| + \frac{1}{n}\Big)^\gamma \\
&> \lambda + C\big(u_{n,\lambda^*}\big)^q\Big(u_{n,\lambda^*} + \frac{1}{n}\Big)^\gamma, \\
&> \lambda + g(u_{n,\lambda^*})\Big(u_{n,\lambda^*} + \frac{1}{n}\Big)^\gamma,
\end{aligned}$$

for $\lambda \leq \lambda_0$. Hence,

$$-\Delta_p u_{n,\lambda^*} = \frac{\lambda^*}{(u_{n,\lambda^*} + \frac{1}{n})^\gamma} + \mu_n > \frac{\lambda}{(u_{n,\lambda^*} + \frac{1}{n})^\gamma} + \mu_n + g(u_{n,\lambda^*}),$$

for $\lambda \leq \lambda^*$ and $n \geq n_0$. Therefore, $u_{n,\lambda^*} \in C^{1,\alpha}(\bar{\Omega})$ is a positive supersolution for some $\alpha > 0$ and u_{n,λ^*} is a supersolution of

$$\begin{gathered}
-\Delta_p u = \frac{\lambda}{(u + \frac{1}{n})^\gamma} + g(u) + \mu_n, \\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{3.10}$$

with $\|u_{n,\lambda^*}\|_\infty \leq \beta_2$.

Claim 2. Problem (2.2) possesses a unique solution. To prove this claim we define

$$f_n(x,r) = \frac{\lambda(r + \frac{1}{n})^{-\gamma} + g(r)}{r^{p-1}}, \quad \text{for } r \in [0,\infty).$$

Now the derivative of f_n w.r.t r is

$$\begin{aligned}
f_n'(x,r) &= \frac{1}{r^p}\Big[\frac{\lambda\{(1-p-\gamma)r + \frac{1-p}{n}\}}{(r+\frac{1}{n})^{1+\gamma}}\Big] + \frac{rg'(r) - g(r)(p-1)}{r^p} \\
&< \frac{1}{r^p}\Big[\frac{\lambda[(1-p-\gamma)r + \frac{1-p}{n}]}{(r+\frac{1}{n})^{1+\gamma}}\Big] + (q-p+1)r^{q-p}.
\end{aligned} \tag{3.11}$$

As the function $r^q(r+\frac{1}{n})^{1+\gamma}$ is convex, so there exists a unique $C_n > 0$, which is increasing with respect to λ such that

$$\lambda\big[(p+\gamma-1)C_n + \frac{p-1}{n}\big] > (q-p+1)C_n^q(C_n + \frac{1}{n})^{1+\gamma}.$$

Now for $r \leq C_n$, we have

$$(q-p+1)r^q(r+\frac{1}{n})^{1+\gamma} \leq \lambda\big[(p+\gamma-1)r + \frac{p-1}{n}\big].$$

Hence, $f_n'(x,r) < 0$. Consider

$$F_n(x,r) = \frac{\lambda(r+\frac{1}{n})^{-\gamma} + g(r) + \mu_n}{r^{p-1}}, \quad \text{for } r \in [0,\infty).$$

Clearly, $F_n'(x,r) = f_n'(x,r) - \frac{\mu_n(p-1)}{r^p} < 0$. Therefore, F_n is decreasing and using the result of Díaz-Saá [13], we guarantee that the problem (2.2) has unique solution and $\|u_n\|_\infty \leq C_n$.

Thus, we have $\beta_2 \leq \delta_0$. So

$$\frac{q-p+1}{\gamma+p-1}\beta_2^{\gamma+q} < \lambda_0, \quad \text{for } \gamma > 1.$$

Choose

$$\lambda_m = \frac{\{(q-p+1)(\beta_2+\epsilon)^q - \mu_m(p-1)\}(\beta_2+\epsilon+\frac{1}{m})^{1+\gamma}}{(p+\gamma-1)(\beta_2+\epsilon)+\frac{p-1}{m}} < \lambda_0,$$

then for all $n \geq m$, we have $C_n(\lambda_0) \geq C_n(\lambda_n) = \beta_2 + \epsilon$. So, $\|u_n\|_\infty \leq \beta_2 + \epsilon$. We can see that using Lemmas 3.3, 3.5 and 3.6, $\bar{J}_0$ and G satisfy all the conditions of Lemma (4.10) taken from [16] for some $0 < r < \beta_2 < R$. Since $\beta_2 < \alpha$, $(I - \bar{J}_0)(u)$ has no solution on ∂B_r. Now considering Lemma 3.5 and using Lemma 4.8 of [2], we can obtain a continuum $A_n \subset A = \{(\lambda, u) \in [0, \bar{\lambda}] \times C(\bar{\Omega}) : u - \bar{J}_\lambda(u) = 0\}$ such that

$$A_n \cap (\{0\} \times B_r) \neq \phi, \quad A_n \cap (\{0\} \times (B_R - B_r)) \neq \phi. \tag{3.12}$$

Next, we define $F : [0, \lambda_0] \to C_0^{1,\alpha}(\bar{\Omega})$ a continuous map such that $F(\lambda) = u_{n,\lambda^*}$. Using Lemma 4.7, we conclude that there exists $u_n \in A_n^{\lambda_0} = \{u \in C(\bar{\Omega}) : (\lambda_0, u) \in A_n\}$ such that $0 < u_n < u_{n,\lambda^*}$. We have $\|u_{n,\lambda^*}\|_\infty \leq \beta_2$ and hence $\|u_n\|_\infty \leq \|u_{n,\lambda^*}\|_\infty \leq \beta_2$. We have $A_n \cap (\{0\} \times (B_R - B_r)) \neq \phi$ by equation (3.12). Hence, for $n \geq \max(n_0, m)$, there exists v_n such that $\|v_n\|_\infty \geq \beta_2 + \epsilon$. For $\lambda = \lambda_0$ we have at least two solutions u_n and v_n to the problem (2.2). As $\lambda_0 < \bar{\lambda}$ is arbitrary, it concludes the proof. □

Theorem 3.8. *Given $\gamma > 0$ there exists $\bar{\lambda} > 0$ such that problem* (1.5) *admits at least two solutions u, v in $W^{1,p}_{\rm loc}(\Omega)$, provided Ω is strictly convex with $1 < p < N$, $p - 1 < q < \frac{p(N-1)}{N-p} - 1$ and for $0 < \lambda < \bar{\lambda}$.*

Proof. From Lemma 3.7, we can conclude the existence of at least two solutions u_n and v_n of problem (2.2). Also for a suitable choice of $c > 0$,

$$\underline{u} = (c\phi_1 + n^{\frac{1+p-\gamma}{p}})^{\frac{p}{\gamma+p-1}} - \frac{1}{n}$$

will be a weak subsolution to (3.1) for $\lambda = \lambda_0$.

Again, using $\frac{\lambda_0}{(r+\frac{1}{n})^\gamma} \leq \frac{\lambda_0}{(r+\frac{1}{n})^\gamma} + g(r) + \mu_n$ for all $r \geq 0$ we can conclude that each solutions of (2.2) with $\lambda = \lambda_0$ is a weak supersolution of (3.1). Now by the strong comparison principle [21], we have

$$\bar{u} \leq u_{n,\lambda_0} \leq u_n \leq \beta_2, \quad \bar{u} \leq u_{n,\lambda_0} \leq v_n, \quad \|v_n\|_\infty \geq \beta_2 + \epsilon. \tag{3.13}$$

Let us take $z_n = u_n$ or v_n, then from (3.13) and the Lemma 3.6 we have

$$\bar{u} \leq z_n \leq M,$$

where M is independent of n. By using the strong comparison principle [21] and Lemma 3.2, we have

$$\forall K \Subset \Omega,\ \exists C_K \text{ such that } z_n \geq C_K > 0 \text{ in } K,\ \forall n \in \mathbb{N}. \tag{3.14}$$

Claim. (z_n) is bounded in $W^{1,p}_{\rm loc}(\Omega)$. Consider $z_n\phi^p$ as a test function in (2.2) for $\phi \in C_0^1(\Omega)$, then we obtain

$$\begin{aligned}\int_\Omega |\nabla z_n|^p \phi^p = -p\int_\omega \phi^{p-1} z_n |\nabla z_n|^{p-2}\nabla\phi\cdot\nabla z_n + \int_\Omega \frac{\lambda_0 z_n \phi^p}{(z_n+\frac{1}{n})^\gamma} \\ + \int_\Omega z_n g(z_n)\phi^p + \int_\Omega z_n\mu_n\end{aligned}$$

By using the modified Young's inequality we have $\int_\Omega |\nabla z_n|^p \phi^p \le C_\phi$ for all $n \in \mathbb{N}$, where C_ϕ is a constant depending only on ϕ. Hence, $z_n \in W^{1,p}_{\rm loc}(\Omega)$ and there exists $z \in W^{1,p}_{\rm loc}(\Omega) \cap L^\infty(\Omega)$ such that $z_n \to z$ a.e. up to a subsequence and $z_n \to z$ weakly in $W^{1,p}(K)$ for all $K \subset\subset \Omega$. From the [10, Theorem 4.4], $\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\phi$ converges to $\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi$. Again, by using dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_\Omega \Big(\frac{\lambda_0\phi}{(z_n+\frac{1}{n})^\gamma} + \phi\ g(z_n)\Big)dx = \lambda_0 \int_\Omega \frac{\phi}{z^\gamma}dx + \int_\Omega \phi\ g(z)dx$$

Since, $\|u_n\|_\infty \le \beta_2$, $\|v_n\|_\infty \ge \beta_2 + \epsilon > \beta_2$ and $u_n \to u, v_n \to v$, we have the existence of two distinct solutions u and v. $\square$

Next we prove the existence result of the problem (1.5).

4. Existence result

4.1. The case of $\gamma < 1$. Let us consider the problem in (2.2) for $\gamma < 1$.

Lemma 4.1. *Let u_n be a solution of (2.2) with $\gamma < 1$. Then (u_n) is bounded in $W_0^{1,r}(\Omega)$ for every $r < \frac{N(p-1)}{N-1}$.*

Proof. We prove the boundedness of (∇u_n) in the Marcinkiewicz space $\mathcal{M}^{\frac{N(p-1)}{N-1}}(\Omega)$. For this, let us take $\varphi = T_k(u_n)$ as a test function in the weak formulation (2.3) and we have

$$\int_\Omega |\nabla T_k(u_n)|^p = \int_\Omega \frac{\lambda}{(u_n+\frac{1}{n})^\gamma} T_k(u_n) + \int_\Omega g(u_n)T_k(u_n) + \int_\Omega T_k(u_n)\mu_n. \tag{4.1}$$

Observe tht

$$\frac{T_k(u_n)}{(u_n+\frac{1}{n})^\gamma} \le \frac{u_n}{(u_n+\frac{1}{n})^\gamma} = \frac{u_n^\gamma}{(u_n+\frac{1}{n})^\gamma u_n^{\gamma-1}} \le u_n^{1-\gamma},$$
$$\int_\Omega T_k(u_n)\mu_n \le k\|\mu_n\|_{L^1(\Omega)} \le Ck.$$

Therefore,

$$\int_\Omega |\nabla T_k(u_n)|^p \le Ck. \tag{4.2}$$

Now consider the following set inclusion

$$\begin{aligned} \{|\nabla u_n| \ge t\} &= \{|\nabla u_n| \ge t, u_n < k\} \cup \{|\nabla u_n| \ge t, u_n \ge k\} \\ &\subset \{|\nabla u_n| \ge t, u_n < k\} \cup \{u_n \ge k\} \subset \Omega. \end{aligned}$$

With the help of the subadditivity property of Lesbegue measure m we have

$$m(\{|\nabla u_n| \ge t\}) \le m(\{|\nabla u_n| \ge t, u_n < k\}) + m(\{u_n \ge k\}). \tag{4.3}$$

By the Sobolev inequality,

$$\frac{1}{\lambda_1}\Big(\int_\Omega |T_k(u_n)|^{p^*}\Big)^{p/p^*} \le \int_\Omega |\nabla T_k(u_n)|^p \le Ck \tag{4.4}$$

where λ_1 is the first eigenvalue of the p-Laplacian operator. Now, on restricting the left hand side of the integral (4.4) on $I = [x \in \Omega : u_n \ge k]$, such that $T_k(u_n) = k$, we obtain

$$k^p m(\{u_n \ge k\})^{p/p^*} \le Ck$$

$$\Rightarrow m(\{u_n \geq k\}) \leq \frac{C}{k^{\frac{N(p-1)}{N-p}}}, \quad \forall k \geq 1.$$

Hence, (u_n) is bounded in $\mathcal{M}^{\frac{N(p-1)}{N-p}}(\Omega)$.

Similarly on restricting (4.4) on $I^{'} = \{|\nabla u_n| \geq t, u_n < k\}$, we have

$$m(\{|\nabla u_n| \geq t, u_n < k\}) \leq \frac{1}{t^p}\int_\Omega |\nabla T_k(u_n)|^p \leq \frac{Ck}{t^p}, \quad \forall k > 1.$$

Now (4.3) becomes

$$m(\{|\nabla u_n| \geq t\}) \leq m(\{|\nabla u_n| \geq t, u_n < k\}) + m(\{u_n \geq k\}) \leq \frac{Ck}{t^p} + \frac{C}{k^{\frac{N(p-1)}{N-p}}}, \quad \forall k > 1.$$

Let us choose, $k = t^{\frac{N-p}{N-1}}$ and hence we obtain

$$m(\{|\nabla u_n| \geq t\}) \leq \frac{C}{t^{\frac{N(p-1)}{N-1}}}, \quad \forall t \geq 1.$$

We have proved that (∇u_n) is bounded in $\mathcal{M}^{\frac{N(p-1)}{N-1}}(\Omega)$. This implies by property (2.1) that (u_n) is bounded in $W_0^{1,r}(\Omega)$, for every $r < \frac{N(p-1)}{N-1}$. □

Theorem 4.2. *Let $\gamma < 1$. Then there exists a weak solution u of* (1.5) *in $W_0^{1,r}(\Omega)$ for every $r < \frac{N(p-1)}{N-1}$.*

Proof. Lemma 4.1, implies that there exists u such that a subsequence of u_n converges weakly to u in $W_0^{1,r}(\Omega)$, for every $r < \frac{N(p-1)}{N-1}$. This implies that for φ in $C_c^1(\Omega)$,

$$\lim_{n\to+\infty}\int_\Omega \nabla u_n.\nabla\varphi = \int_\Omega \nabla u\cdot\nabla\varphi.$$

Also by the compact embeddings we can assume that u_n converges to u both strongly in $L^1(\Omega)$ and a.e. in Ω. Thus, taking φ in $C_c^1(\Omega)$, we obtain

$$0 \leq \Big|\frac{\lambda}{(u_n+\frac{1}{n})^\gamma}\varphi\Big| \leq C\lambda\|\varphi\|_{L^\infty(\Omega)}$$

This is sufficient to apply the dominated convergence theorem to obtain

$$\lim_{n\to+\infty}\int_\Omega \frac{\lambda}{(u_n+\frac{1}{n})^\gamma}\varphi = \int_\Omega \frac{\lambda}{u^\gamma}\varphi.$$

Further, since (u_n) is bounded in $W_0^{1,r}(\Omega)$, we have by the compact embedding that $u_n \to u$ in $L^r(\Omega)$. By the same standard argument, there exists a subsequence that converge to u uniformly except on a set of arbitrarily small Lebesgue measure. Since, by the hypothesis g is continuous, the limit $n\to\infty$ can be passed on. On applying a similar argument as in [26, step 4 of the Theorem 3.2], we have a.e. convergence of the ∇u_n towards ∇u that follows in a standard way by proving that $\nabla T_k(u_n)$ goes to $\nabla T_k(u)$, in $L^r_{\rm loc}(\Omega)$ for $r < p$, for every $k > 0$. Finally, we can pass the limit $n\to\infty$ in the last term of (2.3) involving μ_n. This concludes the proof of the result as it is easy to pass to the limit in (2.3). Therefore, we obtain a weak solution of (1.5) in $W_0^{1,r}(\Omega)$ for every $r < \frac{N(p-1)}{N-1}$. □

4.2. **The case of $\gamma \geq 1$.** Because of the strong singularity we can have some local estimates on u_n in the Sobolev space. We shall give global estimates on $T_k^{\frac{\gamma+p-1}{2}}(u_n)$ in $W_0^{1,2}(\Omega)$ with the aim of giving sense, at least in a weak sense, to the boundary values of u.

Lemma 4.3. *Let u_n be a solution of (2.2) with $\gamma \geq 1$. Then $T_k^{\frac{\gamma+p-1}{p}}(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ for every fixed $k > 0$.*

Proof. Consider $\varphi = T_k^\gamma(u_n)$ as a test function in (2.3). We have

$$\begin{aligned} &\gamma \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla T_k(u_n) T_k^{\gamma-1}(u_n) \\ &= \int_\Omega \frac{\lambda}{(u_n+\frac{1}{n})^\gamma} T_k^\gamma(u_n) + \int_\Omega g(u_n)T_k^\gamma(u_n) + \int_\Omega T_k^\gamma(u_n)\mu_n. \end{aligned} \tag{4.5}$$

We can estimate the term on the left-hand side of (4.5) as

$$\gamma \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla T_k(u_n) T_k^{\gamma-1}(u_n) = \gamma \int_\Omega |\nabla T_k^{\frac{\gamma+p-1}{p}}(u_n)|^p. \tag{4.6}$$

As

$$\frac{T_k^\gamma(u_n)}{(u_n+\frac{1}{n})^\gamma} \leq \frac{u_n^\gamma}{(u_n+\frac{1}{n})^\gamma} \leq 1,$$

the term on the right-hand side of (4.5) can be estimated as

$$\begin{aligned} &\int_\Omega \frac{\lambda}{(u_n+\frac{1}{n})^\gamma} T_k^\gamma(u_n) + \int_\Omega g(u_n)T_k^\gamma(u_n) + \int_\Omega T_k^\gamma(u_n)\mu_n \\ &\leq C\lambda k^\gamma + C\int_\Omega u_n^q T_k^\gamma(u_n) + k^\gamma \int_\Omega \mu_n \\ &\leq C\lambda k^\gamma + CMk^\gamma + k^\gamma \int_\Omega \mu_n \\ &\leq C(k,\gamma)k^\gamma. \end{aligned} \tag{4.7}$$

On combining the inequalities (4.6) and (4.7) we obtain

$$\int_\Omega |\nabla T_k^{\frac{\gamma+p-1}{p}}(u_n)|^p \leq Ck^\gamma\,. \tag{4.8}$$

Then $\big(T_k^{\frac{\gamma+p-1}{p}}(u_n)\big)$ is bounded in $W_0^{1,p}(\Omega)$ for every fixed $k > 0$. □

Now, so as to pass to the limit $n \to \infty$ in the weak formulation (2.3), we require to prove some local estimates on u_n. We first prove the following lemma.

Lemma 4.4. *Let u_n be a solution of (2.2) with $\gamma \geq 1$. Then (u_n) is bounded in $W_{\rm loc}^{1,r}(\Omega)$ for every $r < \frac{N(p-1)}{N-1}$.*

Proof. We prove the theorem in two steps.

Step 1. We claim that $(G_1(u_n))$ is bounded in $W_0^{1,r}(\Omega)$ for every $r < \frac{N(p-1)}{N-1}$. We can see that $G_1(u_n) = 0$ when $0 \leq u_n \leq 1$, $G_1(u_n) = u_n - 1$, otherwise i.e. when $u_n > 1$. So $\nabla G_1(u_n) = \nabla u_n$ for $u_n > 1$. Now, we need to show that $(\nabla G_1(u_n))$ is bounded in $\mathcal{M}^{\frac{N(p-1)}{N-1}}(\Omega)$, where $\mathcal{M}^{\frac{N(p-1)}{N-1}}(\Omega)$ is the Marcinkiewicz space. Then we have

$$\{|\nabla u_n| > t, u_n > 1\} = \{|\nabla u_n| > t, 1 < u_n \leq k+1\} \cup \{|\nabla u_n| > t, u_n > k+1\}$$

$$\subset \{|\nabla u_n| > t, 1 < u_n \le k+1\} \cup \{u_n > k+1\} \subset \Omega.$$

Hence,

$$\begin{aligned} &m(\{|\nabla u_n| > t, u_n > 1\}) \\ &\le m(\{|\nabla u_n| > t, 1 < u_n \le k+1\}) + m(\{u_n > k+1\}). \end{aligned} \tag{4.9}$$

To estimate (4.9) we take $\varphi = T_k(G_1(u_n))$, for $k > 1$, as a test function in (2.2). We observe that $\nabla T_k(G_1(u_n)) = \nabla u_n$ only when $1 < u_n \le k+1$, otherwise is zero, and $T_k(G_1(u_n)) = 0$ on $\{u_n \le 1\}$, we have

$$\begin{aligned} &\int_\Omega |\nabla T_k(G_1(u_n))|^p \\ &= \int_\Omega \frac{\lambda}{(u+\frac{1}{n})^\gamma} T_k(G_1(u_n)) + \int_\Omega g(u_n) T_k(G_1(u_n)) + \int_\Omega T_k(G_1(u_n))\mu_n \\ &\le C\lambda k + Ck \int_\Omega u_n^q + k\int_\Omega \mu_n \le Ck \end{aligned}$$

and by restricting the above integral on $I_1 = [1 < u_n \le k+1]$, we obtain

$$\begin{aligned} \int_{[1<u_n\le k+1]} |\nabla T_k(G_1(u_n))|^p &= \int_{[1<u_n\le k+1]} |\nabla u_n|^p \\ &\ge \int_{[|\nabla u_n|>t, 1<u_n\le k+1]} |\nabla u_n|^p \\ &\ge t^p m(\{|\nabla u_n| > t, 1 < u_n \le k+1\}) \end{aligned}$$

so that

$$m(\{|\nabla u_n| > t, 1 < u_n \le k+1\}) \le \frac{Ck}{t^p} \quad \forall k \ge 1.$$

According to (4.8) in the proof of Lemma 3.2, one can see that

$$\int_\Omega |\nabla T_k^{\frac{\gamma+p-1}{p}}(u_n)|^p \le Ck^\gamma \quad \text{for all } k > 1.$$

Therefore, from the Sobolev inequality

$$\frac{1}{\lambda_1}\Big(\int_\Omega |T_k^{\frac{\gamma+p-1}{p}}(u_n)|^{p^*}\Big)^{p/p^*} \le \int_\Omega |\nabla T_k^{\frac{\gamma+p-1}{p}}(u_n)|^p \le Ck^\gamma,$$

where, λ_1 is the first eigenvalue of the p-Laplacian operator. Now, if we restrict the integral on the left hand side on $I_2 = [u_n > k+1]_{x\in\Omega}$, on which $T_k(u_n) = k$, we then obtain

$$k^{\gamma+p-1} m(\{u_n > k+1\})^{p/p^*} \le Ck^\gamma,$$

so that

$$m(\{u_n > k+1\}) \le \frac{C}{k^{\frac{N(p-1)}{N-p}}}, \quad \forall k \ge 1.$$

So, (u_n) is bounded in $\mathcal{M}^{\frac{N(p-1)}{N-p}}(\Omega)$, i.e. $(G_1(u_n))$ is also bounded in $\mathcal{M}^{\frac{N(p-1)}{N-p}}(\Omega)$. Now (4.9) becomes

$$\begin{aligned} m(\{|\nabla u_n| > t, u_n > 1\}) &\le m(\{|\nabla u_n| > t, 1 < u_n \le k+1\}) + m(\{u_n > k+1\}) \\ &\le \frac{Ck}{t^p} + \frac{C}{k^{\frac{N(p-1)}{N-p}}}, \forall k > 1. \end{aligned}$$

We then choose $k = t^{\frac{N-p}{N-1}}$ and we obtain

$$m(\{|\nabla u_n| > t, u_n > 1\}) \leq \frac{C}{t^{\frac{N(p-1)}{N-1}}} \quad \forall t \geq 1.$$

We just proved that $(\nabla u_n) = (\nabla G_1(u_n))$ is bounded in $\mathcal{M}^{\frac{N(p-1)}{N-1}}(\Omega)$. This implies by property (2.1) that $(G_1(u_n))$ is bounded in $W_0^{1,r}$ for every $r < \frac{N(p-1)}{N-1}$.

Step 2. We claim that $T_1(u_n)$ is bounded in $W^{1,r}_{\rm loc}(\Omega)$. We have to examine the behavior of u_n for small values of u_n for each n. We want to show that for every $K \subset\subset \Omega$,

$$\int_K |\nabla T_1(u_n)|^p \leq C. \tag{4.10}$$

We have already proved that $u_n \geq C_K > 0$ on K in Lemma 3.2. We will use $\varphi = T_1^\gamma(u_n)$ as a test function in (2.3) to obtain

$$\begin{aligned} &\gamma \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla T_k(u_n) T_k^{\gamma-1}(u_n) \\ &= \int_\Omega \frac{\lambda}{(u_n + \frac{1}{n})^\gamma} T_k^\gamma(u_n) + \int_\Omega g(u_n) T_k^\gamma(u_n) + \int_\Omega T_k^\gamma(u_n)\mu_n \leq C. \end{aligned} \tag{4.11}$$

Now observe that

$$\begin{aligned} \gamma \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla T_1(u_n) T_1^{\gamma-1}(u_n) &\geq \int_K |\nabla T_1(u_n)|^p T_1^{\gamma-1}(u_n) \\ &\geq C_K^{\gamma-1} \int_K |\nabla T_1(u_n)|^p. \end{aligned} \tag{4.12}$$

On combining (4.11) and (4.12) we obtain (4.10). We completed the proof as $u_n = T_1(u_n) + G_1(u_n)$. Hence, (u_n) is bounded in $W^{1,r}_{\rm loc}(\Omega)$ for every $r < \frac{N(p-1)}{N-1}$. $\square$

Now, we can finally state and prove the existence result for $\gamma \geq 1$.

Theorem 4.5. *Let $\gamma \geq 1$. Then there exists a weak solution u of (1.5) in $W^{1,r}_{\rm loc}(\Omega)$ for every $r < \frac{N(p-1)}{N-1}$.*

The proof of the above theorem is a straightforward application of the Theorem 4.2 and using the results in Lemmas 4.3 and 4.4.

Other important results. Define, $X = \{u \in C_0^{1,\alpha}(\bar{\Omega}) : u(x) \geq 0 \text{ in } \bar{\Omega}\}$ and let ξ is a unit outward normal at $\partial\Omega$, then define $X_0 = \{u \in C_0^{1,\alpha}(\bar{\Omega}) : u(x) > 0 \text{ and } \frac{\partial u}{\partial \xi}(x) < 0, \forall x \in \partial\Omega\}$. Clearly X_0 is the interior of X.

Lemma 4.6. *If $u_1, u \in C_0^{1,\alpha}(\bar{\Omega})$ with $u_1 \neq u$ and*

$$-\Delta_p u_1 > \frac{\lambda}{(u_1 + \frac{1}{n})^\gamma} + g(u_1) + \mu_n,$$
$$-\Delta_p u = \frac{\lambda}{(u + \frac{1}{n})^\gamma} + g(u) + \mu_n,$$

then $(u_1 - u) \notin \partial X$.

Proof. We prove this Lemma by contradiction. Suppose $(u_1 - u) \in \partial X$. Then $u_1(x) \geq u(x)$. By Strong maximum principle [21], we can obtain $(u_1 - u) \in X_0$. But $X_0 \cap \partial X = \phi$, for which we obtain a contradiction. Therefore, $u_1 - u$ does not belong to ∂X. $\square$

Lemma 4.7. *Assume I is an interval in $\mathbb{R}$ and $A = I \times C_0^{1,\alpha}(\bar{\Omega})$ is a connected set of solutions of (2.2). Define $F : I \to C_0^{1,\alpha}(\bar{\Omega})$ is continuous such that $F(\lambda)$ is a supersolution to (2.2). If $u_1 \leq F(\lambda_1)$ in Ω, $u_1 \neq F(\lambda_1)$ for some $(\lambda_1, u_1) \in A$, then $u < F(\lambda)$ in Ω, for all $(\lambda, u) \in A$.*

Proof. Let $Z : A \to C_0^{1,\alpha}(\bar{\Omega})$ is a continuous map such that $Z(\lambda, u) = F(\lambda) - u$. A is connected, so by continuity $Z(A)$ is connected in $C_0^{1,\alpha}(\bar{\Omega})$. Using Lemma 4.6, $F(\lambda_1) - u_1 = Z(\lambda_1, u_1) \notin \partial X$. Hence, $Z(\lambda_1, u_1) \in X_0$. So, $Z(A) \subset X_0$, as $Z(A)$ is connected. Therefore, $F(\lambda) - u > 0$, which implies $F(\lambda) > u$ for all $(\lambda, u) \in A$. Hence, we obtain our required result. □

Lemma 4.8 (Ambrosetti-Arcoya [2]). *Given X be a real Banach space with $U \subset X$ be open, bounded set. Let $a, b \in \mathbb{R}$ such that the equation $u - T(\lambda, u) = 0$ has no solution on ∂U for all $\lambda \in [a, b]$ and that $u - T(\lambda, u) = 0$ has no solution in $\overline{U}$ for $\lambda = b$. Also let $U_1 \subset U$ be open such that $u - T(\lambda, u) = 0$ has no solution in ∂U_1 for $\lambda = a$ and $\deg(I - K_a, U_1, 0) \neq 0$. Then there exists a continuum C in $\sum = \{(\lambda, u) \in [a, b] \times X : u - T(\lambda, u) = 0\}$ such that*

$$C \cap (\{a\} \times U_1) \neq \emptyset \quad \text{and} \quad C \cap (\{a\} \times (U - U_1)) \neq \emptyset.$$

Theorem 4.9 (Mitidieri-Pohozaev [25]). *If $p - 1 < q < \frac{N(p-1)}{N-p}$, $p < N$ and $C > 0$, then the problem*

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \geq C \int_{\mathbb{R}^n} u^q \phi; \ \phi \in C_c^\infty(\mathbb{R}^n)$$

does not have any positive solution in $C^1(\mathbb{R}^n)$.

Theorem 4.10 (De Figueiredo et al. [16]). *Let C be a cone in a Banach space X and $\phi : C \to C$ be a compact map such that $\phi(0) = 0$. Assume that there exists $0 < r < R$ such that*

1. *$x \neq t\phi(x)$ for $0 \leq t \leq 1$ and $\|x\| = r$*
2. *a compact homotopy $F : \bar{B_R} \times [0, \infty) \to C$ such that $F(x, 0) = \phi(x)$ for $\|x\| = R$, $F(x, t) \neq x$ for $\|x\| = R$ and $0 \leq t < \infty$ and $F(x, t) = x$ has no solution for $x \in \bar{B_R}$ for $t \geq t_0$.*

Then, if $U = \{x \in C : r < \|x\| < R\}$ and $B_\rho = \{x \in C : \|x\| < \rho\}$, we have $\deg(I - \phi, B_R, 0) = 0$, $\deg(I - \phi, B_r, 0) = 1$ and $\deg(I - \phi, U, 0) = -1$.

Acknowledgements. S. Ghosh and A. Panda acknowledge the financial assistantship received to carry out this research from the Council of Scientific and Industrial Research (C.S.I.R.), Govt. of India and Ministry of Human Resource Development (M.H.R.D.), Govt. of India, respectively. The authors want to thank the anonymous referees and the editors for the constructive comments and suggestions.

References

[1] W. Allegretto, Y. X. Huang; *A Picone's identity for the p-Laplace and applications*, Nonlinear Anal., 32 (7), (1998), 819-830.

[2] A. Ambrosetti, D. Arcoya; *An introduction to nonlinear functional analysis and elliptic problems*, Progress in Nonlinear Differential Equations and their Applications, Birkhäuser Boston, Inc., Boston, MA, 82, 2011.

[3] D. Arcoya, L. Moreno-Mérida; *Multiplicity of solutions for a Dirichlet problem with a strongly singular nonlinearities*, Nonlinear Anal., 95, (2014), 281-291.

[4] C. Azizieh, P. Clément; *A priori estimates and continuation methods for positive solutions of p-Laplace equations*, J. Differential Equations, 179 (1), (2001), 213-245.

[5] C. Azizieh, L. Lemaire; *A note on the moving hyperplane method*, Proceedings of the 2001 Luminy Conference on Quasilinear Elliptic and Parabolic Equations and System, Electron. J. Differ. Equ. Conf. 8, (2002), 1-6.
[6] K. Bal; *Generalized Picone's identity and its applications*, Electron. J. Differential Equations, (2013), No. 243, 6 pp.
[7] K. Bal, P. Garain; *Multiplicity results for a quasilinear equation with singular nonlinearities*, Preprint arXiv:1709.05400vl [math.AP].
[8] A. Canino, M. Degiovanni; *A variational approach to a class of singular semilinear elliptic equations*, Journal of Convex Analysis, 11 (1) (2004), 147-162.
[9] A. Canino, M. Grandinetti, B. Sciunzi; *Symmetry of solutions of some semilinear elliptic equations with singular nonlinearities*, Journal of Differential Equations, 255 (12) (2013), 4437-4447.
[10] A. Canino, B. Sciunzi, A. Trombetta; *Existence and uniqueness for p-laplace equations involving singular nonlinearities*, NoDEA Nonlinear Differential Equations Appl., 23 (2) (2016), Art.8, 18.
[11] M. G. Crandall, P. H. Rabinowitz, L. Tartar; *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations, 2 (2) (1977), 193-222.
[12] L. Damascelli, B. Sciunzi; *Regularity, monotonicity and symmetry of positive solutions of m-Laplace equations*, J. Differential Equations, 206 (2) (2004), 483-515.
[13] J. I. Díaz, J. E. Saá; *Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires*, C. R. Acad. Sci. Paris Sér. I Math., 305 (12) (1987), 521-524.
[14] G. Ercole, G. A. Pereira; *Fractional Sobolev inequalities associated with singular problems*, Mathematische Nachrichten, 10.1002/mana.201700302.
[15] L. C. Evans; *Partial Differential Equations*, American Mathematical Society, Vol-19.
[16] D. G. de Figueiredo, P. L. Lions, R. D. Nussbaum; *A priori estimates and existence of positive solutions of semilinear elliptic equations*, J. Math. Purcs Appl., (9), 61 (1) (1982), 41-63.
[17] G. B. Folland; *Real analysis (Modern techniques and their applications)*, Second edition, A Wiley-Interscience publication.
[18] J. Giacomoni, K. Sreeandh; *Multiplicity results for a singular and quazilinear equation.* Conference Publications, 2007 (Special): 429-435. doi: 10.3934/proc.2007.2007.429
[19] J. Giacomoni, I. Schindler, P. Takáě; *Sobolev versus Hölder local minimizers and existence of multiple solutions for a singular quasilinear equation*, Ann Sc. Norm. Super. Pisa Cl. Sci., Série:5, 6 (1) (2007), 117-158.
[20] J. Giacomoni, I. Schindler, P. Takáč; *Sobolev versus Hölder local minimizers and existence of multiple solutions for a singular quasilinear equation*, Ann. Sc. Norm. Super. Pisa Cl. Sci., (5) 6 (1) (2017), 117-158.
[21] M. Guedda, L. Véron; *Quassilinear elliptic equations involving critical Sobolev exponents*, Nonlinear Anal., 13 (8) (1989), 879-902.
[22] Y. Haitao; *Multiplicity and asymptotic behavior of positive solutions for a singular semilinear elliptic problem*, J. Differential Equations, 189 (2) (2003), 487-512.
[23] G. M. Lieberman; *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal., 12 (11) (1988), 1203-1219.
[24] A. C. Lazer, P. J. McKenna; *On a singular nonlinear elliptic boundary-value problem*, Proc. Amer. Math. Soc., 111 (3) (1991), 721-730.
[25] E. Mitidieri, S. I. Pokhozhaev; *A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities*, Tr. Mat. Inst. Steklova, 234 (2001), 1-384.
[26] F. Oliva, F. Petitta; *On singular elliptic equations with measure sources*, ESAIM: Control, Optimisation and Calculus of Variations, 22 (2016), 289-308.
[27] A. Panda, S. Ghosh, D. Choudhuri; *Elliptic Partial Differential Equation Involving a Singularity and a Radon Measure*, The Journal of the Indian Mathematical Society, 86 (1-2) (2019), 95-117.
[28] G. Stampacchia; *Le problème de Dirichlet pour les équations elliptiques du seconde ordre à coefficientes discontinus*, Ann. Inst. Fourier (Grenoble), 15 (1965), 189-258.
[29] S. Taliaferro; *A nonlinear singular boundary value problem*, Nonlinear Analysis, 3 (1979), 897-904.
[30] P. Tolksdorf; *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations, 51 (1) (1984), 126-150.

[31] J. L. Vázquez; *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim., 12 (3) (1984), 191-202.

Sekhar Ghosh
Department of Mathematics, National Institute of Technology Rourkela, India
Email address: `sekharghosh1234@gmail.com`

Akasmika Panda
Department of Mathematics, National Institute of Technology Rourkela, India
Email address: `akasmika44@gmail.com`

Debajyoti Choudhuri
Department of Mathematics, National Institute of Technology Rourkela, India
Email address: `dc.iit12@gmail.com`