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CONTINUITY OF THE SET OF EQUILIBRIA FOR NON-AUTONOMOUS DAMPED WAVE EQUATIONS WITH TERMS CONCENTRATING ON THE BOUNDARY

GLEICIANE DA SILVA ARAGÃO, FLANK DAVID MORAIS BEZERRA

Abstract. In this article we study the behavior of the solutions of non-autonomous damped wave equations when some reaction terms are concentrated in a neighborhood of the boundary and this neighborhood collapses toward the boundary as a parameter approaches zero. We prove the continuity of the set of equilibria for these equations. Moreover, if an equilibrium solution of the limit problem is hyperbolic, then we show that the perturbed equation has only one equilibrium solution nearby.

1. Introduction

In this article we prove the continuity of the set of equilibria for non-autonomous damped wave equations when some reaction terms are concentrated in a neighborhood of the boundary and this neighborhood approaches to boundary as a parameter $\varepsilon \to 0$. To better describe the problem, let Ω be a bounded smooth domain in $\mathbb{R}^3$ with boundary $\Gamma = \partial\Omega$. We define the strip of width ε and base $\partial\Omega$ as

$$\omega_\varepsilon = \{x - \sigma\vec{n}(x) : x \in \Gamma \text{ and } \sigma \in [0,\varepsilon)\},$$

for sufficiently small ε, say $\varepsilon \in (0,\varepsilon_0]$, where $\vec{n}(x)$ denotes the outward normal vector at $x \in \Gamma$. We note that the set ω_ε has Lebesgue measure $|\omega_\varepsilon| = O(\varepsilon)$ with $|\omega_\varepsilon| \leqslant k|\Gamma|\varepsilon$, for some $k > 0$ independent of ε, Also note that for small ε, the set ω_ε is a neighborhood of Γ in $\bar{\Omega}$, that collapses toward the boundary when the parameter ε approaches zero, see Figure 1.

We are interested in the behavior, for small ε, of the solutions of the non-autonomous damped wave equation with concentrated terms:

$$\begin{gathered} u^\varepsilon_{tt} - \operatorname{div}(a(x)\nabla u^\varepsilon) + u^\varepsilon + \beta(t)u^\varepsilon_t = f(u^\varepsilon) + \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g(u^\varepsilon) \quad \text{in } \Omega \times (\tau,+\infty), \\ a(x)\frac{\partial u^\varepsilon}{\partial \vec{n}} = 0 \quad \text{on } \Gamma \times (\tau,+\infty), \\ u^\varepsilon(\tau) = u_0 \in H^1(\Omega), \quad u^\varepsilon_t(\tau) = v_0 \in L^2(\Omega), \end{gathered} \tag{1.1}$$

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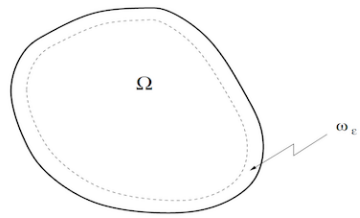

FIGURE 1. The set ω_ϵ.

where $a \in \mathcal{C}^1(\overline{\Omega})$ with

$$0 < a_0 \leqslant a(x) \leqslant a_1, \quad \forall x \in \overline{\Omega},$$

for fixed constants $a_0, a_1 > 0$, and $\chi_{\omega_\varepsilon}$ denotes the characteristic function of the set ω_ε. We refer to $\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g(u^\varepsilon)$ as the concentrating reaction in ω_ε. We assume that $\beta : \mathbb{R} \to \mathbb{R}$ is bounded, globally Lipschitz, and

$$0 < \beta_0 \leqslant \beta(t) \leqslant \beta_1, \quad \forall t \in \mathbb{R},$$

for fixed constants $\beta_0, \beta_1 > 0$ (the assumption that β is globally Lipschitz can be weakened to uniform continuity on $\mathbb{R}$ and continuous differentiability).

In [1] we assumed that $f, g : \mathbb{R} \to \mathbb{R}$ are $\mathcal{C}^2$ and assume the conditions

$$|j'(s)| \leqslant c(1 + |s|^{\rho_j}), \quad \forall s \in \mathbb{R}, \tag{1.2}$$

$$|j(s_1) - j(s_2)| \leqslant c|s_1 - s_2|(1 + |s_1|^{\rho_j} + |s_2|^{\rho_j}), \quad \forall s_1, s_2 \in \mathbb{R}, \tag{1.3}$$

with $j = f$ or $j = g$ and exponents ρ_f and ρ_g, respectively, such that $\rho_f \leqslant 2$ and $\rho_g \leqslant 1$. We note that estimate (1.2) implies (1.3).

Moreover, we assume that

$$\limsup_{|s| \to +\infty} \frac{j(s)}{s} \leqslant 0, \tag{1.4}$$

with $j = f$ or $j = g$. We note that (1.4) is equivalent to saying that for any $\gamma > 0$ there exists $c_\gamma > 0$ such that

$$sj(s) \leqslant \gamma s^2 + c_\gamma, \quad \forall s \in \mathbb{R}.$$

As in (1.1) the nonlinear term $g(u^\varepsilon)$ is only effective on the region ω_ε which collapses to Γ as $\varepsilon \to 0$, then it is reasonable to expect that the family of solutions u^ε of (1.1) will converge to a solution of an equation of the same type with nonlinear boundary condition on Γ. Indeed, under assumptions above, in [1] we prove that the "limit problem" for the non-autonomous singularly wave equation (1.1) is

$$\begin{gathered} u_{tt} - \operatorname{div}(a(x)\nabla u) + u + \beta(t)u_t = f(u) \quad \text{in } \Omega \times (\tau, +\infty), \\ a(x)\frac{\partial u}{\partial \vec{n}} = g(u) \quad \text{on } \Gamma \times (\tau, +\infty), \\ u(\tau) = u_0 \in H^1(\Omega), \quad u_t(\tau) = v_0 \in L^2(\Omega). \end{gathered} \tag{1.5}$$

We also prove the existence and regularity of the pullback attractors of the problems (1.1) and (1.5), and that the family of attractors is upper semicontinuous at $\varepsilon = 0$. Moreover, we show that the attractors are bounded in $H^2(\Omega) \times H^1(\Omega)$, uniformly in ε. In particular, all solutions of (1.1) and (1.5) are bounded in $H^2(\Omega)$ with a bound independent of ε and, using Sobolev imbedding, we obtain that these solutions are also bounded in $L^\infty(\Omega)$, uniformly in ε. This enables us to cut the nonlinearities f and g in such a way that it becomes bounded with bounded derivatives up to second order without changing the solutions of the equations. After these considerations, without loss of generality, in this work we assume the hypothesis

(H1) $f, g : \mathbb{R} \to \mathbb{R}$ are $\mathcal{C}^2$-functions satisfying (1.4) and
$$|j(s)| + |j'(s)| + |j''(s)| \leqslant K, \quad \forall s \in \mathbb{R},$$
for some constant $K > 0$ and with $j = f$ or $j = g$.

In this work we continue the analysis made in [1]. The first step is to study the lower semicontinuity of the family of the pullback attractors of the problems (1.1) and (1.5) at $\varepsilon = 0$ in the simplest elements of the attractors; that is, the equilibrium solutions of (1.1) and (1.5). The equilibrium solutions of (1.1) and (1.5) are those solutions which are independent of time; that is, the solutions of the nonlinear elliptic problems

$$\begin{gathered} -\operatorname{div}(a(x)\nabla u_\varepsilon^*) + u_\varepsilon^* = f(u_\varepsilon^*) + \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g(u_\varepsilon^*) \quad \text{in } \Omega, \\ a(x)\frac{\partial u_\varepsilon^*}{\partial \vec{n}} = 0 \quad \text{on } \Gamma, \end{gathered} \tag{1.6}$$

and

$$\begin{gathered} -\operatorname{div}(a(x)\nabla u_0^*) + u_0^* = f(u_0^*) \quad \text{in } \Omega, \\ a(x)\frac{\partial u_0^*}{\partial \vec{n}} = g(u_0^*) \quad \text{on } \Gamma. \end{gathered} \tag{1.7}$$

More precisely, we will prove the continuity of the set of equilibria for (1.1) and (1.5) at $\varepsilon = 0$. Also, we will prove a "uniqueness result", in the sense that for any hyperbolic equilibrium of the limiting problem (1.5), there exists one and only one equilibrium of (1.1) in its neighborhood. In particular, if all the equilibria of (1.5) are hyperbolic, then there exists only a finite number of them and for all ε small enough, problem (1.1) has exactly the same number of equilibria and they are close to the equilibria of (1.5).

In many theoretical and applied problems, it is important to understand what happens to the solutions as a parameter varies in the model. Wave equations with variable coefficients arise naturally in mathematical modeling of inhomogeneous media (e.g. functionally graded materials or materials with damage induced inhomogeneity) in solid mechanics, electromagnetic, fluid flows trough porous media (e.g. modeling traveling waves in a inhomogeneous gas, see Egorov and Shubin [9] and Suggs [12]), and other areas of physics and engineering. Semilinear wave equation arises in quantum mechanics, whereas variants of the form $u_{tt} - \operatorname{div}(a\nabla u) + F(u, u_t) = 0$ appear in the study of vibrating systems with or without damping, and with or without forcing terms; see e.g. Araruna and Bezerra [4] and Arrieta, Carvalho and Hale [5].

The non-autonomous damped wave equation
$$u_{tt} - \Delta u + \beta(t)u_t = f(u), \quad \text{in } \Omega,$$
with Dirichlet boundary condition $u = 0$ on Γ and initial conditions $u(\tau) = u_0$ and $u_t(\tau) = v_0$ was studied in Carvalho, Langa and Robinson [8, Chapter 15]. Under the same assumptions as above, the authors used the theory of pullback asymptotic

compactness to show that this equation had a pullback attractor, and studied the gradient-like structure of the pullback attractor, see also Carabalho et al. [7].

On the other hand, problems with concentrated terms on the strip $\omega_\varepsilon \subset \overline{\Omega}$ were initially studied in Arrieta, Jiménez-Casas and Rodríguez-Bernal [6], where convergence results were proved. Later, the asymptotic behavior of the attractors of a parabolic problem was analyzed in Jiménez-Casas and Rodríguez-Bernal [10, 11], where the upper semicontinuity of attractors at $\varepsilon = 0$ was proved. In these works the domain Ω is C^2 in $\mathbb{R}^n$, with $n \geqslant 2$. In [2] some results from [6] were adapted to a nonlinear elliptic problem posed on an open square Ω in $\mathbb{R}^2$, considering $\omega_\epsilon \subset \Omega$ and with highly oscillatory behavior in the boundary inside of Ω. Later the continuity of attractors was proved in [3], for a nonlinear parabolic problem posed on a C^2 domain Ω in $\mathbb{R}^2$, when some terms are concentrated in a neighborhood of the boundary and the inner boundary of this neighborhood presents a highly oscillatory behavior. The work [3] was the first to consider the lower semicontinuity of attractors for problems with terms concentrating on the boundary.

In Aragão and Bezerra [1] we considered a class of damped wave equations having Neumann boundary conditions and terms concentrating on the boundary. We prove the existence, regularity and upper semicontinuity of pullback attractors. To the best of our knowledge, there are no results on the continuity of the set equilibria for these non-autonomous wave equations with terms concentrating on the boundary; it is natural because the hyperbolic structure of the equations here brings a further difficulty.

This article is organized as follows. In Section 2, we will give the notation that it will be used in this paper and we will see that the solutions of the nonlinear elliptic equations (1.6) and (1.7) will be obtained as fixed points of appropriate nonlinear maps defined in the space $H^1(\Omega)$. In Section 3, we will prove several important technical results that will be needed in the proof of continuity of the set equilibria. In Section 4, we will show the upper semicontinuity of the set of equilibria of (1.1) and (1.5) at $\varepsilon = 0$. Finally, in Section 5, we will prove the lower semicontinuity of the set of equilibria for (1.1) and (1.5) at $\varepsilon = 0$. Also we will obtain the continuity of this set of equilibria; for this, we will also need to assume that the equilibrium points of (1.5) are stable under perturbation. Moreover, if an equilibrium solution of the limit problem is hyperbolic, then we will show that the perturbed equation has only one equilibrium solution nearby.

2. Abstract setting and solutions as fixed points

In this section we present the functional framework to be used in this article, and define the abstract problems associated with (1.6) and (1.7). To better explain the results in the paper, initially, we will define the abstract problems associated with (1.1) and (1.5). Let us consider the Hilbert space $Y := L^2(\Omega)$, $Y^{1/2} := H^1(\Omega)$ equipped with the inner product

$$\langle u, v\rangle_{Y^{1/2}} = \int_\Omega a(x)\nabla u \nabla v dx + \int_\Omega uv dx,$$

and we consider the unbounded linear operator $\Lambda : D(\Lambda) \subset Y \to Y$, defined by

$$\Lambda u = -\operatorname{div}(a(x)\nabla u) + u, \quad u \in D(\Lambda),$$

with

$$D(\Lambda) := \big\{u \in H^2(\Omega) \;:\; a(x)\frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \Gamma\big\}.$$

Since this operator turns out to be sectorial in Y, associated with it there is a scale of Banach spaces (the fractional power spaces) Y^α, $\alpha \in \mathbb{R}$, denoting the domain of the fractional power operators associated with Λ, that is, $Y^\alpha := D(\Lambda^\alpha)$ for $\alpha \geqslant 0$, with Y^α endowed with the graph norm $\|x\|_{Y^\alpha} = \|\Lambda^\alpha x\|_Y$, $\alpha \geqslant 0$. Let us consider $Y^{-\alpha} = (Y^\alpha)'$ for $\alpha \geqslant 0$. The fractional power spaces are related to the Bessel Potentials spaces $H^s(\Omega)$, $s \in \mathbb{R}$, and it is well know that

$$Y^\alpha \hookrightarrow H^{2\alpha}(\Omega), \quad \alpha \geqslant 0,$$

with $Y^{1/2} = H^1(\Omega)$, $Y^{-1/2} = (H^1(\Omega))'$, $Y = Y^0 = L^2(\Omega)$ and $Y^1 = D(\Lambda)$.

We note that problem (1.5) has a nonlinear term on boundary. Then choosing $1/2 < s \leqslant 1$ and using the standard trace theory results, we have that for any function $v \in H^s(\Omega)$, the trace of v is well defined and lies in $L^2(\Gamma)$. Moreover, the scale of negative exponents $Y^{-\alpha}$, for $\alpha > 0$, is necessary to introduce the nonlinear term of (1.5) in the abstract equation, since we are using the operator Λ with homogeneous boundary conditions.

Considering the realizations of Λ in this scale, the operator $\Lambda_{-1/2} \in \mathcal{L}(Y^{1/2}, Y^{-1/2})$ is given by

$$\langle \Lambda_{-1/2} u, v\rangle = \int_\Omega a(x)\nabla u \nabla v dx + \int_\Omega uv dx, \quad \text{for } u, v \in Y^{1/2}.$$

With some abuse of notation we will identify all different realizations of this operator and we will write them all as Λ. Also, let us consider the Hilbert space

$$X = X^0 = Y^{1/2} \times Y$$

equipped with the inner product

$$\Big\langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \Big\rangle_X = \langle u_1, u_2\rangle_{Y^{1/2}} + \langle v_1, v_2\rangle_Y,$$

where $\langle \cdot, \cdot\rangle_Y$ is the usual inner product in $L^2(\Omega)$.

We define the unbounded linear operator $A : D(A) \subset X \to X$ by

$$A \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -I \\ \Lambda & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -v \\ \Lambda u \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix} \in D(A),$$

with

$$D(A) = D(\Lambda) \times Y = Y^1 \times Y.$$

It is proved in [8, Proposition 6.21] that A generates a strongly continuous semigroup in X.

For each $\varepsilon \in (0, \varepsilon_0]$, we write (1.1) in an abstract form as

$$\begin{gathered} \frac{dw^\varepsilon}{dt} + Aw^\varepsilon = F_\varepsilon(t, w^\varepsilon), \quad t > \tau, \\ w^\varepsilon(\tau) = w_0, \end{gathered} \tag{2.1}$$

with

$$w^\varepsilon = \begin{pmatrix} u^\varepsilon \\ u^\varepsilon_t \end{pmatrix}, \quad w_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in X\,.$$

A nonlinear map $F_\varepsilon(t,\cdot): X \to H^1(\Omega)\times H^{-s}(\Omega)$, with $1/2<s<1$ and $t>\tau$, defined by

$$F_\varepsilon(t,w)=\begin{pmatrix}0\\ -\beta_\Omega(t,v)+f_\Omega(u)+\frac{1}{\varepsilon}\chi_{\omega_\varepsilon}g_\Omega(u)\end{pmatrix},\quad \text{for } w=\begin{pmatrix}u\\ v\end{pmatrix}\in X,$$

where $\beta_\Omega(t,\cdot): L^2(\Omega)\to H^{-s}(\Omega)$ and $f_\Omega, \frac{1}{\varepsilon}\chi_{\omega_\varepsilon}g_\Omega : H^1(\Omega)\to H^{-s}(\Omega)$ are the operators given by

$$\langle \beta_\Omega(t,v),\varphi\rangle=\int_\Omega \beta(t)v\varphi dx,\quad \forall v\in L^2(\Omega),\ \forall\varphi\in H^s(\Omega), \tag{2.2}$$

$$\langle f_\Omega(u),\varphi\rangle=\int_\Omega f(u)\varphi dx,\quad \forall u\in H^1(\Omega),\ \forall\varphi\in H^s(\Omega), \tag{2.3}$$

and

$$\langle \frac{1}{\varepsilon}\chi_{\omega_\varepsilon}g_\Omega(u),\varphi\rangle=\frac{1}{\varepsilon}\int_{\omega_\varepsilon} g(u)\varphi dx,\quad \forall u\in H^1(\Omega),\ \forall\varphi\in H^s(\Omega). \tag{2.4}$$

Also problem (1.5) can be written in an abstract form as

$$\begin{gathered}\frac{dw}{dt}+Aw=F_0(t,w),\quad t>\tau,\\ w(\tau)=w_0,\end{gathered} \tag{2.5}$$

with $w=\begin{pmatrix}u\\ u_t\end{pmatrix}$. The nonlinear map $F_0(t,\cdot): X\to H^1(\Omega)\times H^{-s}(\Omega)$, with $1/2<s<1$ and $t>\tau$, is defined as

$$F_0(t,w)=\begin{pmatrix}0\\ -\beta_\Omega(t,v)+f_\Omega(u)+g_\Gamma(u)\end{pmatrix},\quad \text{for } w=\begin{pmatrix}u\\ v\end{pmatrix}\in X,$$

where $\beta_\Omega(t,\cdot)$ and f_Ω are defined in (2.2) and (2.3), respectively. The operator $g_\Gamma: H^1(\Omega)\to H^{-s}(\Omega)$ is given by

$$\langle g_\Gamma(u),\varphi\rangle=\int_\Gamma \gamma(g(u))\gamma(\varphi)dS,\quad \forall u\in H^1(\Omega),\ \forall\varphi\in H^s(\Omega), \tag{2.6}$$

where $\gamma: H^s(\Omega)\to L^2(\Gamma)$ is the trace operator.

In [1] we showed that for each $\tau\in\mathbb{R}$ and $w_0\in X$, problems (2.1) and (2.5) are global well-posedness, and that for each $\varepsilon\in[0,\varepsilon_0]$, we can define an evolution process $\{S^\varepsilon(t,\tau): t\geqslant\tau\}$ in X by

$$S^\varepsilon(t,\tau)w_0=w^\varepsilon(t,\tau,w_0)\quad\text{and}\quad S^0(t,\tau)w_0=w(t,\tau,w_0),\quad t\geqslant\tau,$$

with $\varepsilon\in(0,\varepsilon_0]$, where w^ε and w are the unique solutions of (2.1) and (2.5), respectively. We proved the existence of pullback attractor $\{\mathcal{A}^\varepsilon(t): t\in\mathbb{R}\}$ for (2.1) and (2.5) in $X=H^1(\Omega)\times L^2(\Omega)$. We also proved the regularity and upper semicontinuity of the pullback attractors at $\varepsilon=0$. Here, we will continue this analysis showing the continuity of the set of equilibria for (2.1) and (2.5) at $\varepsilon=0$.

The equilibrium solutions of (1.1) and (1.5) (or of (2.1) and (2.5)) are the solutions of the nonlinear elliptic problems (1.6) and (1.7), respectively. We will write the elliptic problems (1.6) and (1.7) in abstract forms. For each $\varepsilon\in(0,\varepsilon_0]$, we write (1.6) in an abstract form as

$$\Lambda u_\varepsilon^*=\tilde{F}_\varepsilon(u_\varepsilon^*), \tag{2.7}$$

with $\tilde{F}_\varepsilon = f_\Omega + \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega$, where $f_\Omega, \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega : H^1(\Omega) \to H^{-s}(\Omega)$, $1/2 < s < 1$, are defined by (2.3) and (2.4), respectively. Also problem (1.7) can be written in an abstract form as

$$\Lambda u_0^* = \tilde{F}_0(u_0^*), \tag{2.8}$$

with $\tilde{F}_0 = f_\Omega + g_\Gamma$, where $f_\Omega, g_\Gamma : H^1(\Omega) \to H^{-s}(\Omega)$, $1/2 < s < 1$, are defined by (2.3) and (2.6), respectively.

In particular, for each $\varepsilon \in [0, \varepsilon_0]$, u_ε^* is a solution of (1.6) and (1.7) if and only if $u_\varepsilon^* \in H^1(\Omega)$ satisfies

$$u_\varepsilon^* = \Lambda^{-1}\tilde{F}_\varepsilon(u_\varepsilon^*);$$

that is, u_ε^* is a fixed point of the nonlinear map $\Lambda^{-1}\tilde{F}_\varepsilon : H^1(\Omega) \to H^1(\Omega)$.

For each $\varepsilon \in [0, \varepsilon_0]$, we denote by $\mathcal{E}_\varepsilon$ the set of solutions of (1.6) and (1.7), that is,

$$\mathcal{E}_\varepsilon = \big\{ u_\varepsilon^* \in H^1(\Omega) : \Lambda u_\varepsilon^* - \tilde{F}_\varepsilon(u_\varepsilon^*) = 0 \big\}.$$

Moreover, we denote by E_ε the set of equilibria of (1.1) and (1.5), that is,

$$E_\varepsilon = \Big\{ e_\varepsilon = \begin{pmatrix} u_\varepsilon^* \\ 0 \end{pmatrix} \in H^1(\Omega) \times L^2(\Omega) : \ u_\varepsilon^* \in \mathcal{E}_\varepsilon \Big\}.$$

Next, we will see that the upper semicontinuity of the family of equilibria $\{E_\varepsilon\}_{\varepsilon \in [0,\varepsilon_0]}$ of (1.1) and (1.5) at $\varepsilon = 0$ is an immediate consequence of upper semicontinuity of the family attractors $\{\mathcal{A}^\varepsilon(t) : t \in \mathbb{R}\}_{\varepsilon \in [0,\varepsilon_0]}$ at $\varepsilon = 0$. On the other hand, note that to obtain the lower semicontinuity of the family of equilibria $\{E_\varepsilon\}_{\varepsilon \in [0,\varepsilon_0]}$, it is sufficient to prove the lower semicontinuity of the family of solutions $\{\mathcal{E}_\varepsilon\}_{\varepsilon \in [0,\varepsilon_0]}$ of (1.6) and (1.7) at $\varepsilon = 0$.

3. Some technical results

Initially, we will show that each set $\mathcal{E}_\varepsilon$ is not empty and it is compact.

Lemma 3.1. *Suppose that* (H1) *holds. Then, for each $\varepsilon \in [0, \varepsilon_0]$ fixed, the set $\mathcal{E}_\varepsilon$ of the solutions of* (1.6) *and* (1.7) *is not empty. Moreover, $\mathcal{E}_\varepsilon$ is compact in $H^1(\Omega)$.*

Proof. Initially, we note that the linear operator $\Lambda^{-1} : H^{-s}(\Omega) \to H^{2-s}(\Omega)$ is continuous and using the compact embedding of $H^{2-s}(\Omega)$ in $H^1(\Omega)$, with $2 - s > 1$, we obtain that $\Lambda^{-1} : H^{-s}(\Omega) \to H^1(\Omega)$ is compact. Now, for each $\varepsilon \in [0, \varepsilon_0]$ fixed, using [1, Lemma 2.1], we have that if B is bounded set in $H^1(\Omega)$ then $\tilde{F}_\varepsilon(B)$ is bounded set in $H^{-s}(\Omega)$. Hence, by compactness of $\Lambda^{-1} : H^{-s}(\Omega) \to H^1(\Omega)$, we obtain that $\Lambda^{-1}\tilde{F}_\epsilon : H^1(\Omega) \to H^1(\Omega)$ is compact.

Now, we show that for each $\varepsilon \in [0, \varepsilon_0]$ fixed, the set $\mathcal{E}_\varepsilon$ of the solutions of (1.6) and (1.7) is not empty; that is, that the equations (1.6) and (1.7) have at least one solution in $H^1(\Omega)$. This is equivalent to show that the compact operator $\Lambda^{-1}\tilde{F}_\varepsilon : H^1(\Omega) \to H^1(\Omega)$ has at least one fixed point.

From [1, Lemma 2.1] and (H1) there exists $k > 0$ independent of ε such that

$$\|\tilde{F}_\varepsilon(u)\|_{H^{-s}(\Omega)} \leqslant k, \quad \forall u \in H^1(\Omega),\ \varepsilon \in [0, \varepsilon_0].$$

We consider the closed ball $\bar{B}_r(0)$ in $H^1(\Omega)$, where $r = k\|\Lambda^{-1}\|_{\mathcal{L}(H^{-s}(\Omega), H^1(\Omega))}$. For each $u \in H^1(\Omega)$, we have

$$\|\Lambda^{-1}\tilde{F}_\varepsilon(u)\|_{H^1(\Omega)} \leqslant \|\Lambda^{-1}\|_{\mathcal{L}(H^{-s}(\Omega), H^1(\Omega))} \|\tilde{F}_\varepsilon(u)\|_{H^{-s}(\Omega)} \leqslant r. \tag{3.1}$$

Therefore, the compact operator $\Lambda^{-1}\tilde{F}_\varepsilon : H^1(\Omega) \to H^1(\Omega)$ takes $H^1(\Omega)$ in the ball $\bar{B}_r(0)$; in particular, $\Lambda^{-1}\tilde{F}_\varepsilon$ takes $\bar{B}_r(0)$ into itself. From Schauder Fixed Point

Theorem, we obtain that $\Lambda^{-1}\tilde{F}_\varepsilon$ has at least one fixed point in $H^1(\Omega)$. Thus, equations (1.6) and (1.7) have at least one solution in $H^1(\Omega)$.

Now, for each $\varepsilon \in [0, \varepsilon_0]$ fixed, we prove that $\mathcal{E}_\varepsilon$ is compact in $H^1(\Omega)$. For each $\varepsilon \in [0, \varepsilon_0]$ fixed, let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{E}_\varepsilon$, then $u_n = \Lambda^{-1}\tilde{F}_\varepsilon(u_n)$, for all $n \in \mathbb{N}$. Similarly to (3.1), we obtain that $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^1(\Omega)$. Thus, $\{\Lambda^{-1}\tilde{F}_\varepsilon(u_n)\}_{n\in\mathbb{N}}$ has a convergent subsequence, that we will denote by $\{\Lambda^{-1}\tilde{F}_\varepsilon(u_{n_k})\}_{k\in\mathbb{N}}$, with limit $u \in H^1(\Omega)$; that is,

$$\Lambda^{-1}\tilde{F}_\varepsilon(u_{n_k}) \to u \quad \text{in } H^1(\Omega), \quad \text{as } k \to \infty.$$

Hence, $u_{n_k} \to u$ in $H^1(\Omega)$, as $k \to \infty$.

By continuity of the operator $\Lambda^{-1}\tilde{F}_\varepsilon : H^1(\Omega) \to H^1(\Omega)$, we obtain

$$\Lambda^{-1}\tilde{F}_\varepsilon(u_{n_k}) \to \Lambda^{-1}\tilde{F}_\varepsilon(u) \quad \text{in } H^1(\Omega), \quad \text{as } k \to \infty.$$

By the uniqueness of the limit, $u = \Lambda^{-1}\tilde{F}_\varepsilon(u)$. Thus, $\Lambda u - \tilde{F}_\varepsilon(u) = 0$ and $u \in \mathcal{E}_\varepsilon$. Therefore, $\mathcal{E}_\varepsilon$ is a compact set in $H^1(\Omega)$. □

Now, for each $\varepsilon \in (0, \varepsilon_0]$ and $1/2 < s < 1$, we define the maps $Df_\Omega, \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega, Dg_\Gamma :$ $H^1(\Omega) \to \mathcal{L}(H^1(\Omega), H^{-s}(\Omega))$ by

$$\langle Df_\Omega(u)w, \varphi\rangle = \int_\Omega f'(u)w\varphi dx, \quad \forall u, w \in H^1(\Omega),\ \forall \varphi \in H^s(\Omega), \tag{3.2}$$

$$\langle \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u)w, \varphi\rangle = \frac{1}{\varepsilon}\int_{\omega_\varepsilon} g'(u)w\varphi dx, \quad \forall u, w \in H^1(\Omega),\ \forall \varphi \in H^s(\Omega), \tag{3.3}$$

$$\langle Dg_\Gamma(u)w, \varphi\rangle = \int_\Gamma \gamma(g'(u)w)\gamma(\varphi) dS, \quad \forall u, w \in H^1(\Omega),\ \forall \varphi \in H^s(\Omega), \tag{3.4}$$

where $\gamma : H^s(\Omega) \to L^2(\Gamma)$ is the trace operator.

Now, we prove a result of uniform boundedness and convergence of the Fréchet differential of the nonlinearity $\tilde{F}_\varepsilon$.

Lemma 3.2. *Suppose that* (H1) *holds. We have:*

(1) *For each $\varepsilon \in [0, \varepsilon_0]$, the map $\tilde{F}_\varepsilon : H^1(\Omega) \to H^{-s}(\Omega)$ is Fréchet differentiable, uniformly in ε, and your Fréchet differentials are given by $D\tilde{F}_0 = Df_\Omega + Dg_\Gamma$ and $D\tilde{F}_\varepsilon = Df_\Omega + \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega$, for $\varepsilon \in (0, \varepsilon_0]$, where the maps $Df_\Omega, \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega, Dg_\Gamma : H^1(\Omega) \to \mathcal{L}(H^1(\Omega), H^{-s}(\Omega))$ are given respectively by* (3.2), (3.3) *and* (3.4).

(2) *For each $\varepsilon \in [0, \varepsilon_0]$, the map $D\tilde{F}_\varepsilon : H^1(\Omega) \to \mathcal{L}(H^1(\Omega), H^{-s}(\Omega))$ is globally Lipschitz, uniformly in ε.*

(3) *There exist $k > 0$ independent of ε such that*

$$\|D\tilde{F}_\varepsilon(u^*)\|_{\mathcal{L}(H^1(\Omega), H^{-s}(\Omega))} \leqslant k, \quad \forall u^* \in H^1(\Omega),\ \varepsilon \in [0, \varepsilon_0].$$

(4) *For each $u^* \in H^1(\Omega)$, we have*

$$\|D\tilde{F}_\varepsilon(u^*) - D\tilde{F}_0(u^*)\|_{\mathcal{L}(H^1(\Omega), H^{-s}(\Omega))} \to 0, \quad \text{as } \varepsilon \to 0,$$

and this limit is uniform for $u^* \in H^1(\Omega)$ such that $\|u^*\|_{H^1(\Omega)} \leqslant R$, for some $R > 0$.

(5) *If $u^*_\varepsilon \to u^*$ in $H^1(\Omega)$, as $\varepsilon \to 0$, then*

$$\|D\tilde{F}_\varepsilon(u^*_\varepsilon) - D\tilde{F}_0(u^*)\|_{\mathcal{L}(H^1(\Omega), H^{-s}(\Omega))} \to 0, \quad \text{as } \varepsilon \to 0.$$

(6) *If* $u_\varepsilon^* \to u^*$ *in* $H^1(\Omega)$*, as* $\varepsilon \to 0$*, and* $w_\varepsilon \to w$ *in* $H^1(\Omega)$*, as* $\varepsilon \to 0$*, then*

$$\|D\tilde{F}_\varepsilon(u_\varepsilon^*)w_\varepsilon - D\tilde{F}_0(u^*)w\|_{H^{-s}(\Omega)} \to 0, \quad \text{as } \varepsilon \to 0.$$

Proof. Items (1) and (2) are immediate consequence of [1, Lemmas 2.2 and 2.3].

(3) For each $u^* \in H^1(\Omega)$ and $\varepsilon \in [0, \varepsilon_0]$, we have

$$\|D\tilde{F}_\varepsilon(u^*)\|_{\mathcal{L}(H^1(\Omega), H^{-s}(\Omega))} = \sup_{w \in H^1(\Omega),\, \|w\|_{H^1(\Omega)}=1} \|D\tilde{F}_\varepsilon(u^*)w\|_{H^{-s}(\Omega)}.$$

Note that, for each $w \in H^1(\Omega)$,

$$\begin{gathered}\|D\tilde{F}_\varepsilon(u^*)w\|_{H^{-s}(\Omega)} = \big\|Df_\Omega(u^*)w + \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u^*)w\big\|_{H^{-s}(\Omega)}, \quad \varepsilon \in (0, \varepsilon_0], \\ \|D\tilde{F}_0(u^*)w\|_{H^{-s}(\Omega)} = \|Df_\Omega(u^*)w + Dg_\Gamma(u^*)w\|_{H^{-s}(\Omega)},\end{gathered}$$

where the maps Df_Ω, $\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega$ and Dg_Γ are given respectively by (3.2), (3.3) and (3.4).

Using that f' is bounded and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|\langle Df_\Omega(u^*)w, \varphi\rangle| &\leqslant \int_\Omega |f'(u^*)w||\varphi|dx \leqslant K\int_\Omega |w||\varphi|dx \\ &\leqslant K\Big(\int_\Omega |w|^2dx\Big)^{1/2}\Big(\int_\Omega |\varphi|^2dx\Big)^{1/2} \\ &\leqslant k_1\|w\|_{H^1(\Omega)}\|\varphi\|_{H^s(\Omega)}, \quad \forall \varphi \in H^s(\Omega).\end{aligned}$$

Therefore,

$$\|Df_\Omega(u^*)w\|_{H^{-s}(\Omega)} \leqslant k_1\|w\|_{H^1(\Omega)}, \quad \forall w \in H^1(\Omega). \tag{3.5}$$

Using that g' is bounded, Cauchy-Schwarz inequality and [6, Lemma 2.1], we have

$$\begin{aligned}|\langle \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u^*)w, \varphi\rangle| &\leqslant \frac{1}{\varepsilon}\int_{\omega_\varepsilon} |g'(u^*)w||\varphi|dx \leqslant \frac{K}{\varepsilon}\int_{\omega_\varepsilon} |w||\varphi|dx \\ &\leqslant K\Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |w|^2dx\Big)^{1/2}\Big(\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\varphi|^2dx\Big)^{1/2} \\ &\leqslant k_2\|w\|_{H^1(\Omega)}\|\varphi\|_{H^s(\Omega)}, \quad \forall \varphi \in H^s(\Omega),\end{aligned}$$

where the positive constant k_2 is independent of ε. Thus,

$$\Big\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u^*)w\Big\|_{H^{-s}(\Omega)} \leqslant k_2\|w\|_{H^1(\Omega)}, \quad \forall w \in H^1(\Omega). \tag{3.6}$$

Now, using that g' is bounded, Cauchy-Schwarz inequality and trace theorems, we obtain

$$\begin{aligned}|\langle Dg_\Gamma(u^*)w, \varphi\rangle| &\leqslant \int_\Gamma |\gamma(g'(u^*)w)||\gamma(\varphi)|dS \leqslant K\int_\Gamma |\gamma(w)||\gamma(\varphi)|dS \\ &\leqslant K\Big(\int_\Gamma |\gamma(w)|^2dS\Big)^{1/2}\Big(\int_\Gamma |\gamma(\varphi)|^2dS\Big)^{1/2} \\ &\leqslant k_3\|w\|_{H^1(\Omega)}\|\varphi\|_{H^s(\Omega)}, \quad \forall \varphi \in H^s(\Omega).\end{aligned}$$

Thus,

$$\|Dg_\Gamma(u^*)w\|_{H^{-s}(\Omega)} \leqslant k_3\|w\|_{H^1(\Omega)}, \quad \forall w \in H^1(\Omega). \tag{3.7}$$

Therefore, the result follows from (3.5), (3.6) and (3.7).

(4) For each $u^* \in H^1(\Omega)$, we ahve

$$\|D\tilde{F}_\varepsilon(u^*) - D\tilde{F}_0(u^*)\|_{\mathcal{L}(H^1(\Omega),H^{-s}(\Omega))} = \|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u^*) - Dg_\Gamma(u^*)\|_{\mathcal{L}(H^1(\Omega),H^{-s}(\Omega))}.$$

For each $u^* \in H^1(\Omega)$ with $\|u^*\|_{H^1(\Omega)} \leqslant R$, in [11, Lemma 5.2] has been proven that there exists $M(\varepsilon, R) > 0$ with $M(\varepsilon, R) \to 0$ as $\varepsilon \to 0$ such that

$$\begin{aligned}
&|\langle \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u^*)w - Dg_\Gamma(u^*)w, \varphi\rangle| \\
&= \Big|\frac{1}{\varepsilon}\int_{\omega_\varepsilon} g'(u^*)w\varphi dx - \int_\Gamma \gamma(g'(u^*)w)\gamma(\varphi)dS\Big| \\
&\leqslant M(\varepsilon, R)\|w\|_{H^1(\Omega)}\|\varphi\|_{H^1(\Omega)}, \quad \forall w, \varphi \in H^1(\Omega).
\end{aligned}$$

Therefore,

$$\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u^*) - Dg_\Gamma(u^*)\|_{\mathcal{L}(H^1(\Omega),H^{-1}(\Omega))} \to 0, \quad \text{as } \varepsilon \to 0, \tag{3.8}$$

uniformly for $u^* \in H^1(\Omega)$ such that $\|u^*\|_{H^1(\Omega)} \leqslant R$.

Now, fix $1/2 < s_0 < 1$ and $0 < \theta < 1$. Then for any s such that $-1 < -s < -s_0 < -1/2$, using (3.6), (3.7) and interpolation we have

$$\begin{aligned}
&\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u^*)w - Dg_\Gamma(u^*)w\|_{H^{-s}(\Omega)} \\
&\leqslant \|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u^*)w - Dg_\Gamma(u^*)w\|^\theta_{H^{-s_0}(\Omega)}\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u^*)w - Dg_\Gamma(u^*)w\|^{1-\theta}_{H^{-1}(\Omega)} \\
&\leqslant (k_2 + k_3)^\theta\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u^*) - Dg_\Gamma(u^*)\|^{1-\theta}_{\mathcal{L}(H^1(\Omega),H^{-1}(\Omega))}\|w\|_{H^1(\Omega)}, \forall w \in H^1(\Omega).
\end{aligned}$$

Using (3.8), we obtain

$$\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u^*) - Dg_\Gamma(u^*)\|_{\mathcal{L}(H^1(\Omega),H^{-s}(\Omega))} \to 0, \quad \text{as } \varepsilon \to 0,$$

uniformly for $u^* \in H^1(\Omega)$ such that $\|u^*\|_{H^1(\Omega)} \leqslant R$.

(5) From item (2), we have that there exists $L > 0$ independent of ε such that

$$\begin{aligned}
&\|D\tilde{F}_\varepsilon(u^*_\varepsilon) - D\tilde{F}_0(u^*)\|_{\mathcal{L}(H^1(\Omega),H^{-s}(\Omega))} \\
&\leqslant \|D\tilde{F}_\varepsilon(u^*_\varepsilon) - D\tilde{F}_\varepsilon(u^*)\|_{\mathcal{L}(H^1(\Omega),H^{-s}(\Omega))} + \|D\tilde{F}_\varepsilon(u^*) - D\tilde{F}_0(u^*)\|_{\mathcal{L}(H^1(\Omega),H^{-s}(\Omega))} \\
&\leqslant L\|u^*_\varepsilon - u^*\|_{H^1(\Omega)} + \|D\tilde{F}_\varepsilon(u^*) - D\tilde{F}_0(u^*)\|_{\mathcal{L}(H^1(\Omega),H^{-s}(\Omega))} \to 0, \quad \text{as } \varepsilon \to 0,
\end{aligned}$$

where we used item (4) and $u^*_\varepsilon \to u^*$ in $H^1(\Omega)$, as $\varepsilon \to 0$.

(6) We take $u^*_\varepsilon \to u^*$ in $H^1(\Omega)$, as $\varepsilon \to 0$, and $w_\varepsilon \to w$ in $H^1(\Omega)$, as $\varepsilon \to 0$. Using the items (3) and (5), we obtain

$$\begin{aligned}
&\|D\tilde{F}_\varepsilon(u^*_\varepsilon)w_\varepsilon - D\tilde{F}_0(u^*)w\|_{H^{-s}(\Omega)} \\
&\leqslant \|D\tilde{F}_\varepsilon(u^*_\varepsilon)w_\varepsilon - D\tilde{F}_\varepsilon(u^*_\varepsilon)w\|_{H^{-s}(\Omega)} + \|D\tilde{F}_\varepsilon(u^*_\varepsilon)w - D\tilde{F}_0(u^*)w\|_{H^{-s}(\Omega)} \\
&\leqslant \|D\tilde{F}_\varepsilon(u^*_\varepsilon)\|_{\mathcal{L}(H^1(\Omega),H^{-s}(\Omega))}\|w_\varepsilon - w\|_{H^1(\Omega)} \\
&\quad + \|D\tilde{F}_\varepsilon(u^*_\varepsilon) - D\tilde{F}_0(u^*)\|_{\mathcal{L}(H^1(\Omega),H^{-s}(\Omega))}\|w\|_{H^1(\Omega)} \\
&\leqslant k\|w_\varepsilon - w\|_{H^1(\Omega)} + \|D\tilde{F}_\varepsilon(u^*_\varepsilon) - D\tilde{F}_0(u^*)\|_{\mathcal{L}(H^1(\Omega),H^{-s}(\Omega))}\|w\|_{H^1(\Omega)} \to 0,
\end{aligned}$$

as $\varepsilon \to 0$. $\square$

4. Upper semicontinuity of the set of equilibria

We will prove the upper semicontinuity of the family of equilibria $\{E_\varepsilon\}_{\varepsilon\in[0,\varepsilon_0]}$ of (1.1) and (1.5) at $\varepsilon = 0$. For this, we will that $E_\varepsilon \subset \mathcal{A}^\varepsilon(t)$ and that the family of attractors $\{\mathcal{A}^\varepsilon(t) : t \in \mathbb{R}\}_{\varepsilon\in[0,\varepsilon_0]}$ is upper semicontinuous at $\varepsilon = 0$.

Theorem 4.1. *Suppose that* (H1) *holds. Then the family of equilibria* $\{E_\varepsilon\}_{\varepsilon\in[0,\varepsilon_0]}$ *of* (1.1) *and* (1.5) *is upper semicontinuous at* $\varepsilon = 0$.

Proof. We will prove that for any sequence of $\varepsilon \to 0$ and for any $e_\varepsilon \in E_\varepsilon$ we can extract a subsequence which converges to an element of E_0. From the upper semicontinuity of the attractors and using that $e_\varepsilon \in E_\varepsilon \subset \mathcal{A}^\varepsilon(t)$, we can extract a subsequence $e_{\varepsilon_k} \in E_{\varepsilon_k}$ with $\varepsilon_k \to 0$, as $k \to \infty$, and we obtain the existence of an $e_0 \in \mathcal{A}^0(t)$ such that

$$\|e_{\varepsilon_k} - e_0\|_X \to 0, \quad \text{as } k \to \infty.$$

We need to prove that $e_0 \in E_0$, that is, $S^0(t,\tau)e_0 = e_0$, for any $t \geqslant \tau$. We first observe that for any $t > \tau$,

$$\|e_{\varepsilon_k} - S^0(t,\tau)e_0\|_X \leqslant \|e_{\varepsilon_k} - e_0\|_X + \|e_0 - S^0(t,\tau)e_0\|_X \to \|e_0 - S^0(t,\tau)e_0\|_X,$$

as $k \to \infty$. Moreover, for a fixed $\tau^* > \tau$ and for any $t \in (\tau, \tau^*)$, we obtain

$$\begin{aligned}
&\|e_{\varepsilon_k} - S^0(t,\tau)e_0\|_X \\
&= \|S^{\varepsilon_k}(t,\tau)e_{\varepsilon_k} - S^0(t,\tau)e_0\|_X \\
&\leqslant \|S^{\varepsilon_k}(t,\tau)e_{\varepsilon_k} - S^0(t,\tau)e_{\varepsilon_k}\|_X + \|S^0(t,\tau)e_{\varepsilon_k} - S^0(t,\tau)e_0\|_X \to 0,
\end{aligned}$$

as $k \to \infty$, where we have used the continuity of processes given in [1, Lemma 6.1] and that $\{S^0(t,\tau) : t \geqslant \tau\} \subset \mathcal{L}(X)$. In particular, we have that for each $t \geqslant \tau$, $S^0(t,\tau)e_0 = e_0$, which implies that $e_0 \in E_0$. □

5. Lower semicontinuity of the set of equilibria

In this section we will prove the lower semicontinuity of the family of equilibria $\{E_\varepsilon\}_{\varepsilon\in[0,\varepsilon_0]}$ of (1.1) and (1.5) at $\varepsilon = 0$. Moreover, we will prove a "uniqueness result". To obtain the lower semicontinuity of the family of equilibria $\{E_\varepsilon\}_{\varepsilon\in[0,\varepsilon_0]}$ of (1.1) and (1.5) at $\varepsilon = 0$, we need to prove the lower semicontinuity of the family of solutions $\{\mathcal{E}_\varepsilon\}_{\varepsilon\in[0,\varepsilon_0]}$ of (1.6) and (1.7) at $\varepsilon = 0$ and this proof requires additional assumptions. We need to assume that the solutions of (1.7) are stable under perturbation. This stability under perturbation can be given by hyperbolicity.

Definition 5.1. For each $\varepsilon \in [0, \varepsilon_0]$, we say that the solution u^*_ε of (1.6) and (1.7) is hyperbolic if the spectrum $\sigma(\Lambda - D\tilde{F}_\varepsilon(u^*_\varepsilon))$ is disjoint from the imaginary axis, that is, $\sigma(\Lambda - D\tilde{F}_\varepsilon(u^*_\varepsilon)) \cap \imath\mathbb{R} = \emptyset$.

Theorem 5.2. *Suppose that* (H1) *holds. If* u^*_0 *is a solution of* (1.7) *which satisfies* $0 \notin \sigma(\Lambda - D\tilde{F}_0(u^*_0))$, *then* u^*_0 *is isolated.*

Proof. Since $0 \notin \sigma(\Lambda - D\tilde{F}_0(u^*_0))$ then $0 \in \rho(\Lambda - D\tilde{F}_0(u^*_0))$. Thus, there exists $C > 0$ such that

$$\|(\Lambda - D\tilde{F}_0(u^*_0))^{-1}\|_{\mathcal{L}(H^{-s}(\Omega),H^1(\Omega))} \leqslant C.$$

Now, we note that u is a solution of (1.7) if and only if

$$\begin{aligned}
&0 = \Lambda u - D\tilde{F}_0(u^*_0)u + D\tilde{F}_0(u^*_0)u - \tilde{F}_0(u) \\
&\Leftrightarrow (\Lambda - D\tilde{F}_0(u^*_0))u = \tilde{F}_0(u) - D\tilde{F}_0(u^*_0)u
\end{aligned}$$

$$\Leftrightarrow u = (\Lambda - D\tilde{F}_0(u_0^*))^{-1}(\tilde{F}_0(u) - D\tilde{F}_0(u_0^*)u).$$

So, u is a solution of (1.7) if and only if u is a fixed point of the map $\Phi : H^1(\Omega) \to H^1(\Omega)$ given by

$$\Phi(u) = (\Lambda - D\tilde{F}_0(u_0^*))^{-1}(\tilde{F}_0(u) - D\tilde{F}_0(u_0^*)u).$$

We will show that there exists $r > 0$ such that $\Phi : \bar{B}_r(u_0^*) \to \bar{B}_r(u_0^*)$ is a contraction, where $\bar{B}_r(u_0^*)$ is a closed ball in $H^1(\Omega)$ with radius r centered at u_0^*.

In fact, from item (1) of Lemma 3.2 we have that there exists $\delta > 0$ such that

$$C\|\tilde{F}_0(u) - \tilde{F}_0(v) - D\tilde{F}_0(u_0^*)(u - v)\|_{H^{-s}(\Omega)} \leqslant \frac{1}{2}\|u - v\|_{H^1(\Omega)},$$

for $\|u - v\|_{H^1(\Omega)} \leqslant \delta$. Taking $r = \delta/2$ and $u, v \in \bar{B}_r(u_0^*)$, we have

$$\begin{aligned}
&\|\Phi(u) - \Phi(v)\|_{H^1(\Omega)} \\
&= \|(\Lambda - D\tilde{F}_0(u_0^*))^{-1}[\tilde{F}_0(u) - \tilde{F}_0(v) - D\tilde{F}_0(u_0^*)(u - v)]\|_{H^1(\Omega)} \\
&\leqslant C\|\tilde{F}_0(u) - \tilde{F}_0(v) - D\tilde{F}_0(u_0^*)(u - v)\|_{H^{-s}(\Omega)} \\
&\leqslant \frac{1}{2}\|u - v\|_{H^1(\Omega)}.
\end{aligned}$$

Thus, Φ is a contraction on $\bar{B}_r(u_0^*)$. Moreover, if $u \in \bar{B}_r(u_0^*)$ then

$$\|\Phi(u) - u_0^*\|_{H^1(\Omega)} = \|\Phi(u) - \Phi(u_0^*)\|_{H^1(\Omega)} \leqslant \frac{1}{2}\|u - u_0^*\|_{H^1(\Omega)} \leqslant \frac{r}{2} < r.$$

Hence, $\Phi\left(\bar{B}_r(u_0^*)\right) \subset \bar{B}_r(u_0^*)$.

Therefore, from the Contraction Theorem, Φ has an unique fixed point in $\bar{B}_r(u_0^*)$. Since u_0^* is a fixed point of Φ, then u_0^* is the unique fixed point of Φ in $\bar{B}_r(u_0^*)$. Thus, u_0^* is isolated. □

Corollary 5.3. *Suppose that* (H1) *holds. If u_0^* is a hyperbolic solution of* (1.7), *then u_0^* is isolated.*

Proposition 5.4. *Suppose that* (H1) *holds. If all points in $\mathcal{E}_0$ are isolated, then there is only a finite number of them. Moreover, if $0 \notin \sigma(\Lambda - D\tilde{F}_0(u_0^*))$ for each $u_0^* \in \mathcal{E}_0$, then $\mathcal{E}_0$ is a finite set.*

Proof. We suppose that the number of elements in $\mathcal{E}_0$ is infinite, hence there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{E}_0$. From Lemma 3.1, $\mathcal{E}_0$ is compact, thus there exist a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$ and $u^* \in \mathcal{E}_0$ such that

$$u_{n_k} \to u^* \quad \text{in} \quad H^1(\Omega), \quad \text{as } k \to \infty.$$

Thus, for all $\delta > 0$, there exists $k_0 \in \mathbb{N}$ such that $u_{n_k} \in B_\delta(u^*)$, for all $k > k_0$, which is a contradiction with the fact that each fixed point in $\mathcal{E}_0$ is isolated and $u^* \in \mathcal{E}_0$.

Now, if $0 \notin \sigma(\Lambda - D\tilde{F}_0(u_0^*))$ for each $u_0^* \in \mathcal{E}_0$, then, by Theorem 5.2, u_0^* is isolated. Thus, $\mathcal{E}_0$ is a finite set. □

To prove the lower semicontinuity of the family of solutions $\{\mathcal{E}_\varepsilon\}_{\varepsilon\in[0,\varepsilon_0]}$ at $\varepsilon = 0$, we will need of the following lemmas.

Lemma 5.5. *Suppose that* (H1) *holds and let* $u^* \in H^1(\Omega)$. *Then, for each* $\varepsilon \in [0, \varepsilon_0]$ *fixed, the operator* $\Lambda^{-1} D\tilde{F}_\varepsilon(u^*) : H^1(\Omega) \to H^1(\Omega)$ *is compact. For any bounded family* $\{w_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ *in* $H^1(\Omega)$, *the family* $\{\Lambda^{-1} D\tilde{F}_\varepsilon(u^*) w_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ *is relatively compact in* $H^1(\Omega)$. *Moreover, if* $w_\varepsilon \to w$ *in* $H^1(\Omega)$, *as* $\varepsilon \to 0$, *then*

$$\Lambda^{-1} D\tilde{F}_\varepsilon(u^*) w_\varepsilon \to \Lambda^{-1} D\tilde{F}_0(u^*) w \quad \textit{in } H^1(\Omega), \quad \textit{as } \varepsilon \to 0.$$

Proof. For each $\varepsilon \in [0, \varepsilon_0]$ fixed, the compactness of linear operator $\Lambda^{-1} D\tilde{F}_\varepsilon(u^*) : H^1(\Omega) \to H^1(\Omega)$ follows from item (3) of Lemma 3.2 and of compactness of linear operator $\Lambda^{-1} : H^{-s}(\Omega) \to H^1(\Omega)$.

Let $\{w_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ be a bounded family in $H^1(\Omega)$. Since

$$\|D\tilde{F}_\varepsilon(u^*) w_\varepsilon\|_{H^{-s}(\Omega)} \leqslant \|D\tilde{F}_\varepsilon(u^*)\|_{\mathcal{L}(H^1(\Omega), H^{-s}(\Omega))} \|w_\varepsilon\|_{H^1(\Omega)}, \quad \forall \varepsilon \in (0, \varepsilon_0],$$

and from item (3) of Lemma 3.2, $\{D\tilde{F}_\varepsilon(u^*)\}_{\varepsilon \in (0,\varepsilon_0]}$ is a bounded family in the space $\mathcal{L}(H^1(\Omega), H^{-s}(\Omega))$, uniformly in ε, then $\{D\tilde{F}_\varepsilon(u^*) w_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ is a bounded family in $H^{-s}(\Omega)$. By compactness of $\Lambda^{-1} : H^{-s}(\Omega) \to H^1(\Omega)$, we have that $\{\Lambda^{-1} D\tilde{F}_\varepsilon(u^*) w_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ has a convergent subsequence in $H^1(\Omega)$. Therefore, the family $\{\Lambda^{-1} D\tilde{F}_\varepsilon(u^*) w_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ is relatively compact.

Now, when $w_\varepsilon \to w$ in $H^1(\Omega)$, as $\varepsilon \to 0$. Thus, from item (6) of Lemma 3.2,

$$D\tilde{F}_\varepsilon(u^*) w_\varepsilon \to D\tilde{F}_0(u^*) w \quad \text{in } H^{-s}(\Omega), \quad \text{as } \varepsilon \to 0.$$

By continuity of the operator $\Lambda^{-1} : H^{-s}(\Omega) \to H^1(\Omega)$, we obtain

$$\Lambda^{-1} D\tilde{F}_\varepsilon(u^*) w_\varepsilon \to \Lambda^{-1} D\tilde{F}_0(u^*) w \quad \text{in } H^1(\Omega), \quad \text{as } \varepsilon \to 0.$$

□

Lemma 5.6. *Suppose that* (H1) *holds and let* $u^* \in H^1(\Omega)$ *such that* $0 \notin \sigma(\Lambda - D\tilde{F}_0(u^*))$. *Then, there exist* $\varepsilon_0 > 0$ *and* $C > 0$ *independent of* ε *such that* $0 \notin \sigma(\Lambda - D\tilde{F}_\varepsilon(u^*))$ *and*

$$\|(\Lambda - D\tilde{F}_\varepsilon(u^*))^{-1}\|_{\mathcal{L}(H^{-s}(\Omega), H^1(\Omega))} \leqslant C, \quad \forall \varepsilon \in [0, \varepsilon_0]. \tag{5.1}$$

Furthermore, for each $\varepsilon \in [0, \varepsilon_0]$ *fixed, the operator* $(\Lambda - D\tilde{F}_\varepsilon(u^*))^{-1} : H^{-s}(\Omega) \to H^1(\Omega)$ *is compact. For any bounded family* $\{w_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ *in* $H^{-s}(\Omega)$, *the family* $\{(\Lambda - D\tilde{F}_\varepsilon(u^*))^{-1} w_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ *is relatively compact in* $H^1(\Omega)$. *Moreover, if* $w_\varepsilon \to w$ *in* $H^{-s}(\Omega)$, *as* $\varepsilon \to 0$, *then*

$$(\Lambda - D\tilde{F}_\varepsilon(u^*))^{-1} w_\varepsilon \to (\Lambda - D\tilde{F}_0(u^*))^{-1} w \quad \textit{in } H^1(\Omega), \quad \textit{as } \varepsilon \to 0.$$

Proof. Initially, for each $\varepsilon \in [0, \varepsilon_0]$, we note that

$$(\Lambda - D\tilde{F}_\varepsilon(u^*))^{-1} = [\Lambda(I - \Lambda^{-1} D\tilde{F}_\varepsilon(u^*))]^{-1} = (I - \Lambda^{-1} D\tilde{F}_\varepsilon(u^*))^{-1} \Lambda^{-1}.$$

Then, proving that $0 \notin \sigma(\Lambda - D\tilde{F}_\varepsilon(u^*))$ is equivalent to proving that $1 \in \rho(\Lambda^{-1} D\tilde{F}_\varepsilon(u^*))$. Moreover, to prove that there exist $\varepsilon_0 > 0$ and $C > 0$ independent of ε such that (5.1) holds, it is sufficient to prove that there exist $\varepsilon_0 > 0$ and $M > 0$ independent of ε such that

$$\|(I - \Lambda^{-1} D\tilde{F}_\varepsilon(u^*))^{-1}\|_{\mathcal{L}(H^1(\Omega))} \leqslant M, \quad \forall \varepsilon \in [0, \varepsilon_0]. \tag{5.2}$$

In fact, we note that

$$\begin{aligned} &\|(\Lambda - D\tilde{F}_\varepsilon(u^*))^{-1}\|_{\mathcal{L}(H^{-s}(\Omega), H^1(\Omega))} \\ &\leqslant \|(I - \Lambda^{-1} D\tilde{F}_\varepsilon(u^*))^{-1}\|_{\mathcal{L}(H^1(\Omega))} \|\Lambda^{-1}\|_{\mathcal{L}(H^{-s}(\Omega), H^1(\Omega))} \end{aligned}$$

$$\leqslant M\|\Lambda^{-1}\|_{\mathcal{L}(H^{-s}(\Omega),H^1(\Omega))} = C, \quad \forall \varepsilon \in [0,\varepsilon_0],$$

where $C > 0$ does not depend of ε.

Next we show (5.2). Initially, from hypothesis $0 \notin \sigma(\Lambda - D\tilde{F}_0(u^*))$ we have $1 \in \rho(\Lambda^{-1}D\tilde{F}_0(u^*))$. Hence, there exists the inverse $(I-\Lambda^{-1}D\tilde{F}_0(u^*))^{-1} : H^1(\Omega) \to H^1(\Omega)$ and, in particular, the kernel $\mathcal{N}(I - \Lambda^{-1}D\tilde{F}_0(u^*)) = \{0\}$.

Now, let $B_\varepsilon = \Lambda^{-1}D\tilde{F}_\varepsilon(u^*)$, for all $\varepsilon \in [0,\varepsilon_0]$. From Lemma 5.5 we have that, for each $\varepsilon \in [0,\varepsilon_0]$ fixed, the operator $B_\varepsilon : H^1(\Omega) \to H^1(\Omega)$ is compact. Using the compactness of B_ε, we can show that the estimate (5.2) is equivalent to

$$\|(I - B_\varepsilon)u_\varepsilon\|_{H^1(\Omega)} \geqslant \frac{1}{M}, \quad \forall \varepsilon \in [0,\varepsilon_0],\ \forall u_\varepsilon \in H^1(\Omega), \tag{5.3}$$

with $\|u_\varepsilon\|_{H^1(\Omega)} = 1$. In fact, suppose that (5.2) holds, then there exists the inverse $(I - B_\varepsilon)^{-1} : H^1(\Omega) \to H^1(\Omega)$ and it is continuous. Moreover,

$$\|(I - B_\varepsilon)^{-1}v_\varepsilon\|_{H^1(\Omega)} \leqslant M\|v_\varepsilon\|_{H^1(\Omega)}, \quad \forall \varepsilon \in [0,\varepsilon_0],\ \forall v_\varepsilon \in H^1(\Omega).$$

Let $u_\varepsilon \in H^1(\Omega)$ such that $\|u_\varepsilon\|_{H^1(\Omega)} = 1$ and taking $v_\varepsilon = (I - B_\varepsilon)u_\varepsilon$, we have

$$\|(I - B_\varepsilon)^{-1}(I - B_\varepsilon)u_\varepsilon\|_{H^1(\Omega)} \leqslant M\|(I - B_\varepsilon)u_\varepsilon\|_{H^1(\Omega)}$$

which implies

$$1 = \|u_\varepsilon\|_{H^1(\Omega)} \leqslant M\|(I - B_\varepsilon)u_\varepsilon\|_{H^1(\Omega)} \Rightarrow \|(I - B_\varepsilon)u_\varepsilon\|_{H^1(\Omega)} \geqslant \frac{1}{M}.$$

Therefore, (5.3) holds. Conversely, suppose that (5.3) holds. We want to prove that there exists the inverse $(I - B_\varepsilon)^{-1} : H^1(\Omega) \to H^1(\Omega)$, it is continuous and satisfies (5.2). For this, we will prove the estimative

$$\|(I - B_\varepsilon)u_\varepsilon\|_{H^1(\Omega)} \geqslant \frac{1}{M}\|u_\varepsilon\|_{H^1(\Omega)}, \quad \forall \varepsilon \in [0,\varepsilon_0],\ \forall u_\varepsilon \in H^1(\Omega). \tag{5.4}$$

We note that (5.4) is immediate for $u_\varepsilon = 0$. Let $u_\varepsilon \in H^1(\Omega)$, $u_\varepsilon \neq 0$, and we take $v_\varepsilon = \frac{u_\varepsilon}{\|u_\varepsilon\|_{H^1(\Omega)}}$. Thus, $\|v_\varepsilon\|_{H^1(\Omega)} = 1$ and using (5.3), we obtain

$$\begin{gathered}
\|(I - B_\varepsilon)v_\varepsilon\|_{H^1(\Omega)} \geqslant \frac{1}{M} \\
\Rightarrow \|(I - B_\varepsilon)\frac{u_\varepsilon}{\|u_\varepsilon\|_{H^1(\Omega)}}\|_{H^1(\Omega)} \geqslant \frac{1}{M} \\
\Rightarrow \frac{1}{\|u_\varepsilon\|_{H^1(\Omega)}}\|(I - B_\varepsilon)u_\varepsilon\|_{H^1(\Omega)} \geqslant \frac{1}{M} \\
\Rightarrow \|(I - B_\varepsilon)u_\varepsilon\|_{H^1(\Omega)} \geqslant \frac{1}{M}\|u_\varepsilon\|_{H^1(\Omega)}.
\end{gathered}$$

Now, let $u_\varepsilon \in H^1(\Omega)$ be such that $(I - B_\varepsilon)u_\varepsilon = 0$. From (5.4) it follows that $u_\varepsilon = 0$. Thus, for each $\varepsilon \in [0,\varepsilon_0]$, $\mathcal{N}(I - B_\varepsilon) = \{0\}$ and the operator $I - B_\varepsilon$ is injective. So there exists the inverse $(I - B_\varepsilon)^{-1} : \mathcal{R}(I - B_\varepsilon) \to H^1(\Omega)$, where $\mathcal{R}(I - B_\varepsilon)$ denotes the image of the operator $I - B_\varepsilon$.

Since B_ε is compact, for all $\varepsilon \in [0,\varepsilon_0]$, then by Fredholm Alternative Theorem, we have

$$\mathcal{N}(I - B_\varepsilon) = \{0\} \Leftrightarrow \mathcal{R}(I - B_\varepsilon) = H^1(\Omega).$$

Hence, $\mathcal{R}(I - B_\varepsilon) = H^1(\Omega)$ and $I - B_\varepsilon$ is bijective, thus there exists the inverse $(I - B_\varepsilon)^{-1} : H^1(\Omega) \to H^1(\Omega)$.

Now, taking $v_\varepsilon \in H^1(\Omega)$ there exists $u_\varepsilon \in H^1(\Omega)$ such that $(I - B_\varepsilon)u_\varepsilon = v_\varepsilon$ and $u_\varepsilon = (I - B_\varepsilon)^{-1}v_\varepsilon$. From (5.4) we have

$$\|(I - B_\varepsilon)^{-1}v_\varepsilon\|_{H^1(\Omega)} = \|u_\varepsilon\|_{H^1(\Omega)} \leqslant M\|(I - B_\varepsilon)u_\varepsilon\|_{H^1(\Omega)} = M\|v_\varepsilon\|_{H^1(\Omega)}$$
$$\Rightarrow \|(I - B_\varepsilon)^{-1}\|_{\mathcal{L}(H^1(\Omega))} \leqslant M, \quad \forall \varepsilon \in [0, \varepsilon_0].$$

Therefore, (5.2) holds.

Since (5.2) and (5.3) are equivalent, it is sufficient to show (5.3). Suppose that (5.3) is not true, that is, there exist a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $H^1(\Omega)$, with $\|u_n\|_{H^1(\Omega)} = 1$ and $\varepsilon_n \to 0$, as $n \to \infty$, such that

$$\|(I - B_{\varepsilon_n})u_n\|_{H^1(\Omega)} \to 0, \quad \text{as } n \to \infty.$$

From Lemma 5.5 we obtain that $\{B_{\varepsilon_n}u_n\}_{n\in\mathbb{N}}$ is relatively compact. Thus, $\{B_{\varepsilon_n}u_n\}_{n\in\mathbb{N}}$ has a convergent subsequence, which we again denote by $\{B_{\varepsilon_n}u_n\}_{n\in\mathbb{N}}$, with limit $u \in H^1(\Omega)$; that is,

$$B_{\varepsilon_n}u_n \to u \quad \text{in} \quad H^1(\Omega), \quad \text{as } n \to \infty.$$

Since $u_n - B_{\varepsilon_n}u_n \to 0$ in $H^1(\Omega)$, as $n \to \infty$, it follows that $u_n \to u$ in $H^1(\Omega)$, as $n \to \infty$. Hence, $\|u\|_{H^1(\Omega)} = 1$. Moreover, since $u_n \to u$ in $H^1(\Omega)$, as $n \to \infty$, using the Lemma 5.5, we have $B_{\varepsilon_n}u_n \to B_0u$ in $H^1(\Omega)$, as $n \to \infty$. Thus,

$$u_n - B_{\varepsilon_n}u_n \to u - B_0u \quad \text{in } H^1(\Omega), \quad \text{as } n \to \infty.$$

By the uniqueness of the limit, $u - B_0u = 0$. This implies that $(I - B_0)u = 0$, with $u \neq 0$, contradicting our hypothesis. Therefore, (5.3) holds.

With this we conclude that there exist $\varepsilon_0 > 0$ and $C > 0$ independent of ε such that (5.1) holds. Now, for each $\varepsilon \in [0, \varepsilon_0]$, the operator $(\Lambda - D\tilde{F}_\varepsilon(u^*))^{-1}$ is compact and the prove of this compactness follows similarly.

Let $\{w_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ be a bounded family in $H^{-s}(\Omega)$. For each $\varepsilon \in (0, \varepsilon_0]$, let $v_\varepsilon = (\Lambda - D\tilde{F}_\varepsilon(u^*))^{-1}w_\varepsilon$. From (5.1) we have

$$\begin{aligned}
\|v_\varepsilon\|_{H^1(\Omega)} &\leqslant \|(\Lambda - D\tilde{F}_\varepsilon(u^*))^{-1}w_\varepsilon\|_{H^1(\Omega)} \\
&\leqslant \|(\Lambda - D\tilde{F}_\varepsilon(u^*))^{-1}\|_{\mathcal{L}(H^{-s}(\Omega),H^1(\Omega))}\|w_\varepsilon\|_{H^{-s}(\Omega)} \\
&\leqslant C\|w_\varepsilon\|_{H^{-s}(\Omega)}.
\end{aligned}$$

Hence, $\{v_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is a bounded family in $H^1(\Omega)$. Moreover,

$$v_\varepsilon = (\Lambda - D\tilde{F}_\varepsilon(u^*))^{-1}w_\varepsilon = (I - \Lambda^{-1}D\tilde{F}_\varepsilon(u^*))^{-1}\Lambda^{-1}w_\varepsilon$$

which is equivalent to

$$(I - \Lambda^{-1}D\tilde{F}_\varepsilon(u^*))v_\varepsilon = \Lambda^{-1}w_\varepsilon \Leftrightarrow v_\varepsilon = \Lambda^{-1}D\tilde{F}_\varepsilon(u^*)v_\varepsilon + \Lambda^{-1}w_\varepsilon.$$

By compactness of $\Lambda^{-1} : H^{-s}(\Omega) \to H^1(\Omega)$, we obtain that $\{\Lambda^{-1}w_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ has a convergent subsequence in $H^1(\Omega)$. Moreover, using Lemma 5.5, we have that $\{\Lambda^{-1}D\tilde{F}_\varepsilon(u^*)v_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is relatively compact in $H^1(\Omega)$, then $\{\Lambda^{-1}D\tilde{F}_\varepsilon(u^*)v_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ has a convergent subsequence in $H^1(\Omega)$. Therefore, $\{v_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ has a convergent subsequence in $H^1(\Omega)$; that is, the family $\{(\Lambda - D\tilde{F}_\varepsilon(u^*))^{-1}w_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ has a convergent subsequence in $H^1(\Omega)$, thus it is relatively compact in $H^1(\Omega)$.

Now, we take $w_\varepsilon \to w$ in $H^{-s}(\Omega)$, as $\varepsilon \to 0$. By continuity of operator $\Lambda^{-1} : H^{-s}(\Omega) \to H^1(\Omega)$, we have

$$\Lambda^{-1}w_\varepsilon \to \Lambda^{-1}w \quad \text{in } H^1(\Omega), \quad \text{as } \varepsilon \to 0.$$

Moreover, $\{w_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is bounded in $H^{-s}(\Omega)$, for some $\varepsilon_0>0$ sufficiently small, and $\{v_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ has a convergent subsequence, which we again denote by $\{v_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$, with limit $v\in H^1(\Omega)$; that is,

$$v_\varepsilon\to v \quad \text{in} \quad H^1(\Omega), \quad \text{as } \varepsilon\to 0.$$

From Lemma 5.5 we obtain

$$\Lambda^{-1}D\tilde{F}_\varepsilon(u^*)v_\varepsilon\to\Lambda^{-1}D\tilde{F}_0(u^*)v \quad \text{in } H^1(\Omega), \quad \text{as } \varepsilon\to 0.$$

Thus, v satisfies $v=\Lambda^{-1}D\tilde{F}_0(u^*)v+\Lambda^{-1}w$, and so $v=(\Lambda-D\tilde{F}_0(u^*))^{-1}w$. Therefore,

$$(\Lambda-D\tilde{F}_\varepsilon(u^*))^{-1}w_\varepsilon\to(\Lambda-D\tilde{F}_0(u^*))^{-1}w \quad \text{in} \quad H^1(\Omega), \quad \text{as } \varepsilon\to 0.$$

The limit above is independent of the subsequence, thus whole family $\{(\Lambda-D\tilde{F}_\varepsilon(u^*))^{-1}w_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ converges to $(\Lambda-D\tilde{F}_0(u^*))^{-1}w$ in $H^1(\Omega)$, as $\varepsilon\to 0$. ☐

Finally, we can prove our main results.

Theorem 5.7. *Suppose that* (H1) *holds and that* u_0^* *is a solution of* (1.7) *which satisfies* $0\notin\sigma(\Lambda-D\tilde{F}_0(u_0^*))$. *Then, there exist* $\varepsilon_0>0$ *and* $\delta>0$ *such that, for each* $\varepsilon\in(0,\varepsilon_0]$, *the equation* (1.6) *has exactly one solution,* u_ε^*, *in*

$$\{v_\varepsilon\in H^1(\Omega):\|v_\varepsilon-u_0^*\|_{H^1(\Omega)}\leqslant\delta\}.$$

Furthermore,

$$u_\varepsilon^*\to u_0^* \quad \text{in } H^1(\Omega), \quad \text{as } \varepsilon\to 0.$$

In particular, the family of solutions $\{\mathcal{E}_\varepsilon\}_{\varepsilon\in[0,\varepsilon_0]}$ *of* (1.6) *and* (1.7) *is lower semicontinuous at* $\varepsilon=0$.

Proof. Initially, using Lemma 5.6, we have that there exist $\varepsilon_0>0$ and $C>0$, independent of ε, such that $0\notin\sigma(\Lambda-D\tilde{F}_\varepsilon(u_0^*))$ and

$$\|(\Lambda-D\tilde{F}_\epsilon(u_0^*))^{-1}\|_{\mathcal{L}(H^{-s}(\Omega),H^1(\Omega))}\leqslant C, \quad \forall\varepsilon\in(0,\varepsilon_0]. \tag{5.5}$$

By item (1) of Lemma 3.2, there exists $\tilde{\delta}=\tilde{\delta}(C)>0$ independent of ε such that

$$C\|\tilde{F}_\varepsilon(u_\varepsilon)-\tilde{F}_\varepsilon(v_\varepsilon)-D\tilde{F}_\varepsilon(u_0^*)(u_\varepsilon-v_\varepsilon)\|_{H^{-s}(\Omega)}\leqslant\frac{1}{2}\|u_\varepsilon-v_\varepsilon\|_{H^1(\Omega)}, \tag{5.6}$$

for all $\varepsilon\in(0,\varepsilon_0]$ and for $\|u_\varepsilon-v_\varepsilon\|\leqslant\tilde{\delta}$.

We note that u_ε, $\varepsilon\in(0,\varepsilon_0]$, is a solution of (1.6) if and only if u_ε is a fixed point of the map $\Phi_\varepsilon:H^1(\Omega)\to H^1(\Omega)$ given by

$$\Phi_\varepsilon(u_\varepsilon)=(\Lambda-D\tilde{F}_\varepsilon(u_0^*))^{-1}(\tilde{F}_\varepsilon(u_\varepsilon)-D\tilde{F}_\varepsilon(u_0^*)u_\varepsilon).$$

Initially, we affirm that

$$\Phi_\varepsilon(u_0^*)\to u_0^* \quad \text{in } H^1(\Omega), \quad \text{as } \varepsilon\to 0. \tag{5.7}$$

In fact, using (5.5), for $\varepsilon\in(0,\varepsilon_0]$, we have

$$\begin{aligned}
&\|\Phi_\varepsilon(u_0^*)-u_0^*\|_{H^1(\Omega)}\\
&\leqslant\|(\Lambda-D\tilde{F}_\varepsilon(u_0^*))^{-1}[(\tilde{F}_\varepsilon(u_0^*)-D\tilde{F}_\varepsilon(u_0^*)u_0^*)-(\tilde{F}_0(u_0^*)-D\tilde{F}_0(u_0^*)u_0^*)]\|_{H^1(\Omega)}\\
&\quad+\|[(\Lambda-D\tilde{F}_\varepsilon(u_0^*))^{-1}-(\Lambda-D\tilde{F}_0(u_0^*))^{-1}](\tilde{F}_0(u_0^*)-D\tilde{F}_0(u_0^*)u_0^*)\|_{H^1(\Omega)}\\
&\leqslant C(\|\tilde{F}_\varepsilon(u_0^*)-\tilde{F}_0(u_0^*)\|_{H^{-s}(\Omega)}+\|D\tilde{F}_\varepsilon(u_0^*)u_0^*-D\tilde{F}_0(u_0^*)u_0^*\|_{H^{-s}(\Omega)})\\
&\quad+\|[(\Lambda-D\tilde{F}_\varepsilon(u_0^*))^{-1}-(\Lambda-D\tilde{F}_0(u_0^*))^{-1}](\tilde{F}_0(u_0^*)-D\tilde{F}_0(u_0^*)u_0^*)\|_{H^1(\Omega)}\to 0,
\end{aligned}$$

as $\varepsilon \to 0$. This follows from [1, Lemma 2.1], item (6) of Lemma 3.2 and Lemma 5.6.

Next, we show that, for $\varepsilon \in (0, \varepsilon_0]$, for some $\varepsilon_0 > 0$ sufficiently small, Φ_ε is a contraction map from the closed ball $\bar{B}_\delta(u_0^*) = \{v_\varepsilon \in H^1(\Omega) : \|v_\varepsilon - u_0^*\|_{H^1(\Omega)} \leqslant \delta\}$ into itself, where $\delta = \tilde{\delta}/2$. First, we show that Φ_ε is a contraction on the $\bar{B}_\delta(u_0^*)$ (uniformly in ε). Let u_ε, $v_\varepsilon \in \bar{B}_\delta(u_0^*)$ and using (5.5) and (5.6), for $\varepsilon \in (0, \varepsilon_0]$, we have

$$
\begin{aligned}
&\|\Phi_\varepsilon(u_\varepsilon) - \Phi_\varepsilon(v_\varepsilon)\|_{H^1(\Omega)} \\
&= \|(\Lambda - D\tilde{F}_\varepsilon(u_0^*))^{-1}[\tilde{F}_\varepsilon(u_\varepsilon) - \tilde{F}_\varepsilon(v_\varepsilon) - D\tilde{F}_\varepsilon(u_0^*)(u_\varepsilon - v_\varepsilon)]\|_{H^1(\Omega)} \\
&\leqslant C\|\tilde{F}_\varepsilon(u_\varepsilon) - \tilde{F}_\varepsilon(v_\varepsilon) - D\tilde{F}_\varepsilon(u_0^*)(u_\varepsilon - v_\varepsilon)\|_{H^{-s}(\Omega)} \\
&\leqslant \frac{1}{2}\|u_\varepsilon - v_\varepsilon\|_{H^1(\Omega)}, \quad \text{for } \varepsilon \in (0, \varepsilon_0].
\end{aligned}
$$

To show that Φ_ε maps $\bar{B}_\delta(u_0^*)$ into itself, we observe that if $u_\varepsilon \in \bar{B}_\delta(u_0^*)$, then

$$
\begin{aligned}
\|\Phi_\varepsilon(u_\varepsilon) - u_0^*\|_{H^1(\Omega)} &\leqslant \|\Phi_\varepsilon(u_\varepsilon) - \Phi_\varepsilon(u_0^*)\|_{H^1(\Omega)} + \|\Phi_\varepsilon(u_0^*) - u_0^*\|_{H^1(\Omega)} \\
&\leqslant \frac{\delta}{2} + \|\Phi_\varepsilon(u_0^*) - u_0^*\|_{H^1(\Omega)}, \quad \text{for } \varepsilon \in (0, \varepsilon_0].
\end{aligned}
$$

By convergence in (5.7), we have that there exists $\varepsilon_0 > 0$ such that

$$
\|\Phi_\varepsilon(u_\varepsilon) - u_0^*\|_{H^1(\Omega)} \leqslant \frac{\delta}{2} + \frac{\delta}{2} = \delta, \quad \text{for } \varepsilon \in (0, \varepsilon_0].
$$

Hence, $\Phi_\varepsilon : \bar{B}_\delta(u_0^*) \to \bar{B}_\delta(u_0^*)$ is a contraction for all $\varepsilon \in (0, \varepsilon_0]$. By the Contraction Theorem follows that, for each $\varepsilon \in (0, \varepsilon_0]$, Φ_ε has an unique fixed point, u_ε^*, in the $\bar{B}_\delta(u_0^*)$.

To show that $u_\varepsilon^* \to u_0^*$ in $H^1(\Omega)$, as $\varepsilon \to 0$, we proceed in the following manner: since Φ_ε is a contraction map from $\bar{B}_\delta(u_0^*)$ into itself, it follows that

$$
\begin{aligned}
\|u_\varepsilon^* - u_0^*\|_{H^1(\Omega)} &= \|\Phi_\varepsilon(u_\varepsilon^*) - u_0^*\|_{H^1(\Omega)} \\
&\leqslant \|\Phi_\varepsilon(u_\varepsilon^*) - \Phi_\varepsilon(u_0^*)\|_{H^1(\Omega)} + \|\Phi_\varepsilon(u_0^*) - u_0^*\|_{H^1(\Omega)} \\
&\leqslant \frac{1}{2}\|u_\varepsilon^* - u_0^*\|_{H^1(\Omega)} + \|\Phi_\varepsilon(u_0^*) - u_0^*\|_{H^1(\Omega)}.
\end{aligned}
$$

Thus, using (5.7),

$$
\|u_\varepsilon^* - u_0^*\|_{H^1(\Omega)} \leqslant 2\|\Phi_\varepsilon(u_0^*) - u_0^*\|_{H^1(\Omega)} \to 0, \quad \text{as } \varepsilon \to 0.
$$

Hence and by compactness of $\mathcal{E}_0$ (Lemma 3.1), we have that the family $\{\mathcal{E}_\varepsilon\}_{\varepsilon \in [0, \varepsilon_0]}$ is lower semicontinuous at $\varepsilon = 0$. ☐

Corollary 5.8. *Suppose that* (H1) *holds and that* u_0^* *is a hyperbolic solution of* (1.7). *Then, there exist* $\varepsilon_0 > 0$ *and* $\delta > 0$ *such that, for each* $\varepsilon \in (0, \varepsilon_0]$, *the equation* (1.6) *has exactly one solution,* u_ε^*, *in*

$$
\{v_\varepsilon \in H^1(\Omega) : \|v_\varepsilon - u_0^*\|_{H^1(\Omega)} \leqslant \delta\}.
$$

Furthermore, $u_\varepsilon^* \to u_0^*$ *in* $H^1(\Omega)$, *as* $\varepsilon \to 0$.

Remark 5.9. Now that we have obtained an unique solution u_ε^* for (1.6) in a small neighborhood of the hyperbolic solution u_0^* for (1.7), we can consider the linearization $\Lambda - D\tilde{F}_\varepsilon(u_\varepsilon^*)$ and from the convergence of u_ε^* to u_0^* in $H^1(\Omega)$ it is easy to obtain that $(\Lambda - D\tilde{F}_\varepsilon(u_\varepsilon^*))^{-1}w_\varepsilon$ converges to $(\Lambda - D\tilde{F}_0(u_0^*))^{-1}w$ in $H^1(\Omega)$, whenever $w_\varepsilon \to w$ in $H^{-s}(\Omega)$, as $\varepsilon \to 0$. Consequently, the hyperbolicity of u_0^* implies the hyperbolicity of u_ε^*, for suitably small ε.

Theorem 5.10. *Suppose that* (H1) *holds. If all solutions u_0^* of* (1.7) *satisfy $0 \notin \sigma(\Lambda - D\tilde{F}_0(u_0^*))$, then* (1.7) *has a finite number k of solutions, $u_{0,1}^*, \dots, u_{0,k}^*$, and there exists $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (0, \varepsilon_0]$, the equation* (1.6) *has exactly k solutions, $u_{\varepsilon,1}^*, \dots, u_{\varepsilon,k}^*$. Moreover, for all $i = 1, \dots, k$,*

$$u_{\varepsilon,i}^* \to u_{0,i}^* \quad \text{in} \quad H^1(\Omega), \quad \text{as } \varepsilon \to 0.$$

The proof follows of Proposition 5.4 and Theorem 5.7.

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Gleiciane da Silva Aragão
Departamento de Ciências Exatas e da Terra, Universidade Federal de São Paulo, Av. Conceição, 515, Centro, Cep 09920-000, Diadema-SP, Brazil
Email address: gleiciane.aragao@unifesp.br, Phone +55 (11) 4044-0500

Flank David Morais Bezerra
Departamento de Matemática, Universidade Federal da Paraíba, Cidade Universitária, Campus I, Via Expressa Padre Zé-Castelo Branco III, Cep 58051-900, João Pessoa-PB, Brazil
Email address: flank@mat.ufpb.br, Phone +55 (83) 3216-7434