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STABILITY OF RIEMANN SOLUTIONS TO PRESSURELESS EULER EQUATIONS WITH COULOMB-TYPE FRICTION BY FLUX APPROXIMATION

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Abstract. We study the stability of Riemann solutions to pressureless Euler equations with Coulomb-type friction under the nonlinear approximation of flux functions with one parameter. The approximated system can be seen as the generalized Chaplygin pressure Aw-Rascle model with Coulomb-type friction, which is also equivalent to the nonsymmetric system of Keyfitz-Kranzer type with generalized Chaplygin pressure and Coulomb-type friction. Compared with the original system. The approximated system is strictly hyperbolic, which has one eigenvalue genuinely nonlinear and the other linearly degenerate. Hence, the structure of its Riemann solutions is much different from the ones of the original system. However, it is proven that the Riemann solutions to the approximated system converge to the corresponding ones to the original system as the perturbation parameter tends to zero, which shows that the Riemann solutions to the nonhomogeneous pressureless Euler equations is stable under such kind of flux approximation. In a word, we not only analyze the mechanism of the occurrence of the delta shocks, but also generalize the result about the stability of Riemann solutions with respect to flux perturbation from the well-known homogeneous case to the nonhomogeneous case.

1. Introduction

Non-strictly hyperbolic systems have important physical background, which are also difficult and interesting in mathematics. Many people have studied them and is well known that their Cauchy problem usually does not have a weak L^∞-solution. A typical example of this is the Cauchy problem for pressureless Euler equations (which is also called as zero pressure flow or transportation equations) [15, 36]. Therefore, the measure-value solution should be introduced to this nonclassical situation, such as delta shock wave [4, 30, 33] and singular shock [18, 21], which can also provide a reasonable explanation for some physical phenomena. However, the mechanism for the formation of delta shock wave cannot be fully understood, although the necessity of delta shock wave is obvious for Riemann solutions to some

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non-strictly hyperbolic systems. Now there are some related results for homogenous equations [4, 27], but few results have been shown for nonhomogeneous equations.

In this article, we are mainly concerned with zero pressure flow with Coulomb-type friction

$$\begin{aligned} &\rho_t + (\rho u)_x = 0, \\ &(\rho u)_t + (\rho u^2)_x = \beta\rho, \end{aligned} \tag{1.1}$$

where the state variable $\rho > 0$, u denote the density and velocity, respectively, and β is a frictional constant.

The motivation for studying (1.1) comes from the violent discontinuities in shallow flows with large Froude number [11]. It can also be derived directly from the so-called pressureless Euler/Euler-Possion systems [22]. Moreover, system (1.1) can also be obtained formally from the model proposed by Brenier et al. [3] to describe the sticky particle dynamics with interactions. Recently, the Riemann problem and shadow wave for (1.1) have been studied in [25] and [10]. Remarkably, in [25], it is shown that the Riemann problem for the nonhomogeneous equations (1.1) has delta shock wave solutions in some situations.

Delta shock wave is a kind of nonclassical nonlinear wave on which at least one of the state variables becomes a singular measure. Korchinski [19] firstly introduced the concept of the δ-function into the classical weak solution in his unpublished Ph.D. thesis. In 1994, Tan, Zhang and Zheng [33] considered some 1-D reduced system and discovered that the form of δ-functions supported on shocks was used as parts in their Riemann solutions for certain initial data. Since then, delta shock wave has been widely investigated, see [2, 20, 30] and references cited therein.

The formation of delta shock wave has been extensively studied by the vanishing pressure approximation for zero pressure flow [4, 27] and Chaplygin gas dynamics [7, 29, 38]. Recently, the flux approximation with two parameters [39] and three parameters [37] has also been carried out for zero pressure flow. In the present paper, we consider the nonlinear approximation of flux functions for zero pressure flow with Coulomb-type friction which has not attracted much attention before.

Specifically, we introduce the nonlinear approximation of flux functions for (1.1) as follows:

$$\begin{aligned} &\rho_t + (\rho u)_x = 0, \\ &(\rho(u+P))_t + (\rho u(u+P))_x = \beta\rho, \end{aligned} \tag{1.2}$$

where P is given by the state equation for generalized Chaplygin gas [1, 24, 29, 34]

$$P = -\frac{A}{\rho^\alpha}, \quad A > 0,\ 0 < \alpha < 1, \tag{1.3}$$

with α a real constant and the parameter A sufficiently small. System (1.2) and (1.3) can be seen as the generalized Chaplygin pressure Aw-Rascle model with Coulomb-type friction. By taking $u = w - P$, (1.2) can be written as

$$\begin{aligned} &\rho_t + (\rho(w-P))_x = 0, \\ &(\rho w)_t + (\rho w(w-P))_x = \beta\rho, \end{aligned} \tag{1.4}$$

with a pure flux approximation. Systems (1.4) and (1.3) can also be seen as the nonsymmetric system of Keyfitz-Kranzer type with generalized Chaplygin pressure and Coulomb-type friction [13]. Recently, for $\beta = 0$, Cheng has showed that the structure of Riemann solutions to (1.2) and (1.4) were very similar [5, 6].

More precisely, we are only concerned with the Riemann problem, i.e. the initial data

$$(\rho, u)(x, 0) = \begin{cases} (\rho_-, u_-), & x < 0, \\ (\rho_+, u_+), & x > 0, \end{cases} \tag{1.5}$$

where $\rho_\pm$ and $u_\pm$ are all given constants.

In this article, we will find that the delta shock wave also appears in the Riemann solutions to (1.2) for some specific initial data. We are interested in how the delta-shock solution of (1.2) and (1.5) develops under the influence of the Coulomb-type friction. The advantage of this kind source term is in that (1.2) can be written in a conservative form such that exact solutions to the Riemann problem (1.2) and (1.5) can be constructed explicitly. We shall see that the Riemann solutions to (1.2) and (1.5) are not self-similar any more, in which the state variable u varies linearly along with the time t under the influence of the Coulomb-type friction. In other words, the state variable $u - \beta t$ remains unchanged in the left, intermediate and right states. In some situations, the delta-shock wave appears in Riemann solutions to (1.2) and (1.5). In order to describe the delta-shock wave, the generalized Rankine-Hugoniot conditions are derived and the exact position, propagation speed and strength of the delta shock wave are obtained completely. It is shown that the Coulomb-type friction term make contact discontinuities, shock waves, rarefaction waves and delta shock waves for Riemann solutions bend into parabolic shapes.

Furthermore, it is proven rigorously that the limits of Riemann solutions to (1.2) and (1.5) converge to the corresponding ones to (1.1) and (1.5) when the perturbation parameter A tends to zero. In other words, the Riemann solutions (1.1) and (1.5) is stable with respect to the nonlinear approximations of flux functions in the form of (1.2). Actually, for the case $\alpha = 1$ in (1.3), system (1.2) becomes the Chaplygin pressure Aw-Rascle model with Coulomb-type friction [23]. Similar result can be easily got, so we do not focus on it here. Moreover, the results got in this paper can also be generalized to the nonsymmetric system of Keyfitz-Kranzer type (1.4) with the same generalized Chaplygin pressure and Coulomb-type friction.

This article is organized as follows. In section 2, we describe simply the solutions to the Riemann problem (1.1) and (1.5) for completeness. In Section 3, the approximated system (1.2) is reformulated into a conservative form and some general properties of the conservative form are obtained. Then, the exact solution to the Riemann problem for the conservative form is constructed explicitly, which involves the delta shock wave. Furthermore, the generalized Rankine-Hugoniot conditions are established and the exact position, propagation speed and strength of the delta shock wave are given explicitly. In Section 4, the generalized Rankine-Hugoniot conditions and three kinds of Riemann solutions to the approximated system (1.2) and (1.5) are given. Furthermore, it is proven rigorously that the delta-shock wave is indeed a week solution to the Riemann problem (1.2) and (1.5) in the sense of distributions. In Section 5, the limit of Riemann solutions to the approximated system (1.2) is taken by letting the perturbation parameter A tends to zero, which is identical with the corresponding ones to the original system. Finally, conclusions and discussions are drawn in Section 6.

2. Preliminaries

In this section, we simply describe the results on the Riemann problem (1.1) and (1.5), which can be referred to [25] for details.

Let us first state some known facts about elementary waves of the given system. The system (1.1) is weakly hyperbolic with double eigenvalues $\lambda_1 = \lambda_2 = u$. Let us first look for a solution to (1.1) when initial data are constants, $(\rho(x, 0), u(x, 0)) = (\rho_0, u_0)$. For smooth solutions, one can substitute ρ_t from the first equation of (1.1) into the second one and eliminate ρ from it by division (provided that we are away from a vacuum state). So, we have now the equation $u_t + uu_x = \beta$ that can be solved by the method of characteristics: $u = u_0 + \beta t, x = x_0 + u_0 t + \frac{1}{2}\beta t^2$. The first equation then becomes $\rho_t + (u_0 + \beta t)\rho_x = 0$ with a solution $\rho = \rho_0$ on each curve $x = x_0 + u_0 t + \frac{1}{2}\beta t^2$. So, the solution for constant initial data is $(\rho, u) = (\rho_0, u_0 + \beta t)$.

For the case $u_- < u_+$, there is no characteristic passing through the region $\{(x, t) : u_- t + \frac{1}{2}\beta t^2 < x < u_+ t + \frac{1}{2}\beta t^2\}$, so the vacuum should appear in the region. The solution can be expressed as

$$(\rho, u)(x, t) = \begin{cases} (\rho_-, u_- + \beta t), & -\infty < x < u_- t + \frac{1}{2}\beta t^2, \\ \text{vacuum}, & u_- t + \frac{1}{2}\beta t^2 < x < u_+ t + \frac{1}{2}\beta t^2, \\ (\rho_+, u_+ + \beta t), & u_+ t + \frac{1}{2}\beta t^2 < x < \infty. \end{cases} \tag{2.1}$$

For the case $u_- = u_+$, it is easy to see that the two states $(\rho_\pm, u_\pm + \beta t)$ can be connected by a contact discontinuity $x = u_\pm t + \frac{1}{2}\beta t^2$. So the solution can be expressed as

$$(\rho, u)(x, t) = \begin{cases} (\rho_-, u_- + \beta t), & x < u_- t + \frac{1}{2}\beta t^2, \\ (\rho_+, u_+ + \beta t), & x > u_+ t + \frac{1}{2}\beta t^2, \end{cases} \tag{2.2}$$

For the case $u_- > u_+$, the characteristics originating from the origin overlap in the domain $\{(x, t) : u_+ t + \frac{1}{2}\beta t^2 < x < u_- t + \frac{1}{2}\beta t^2\}$, which means that there exists singularity. A solution containing a weighted δ-measure supported on a curve will be constructed.

To define the measure solution as above, like as in [4, 30], the two-dimensional weighted δ-measure $p(s)\delta_S$ supported on a smooth curve $S = \{(x(s), t(s)) : a \leq s \leq b\}$ should be introduced as

$$\langle p(s)\delta_S, \psi(x(s), t(s))\rangle = \int_a^b p(s)\psi(x(s), t(s))\sqrt{x'(s)^2 + t'(s)^2}\,ds, \tag{2.3}$$

for any $\psi \in C_0^\infty(R \times R_+)$. For convenience, we usually select the parameter $s = t$ and use $w(t) = \sqrt{1 + x'(t)^2}p(t)$ to denote the strength of delta shock wave from now on.

Let $x = x(t)$ be a discontinuity curve, we consider a piecewise smooth solution of (1.1) in the form

$$(\rho, u)(x, t) = \begin{cases} (\rho_-, u_- + \beta t), & x < x(t), \\ (w(t)\delta(x - x(t)), u_\delta(t)), & x = x(t), \\ (\rho_+, u_+ + \beta t), & x > x(t), \end{cases} \tag{2.4}$$

in which $u_\delta(t)$ is the assignment of u on this delta shock wave curve and $u_\delta(t) - \beta t$ is assumed to be a constant. The delta shock wave solution of the Riemann problem

(1.1) and (1.5) must obey the following generalized Ranking-Hugoniot conditions:

$$\begin{aligned}\frac{dx(t)}{dt}&=\sigma(t)=u_\delta(t),\\ \frac{dw(t)}{dt}&=\sigma(t)[\rho]-[\rho u],\\ \frac{d(w(t)u_\delta(t))}{dt}&=\sigma(t)[\rho u]-[\rho u^2]+\beta w(t),\end{aligned}\tag{2.5}$$

and the over-compressive entropy condition

$$\lambda(\rho_+,u_+)<\sigma(t)<\lambda(\rho_-,u_-),\quad \text{namely } u_++\beta t<u_\delta(t)<u_-+\beta t.\tag{2.6}$$

In (2.5), it should be remarkable that

$$[\rho u]=\rho_+(u_++\beta t)-\rho_-(u_-+\beta t),\quad [\rho u^2]=\rho_+(u_++\beta t)^2-\rho_-(u_-+\beta t)^2.$$

Through solving (2.5) with $x(0)=0$, $w(0)=0$, we obtain

$$\begin{aligned}u_\delta(t)&=\sigma(t)=\sigma_0+\beta t,\\ x(t)&=\sigma_0 t+\frac12\beta t^2,\\ w(t)&=-\sqrt{\rho_-\rho_+}(u_+-u_-)t,\end{aligned}\tag{2.7}$$

with

$$\sigma_0=\frac{\sqrt{\rho_-}u_-+\sqrt{\rho_+}u_+}{\sqrt{\rho_-}+\sqrt{\rho_+}}.$$

It is easy to prove that the delta shock wave solution (2.4) with (2.7) satisfy the system (1.1) in the distributional sense. That is to say, the equalities

$$\begin{aligned}\langle\rho,\psi_t\rangle+\langle\rho u,\psi_x\rangle&=0,\\ \langle\rho u,\psi_t\rangle+\langle\rho u^2,\psi_x\rangle&=-\langle\beta\rho,\psi\rangle,\end{aligned}\tag{2.8}$$

hold for any test function $\psi\in C_0^\infty(R\times R_+)$, in which

$$\langle\rho u,\psi\rangle=\int_0^\infty\int_{-\infty}^\infty\hat\rho_0\hat u_0\psi\,dx\,dt+\langle w(t)u_\delta(t)\delta_S,\psi\rangle,$$

with

$$\hat\rho_0=\rho_-+[\rho]H(x-\sigma t),\quad \hat u_0=u_--\beta t+[u]H(x-\sigma t).$$

From the above discussions, we can conclude that the Riemann problem (1.1) and (1.5) can be solved by three kinds of solutions: one contact discontinuity, two contact discontinuities with the vacuum state between them (see Figure 1), or the delta shock wave connecting two states $(\rho_\pm,u_\pm+\beta t)$ (see Figure 2).

3. Riemann problem for a modified conservative system of (1.2)

In this section, we are devoted to the study of the Riemann problem for a conservative system of (1.2) in detail. Let us introduce the new velocity $v(x,t)=u(x,t)-\beta t$, then the system (1.2) can be reformulated into a conservative form as

$$\begin{aligned}\rho_t+(\rho(v+\beta t))_x&=0,\\ (\rho(v+P))_t+(\rho(v+P)(v+\beta t))_x&=0.\end{aligned}\tag{3.1}$$

In fact, the change of variable was introduced by Faccanoni and Mangeney [12] to study the shock and rarefaction waves of the Riemann problem for the shallow

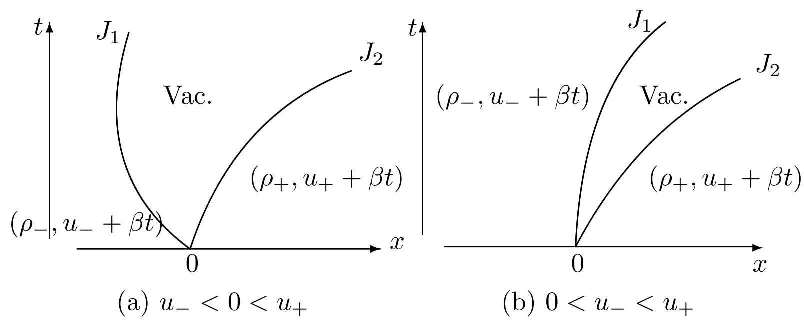


FIGURE 1. The Riemann solution to (1.1) and (1.5) when $\beta > 0$.

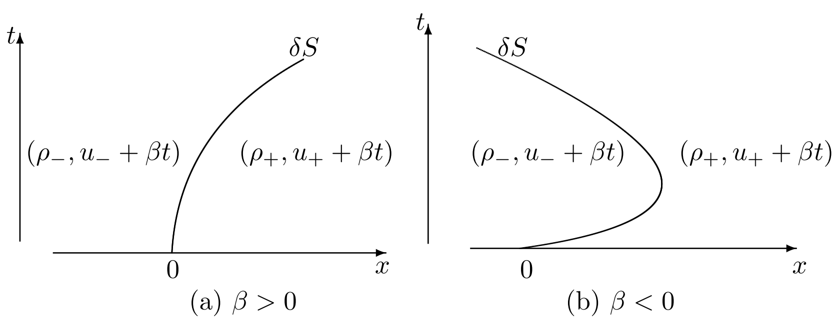


FIGURE 2. The delta shock wave solution to (1.1) and (1.5) when $u_+ < u_-$ and $\sigma_0 > 0$.

water equations with a Coulomb-type friction. Here, we use this transformation to study the delta shock wave for the system (1.2).

Now we want to deal with the Riemann problem for the conservative system (3.1) with the same Riemann initial data (1.5),

$$(\rho, v)(x, 0) = \begin{cases} (\rho_-, u_-), & x < 0, \\ (\rho_+, u_+), & x > 0. \end{cases} \tag{3.2}$$

We shall see hereafter that the Riemann solutions to (1.2) and (1.5) can be obtained immediately from the Riemann solutions to (3.1) and (3.2) by using the transformation of state variables $(\rho, u)(x, t) = (\rho, v + \beta t)(x, t)$.

System (3.1) can be rewritten in the quasi-linear form

$$\begin{aligned} &\begin{pmatrix} 1 & 0 \\ v + P + \rho P' & \rho \end{pmatrix} \begin{pmatrix} \rho \\ v \end{pmatrix}_t \\ &+ \begin{pmatrix} v + \beta t & \rho \\ (v + P + \rho P')(v + \beta t) & \rho(2v + \beta t + P) \end{pmatrix} \begin{pmatrix} \rho \\ v \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{3.3}$$

It can be derived directly from (3.3) that the conservative system (3.1) has two eigenvalues

$$\lambda_1(\rho, v) = v + \beta t - \frac{A\alpha}{\rho^\alpha}, \quad \lambda_2(\rho, v) = v + \beta t,$$

whose corresponding right eigenvectors are by

$$r_1 = (\rho, -\frac{A\alpha}{\rho^\alpha})^T, \quad r_2 = (1,0)^T.$$

So (3.1) is strictly hyperbolic for $\rho > 0$. Moreover, $\nabla\lambda_1 \cdot r_1 \neq 0$ and $\nabla\lambda_2 \cdot r_2 = 0$. Then it can be concluded that λ_1 is genuinely nonlinear whose associated waves are shock waves denoted by S_1 or rarefaction waves denoted by R_1. Then the Riemann invariants along the characteristic fields may be chosen as

$$w = v - \frac{A}{\rho^\alpha}, \quad z = v,$$

which should satisfy $\nabla w \cdot r_1 = 0$ and $\nabla z \cdot r_2 = 0$, respectively. For details about above elementary waves, one can refer to [28, 32] to see how to solve the Riemann problem.

Let us draw our attention on the elementary waves for the system (3.1) in detail. We first consider the rarefaction wave which is a one-parameter family of states connecting a given state. This kind of continuous solution satisfying the system (3.1) can be obtained by determining the integral curves of the first characteristic fields. It is worthwhile to notice that the 1-Riemann invariant is conserved in the 1-rarefaction wave.

For a given left state (ρ_-, u_-), the 1-rarefaction wave curve $R_1(\rho_-, v_-)$ in the phase plane, the set of states connected on the right, should satisfy $R_1(\rho_-, u_-)$:

$$\begin{gathered} \frac{dx}{dt} = \lambda_1(\rho, v) = v + \beta t - \frac{A\alpha}{\rho^\alpha}, \\ v - \frac{A}{\rho^\alpha} = u_- - \frac{A}{\rho_-^\alpha} = w_-, \\ \lambda_1(\rho_-, u_-) \leq \lambda_1(\rho, v). \end{gathered} \tag{3.4}$$

By differentiating v with respect to ρ in the second equation in (3.4), we have

$$\begin{gathered} \frac{dv}{d\rho} = -\frac{A\alpha}{\rho^{\alpha+1}} < 0, \\ \frac{d^2v}{d\rho^2} = \frac{A\alpha(\alpha+1)}{\rho^{\alpha+2}} > 0. \end{gathered}$$

Thus, the 1-rarefaction wave is made up of the half-branch of $R_1(\rho_-, u_-)$ satisfying $v \geq u_-$ and $\rho \leq \rho_-$, which is convex in the (ρ, v) plane.

Let us compute the solution (ρ, v) at a point in the interior of the 1-rarefaction wave, then it follows from the first equation in (3.4), we have

$$v - \frac{A\alpha}{\rho^\alpha} = \frac{x}{t} - \beta t. \tag{3.5}$$

By combining (3.5) with the second equation in (3.4), we obtain

$$(\rho, v)(x,t) = \Big(\big(\frac{A(1-\alpha)}{\frac{x}{t} - \beta t - w_-} \big)^{1/\alpha}, \frac{\frac{x}{t} - \beta t - \alpha w_-}{1-\alpha} \Big). \tag{3.6}$$

Let us return our attention on the shock wave which is a piecewise constant discontinuous solution, satisfying the Rankine-Hugoniot conditions and the entropy condition. Here the Ranking-Hugoniot conditions can be derived in a standard method as in [28], since the parameter t only appears in the flux functions in the conservative system (3.1). For a bounded discontinuity at $x = x(t)$, let us denote

$\sigma(t) = x'(t)$, then the Ranking-Hugoniot conditions for the conservative system (3.1) can be expressed as

$$\begin{aligned} -\sigma(t)\rho + [\rho(v+\beta t)] &= 0, \\ -\sigma(t)[\rho(v+P)] + [\rho(v+P)(v+\beta t)] &= 0, \end{aligned} \tag{3.7}$$

where $[\rho] = \rho_r - \rho_l$ with $\rho_l = \rho(x(t)-0,t)$, $\rho_r = \rho(x(t)+0,t)$, in which $[\rho]$ denote the jump of ρ across the discontinuity, etc. It is clear that the propagation speed of the discontinuity depends on the parameter t, which is obviously different from classical hyperbolic conservation laws.

If $\sigma(t) \neq 0$, then it follows from (3.7) that

$$\rho_r\rho_l(v_r - v_l)\Big((v_r - \frac{A}{\rho_r^\alpha}) - (v_l - \frac{A}{\rho_l^\alpha})\Big) = 0, \tag{3.8}$$

from which we have $v_r = v_l$ or $v_r - \frac{A}{\rho_r^\alpha} = v_l - \frac{A}{\rho_l^\alpha}$.

Thus, for a given left state (ρ_-, u_-), with the latex entropy condition in mind, the 1-shock wave curve $S_1(\rho_-, u_-)$ in the (ρ, v) plane which is the set of states connected on the right, should satisfy $S_1(\rho_-, u_-)$:

$$\begin{gathered} \sigma_1(t) = \frac{\rho v - \rho u_-}{\rho - \rho_-} + \beta t, \\ v - \frac{A}{\rho^\alpha} = u_- - \frac{A}{\rho_-^\alpha} = w_-, \\ \rho > \rho_-, \quad v < u_-, \end{gathered} \tag{3.9}$$

which indicates the 1-rarefaction wave and 1-shock wave are different branch of the same curve.

Moreover, from (3.8), for a given left state (ρ_-, u_-), the 2-contact discontinuity curve $J(\rho_-, u_-)$ in the (ρ, v) plane which is the set of states connected on the right, should satisfy $J(\rho_-, u_-)$:

$$\sigma(t) = v + \beta t = u_- + \beta t. \tag{3.10}$$

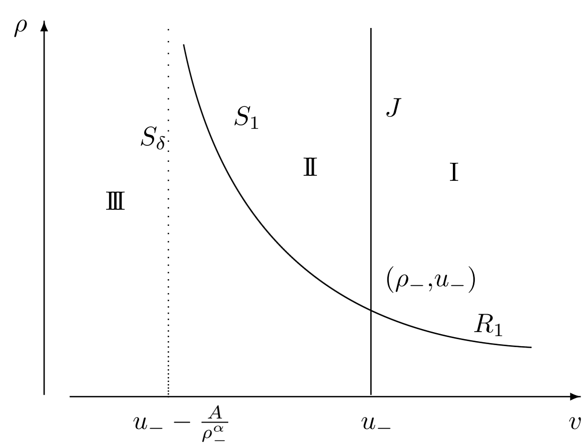


FIGURE 3. The (ρ, v) phase plane for the conservative system (3.1).

Let us now consider the Riemann problem (3.1) and (3.2). In the (ρ, v) phase plane, for a given left state (ρ_-, u_-), the set of states connected on the right consist of the 1-rarefaction wave $R_1(\rho_-, u_-)$, the 1-shock wave $S_1(\rho_-, u_-)$ and the 2-contact discontinuity curve $J(\rho_-, u_-)$. It is clear to see that $R_1(\rho_-, u_-)$ has the line S_δ : $v = u_- - \frac{A}{\rho_-^\alpha}$ and $S_1(\rho_-, u_-)$ has the positive v-axis as their asymptotic lines, respectively.

In view of the right state (ρ_+, u_+) in different positions, one wants to construct the unique global Riemann solution of (3.1) and (3.2). However, as in [13], if $u_+ \leq u_- - \frac{A}{\rho_-^\alpha}$ is satisfied, the Riemann solution of (3.1) and (3.2) can not be constructed by using only the elementary waves including shocks, rarefaction waves and contact discontinuities. In this nonclassical situation, the concept of delta shock wave should be introduced such as in [13, 14, 34] and be discussed later.

Draw all the curves $R_1(\rho_-, u_-)$, $S_1(\rho_-, u_-)$ $J(\rho_-, u_-)$ and S_δ in the the (ρ, v) phase plane, thus the phase plane is divided into three regions I, Ⅱ and Ⅲ (See Figure 3), where

$$
\begin{gathered}
\mathrm{I} = \{(\rho, v) | v \geq u_-\}, \\
\mathrm{II} = \{(\rho, v) | u_- - \frac{A}{\rho_-^\alpha} < v < u_-\}, \\
\mathrm{III} = \{(\rho, v) | v \leq u_- - \frac{A}{\rho_-^\alpha}\}.
\end{gathered}
$$

According to the right state (ρ_+, u_+) in different regions, the unique global Riemann solution of (3.1) and (3.2) can be constructed connecting two constant states (ρ_-, u_-) and (ρ_+, u_+)

If $(\rho_+, u_+) \in \mathrm{I}$, namely $u_+ > u_-$, then the Riemann solution consists of 1-rarefaction wave R_1 and a 2-contact discontinuity J with an intermediate constant state (ρ_*, v_*) determined uniquely by

$$
\begin{gathered}
v_* - \frac{A}{\rho_*^\alpha} = u_- - \frac{A}{\rho_-^\alpha} = w_-, \\
u_+ = v_*.
\end{gathered} \tag{3.11}
$$

which immediately leads to

$$
(\frac{A}{\rho_*^\alpha}, v_*) = (u_+ - u_- + \frac{A}{\rho_-^\alpha}, u_+), \tag{3.12}
$$

or

$$
(\rho_*, v_*) = \big((\frac{A}{(u_+ - u_- + \frac{A}{\rho_-^\alpha}})^{1/\alpha}, u_+\big), \tag{3.13}
$$

Thus, the Riemann solution of (3.1) and (3.2) can be expressed as

$$
(\rho, v)(x,t) = \begin{cases} (\rho_-, u_-), & x < x_1^-(t), \\ R_1, & x_1^-(t) < x < x_1^+(t), \\ (\rho_*, v_*), & x_1^+(t) < x < x_2(t), \\ (\rho_+, u_+), & x > x_2(t), \end{cases} \tag{3.14}
$$

in which

$$
x_1^-(t) = (u_- - \frac{A}{\rho_-^\alpha})t + \frac{1}{2}\beta t^2, \quad x_1^+(t) = (v_* - \frac{A}{\rho_*^\alpha})t + \frac{1}{2}\beta t^2, \tag{3.15}
$$

$$x_2(t) = u_+ t + \frac{1}{2}\beta t^2, \tag{3.16}$$

and the state (ρ_1, u_1) in R_1 can be calculated by (3.6).

If $(\rho_+, u_+) \in \mathrm{II}$, namely $u_- - \frac{A}{\rho_-^\alpha} < u_+ < u_-$, then the Riemann solution consists of a 1-shock wave S_1 and a 2-contact discontinuity J with an intermediate constant state (ρ_*, v_*) determined uniquely by (3.13). Thus, the Riemann solution of (3.1) and (3.2) can be expressed as

$$(\rho, v)(x,t) = \begin{cases} (\rho_-, u_-), & x < x_1(t), \\ (\rho_*, v_*), & x_1(t) < x < x_2(t), \\ (\rho_+, u_+), & x > x_2(t), \end{cases} \tag{3.17}$$

in which the position of S_1 is given by

$$x_1(t) = \frac{\rho_* v_* - \rho_- u_-}{\rho_* - \rho_-} t + \frac{1}{2}\beta t^2, \tag{3.18}$$

and $x_2(t)$ is given by (3.16).

On the other hand, when $(\rho_+, u_+) \in \mathrm{III}$, namely $u_+ \le u_- - \frac{A}{\rho_-^\alpha}$, then there exist a nonclassical situation where the Cauchy problem does not own a weak L^∞-solution. In order to solve the Riemann problem (3.1) and (3.2) in the framework of nonclassical solution, a solution containing a weighted δ-measure supported on a curve should be defined such as in [4, 23, 30]. In what follows, let us provide the definition of delta shock wave solution to the Riemann problem (3.1) and (3.2). One can also refer to [8, 9, 16, 17] about the more exact definition of generalized delta shock wave solution for related systems with delta measure initial data.

Definition 3.1. Let (ρ, v) be a pair of distributions in which ρ has the form of

$$\rho(x,t) = \hat{\rho}(x,t) + w(x,t)\delta_S, \tag{3.19}$$

in which $\hat{\rho}, v \in L^\infty(R \times R_+)$. Then (ρ, v) is called as the delta shock wave solution to the Riemann problem (3.1) and (3.2) if it satisfies

$$\begin{gathered} \langle \rho, \psi_t \rangle + \langle \rho(v + \beta t), \psi_x \rangle = 0, \\ \langle \rho(v + P)), \psi_t \rangle + \langle \rho(v + P)(v + \beta t)), \psi_x \rangle = 0, \end{gathered}$$

for any $\psi \in C_0^\infty(R \times R^+)$. Here we take

$$\begin{aligned} &\langle \rho(v + P)(v + \beta t)), \psi \rangle \\ &= \int_0^\infty \int_{-\infty}^\infty (\widehat{\rho}(v - \frac{A}{\widehat{\rho}^\alpha})(v + \beta t))\psi \, dx \, dt + \langle w(t) v_\delta(t)(v_\delta(t) + \beta t)\delta_S, \psi \rangle, \end{aligned}$$

as an example to explain the inner product, in which we use the symbol S to express the smooth curve with the Dirac delta function supported on it, v_δ is the value of v and $\frac{A}{\rho^\alpha}$ is equal to zero on this delta shock wave S.

With the above definition, if $(\rho_+, u_+) \in \mathrm{III}$ and $u_+ < u_- - \frac{A}{\rho_-^\alpha}$, a piecewise smooth solution of the Riemann problem (3.1) and (3.2) should be introduced in the form

$$(\rho, v)(x,t) = \begin{cases} (\rho_-, u_-), & x < x(t), \\ (w(t)\delta(x - x(t)), v_\delta), & x = x(t), \\ (\rho_+, u_+), & x > x(t), \end{cases} \tag{3.20}$$

where $x(t)$, $w(t)$ and $\sigma(t) = x'(t)$ denote respectively the location, weight and propagation speed of the delta shock, and v_δ indicates the assignment of v on this delta shock wave. It is remarkable that the value of v should be given on the delta shock curve $x = x(t)$ such that the product of ρ and v can be defined in the sense of distributions. When $u_+ = u_- - \frac{A}{\rho_-^\alpha}$, it can be discussed similarly and we omit it.

The delta shock wave solution of the form (3.20) to the Riemann problem (3.1) and (3.2) should obey the generalized Rankine-Hugoniot conditions

$$\begin{gathered}\frac{dx(t)}{dt} = \sigma(t) = v_\delta + \beta t,\\ \frac{dw(t)}{dt} = \sigma(t)[\rho] - [\rho(v+\beta t)],\\ \frac{d(w(t)v_\delta)}{dt} = \sigma(t)[\rho(v - \frac{A}{\rho^\alpha})] - [\rho(v - \frac{A}{\rho^\alpha})(v+\beta t)],\end{gathered} \tag{3.21}$$

with initial data $x(0) = 0$ and $w(0) = 0$. In addition, for the unique solvability of the above Cauchy problem, it is necessary to require that the value of v_δ to be a constant along the trajectory of delta shock wave (see [9] for details). The derivation process of the generalized Rankine-Hugoniot conditions is similar to that in [25, 26, 31] and we omit it here. To ensure the uniqueness of Riemann solutions, an over-compressive entropy condition for the delta shock wave should be assumed as

$$\lambda_1(\rho_+, u_+) < \lambda_2(\rho_+, u_+) < \sigma(t) < \lambda_1(\rho_-, u_-) < \lambda_2(\rho_-, u_-), \tag{3.22}$$

such that we have

$$u_+ < v_\delta < u_- - \frac{A}{\rho_-^\alpha}, \tag{3.23}$$

which implies that all the characteristics on both sides of the delta shock are incoming.

It follows from (3.21) that

$$\frac{dw(t)}{dt} = v_\delta(\rho_+ - \rho_-) - (\rho_+ u_+ - \rho_- u_-), \tag{3.24}$$

$$\begin{aligned}v_\delta \frac{dw(t)}{dt} &= v_\delta\big((\rho_+ u_+ - \rho_- u_-) - (\frac{A}{\rho_+^{\alpha-1}} - \frac{A}{\rho_-^{\alpha-1}})\big)\\ &\quad - (\rho_+ u_+^2 - \rho_- u_-^2) + \big(\frac{Au_+}{\rho_+^{\alpha-1}} - \frac{Au_-}{\rho_-^{\alpha-1}}\big),\end{aligned} \tag{3.25}$$

Therefore,

$$\begin{aligned}&(\rho_+ - \rho_-)v_\delta^2 - \big(2(\rho_+ u_+ - \rho_- u_-) - (\frac{A}{\rho_+^{\alpha-1}} - \frac{A}{\rho_-^{\alpha-1}})\big)v_\delta + (\rho_+ u_+^2 - \rho_- u_-^2)\\ &- \big(\frac{Au_+}{\rho_+^{\alpha-1}} - \frac{Au_-}{\rho_-^{\alpha-1}}\big) = 0,\end{aligned} \tag{3.26}$$

For convenience, let us denote

$$\begin{aligned}w_0 = &\sqrt{\rho_+\rho_-(u_+ - u_-)\big((u_+ - u_-) - (\frac{A}{\rho_+^\alpha} - \frac{A}{\rho_-^\alpha})\big) + \frac{1}{4}\big(\frac{A}{\rho_+^{\alpha-1}} - \frac{A}{\rho_-^{\alpha-1}}\big)^2}\\ &- \frac{1}{2}\big(\frac{A}{\rho_+^{\alpha-1}} - \frac{A}{\rho_-^{\alpha-1}}\big) > 0,\end{aligned} \tag{3.27}$$

If $\rho_+ \neq \rho_-$, with the entropy condition (3.22) in mind, one can obtain directly from (3.26) that

$$v_\delta = \frac{\rho_+ u_+ - \rho_- u_- + w_0}{\rho_+ - \rho_-}, \tag{3.28}$$

which enables us to obtain

$$\sigma(t) = v_\delta + \beta t, \quad x(t) = v_\delta t + \frac{1}{2}\beta t^2, \quad w(t) = w_0 t. \tag{3.29}$$

Otherwise, if $\rho_+ = \rho_-$, then

$$v_\delta = \frac{1}{2}(u_+ + u_- - \frac{A}{\rho_-^\alpha}). \tag{3.30}$$

In this particular case, we obtain

$$\begin{gathered} \sigma(t) = \frac{1}{2}(u_+ + u_- - \frac{A}{\rho_-^\alpha}) + \beta t, \quad x(t) = \frac{1}{2}(u_+ + u_- - \frac{A}{\rho_-^\alpha})t + \frac{1}{2}\beta t^2, \\ w(t) = (\rho_- u_- - \rho_+ u_+)t. \end{gathered} \tag{3.31}$$

4. Riemann problem for the approximated system (1.2)

In this section, let us return to the Riemann problem (1.2) and (1.5). If $(\rho_+, u_+) \in$ I, the Riemann solutions to (1.2) and (1.5) $R_1 + J$ can be represented as

$$(\rho, u)(x,t) = \begin{cases} (\rho_-, u_- + \beta t), & x < x_1^-(t), \\ (\rho_1, v_1 + \beta t), & x_1^-(t) < x < x_1^+(t), \\ (\rho_*, v_* + \beta t), & x_1^+(t) < x < x_2(t), \\ (\rho_+, u_+ + \beta t), & x > x_2(t), \end{cases} \tag{4.1}$$

where $x_1^-(t)$, $x_1^+(t)$ and $x_2(t)$ are given by (3.15) and (3.16) respectively, and the states (ρ_1, v_1) and (ρ_*, v_*) can be calculated as (3.6) and (3.13). Let us use Figure 4(a) to illustrate this situation in detail, where all the characteristics in the rarefaction wave fans R_1 and the contact discontinuity curve J are curved into parabolic shapes.

If $(\rho_+, u_+) \in$ II, the Riemann solutions to (1.2) and (1.5) $S_1 + J$ can be represented as

$$(\rho, u)(x,t) = \begin{cases} (\rho_-, u_- + \beta t), & x < x_1(t), \\ (\rho_*, v_* + \beta t), & x_1(t) < x < x_2(t), \\ (\rho_+, u_+ + \beta t), & x > x_2(t), \end{cases} \tag{4.2}$$

where $x_1(t)$ and $x_2(t)$ are given by (3.18) and (3.16) respectively and the states (ρ_*, v_*) can be calculated as (3.13). Let us use Figure 4(b) to illustrate this situation in detail, where both the shock wave curve S_1 and the contact discontinuity curve J are curved into parabolic shapes.

Analogously, if $(\rho_+, u_+) \in$ III, namely $u_+ \leq u_- - \frac{A}{\rho_-^\alpha}$, then we can also define the weak solutions to the Riemann problem (1.2) and (1.5) in the sense of distributions below.

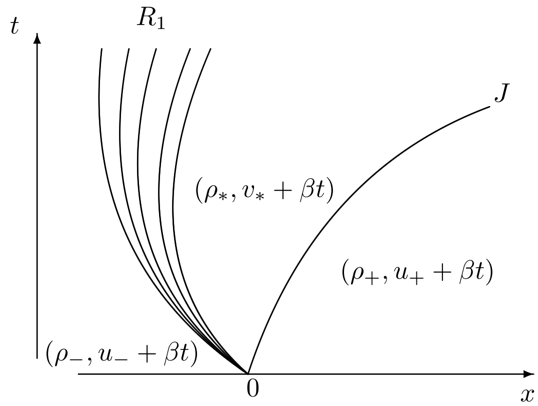
Definition 4.1. Let (ρ, u) be a pair of distributions in which ρ has the form of (3.19), then it is called as the delta shock wave solution to the Riemann problem (1.2) and (1.5) if it satisfies

$$\begin{gathered} \langle \rho, \psi_t \rangle + \langle \rho u, \psi_x \rangle = 0, \\ \langle \rho(u + P)), \psi_t \rangle + \langle \rho u(u + P)), \psi_x \rangle = -\langle \beta\rho, \psi \rangle, \end{gathered} \tag{4.3}$$

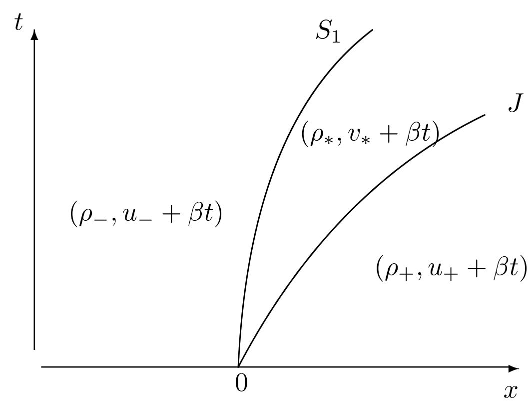
for any $\psi \in C_0^\infty(R \times R^+)$, in which

$$\langle \rho u(u+P)), \psi\rangle = \int_0^\infty \int_{-\infty}^\infty (\widehat{\rho} u(u - \frac{A}{\widehat{\rho}^\alpha}))\psi \, dx \, dt + \langle w(t)(u_\delta(t))^2 \delta_S, \psi\rangle,$$

and $u_\delta(t)$ is the assignment of u on this delta shock wave curve.



(a) $u_- - \frac{A}{\rho_-^\alpha} < u_- < u_+$



(b) $u_- - \frac{A}{\rho_-^\alpha} < u_+ < u_-$

FIGURE 4. The Riemann solution to (1.2) and (1.5) when $u_- - \frac{A}{\rho_-^\alpha} < u_+$ and $\beta > 0$, where (ρ_*, v_*) is given by (3.13).

With the above definition in mind, if $u_+ < u_- - \frac{A}{\rho_-^\alpha}$ is satisfied, then we look for a piecewise smooth solution to the Riemann problem (1.2) and (1.5) in the form

$$(\rho, u)(x,t) = \begin{cases} (\rho_-, u_- + \beta t), & x < x(t), \\ (w(t)\delta(x - x(t)), u_\delta(t)), & x = x(t), \\ (\rho_+, u_+ + \beta t), & x > x(t), \end{cases} \tag{4.4}$$

It is worthwhile to notice that $u_\delta(t) - \beta t$ is assumed to be a constant based on the result in Sect.2. With the similar analysis and derivation as before, the delta shock wave solution of the form (4.4) to the Riemann problem (1.2) and (1.5) should also satisfy the following generalized Rankine-Hugoniot conditions

$$\begin{aligned} \frac{dx(t)}{dt} &= \sigma(t) = u_\delta(t), \\ \frac{dw(t)}{dt} &= \sigma(t)[\rho] - [\rho u], \\ \frac{d(w(t)u_\delta(t))}{dt} &= \sigma(t)[\rho(u - \frac{A}{\rho^\alpha})] - [\rho u(u - \frac{A}{\rho^\alpha})] + \beta w(t). \end{aligned} \tag{4.5}$$

in which the jumps across the discontinuity are

$$[\rho u] = \rho_+(u_+ + \beta t) - \rho_-(u_- + \beta t), \tag{4.6}$$

$$[\rho u(u - \frac{A}{\rho^\alpha})] = \rho_+(u_+ + \beta t)(u_+ + \beta t - \frac{A}{\rho_+^\alpha}) - \rho_-(u_- + \beta t)(u_- + \beta t - \frac{A}{\rho_-^\alpha}). \tag{4.7}$$

To ensure the uniqueness of a solution to the Riemann problem (1.2) and (1.5), the over-compressive entropy condition for the delta shock wave

$$u_+ + \beta t < u_\delta(t) < u_- - \frac{A}{\rho_-^\alpha} + \beta t. \tag{4.8}$$

should also be assumed when $u_+ < u_- - \frac{A}{\rho_-^\alpha}$.

As before, we can also obtain $x(t), \sigma(t)$ and $w(t)$ from (4.5) and (4.8) together. In brief, we have the following theorem to depict the Riemann solution to (1.2) and (1.5) when the Riemann initial data (1.5) satisfy $u_+ < u_- - \frac{A}{\rho_-^\alpha}$ and $\rho_+ \neq \rho_-$.

Theorem 4.2. *If both $u_+ < u_- - \frac{A}{\rho_-^\alpha}$ and $\rho_+ \neq \rho_-$ are satisfied, then the delta shock solution to the Riemann solutions to* (1.2) *and* (1.5) *can be expressed as*

$$\begin{aligned} \frac{dx(t)}{dt} &= \sigma(t) = u_\delta(t), \\ \frac{dw(t)}{dt} &= \sigma(t)[\rho] - [\rho u], \\ \frac{d(w(t)u_\delta(t))}{dt} &= \sigma(t)[\rho(u - \frac{A}{\rho^\alpha})] - [\rho u(u - \frac{A}{\rho^\alpha})] + \beta w(t). \end{aligned} \tag{4.9}$$

in which

$$\sigma(t) = u_\delta(t) = v_\delta + \beta t, \quad x(t) = v_\delta t + \frac{1}{2}\beta t^2, \quad w(t) = w_0 t, \tag{4.10}$$

where w_0 and v_δ are given by (3.27) *and* (3.28) *respectively.*

Let us check briefly that the above constructed delta shock wave solution (4.9) and (4.10) satisfy (1.2) in the sense of distributions. The proof of this theorem is completely analogs to the one in [25, 26]. Therefore, we only state the main steps for the proof of the second equality in (4.3) for completeness. Actually, one can deduce that

$$\begin{aligned} I &= \int_0^\infty \int_{-\infty}^\infty (\rho(u - \frac{A}{\rho^\alpha})\psi_t + \rho u(u - \frac{A}{\rho^\alpha})\psi_x)\,dx\,dt \\ &= \int_0^\infty \int_{-\infty}^{x(t)} (\rho_-(u_- + \beta t - \frac{A}{\rho_-^\alpha})\psi_t + \rho_-(u_- + \beta t)(u_- + \beta t - \frac{A}{\rho_-^\alpha})\psi_x) dx dt \end{aligned}$$

$$+\int_0^\infty \int_{x(t)}^\infty (\rho_+(u_+ + \beta t - \frac{A}{\rho_+^\alpha})\psi_t + \rho_+(u_+ + \beta t)(u_+ + \beta t - \frac{A}{\rho_+^\alpha})\psi_x)dxdt$$
$$+\int_0^\infty w_0 t(v_\delta + \beta t)(\psi_t(x(t),t) + (v_\delta + \beta t)\psi_x(x(t),t))dt.$$

It can be derived from (4.10) that the curve of delta shock wave is given by

$$x(t) = v_\delta t + \frac{1}{2}\beta t^2. \tag{4.11}$$

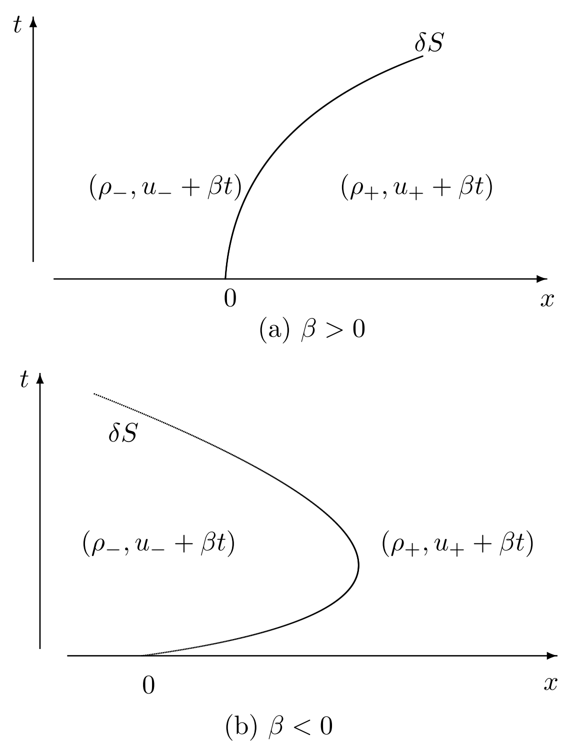


FIGURE 5. The delta shock wave solution to (1.1) and (1.2) when $u_+ < u_- - \frac{A}{\rho_-^\alpha}$ and $v_\delta > 0$, where v_δ is given by (3.28) for $\rho_- \neq \rho_+$ and (3.30) for $\rho_- = \rho_+$.

For $\beta > 0$ (see Figure 5(a)), there exists an inverse function of $x(t)$ globally in the time t, which may be written in the form

$$t(x) = \sqrt{\frac{v_\delta^2}{\beta^2} + \frac{2x}{\beta}} - \frac{v_\delta}{\beta}.$$

Otherwise, for $\beta < 0$ (see Figure 5(b)), there is a critical point $(-\frac{v_\delta^2}{2\beta}, -\frac{v_\delta}{\beta})$ on the delta shock wave curve such that $x'(t)$ change its sign when across the critical point. Thus, the inverse function of $x(t)$ is needed to find respectively for $t \leq -\frac{v_\delta}{\beta}$ and

$t > -\frac{v_\delta}{\beta}$, which enable us to have

$$t(x) = \begin{cases} -\sqrt{\frac{v_\delta^2}{\beta^2} + \frac{2x}{\beta}} - \frac{v_\delta}{\beta}, & t \le -\frac{v_\delta}{\beta}, \\ \sqrt{\frac{v_\delta^2}{\beta^2} + \frac{2x}{\beta}} - \frac{v_\delta}{\beta}, & t > -\frac{v_\delta}{\beta}. \end{cases} \tag{4.12}$$

Without loss of generality, let us assume that $\beta > 0$ for simplicity. Actually, the other situation can be dealt with similarly. Under our assumption, it follows from (4.11) that the position of delta shock wave satisfies $x = x(t) > 0$ for all the time. It follows from (4.10) that

$$\begin{aligned} \frac{d\psi(x(t),t)}{dt} &= \psi_t(x(t),t) + \frac{dx(t)}{dt}\psi_x(x(t),t) \\ &= \psi_t(x(t),t) + (v_\delta + \beta t)\psi_x(x(t),t) \\ &= \psi_t(x(t),t) + u_\delta(t)\psi_x(x(t),t). \end{aligned}$$

By interchanging the order of integration and using integration by parts, we have

$$\begin{aligned} I &= \int_0^\infty \int_{t(x)}^\infty \rho_-(u_- + \beta t - \frac{A}{\rho_-^\alpha})\psi_t \, dt \, dx \\ &\quad + \int_0^\infty \int_{t(x)}^\infty \rho_-(u_- + \beta t)(u_- + \beta t - \frac{A}{\rho_-^\alpha})\psi_x \, dt \, dx \\ &\quad + \int_0^\infty \int_0^{t(x)} \rho_+(u_+ + \beta t - \frac{A}{\rho_+^\alpha})\psi_t \, dt \, dx \\ &\quad + \int_0^\infty \int_0^{t(x)} \rho_+(u_+ + \beta t)(u_+ + \beta t - \frac{A}{\rho_+^\alpha})\psi_x dt dx \\ &\quad + \int_0^\infty w_0 t(v_\delta + \beta t) d\psi(x(t),t) \\ &= \int_0^\infty \big(\rho_+(u_+ + \beta t(x) - \frac{A}{\rho_+^\alpha}) - \rho_-(u_- + \beta t(x) - \frac{A}{\rho_-^\alpha})\big)\psi(x,t(x))dx \\ &\quad + \int_0^\infty \Big(\rho_-(u_- + \beta t)(u_- + \beta t - \frac{A}{\rho_-^\alpha}) \\ &\quad - \rho_+(u_+ + \beta t)(u_+ + \beta t - \frac{A}{\rho_+^\alpha})\Big)\psi(x(t),t)dt \\ &\quad - \int_0^\infty \int_{t(x)}^\infty \beta\rho_-\psi \, dt \, dx - \int_0^\infty \int_0^{t(x)} \beta\rho_+\psi \, dt \, dx \\ &\quad - \int_0^\infty w_0(v_\delta + 2\beta t)\psi(x(t),t)dt \\ &= \int_0^\infty C(t)\psi(x(t),t)dt - \beta(\int_0^\infty \int_{-\infty}^{x(t)} \rho_-\psi dx dt + \int_0^\infty \int_{x(t)}^\infty \rho_+\psi dx dt), \end{aligned} \tag{4.13}$$

in which

$$\begin{aligned} C(t) &= (\rho_+(u_+ + \beta t - \frac{A}{\rho_+^\alpha}) - \rho_-(u_- + \beta t - \frac{A}{\rho_-^\alpha}))(v_\delta + \beta t) \\ &\quad + (\rho_-(u_- + \beta t)(u_- + \beta t - \frac{A}{\rho_-^\alpha}) - \rho_+(u_+ + \beta t)(u_+ + \beta t - \frac{A}{\rho_+^\alpha})) \\ &\quad - w_0(v_\delta + 2\beta t). \end{aligned} \tag{4.14}$$

By tedious calculations, we have

$$C(t) = -\beta w_0 t = -\beta w(t). \tag{4.15}$$

Thus, it can be concluded from (4.13) and (4.15) together that the second equality in (4.3) holds in the sense of distributions. The proof is complete.

Remark 4.3. If both $u_+ < u_- - \frac{A}{\rho_-^\alpha}$ and $\rho_+ = \rho_-$ are satisfied, then the delta shock solution to the Riemann problem (1.2) and (1.5) can be expressed in the form (4.4) where

$$\begin{gathered}\sigma(t) = u_\delta(t) = \frac{1}{2}(u_+ + u_- - \frac{A}{\rho_-^\alpha}) + \beta t, \quad x(t) = \frac{1}{2}(u_+ + u_- - \frac{A}{\rho_-^\alpha})t + \frac{1}{2}\beta t^2, \\ w(t) = (\rho_- u_- - \rho_+ u_+)t.\end{gathered} \tag{4.16}$$

The proof is completely similar to the above proofs, so we omit it.

Remark 4.4. If $u_+ = u_- - \frac{A}{\rho_-^\alpha}$, then the delta shock solution to the Riemann problem (1.2) and (1.5) can also be expressed as the form in Theorem 4.2 and Remark 4.3. The process of proof is easy and we omit it.

5. Flux approximation limits of Riemann solutions to (1.2)

In this section, we are concerned that the flux approximation limits of Riemann solutions to (1.2) and (1.5) converge to the corresponding ones to (1.1) and (1.5) or not when the perturbation parameter A tends to zero. According to the relations between u_- and u_+, we will divide our discussion into the following three cases: $u_- < u_+$, $u_- = u_+$, and $u_- > u_+$.

Case 1: $u_- < u_+$. In this case, $(\rho_+, u_+) \in \mathrm{I}$ in the (ρ, v) plane, so the Riemann solutions to (1.2) and (1.5) $R_1 + J$ is given by (4.1), where $x_1^-(t)$, $x_1^+(t)$ and $x_2(t)$ are given by (3.15) and (3.16) respectively and the states (ρ_1, v_1) and (ρ_*, v_*) can be calculated as (3.6) and (3.13). From (3.6) and (3.13) we have

$$\begin{gathered}\lim_{A\to 0} \rho_1 = \lim_{A\to 0} \big(\frac{A(1-\alpha)}{\frac{x}{t} - \beta t - w_-}\big)^{1/\alpha} = 0, \\ \lim_{A\to 0} \rho_* = \lim_{A\to 0} \big(\frac{A}{u_+ - u_- + \frac{A}{\rho_-^\alpha}}\big)^{1/\alpha} = 0,\end{gathered}$$

which indicate the occurrence of the vacuum states. Furthermore, the Riemann solutions to (1.2) and (1.5) converge to

$$\lim_{A\to 0} (\rho, u)(x,t) = \begin{cases} (\rho_-, u_- + \beta t), & x < u_- t + \frac{1}{2}\beta t^2, \\ \text{vacuum}, & u_- t + \frac{1}{2}\beta t^2 < x < u_+ t + \frac{1}{2}\beta t^2, \\ (\rho_+, u_+ + \beta t), & x > u_+ t + \frac{1}{2}\beta t^2, \end{cases} \tag{5.1}$$

which is exactly the corresponding Riemann solutions to the pressureless Euler equations with the same source term and the same initial data.

Case 2: $u_- = u_+$. In this case, (ρ_+, u_+) is on the J curve in the (ρ, v) plane, so the Riemann solutions to (1.2) and (1.5) is

$$(\rho, u)(x,t) = \begin{cases} (\rho_-, u_- + \beta t), & x < u_- t + \frac{1}{2}\beta t^2, \\ (\rho_+, u_+ + \beta t), & x > u_+ t + \frac{1}{2}\beta t^2, \end{cases} \tag{5.2}$$

which is exactly the corresponding Riemann solutions to the pressureless Euler equations with the same source term and the same initial data.

Case 3: $u_- > u_+$.

Lemma 5.1. *If $u_- > u_+$, there exists $A_1 > A_0 > 0$, such that $(\rho_+, u_+) \in \mathrm{II}$ as $A_0 < A < A_1$, and $(\rho_+, u_+) \in \mathrm{III}$ as $A \le A_0$.*

Proof. If $(\rho_+, u_+) \in \mathrm{II}$, then $0 < u_- - \frac{A}{\rho_-^\alpha} < u_+ < u_-$, which gives $\rho_-^\alpha(u_- - u_+) < A < \rho_-^\alpha u_-$. Thus we take $A_0 = \rho_-^\alpha(u_- - u_+)$ and $A_1 = \rho_-^\alpha u_-$, then $(\rho_+, u_+) \in \mathrm{II}$ as $A_0 < A < A_1$ and $(\rho_+, u_+) \in \mathrm{III}$ as $A \le A_0$.

When $A_0 < A < A_1$, $(\rho_+, u_+) \in \mathrm{II}$ in the (ρ, v) plane, so the Riemann solution to (1.2) and (1.5) is given by (4.2), where $x_1(t)$ and $x_2(t)$ are given by (3.18) and (3.16) respectively and the states (ρ_*, v_*) can be calculated as (3.13). From (3.13) we have

$$\lim_{A\to A_0} \rho_* = \lim_{A\to A_0} \big(\frac{A}{u_+ - u_- + \frac{A}{\rho_-^\alpha}}\big)^{1/\alpha} = \lim_{A\to A_0} \big(\frac{\rho_- A}{A - A_0}\big)^{1/\alpha} = \infty.$$

□

Furthermore, we have the following result.

Lemma 5.2. *Let $\frac{dx_1(t)}{dt} = \sigma_1(t)$, $\frac{dx_2(t)}{dt} = \sigma_2(t)$, then*

$$\begin{aligned} \lim_{A\to A_0} v_* + \beta t = \lim_{A\to A_0} \sigma_1(t) &= \lim_{A\to A_0} \sigma_2(t) \\ &= (u_- - \frac{A_0}{\rho_-^\alpha})t + \beta t = u_+ + \beta t =: \sigma(t), \end{aligned} \tag{5.3}$$

$$\lim_{A\to A_0} \int_{x_1(t)}^{x_2(t)} \rho_* dx = \rho_-(u_- - u_+)t, \tag{5.4}$$

$$\lim_{A\to A_0} \int_{x_1(t)}^{x_2(t)} \rho_*(v_* + \beta t) dx = \rho_-(u_- - u_+)(u_+ + \beta t)t. \tag{5.5}$$

Proof. Equality (5.3) is obviously true. We will only prove (5.4) and (5.5). Note that

$$\begin{aligned} \lim_{A\to A_0} \int_{x_1(t)}^{x_2(t)} \rho_* dx &= \lim_{A\to A_0} \rho_*(x_2(t) - x_1(t)) \\ &= \lim_{A\to A_0} \rho_*(u_+ - \frac{\rho_* v_* - \rho_- u_-}{\rho_* - \rho_-})t = \rho_-(u_- - u_+)t, \end{aligned}$$

$$\begin{aligned} \lim_{A\to A_0} \int_{x_1(t)}^{x_2(t)} \rho_*(v_* + \beta t) dx &= (u_+ + \beta t) \lim_{A\to A_0} \int_{x_1(t)}^{x_2(t)} \rho_* dx \\ &= \rho_-(u_- - u_+)(u_+ + \beta t)t. \end{aligned}$$

The proof is complete. □

From Lemma 5.2 it follows that the curves of the shock wave S_1 and the contact discontinuity J will coincide when A tends to A_0 and the delta shock wave will form. Next we arrange the values which gives the exact position, propagation speed and strength of the delta shock wave according to Lemma 5.2.

Using (5.4) and (5.5), we let

$$w(t) = \rho_-(u_- - u_+)t, \tag{5.6}$$

$$w(t)u_\delta(t) = \rho_-(u_- - u_+)(u_+ + \beta t)t. \tag{5.7}$$

Then

$$u_\delta(t) = (u_+ + \beta t), \tag{5.8}$$

which is equal to $\sigma(t)$. Furthermore, by letting $\frac{dx(t)}{dt} = \sigma(t)$, we have

$$x(t) = u_+ t + \frac{1}{2}\beta t^2. \tag{5.9}$$

From (5.6)-(5.9), we can see that the quantities defined above are exactly consistent with those given by (3.27)-(3.31) or (4.10) in which we take $A = A_0$. Thus, it uniquely determines that the limits of the Riemann solutions to the system (1.2) and (1.5) when $A \to A_0$ in the case $(\rho_+, u_+) \in \mathrm{II}$ is just the delta shock solution of (1.2) and (1.5) in the case $(\rho_+, u_+) \in S_\delta$, where S_δ is actually the boundary between the regions II and III. So we obtain the following results in the case $u_+ < u_-$.

Theorem 5.3. *If $u_+ < u_-$, for each fixed A with $A_0 < A < A_1$, $(\rho_+, u_+) \in \mathrm{II}$, assuming that (ρ, u) is a solution containing a shock wave S_1 and a contact discontinuity J of (1.2) and (1.5) constructed in Section 4, it follows that when $A \to A_0$, the solution (ρ, u) converges to a delta shock wave solution of (1.2), and (1.5) when $A = A_0$.*

When $A \le A_0$, $(\rho_+, u_+) \in \mathrm{III}$, so the Riemann solutions to (1.2) and (1.5) is given by (4.4) with (4.10) or (4.16), which is a delta shock wave solution. It is easy to see that when $A \to 0$, for $\rho_+ \neq \rho_-$,

$$x(t) \to \sigma_0 t + \frac{1}{2}\beta t^2, \quad w(t) \to \sqrt{\rho_+\rho_-}(u_- - u_+)t, \quad \sigma(t) = u_\delta(t) \to \sigma_0 + \beta t,$$

where

$$\sigma_0 = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}},$$

for $\rho_+ = \rho_-$,

$$x(t) \to \frac{1}{2}(u_+ + u_-)t + \frac{1}{2}\beta t^2, \quad w(t) \to \rho_+(u_- - u_+)t,$$
$$\sigma(t) = u_\delta(t) \to \frac{1}{2}(u_+ + u_-) + \beta t,$$

which is exactly the corresponding Riemann solution to the pressureless Euler equations with the same source term and the same initial data [25]. Thus, we have the following result.

Theorem 5.4. *If $u_+ < u_-$, for each fixed $A < A_0$, $(\rho_+, u_+) \in \mathrm{III}$, assuming that (ρ, u) is a delta shock wave solution of (1.2) and (1.5) which is constructed in Section 4, it is obtained that when $A \to 0$, (ρ, u) converges to a delta shock wave solution to the pressureless Euler equations with the same source term and the same initial data [25].*

We summarize the main result in this section as follows.

Theorem 5.5. *As the perturbed parameter $A \to 0$, the Riemann solutions to the approximated nonhomogeneous system (1.2) tend to the three kinds of Riemann solutions to the nonhomogeneous pressureless Euler equations with the same source term and the same initial data, which include a delta shock wave and a vacuum state. That is to say, the Riemann solutions to the transportation equations with Coulomb-type friction is stable under this kind of flux perturbation.*

6. Conclusions and discussions

It can be seen from the above discussions that the limits of solutions to the Riemann problem (1.2) and (1.5) converge to the corresponding ones of the Riemann problem (1.1) and (1.5) as $A \to 0$. The approximated system (1.2) is strictly hyperbolic. Although the characteristic field for λ_1 is genuinely nonlinear, the characteristic field for λ_2 is still linearly degenerate and (1.2) still belongs to the Temple class. Thus, this perturbation does not totally change the structure of Riemann solutions to (1.1).

If we also consider the approximation of the flux functions for (1.1) in the form

$$\begin{gathered} \rho_t + (\rho u)_x = 0, \\ (\rho(u + \frac{1}{1-\alpha}P))_t + (\rho u(u+P))_x = \beta\rho, \end{gathered} \tag{6.1}$$

where P is also given by (1.3). We can check that (6.1) has two different eigenvalues $\lambda = u \pm \sqrt{\alpha A \rho^{-\alpha} u}$, and the characteristic fields for both the two eigenvalues are genuinely nonlinear. Hence, (6.1) is strictly hyperbolic and by simple calculation, it can be seen that (6.1) does not belong to the Temple class anymore. It is clear to see that the Riemann solutions for the approximated system (6.1) have completely different structures from those for the original system (1.1). Similar to [26, 27, 29, 31, 34], we can construct the Riemann solutions to the Riemann problem (6.1) and (1.5) in all situations and prove them converge to the corresponding ones to the Riemann problem (1.1) and (1.5) as $A \to 0$.

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References

[1] N. Bilic, G. Tupper, R.Viollier; Unification of dark matter and dark energy: the inhomogeneous Chaplygin gas. *Phys. Lett. B*, 2002, **535**: 17-21.

[2] F. Bouchut; On zero-pressure gas dynamics, *Advances in Kinetic Theory and Computing.* Ser Adv Math Appl Sci **22**. River Edge, NJ: World Scientific, 1994: 171-190.

[3] Y. Brenier, W. Gangbo, G. Savage, M. Westdickenberg; The sticky particle dynamics with interactions. *J. Math. Pure Appl.*, 2013, **99**: 577-617.

[4] G. Q. Chen, H. Liu; Formation of $\delta-$shocks and vacuum states in the vanishing pressure limit of solutions to the Euler equations for isentropic fluids. *SIAM J. Math. Anal.* **34** (2003), 925-938.

[5] H. Cheng; Delta shock waves for a linearly dengenerate hyperbolic system of conservation laws of Keyfitz-Kranzer type. *Advances in mathmatical Physics*, 2013, Article ID 958120, 10 pages.

[6] H. Cheng; On a nonsymmetric Keyfitz-Kranzer system of conservation laws with generalized and modified Chaplygin gas pressure law. *Advances in mathmatical Physics*, 2013, 14 pages.

[7] H. Cheng, H.Yang; Approaching Chaplygin pressure limit to the Aw-Rascle model. *J. Math. Anal. Appl.*, 2014, **416**: 839-854.

[8] V. G. Danilvo, V. M. Shelkovich; Dynamics of propagation and interaction of δ-shock waves in conservation law system. *J. Differential Equations*, **221** (2005), 333-381.

[9] V. G. Danilvo, V. M. Shelkovich; Delta-shock waves type solution of hyperbolic systems of conservation laws. *Q. Appl. Math.*, **63** (2005), 401-427.

[10] D. A. E. Daw, M. Nedeljkov; Shadow waves for pressureless gas balance laws. *Appl. Math. Lett.*, **57** (2016), 54-59.

[11] C. M. Edwards, S. D. Howison, H. Ockendon, J. R. Ockendon; Non-classical shallow water flows. *IMA J. Appl. Math.*, 2008, **73**: 137-157.
[12] G. Faccanoni, A. Mangeney; Exact solution for granular flows. *Int. J. Numer. Anal. Mech. Geomech.*, 2012, **37**: 1408-1433.
[13] L. Guo; The Riemann problem of the transport equations for the generalized Chaplygin gas. *J. Xinjiang University (Natuatural Science Edition)*, 2013, **30** (2): 170-176.
[14] L. Guo, W. Sheng, T. Zhang; The two-dimensional Riemann problelm for isentropic Chaplygin gas dynamic system. *Comm. Pure Appl. Anal.*, 2010, **9** (2): 431-458.
[15] F. Huang, Z. Wang; Well posedness for pressureless flow. *Comm. Math. Phys.*, 2001, **222**: 117-146.
[16] H. Kalisch, D. Mitrovic; Singular solutions of a fully nonlinear 2×2 system of conservation laws. *Proceedings of the Edinburgh Mathematical Society*, 2012, **55**: 711-729.
[17] H. Kalisch, D.Mitrovic; Singular solutions for shallow water equations. *IMA J. Appl. Math.*, 2012, **77**: 340-350.
[18] B. L. Keyfitz, H. C. Kranzer; Spaces of weighted measures for conservation laws with singular shock solutions. *J. Differential Equations*, 1995, **118**: 420-451.
[19] D. Korchinski; Solutions of a Riemann problem for a system of conservation laws possessing no classical weak solution. Thesis, Adelphi University, 1977.
[20] J. Li, T. Zhang, S. Yang; The Two-dimensional Riemann Prolem in Gas Dynamics. Pitman Monogr Surv Pure Appl Math **98**. Longman Scientific and Technical, 1998.
[21] M. Nedelikov; Shadow waves: entropies and interactions for delta and singular shocks. *Arch.Rational Mech. Anal.*, 2013, **264**: 2457-2468.
[22] T. Nguyen. A. Tudorascu; Pressureless Euler/Euler-Poisson systems wia adhesion dynamics and scalar conservation laws. *SIAM J. Math. Anal.*, 2008, **40**: 754-775.
[23] L. Pan, X. Han; The Aw-Rascle traffic model with Chaplygin pressure. *J. Math. Anal. Appl.*, 2013, **401**: 379-387.
[24] M. Setare; Interacting holographic generalized Chaplygin gas model. *Phys. Lett. B*, **654** (2007), 1-6.
[25] C. Shen; The Riemann problem for the pressureless Euler system with the Coulomb-like friction term. *IAM J. Appl. Math.*, 2015: 1-24,doi:10.1039/imamat/hxv028.
[26] C. Shen; The Riemann problem for the Chaplygin gas equations with a source term. *Z. Angew. Math. Mech.*, 2015:1-15, doi:10.1002/zamm.201500015.
[27] C. Shen, M. Sun; Formation of delta-shocks and vacuum states in the vanishing pressue limit of solutions to the Aw-Rascle model. *J. Differential Equations*, 2010, **249**: 3024-3051.
[28] J. Smoller; Shock Waves and Reaction-Diffusion Equation. New York: Springer-Verlag, 1994.
[29] W. Sheng, G. Wang, G.Yin; Delta wave and vacuum state for generalized Chaplygin gas dynamics system as pressure vanishes. *Nonlinear analysis: real world Appl.*, 2015, **22**: 115-128.
[30] W. Sheng, T. Zhang; The Riemann problem for transportation equations in gas dynamics. *Mem. Amer. Math. Soc.*, 1999, **137** (654).
[31] M. Sun; The exact Riemann solutions to the generalized Chaplygin gas equations with friction. *Commun. Nonlinear Sci. Numer. Simulat.*, 2016, **36**: 342-353.
[32] W. A. Strauss; Partial Differential Equations: An Introduction. Wiley, New York, 2007.
[33] D. Tan, T. Zhang, Y. Zheng; Delta-shock wave as limits of vanishing viscosity for hyperbolic system of conservation laws. *J. Differential Equations*, 1994, **112**: 1-32.
[34] G. Wang. The Riemann problem for one dimensional generalized Chaplygin gas dynamics. J. Math. Anal. Appl., 2013, **403**: 403-450.
[35] Z. Wang, X. Ding; Uniqueness of generalized solution for the Cauchy problem of transportation equations. *Acta Math Scientia*, 1997, **17** (3): 341-352.
[36] Z. Wang, F. Huang, X. Ding. On the Cauchy problem of transportation equations. *Acta Math. Appl. Sinica*, 1997, **13** (2): 113-122.
[37] H. Yang, J. Liu; Delta-shocks and vacuum states in zero-pressue gas dynamics by the flux approximation. *Science China (Mathematics)*, 2015, **58**: 2329-2346.
[38] H. Yang, J. Wang; Delta-shocks and vacuum states in the vanishinig pressure limit of solutions to the isentopic Euler equations for modified Chaplygin gas. *J. Math. Anal. Appl.*, 2014, **413**: 800-820.
[39] H. Yang, J. Wang; Concentration in vanishinig pressure limit of solutions to the modified Chaplygin gas equations. *J. Math.Phys.*, 2016, **57**, doi:10.1063/1.4967299.

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