Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 72, pp. 1–19.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

NON-AUTONOMOUS APPROXIMATIONS GOVERNED BY THE FRACTIONAL POWERS OF DAMPED WAVE OPERATORS

MARCELO J. D. NASCIMENTO, FLANK D. M. BEZERRA

Abstract. In this article we study non-autonomous approximations governed by the fractional powers of damped wave operators of order $\alpha \in (0,1)$ subject to Dirichlet boundary conditions in an n-dimensional bounded domain with smooth boundary. We give explicitly expressions for the fractional powers of the wave operator, we compute their resolvent operators and their eigenvalues. Moreover, we study the convergence as $\alpha \nearrow 1$ with rate $1-\alpha$.

1. Introduction

This article concerns the fractional powers of order $\alpha \in (0,1)$ of the wave operators with time-dependent propagation speed subject to Dirichlet boundary conditions in an n-dimensional bounded domain with smooth boundary, in sense of Amann [1, pg. 148] and Henry [22, pg. 25]. We study the approximation via fractional powers of the following initial-boundary value problem associated with a non-autonomous damped wave equation

$$\begin{gathered} u_{tt} + a(t)(-\Delta_D)u + \eta(t)u_t = 0, \quad x \in \Omega,\ t > \tau, \\ u(x,t) = 0, \quad x \in \partial\Omega,\ t \geqslant \tau, \\ u(x,\tau) = u_0(x), \quad u_t(x,\tau) = v_0(x), \quad x \in \bar{\Omega},\ \tau \in \mathbb{R}, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $n \geqslant 1$, a is a positive and bounded real-valued functions defined in $\mathbb{R}$ such that there are positive constants $a_{\min}$ and $a_{\max}$ satisfying

$$0 < a_{\min} \leqslant a(t) \leqslant a_{\max}, \quad \forall t \in \mathbb{R}, \tag{1.2}$$

We suppose that a is a Hölder continuous function with exponent $1/2 \leqslant \gamma \leqslant 1$ and constant $\kappa > 0$; that is,

$$\forall x, y \in \mathbb{R},\ |a(x) - a(y)| \leqslant \kappa |x-y|^\gamma. \tag{1.3}$$

In this case we say that the function a is (γ,κ)-Hölder continuous in $\mathbb{R}$, or simply $a \in \mathcal{C}^{0,\gamma}(\mathbb{R})$.

We also assume that η is continuously differentiable, positive, decreasing in $\mathbb{R}$ and is Hölder continuous function with some exponent less than 1.

We give explicitly expressions for the fractional powers of the wave operator, compute their resolvent operators, and their eigenvalues. Moreover, we study the

2010 *Mathematics Subject Classification.* 35L05, 35B40.
Key words and phrases. Non-autonomous damped wave equations; fractional powers; rate of convergence; eigenvalues.

Submitted January 24, 2019. Published May 17, 2019.

convergence as $\alpha \nearrow 1$ with rate $1-\alpha$. Our motivation for considering this problem is based on the fact that, if for each $t \in \mathbb{R}$, the linear operator $\Lambda(t)$ is the infinitesimal generator of a semigroup which is strongly continuous (not necessarily analytic, in sense of Henry [22]) for which we can define their fractional powers of order $0 < \alpha < 1$, $\Lambda(t)^\alpha$, the fractional powers are positive sectorial operators, see e.g. Kato [23] and Henry [22]. With this, the analytic semigroup to which infinitesimal generator is the fractional power $\Lambda(t)^\alpha$ is associated with a fractional approximation more regular to the original problem, and this can be used to get more information to original problem in the passage to the limit $\alpha \nearrow 1$.

In recent years many researchers has been studying spatial fractional models in bounded smooth domains and its connection with classical models involving the problem of solvability and analysis of asymptotic dynamics, in the sense of global attractors. Benson, Wheatcraft and Meerschaert [2] studied a fractional advection-dispersion equation. Bezerra, Carvalho, Cholewa and Nascimento [3] considered autonomous approximations via fractional powers of semilinear wave equations with subcritical nonlinear term. Bezerra, Carvalho, Dłotko and Nascimento [4] considered approximations via fractional powers of Schröndiger equations with subcritical nonlinear term. Cholewa and Dłotko [6] and Dłotko [20] studied approximations via fractional powers of Navier-Stokes equations. Fazli and Bahrami [21] studied steady solutions of fractional reaction-diffusion equations. Pan et al. [24] studied the fractional approximations of a thermal transport model for nanofluid in porous media.

Non-autonomous damped wave equations have been considered before by many authors, see e.g. Caraballo et al. [8, 9, 10] and Carvalho, Langa and Robinson [12, Chapter 15] where the dynamics and their continuity is studied under perturbations in the parameters of the equations. Chen and Triggiani [16, 17, 18, 19] studied the characterization and properties of the fractional powers of certain operators arising in elastic systems. Sun, Cao and Duan [27] studied the dynamics (existence of pullback attractors) for a class of non-autonomous damped wave equations.

Carvalho et al. [11] considered the semilinear problem correspondent to (1.1), with η and a positive constants functions, through a limit of a strongly damped wave equation, adding the term $2\rho(-\Delta_D)^{1/2}$ with $\rho > 0$ to the equation, so that the equation becomes 'parabolic' in nature (see Chen and Triggiani [17]), and passing to the limit as $\rho \to 0^+$. With the 'parabolic' structure ($\rho > 0$), they obtain local well posedness for the perturbed problem with the usual semigroup approach. Under a dissipative condition in the nonlinearity they obtain global well posedness, existence of global attractors and some uniform (with respect to ρ) bounds that allow a passage to the limit ($\rho = 0$). After this the authors obtain global solutions of (1.1) that satisfy the variation of constants formula and are able to establish the existence of global attractors.

To express our results better let us first introduce some terminology. If $X = L^2(\Omega)$ and $A : D(A) \subset X \to X$ is defined by $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $Au = -\Delta_D u$ for all $u \in D(A)$, then A is a positive self-adjoint operator and $-A$ generates a compact analytic semigroup on X. Denote by X^α the fractional power spaces associated with operator A; that is, $X^\alpha = D(A^\alpha)$ with the norm $\|A^\alpha \cdot\|_X : X^\alpha \to \mathbb{R}^+$.

Since A is positive self-adjoint operator on X, then the characterization of the scale $\{X^\alpha : 0 \leqslant \alpha \leqslant 1\}$ is quite complete, see for instance Cholewa and Dłotko

[7], Triebel [28] and references therein. In this case the imaginary powers of A are bounded and X^α may be described as the intermediate spaces between $L^2(\Omega)$ and $H^2(\Omega)\cap H_0^1(\Omega)$ based on the complex interpolation method, see for instance Triebel [28] and references therein. For $\alpha > 0$ we define $X^{-\alpha}$ as the completion of X with the norm $\|A^{-\alpha}\cdot\|_X$. With this notation $X^{1/2} = H_0^1(\Omega)$, $X^1 = H^2(\Omega)\cap H_0^1(\Omega)$ and $X^{-\alpha} = (X^\alpha)'$ (see Amann [1] for the characterization of the negative scale).

The initial-boundary value problem (1.1) can be written as a non-autonomous abstract Cauchy problem in the product space $Y = X^{1/2}\times X$ as

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} + \Lambda(t)\begin{bmatrix} u \\ v \end{bmatrix} = F\Big(t, \begin{bmatrix} u \\ v \end{bmatrix}\Big),\ t>\tau, \quad \text{and} \quad \begin{bmatrix} u \\ v \end{bmatrix}_{t=\tau} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \tag{1.4}$$

where the non-autonomous damped wave operator $\Lambda(t) : D(\Lambda(t)) \subset Y \to Y$ is defined by

$$\Lambda(t)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & -I \\ a(t)A & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

where

$$D(\Lambda(t)) = X^1 \times X^{1/2} =: Y^1,$$

and

$$F\Big(t, \begin{bmatrix} u \\ v \end{bmatrix}\Big) := \begin{bmatrix} 0 \\ -\eta(t)v \end{bmatrix},\ t\in\mathbb{R}, \quad \begin{bmatrix} u \\ v \end{bmatrix} \in X^1\times X^{1/2}. \tag{1.5}$$

The aim of this paper is to consider problem (1.1) using an approximation by ‘parabolic type’ problems of ‘lower order’ which we begin to describe. If $-\Lambda(t)$ denotes the wave operator (generator of a C_0-semigroup), we use the fractional power operators $-\Lambda(t)^\alpha$, $\alpha\in(0,1)$, (generator of an analytic semigroup) to approximate $-\Lambda(t)$. This type of approximation (though defined by a lower order operator) has the effect of regularity and ensures properties of smoothing to the operator solution of the perturbed problem that the limit does not have.

With this, we consider the non-autonomous abstract Cauchy problem

$$\frac{d}{dt}\begin{bmatrix} u^\alpha \\ v^\alpha \end{bmatrix} + \Lambda(t)^\alpha\begin{bmatrix} u^\alpha \\ v^\alpha \end{bmatrix} = F(t, \begin{bmatrix} u^\alpha \\ v^\alpha \end{bmatrix}),\ t>\tau, \quad \text{and} \quad \begin{bmatrix} u^\alpha \\ v^\alpha \end{bmatrix}_{t=\tau} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}. \tag{1.6}$$

We will see that the operator $\Lambda(t)^\alpha$ is a positive sectorial operator (see Kato [23]). With this, the system (1.6) can be seen as a parabolic type perturbation of the system (1.4) and we approximate solutions of (1.4) by solutions of (1.6), $\alpha_0 < \alpha < 1$, with suitably chosen initial data, for some $0 < \alpha_0 < 1$.

We emphasize that, though it may appear cumbersome at the moment, we will be able to give explicit expressions to the fractional powers of $\Lambda(t)$ (in terms of the fractional powers of $-\Delta_D$). Exploiting the parabolic structure of (1.6), the local well posedness for (1.6) is obtained for α suitably close do 1.

This paper is organized as follows. In Section 2 we will remember some definitions and results about theory of non-autonomous semilinear parabolic problems, according to Carvalho, Langa and Robinson [12], Carvalho and Nascimento [13], and Sobolevskiĭ [26]. In Section 3 we solve the linear problem associated with (1.6); namely we prove the following theorem.

Theorem 1.1. (i) *For each $\alpha\in(0,1)$, the operators $\Lambda(t)^\alpha$ are uniformly sectorial and the map $t\mapsto\Lambda(t)^\alpha$ is uniformly Hölder continuous in Y;*

(ii) *There exist a linear evolution process $\{U_\alpha(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ that solves the linear homogeneous problem*

$$\frac{d}{dt}\begin{bmatrix} u^\alpha \\ v^\alpha \end{bmatrix} + \Lambda(t)^\alpha \begin{bmatrix} u^\alpha \\ v^\alpha \end{bmatrix} = 0,\ t > \tau, \quad \begin{bmatrix} u^\alpha \\ v^\alpha \end{bmatrix}_{t=\tau} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \tag{1.7}$$

for each $\alpha \in (0,1)$; namely, for any $t \geqslant \tau \in \mathbb{R}$,

$$U_\alpha(t,\tau)\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} u^\alpha \\ v^\alpha \end{bmatrix},$$

where

$$\begin{gathered}(\tau,\infty) \ni t \mapsto \begin{bmatrix} u^\alpha \\ v^\alpha \end{bmatrix}(t) = U_\alpha(t,\tau)\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in X^{1/2} \times X \text{ is continuously differentiable,} \\ \begin{bmatrix} u^\alpha \\ v^\alpha \end{bmatrix}(t) \in X^{\frac{1+\alpha}{2}} \times X^{\alpha/2}\, \forall t \in (\tau,+\infty) \text{ and satisfies (1.7).}\end{gathered} \tag{1.8}$$

(iii) *There exist a linear evolution process $\{S_\alpha(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ that solves the linear problem (1.6) for each $\alpha \in (0,1)$; namely, for any $t \geqslant \tau \in \mathbb{R}$, $S_\alpha(t,\tau)\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} u^\alpha \\ v^\alpha \end{bmatrix}$, where*

$$\begin{bmatrix} u^\alpha \\ v^\alpha \end{bmatrix} \in C^1([\tau,\infty); X^{1/2} \times X) \cap C((\tau,\infty); X^{\frac{1+\alpha}{2}} \times X^{\alpha/2}).$$

In Section 4 we study the spectral properties of the operators $\Lambda(t)$ and $\Lambda(t)^\alpha$, studying the convergence with rate of the spectral projections and compute the eigenvalues of this operators, in terms of α. Namely, we obtain the convergence with rate $1-\alpha$ of the spectral projections associated with $\Lambda(t)^\alpha$.

2. Singularly non-autonomous abstract linear problem

Throughout this article, $L(\mathcal{Z})$ denotes the space of linear and bounded operators defined in a Banach space $\mathcal{Z}$. Let $\mathcal{A}(t)$, $t \in \mathbb{R}$, be a family of unbounded closed linear operators defined on a fixed dense subspace D of $\mathcal{Z}$.

Consider the singularly non-autonomous abstract linear parabolic problem

$$\begin{aligned}\frac{du}{dt} &= -\mathcal{A}(t)u, \quad t > \tau, \\ u(\tau) &= u_0 \in D.\end{aligned}$$

We assume that

(a) The operator $\mathcal{A}(t) : D \subset \mathcal{Z} \to \mathcal{Z}$ is a closed densely defined operator (the domain D is fixed) and there is a constant $C > 0$ (independent of $t \in \mathbb{R}$) such that all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda \geqslant 0$,

$$\|(\mathcal{A}(t) + \lambda I)^{-1}\|_{L(\mathcal{Z})} \leqslant \frac{C}{|\lambda|+1}.$$

To express this fact we will say that the family $\mathcal{A}(t)$ is *uniformly sectorial*, see e.g. [13] and [26].

(b) There are constants $C > 0$ and $\epsilon_0 > 0$ such that, for any $t, \tau, s \in \mathbb{R}$,

$$\|[\mathcal{A}(t) - \mathcal{A}(\tau)]\mathcal{A}^{-1}(s)\|_{L(\mathcal{Z})} \leqslant C(t-\tau)^{\epsilon_0}, \quad \epsilon_0 \in (0,1].$$

To express this fact we will say that the function $\mathcal{A}(t)$ is *uniformly Hölder continuous*, see e.g. [13] and [26].

Denote by $\mathcal{A}_0$ the operator $\mathcal{A}(t_0)$ for some $t_0 \in \mathbb{R}$ fixed. If $\mathcal{Z}^\alpha$ denotes the domain of $\mathcal{A}_0^\alpha$ with the graph norm, $\alpha > 0$, $\mathcal{Z}^0 := \mathcal{Z}$, denote by $\{\mathcal{Z}^\alpha; \alpha \geqslant 0\}$ the fractional power scale associated with $\mathcal{A}_0$ (see Henry [22]).

From (a), $-\mathcal{A}(t)$ is the generator of an analytic semigroup $\{e^{-\tau\mathcal{A}(t)} \in L(\mathcal{Z}) : \tau \geqslant 0\}$. With this and the fact that $0 \in \rho(\mathcal{A}(t))$, it follows that

$$\|e^{-\tau\mathcal{A}(t)}\|_{L(\mathcal{Z})} \leqslant C, \quad \tau \geqslant 0, \quad t \in \mathbb{R},$$

and

$$\|\mathcal{A}(t)e^{-\tau\mathcal{A}(t)}\|_{L(\mathcal{Z})} \leqslant C\tau^{-1},\ \tau > 0,\ t \in \mathbb{R}.$$

From (b) it follows that for all $T > 0$,

$$\|\mathcal{A}(t)\mathcal{A}^{-1}(\tau)\|_{L(\mathcal{Z})} \leqslant C,$$

for any $t, \tau \in [-T, T]$. Also, the semigroup $e^{-\tau\mathcal{A}(t)}$ generated by $-\mathcal{A}(t)$ satisfies the estimate

$$\|e^{-\tau\mathcal{A}(t)}\|_{L(\mathcal{Z}^\beta,\mathcal{Z}^\alpha)} \leqslant M\tau^{\beta-\alpha}, \tag{2.1}$$

where $0 \leqslant \beta \leqslant \alpha < 1 + \epsilon_0$.

Next we recall the definition of evolution process associated with an operator family $\{\mathcal{A}(t) : t \in \mathbb{R}\}$.

Definition 2.1. A family $\{U(t,\tau) : t \geqslant \tau \in \mathbb{R}\} \subset L(\mathcal{Z})$ satisfying

(1) $U(\tau,\tau) = I$,
(2) $U(t,\sigma)U(\sigma,\tau) = U(t,\tau)$ for any $t \geqslant \sigma \geqslant \tau$,
(3) $\mathcal{P} \times \mathcal{Z} \ni ((t,\tau), u_0) \mapsto U(t,\tau)v_0 \in \mathcal{Z}$ is continuous, where $\mathcal{P} = \{(t,\tau) \in \mathbb{R}^2 : t \geqslant \tau\}$.

it is called a *linear evolution process* (process for short) or *family of evolution operators*, see [12].

If the operator $\mathcal{A}(t)$ is uniformly sectorial and uniformly Hölder continuous, then we obtain that there exist a process $\{U(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ associated with operator $\mathcal{A}(t)$, that is give by

$$U(t,\tau) = e^{-(t-\tau)\mathcal{A}(\tau)} + \int_\tau^t U(t,s)[\mathcal{A}(\tau) - \mathcal{A}(s)]e^{-(s-\tau)\mathcal{A}(\tau)}ds.$$

The evolution operator $\{U(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ satisfies the condition

$$\|U(t,\tau)\|_{L(\mathcal{Z}^\beta,\mathcal{Z}^\alpha)} \leqslant C(\alpha,\beta)(t-\tau)^{\beta-\alpha},$$

where $0 \leqslant \beta \leqslant \alpha < 1 + \epsilon_0$. For more details see Sobolevskiĭ [26], and Carvalho and Nascimento [13].

3. Equation governed by fractional powers

In this section we obtain a description of the operator $\Lambda(t)^\alpha$ in terms of the fractional Laplacian operator in bounded smooth domains of $\mathbb{R}^n$ and we prove the Theorem 1.1.

3.1. **Fractional powers of the damped wave operator.** To arrive at (1.6) and apply to it the above results, we need to compute the fractional powers of $\Lambda(t)$ and to understand the fractional power spaces associated with it. This is what we do next.

Remark 3.1. Thanks to (1.2), for any $0 < \alpha < 1$ we have the following identity

$$(a(t)A)^{\alpha} = a(t)^{\alpha} A^{\alpha}, \quad \text{for all } t \in \mathbb{R}.$$

Indeed, for any $t \in \mathbb{R}$ and $0 < \alpha < 1$,

$$\begin{aligned}(a(t)A)^{-\alpha} &= \frac{\sin \pi\alpha}{\pi} \int_0^{\infty} \lambda^{-\alpha}(\lambda I + a(t)A)^{-1} d\lambda \\ &= \frac{\sin \pi\alpha}{\pi} \int_0^{\infty} \lambda^{-\alpha} a(t)^{-1}(a(t)^{-1}\lambda I + A)^{-1} d\lambda,\end{aligned}$$

and a change of variable allows us to obtain

$$\begin{aligned}(a(t)A)^{-\alpha} &= a(t)^{-\alpha} \frac{\sin \pi\alpha}{\pi} \int_0^{\infty} \mu^{-\alpha}(\mu I + A)^{-1} d\mu \\ &= a(t)^{-\alpha} A^{-\alpha}.\end{aligned}$$

Consequently, for any $t \in \mathbb{R}$ and $0 < \alpha < 1$, we have

$$(a(t)A)^{\alpha} = [(a(t)A)^{-\alpha}]^{-1} = [a(t)^{-\alpha} A^{-\alpha}]^{-1} = a(t)^{\alpha} A^{\alpha}.$$

Theorem 3.2. *For any $\alpha \in [0, 1]$ and $t \in \mathbb{R}$, we have*

(i)
$$\Lambda(t)^{-\alpha} = \begin{bmatrix} \cos \frac{\pi\alpha}{2} a(t)^{-\alpha/2} A^{-\alpha/2} & \sin \frac{\pi\alpha}{2} a(t)^{\frac{-1-\alpha}{2}} A^{\frac{-1-\alpha}{2}} \\ -\sin \frac{\pi\alpha}{2} a(t)^{\frac{1-\alpha}{2}} A^{\frac{1-\alpha}{2}} & \cos \frac{\pi\alpha}{2} a(t)^{-\alpha/2} A^{-\alpha/2} \end{bmatrix}. \tag{3.1}$$

(ii) *Zero is in the continuous spectrum of $\Lambda^{-\alpha}(t)$, $\alpha \in (0, 1]$ and the unbounded linear operator $\Lambda(t)^{\alpha} : D(\Lambda(t)^{\alpha}) \subset Y \to Y$ is given by*

$$D(\Lambda(t)^{\alpha}) = X^{\frac{1+\alpha}{2}} \times X^{\alpha/2}$$

and

$$\Lambda(t)^{\alpha} = \begin{bmatrix} \cos \frac{\pi\alpha}{2} a(t)^{\alpha/2} A^{\alpha/2} & -\sin \frac{\pi\alpha}{2} a(t)^{\frac{-1+\alpha}{2}} A^{\frac{-1+\alpha}{2}} \\ \sin \frac{\pi\alpha}{2} a(t)^{\frac{1+\alpha}{2}} A^{\frac{1+\alpha}{2}} & \cos \frac{\pi\alpha}{2} a(t)^{\alpha/2} A^{\alpha/2} \end{bmatrix}. \tag{3.2}$$

Proof. (i) Note that for all $t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$,

$$\lambda I + \Lambda(t) = \begin{bmatrix} \lambda I & -I \\ a(t)A & \lambda I \end{bmatrix},$$

and therefore, for all $\lambda \in \rho(-\Lambda(t))$, we have

$$(\lambda I + \Lambda(t))^{-1} = \begin{bmatrix} \lambda(\lambda^2 I + a(t)A)^{-1} & (\lambda^2 I + a(t)A)^{-1} \\ -a(t)A(\lambda^2 I + a(t)A)^{-1} & \lambda(\lambda^2 I + a(t)A)^{-1} \end{bmatrix}.$$

For any $0 < \alpha < 1$ and $t \in \mathbb{R}$, we can compute the fractional $\Lambda(t)^{-\alpha}$ by the formula

$$\Lambda(t)^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^{\infty} \lambda^{-\alpha}(\lambda I + \Lambda(t))^{-1} d\lambda$$

see Amann [1, pg. 148] or Henry [22, pg. 25]. With this, for any $0 < \alpha < 1$ and $t \in \mathbb{R}$, we can obtain (3.1).

(ii) Also, it is not difficult to see that 0 is in the continuous spectrum of $\Lambda^{-\alpha}(t)$ and (3.2) for every $\alpha \in (0, 1]$ and $t \in \mathbb{R}$. $\square$

The next theorem ensures that the rate of convergence of resolvents $\Lambda(t)^{-\alpha}$ at $\alpha = 1$ it is $1-\alpha$. Before proving the theorem, we have two Lemmas.

Theorem 3.3. *For every $t \in \mathbb{R}$ the operators $\Lambda(t)^{-\alpha}$ converges in the uniform operator topology of $L(X^{1/2} \times X)$ to $\Lambda(t)^{-1}$ as $\alpha \nearrow 1$, with rate $1-\alpha$.*

Proof. Let $\begin{bmatrix} u \\ v \end{bmatrix} \in X^{1/2} \times X$, $t \in \mathbb{R}$ and $\alpha \in (0,1)$ then

$$
\begin{aligned}
&(\Lambda(t)^{-\alpha} - \Lambda(t)^{-1}) \begin{bmatrix} u \\ v \end{bmatrix} \\
&= \begin{bmatrix} \cos\frac{\pi\alpha}{2} a(t)^{-\alpha/2} A^{-\alpha/2} u + (\sin\frac{\pi\alpha}{2} a(t)^{\frac{-1-\alpha}{2}} A^{\frac{-1-\alpha}{2}} - a(t)^{-1} A^{-1}) v \\ (-\sin\frac{\pi\alpha}{2} a(t)^{\frac{1-\alpha}{2}} A^{\frac{1-\alpha}{2}} + I) u + \cos\frac{\pi\alpha}{2} a(t)^{-\alpha/2} A^{-\alpha/2} v \end{bmatrix},
\end{aligned}
$$

in other words,

$$
\begin{aligned}
&\|(\Lambda(t)^{-\alpha} - \Lambda(t)^{-1}) \begin{bmatrix} u \\ v \end{bmatrix}\|_{X^{1/2} \times X} \\
&= \|\cos\frac{\pi\alpha}{2} a(t)^{\frac{-\alpha}{2}} A^{-\alpha/2} u + (\sin\frac{\pi\alpha}{2} a(t)^{\frac{-1-\alpha}{2}} A^{\frac{-1-\alpha}{2}} - a(t)^{-1} A^{-1}) v\|_{X^{1/2}} \\
&\quad + \|(-\sin\frac{\pi\alpha}{2} a(t)^{\frac{1-\alpha}{2}} A^{\frac{1-\alpha}{2}} + I) u + \cos\frac{\pi\alpha}{2} a(t)^{-\alpha/2} A^{-\alpha/2} v\|_X,
\end{aligned}
$$

and by the triangle inequality and by the fact that $\|\cdot\|_{X^{1/2}} = \|A^{1/2} \cdot\|_X$, we obtain

$$
\begin{aligned}
&\|(\Lambda(t)^{-\alpha} - \Lambda(t)^{-1}) \begin{bmatrix} u \\ v \end{bmatrix}\|_{X^{1/2} \times X} \\
&\leqslant \|\cos\frac{\pi\alpha}{2} a(t)^{-\alpha/2} A^{-\alpha/2} u\|_{X^{1/2}} + \|(\sin\frac{\pi\alpha}{2} a(t)^{\frac{-1-\alpha}{2}} A^{\frac{-1-\alpha}{2}} - a(t)^{-1} A^{-1}) v\|_{X^{1/2}} \\
&\quad + \|(-\sin\frac{\pi\alpha}{2} a(t)^{\frac{1-\alpha}{2}} A^{\frac{1-\alpha}{2}} + I) u\|_X + \|\cos\frac{\pi\alpha}{2} a(t)^{-\alpha/2} A^{-\alpha/2} v\|_X \\
&= \|\cos\frac{\pi\alpha}{2} a(t)^{-\alpha/2} A^{-\alpha/2} u\|_{X^{1/2}} + \|(-\sin\frac{\pi\alpha}{2} a(t)^{\frac{-1-\alpha}{2}} A^{-\alpha/2} + a(t)^{-1} A^{-1/2}) v\|_X \\
&\quad + \|(-\sin\frac{\pi\alpha}{2} a(t)^{\frac{1-\alpha}{2}} A^{-\alpha/2} + A^{-1/2}) u\|_{X^{1/2}} + \|\cos\frac{\pi\alpha}{2} a(t)^{-\alpha/2} A^{-\alpha/2} v\|_X.
\end{aligned}
$$

Since $u = A^{-1/2}\bar{u}$ for some $\bar{u} \in X$, it follows that

$$
\begin{aligned}
&\|(\Lambda(t)^{-\alpha} - \Lambda(t)^{-1}) \begin{bmatrix} u \\ v \end{bmatrix}\|_{X^{1/2} \times X} \\
&\leqslant \|\cos\frac{\pi\alpha}{2} a(t)^{-\alpha/2} A^{-\alpha/2} \bar{u}\|_X + \|(-\sin\frac{\pi\alpha}{2} a(t)^{\frac{-1-\alpha}{2}} A^{-\alpha/2} + a(t)^{-1} A^{-1/2}) v\|_X \\
&\quad + \|(-\sin\frac{\pi\alpha}{2} a(t)^{\frac{1-\alpha}{2}} A^{-\alpha/2} + A^{-1/2}) \bar{u}\|_X + \|\cos\frac{\pi\alpha}{2} a(t)^{-\alpha/2} A^{-\alpha/2} v\|_X.
\end{aligned}
\tag{3.3}
$$

Now we recall that the fractional powers of the Laplacian can to be calculated through the spectral decomposition: since $X = L^2(\Omega)$ is a Hilbert space and $A = -\Delta_D$ with zero Dirichlet boundary condition in Ω is a self-adjoint operator and is the infinitesimal generator of a C_0-semigroup of contractions on X, it follows that there exists an orthonormal basis composed by eigenfunctions $\{\varphi_n, n \geqslant 1\}$ of A. Let μ_n be the eigenvalues of $A = -\Delta_D$ in Ω, then $(\mu_n^\alpha, \varphi_n)$ are the eigenvalues and eigenfunctions of $A^\alpha = (-\Delta_D)^\alpha$, also with zero Dirichlet boundary condition.

The fractional Laplacian $A^\alpha : D(A^\alpha) \subset X \to X$ is well defined in the space

$$D(A^\alpha) = \big\{u = \sum_{n=1}^{\infty} a_n\varphi_n \in L^2(\Omega) : \sum_{n=1}^{\infty} a_n^2\mu_n^\alpha < \infty\big\},$$

where

$$A^\alpha u = \sum_{n=1}^{\infty} \mu_n^\alpha a_n \varphi_n, \quad u \in D(A^\alpha) = X^\alpha.$$

Note that $\|u\|_{X^\alpha} = \|A^\alpha u\|_X$.

To study $\cos\frac{\pi\alpha}{2} a(t)^{-\alpha/2}A^{-\alpha/2}w$, $(-\sin\frac{\pi\alpha}{2}a(t)^{\frac{-1-\alpha}{2}}A^{-\alpha/2} + a(t)^{-1}A^{-1/2})w$ and $(-\sin\frac{\pi\alpha}{2}a(t)^{\frac{1-\alpha}{2}}A^{-\alpha/2} + A^{-1/2})w$ for $w \in X$. Let us denote $w = \sum a_n\varphi_n$, then

(i) The norm $\|\cos\frac{\pi\alpha}{2}a(t)^{-\alpha/2}A^{-\alpha/2}w\|_X$ satisfies

$$\|\cos\frac{\pi\alpha}{2}a(t)^{-\alpha/2}A^{-\alpha/2}w\|_X \leqslant \cos\frac{\pi\alpha}{2}a_{\min}^{-\alpha/2}\max_n|\mu_n|^{-\alpha/2}\|w\|_X, \tag{3.4}$$

(ii) The norm $\|(-\sin\frac{\pi\alpha}{2}a(t)^{\frac{-1-\alpha}{2}}A^{-\alpha/2} + a(t)^{-1}A^{-1/2})w\|_X$ satisfies

$$\begin{aligned}
&\|(-\sin\frac{\pi\alpha}{2}a(t)^{\frac{-1-\alpha}{2}}A^{-\alpha/2} + a(t)^{-1}A^{-1/2})w\|_X \\
&\leqslant \max_n|-\sin\frac{\pi\alpha}{2}a(t)^{\frac{-1-\alpha}{2}}\mu_n^{-\alpha/2} + a(t)^{-1}\mu_n^{-1/2}|\|w\|_X \\
&\leqslant a_{\min}^{-1/2}\max_n|-\sin\frac{\pi\alpha}{2}a(t)^{-\alpha/2}\mu_n^{-\alpha/2} + a(t)^{-1/2}\mu_n^{-1/2}|\|w\|_X.
\end{aligned}\tag{3.5}$$

(iii) The norm $\|(-\sin\frac{\pi\alpha}{2}a(t)^{\frac{1-\alpha}{2}}A^{-\alpha/2} + A^{-1/2})w\|_X$ satisfies

$$\begin{aligned}
&\|(-\sin\frac{\pi\alpha}{2}a(t)^{\frac{1-\alpha}{2}}A^{-\alpha/2} + A^{-1/2})w\|_X \\
&\leqslant \max_n|-\sin\frac{\pi\alpha}{2}a(t)^{\frac{1-\alpha}{2}}\mu_n^{-\alpha/2} + \mu_n^{-1/2}|\|w\|_X \\
&\leqslant a_{\max}^{1/2}\max_n|-\sin\frac{\pi\alpha}{2}a(t)^{-\alpha/2}\mu_n^{-\alpha/2} + a(t)^{-1/2}\mu_n^{-1/2}|\|w\|_X.
\end{aligned}\tag{3.6}$$

From (3.4) we have

$$\|\cos\frac{\pi\alpha}{2}a(t)^{-\alpha/2}A^{-\alpha/2}w\|_X \leqslant C_1(1-\alpha)\|w\|_X,$$

for some constant $C_1 > 0$ independent of α.

Using (3.5) we obtain positive constants C_2 and C_3 independents of α such that

$$\|(-\sin\frac{\pi\alpha}{2}a(t)^{\frac{-1-\alpha}{2}}A^{-\alpha/2} + a(t)^{-1}A^{-1/2})w\|_X \leqslant C_2(1-\alpha)|\|w\|_X,$$

and from (3.6) we obtain

$$\|(-\sin\frac{\pi\alpha}{2}a(t)^{\frac{1-\alpha}{2}}A^{-\alpha/2} + A^{-1/2})w\|_X \leqslant C_3(1-\alpha)|\|w\|_X,$$

from this and (3.3) we conclude that the operators $\Lambda(t)^{-\alpha}$ converges in the uniform topology operators (of $L(X^{1/2}\times X)$) to $\Lambda(t)^{-1}$ as $\alpha \nearrow 1$ with rate $1-\alpha$, for any $t \in \mathbb{R}$. $\square$

3.2. **Proof of main theorem.** In the following discussion we will develop properties of Hölder continuity for the function $a^{\alpha/2}(\cdot)$ for $\alpha \in (0,1]$. Our main objective in this section is to prove the Theorem 1.1.

Lemma 3.4. *Let $I \subset \mathbb{R}$ be an interval with nonempty interior (i.e., I has endpoints p_1, p_2 with $p_1 < p_2$). We recall that a function $f : I \to \mathbb{R}$ is called (α, C)-Hölder continuous with exponent α and constant $C > 0$ if there exist real constants $0 < \alpha \leqslant 1$ and $C > 0$ such that*

$$|f(x) - f(y)| \leqslant C|x - y|^\alpha, \quad \text{for all } x, y \in I.$$

For any bounded interval $I \subset \mathbb{R}$, we have

$$\mathcal{C}^{0,\beta}(I) \supset \mathcal{C}^{0,\alpha}(I), \quad \text{for } 0 < \beta \leqslant \alpha \leqslant 1.$$

More specifically, if I has length $\ell < \infty$ and f is (α, C)-Hölder continuous, then f is $(\beta, \ell^{\alpha-\beta} C)$-Hölder continuous.

Proof. Using that f is (β, C)-Hölder continuous, then for all $x, y \in I$, we have

$$|f(x) - f(y)| \leqslant C|x - y|^\beta,$$

and if I has length $\ell < \infty$

$$|f(x) - f(y)| \leqslant C|x - y|^{\beta-\alpha}|y - x|^\alpha \leqslant C\ell^{\beta-\alpha}|x - y|^\alpha,$$

that is f is $(\alpha, \ell^{\beta-\alpha} C)$-Hölder continuous. □

Lemma 3.5. *Let $0 < \alpha \leqslant 1$. The function $[0, \infty) \ni x \mapsto x^\alpha \in \mathbb{R}$ is $(\alpha, 1)$-Hölder continuous. Moreover, if $I \subset [0, \infty)$ is a bounded interval with length $\ell < \infty$, then the function $I \ni x \mapsto x^\alpha \in \mathbb{R}$ is $(\beta, \ell^{\alpha-\beta})$-Hölder continuous.*

Proof. Initially we note that the function $[0, \infty) \ni x \mapsto x^\alpha \in \mathbb{R}$ is subadditive, namely

$$(x + y)^\alpha \leqslant x^\alpha + y^\alpha, \quad \text{for all } x, y \geqslant 0.$$

It is also monotonically increasing. From this, we obtain

$$y^\alpha = (y - x + x)^\alpha \leqslant (y - x)^\alpha + x^\alpha,$$

and this implies

$$0 \leqslant y^\alpha - x^\alpha \leqslant (y - x)^\alpha,$$

whenever $0 \leqslant x \leqslant y$.

Now consider arbitrary nonnegative x and y. If either of them is zero, there is nothing to prove. Otherwise, we may suppose that $x \leqslant y$ (if not, interchange x and y). Then, from the above

$$|y^\alpha - x^\alpha| = y^\alpha - x^\alpha \leqslant (y - x)^\alpha = 1 \cdot |y - x|^\alpha.$$

The second part of the proposition is an immediate consequence of the Lemma 3.4. □

Lemma 3.6. *The function $\mathbb{R} \ni t \mapsto a(t)^{\alpha/2} \in \mathbb{R}$ is $(\frac{1}{4}, C)$-Hölder continuous, for all $\alpha \in (\frac{1}{2}, 1]$, where $C = \max\{(a_{\max} - a_{\min})^{1/2}, 1, \kappa^{\frac{1}{4\gamma}}(2a_{\max})^{\frac{1}{2}(1-\frac{1}{2\gamma})}\}$ is independent of α.*

Proof. For $t,\tau\in\mathbb{R}$, using (1.3) we have (for $\gamma\in[1/2,1]$)

$$|a(t)^{1/2}-a(\tau)^{1/2}|\leqslant|a(t)-a(\tau)|^{1/2}=|a(t)-a(\tau)|^{\frac{1}{2}(1-\theta)}|a(t)-a(\tau)|^{\frac{1}{2}\theta},$$

for any $\theta\in[0,1]$. From this and using (1.2) we obtain

$$|a(t)^{1/2}-a(\tau)^{1/2}|\leqslant C_\gamma|a(t)-a(\tau)|^{\frac{1}{2}\theta}, \tag{3.7}$$

where $C_\gamma=(2a_{\max})^{\frac{1}{2}(1-\theta)}$, for all $\theta\in[0,1]$.

For $\alpha\in(1/2,1)$, it follows from Lemma 3.4 that

$$[a_{\min},a_{\max}]\ni x\mapsto x^{\alpha/2}\in\mathbb{R}$$

is $(\beta,(a_{\max}-a_{\min})^{\frac{\alpha}{2}-\beta})$-Hölder continuous, for any $0<\beta\leqslant\alpha/2<1/2$. Take $\beta=1/4$. Then, for all $t,\tau\in\mathbb{R}$

$$|a(t)^{\alpha/2}-a(\tau)^{\alpha/2}|\leqslant(a_{\max}-a_{\min})^{\frac{\alpha}{2}-\frac{1}{4}}|t-\tau|^{1/4}\leqslant C_0|t-\tau|^{1/4}, \tag{3.8}$$

where $C_0=\max\{(a_{\max}-a_{\min})^{1/2},1\}$ is independent of α. Finally, choose

$$\theta=\frac{1}{2\gamma},$$

where γ is given by (1.3). Since $\gamma\in[\frac{1}{2},1]$ we obtain $\theta\in[0,1]$. From this and using (3.7) we conclude that

$$|a(t)^{1/2}-a(\tau)^{1/2}|\leqslant C_\gamma\kappa^{\frac{1}{4\gamma}}|t-\tau|^{1/4}. \tag{3.9}$$

Thus, it follows from (3.8) and (3.9) that the function $\mathbb{R}\ni t\mapsto a(t)^{\alpha/2}\in\mathbb{R}$ is $(1/4,C)$-Hölder continuous, for all $\alpha\in(\frac{1}{2},1]$, where $C=\max\{C_0,\kappa^{\frac{1}{4\gamma}}C_\gamma\}$. □

Proof of the Theorem 1.1. From (1.2) and sectoriallity of the operators $A(t)$ it follows that $\Lambda(t)^\alpha$ is uniformly sectorial (in Y); that is, there is a constant $C>0$ (independent of t) such that

$$\|(\lambda I+\Lambda(t)^\alpha)^{-1}\|_{L(Y)}\leqslant\frac{C}{|\lambda|+1},\quad\text{for all }\lambda\in\mathbb{C}\text{ with }\operatorname{Re}\lambda\geqslant0.$$

Also, it is not difficult to see that for any $t,\tau,s\in\mathbb{R}$,

$$[\Lambda(t)^\alpha-\Lambda(\tau)^\alpha]\Lambda(s)^{-\alpha}=\begin{bmatrix}E_{11}&E_{12}\\E_{21}&E_{22}\end{bmatrix},$$

where

$$E_{11}=\cos^2\frac{\pi\alpha}{2}[a(t)^{\alpha/2}-a(\tau)^{\alpha/2}]a(s)^{-\alpha/2}+\sin^2\frac{\pi\alpha}{2}[a(t)^{\frac{-1+\alpha}{2}}-a(\tau)^{\frac{-1+\alpha}{2}}]a(s)^{\frac{1-\alpha}{2}},$$

$$\begin{aligned}E_{12}&=\cos\frac{\pi\alpha}{2}\sin\frac{\pi\alpha}{2}[a(t)^{\alpha/2}-a(\tau)^{\alpha/2}]a(s)^{\frac{-1-\alpha}{2}}A^{-1/2}\\&\quad+\cos\frac{\pi\alpha}{2}\sin\frac{\pi\alpha}{2}[a(t)^{\frac{-1+\alpha}{2}}-a(\tau)^{\frac{-1+\alpha}{2}}]a(s)^{-\alpha/2}A^{-1/2},\\E_{21}&=\cos\frac{\pi\alpha}{2}\sin\frac{\pi\alpha}{2}[a(t)^{\frac{1+\alpha}{2}}-a(\tau)^{\frac{1+\alpha}{2}}]a(s)^{-\alpha/2}A^{1/2}\\&\quad-\cos\frac{\pi\alpha}{2}\sin\frac{\pi\alpha}{2}[a(t)^{\alpha/2}-a(\tau)^{\alpha/2}]a(s)^{-\alpha/2}A^{1/2},\end{aligned}$$

$$E_{22}=\sin^2\frac{\pi\alpha}{2}[a(t)^{\frac{1+\alpha}{2}}-a(\tau)^{\frac{1+\alpha}{2}}]a(s)^{\frac{-1-\alpha}{2}}+\cos^2\frac{\pi\alpha}{2}[a(t)^{\alpha/2}-a(\tau)^{\alpha/2}]a(s)^{-\alpha/2}.$$

Since the function a is bounded below by a positive constant (see (1.2)), from Lemma 3.6 we obtain that the map $\mathbb{R} \ni t \mapsto \Lambda(t)^\alpha$ is uniformly Hölder continuous in $X^{1/2} \times X$. Indeed,

$$\Big\|[\Lambda(t)^\alpha - \Lambda(\tau)^\alpha]\Lambda(s)^{-\alpha} \begin{bmatrix} u \\ v \end{bmatrix} \Big\|_{X^{1/2}\times X} = \Big\| \begin{bmatrix} E_{11}u + E_{12}v \\ E_{21}u + E_{22}v \end{bmatrix} \Big\|_{X^{1/2}\times X}, \tag{3.10}$$

where

$$\Big\| \begin{bmatrix} E_{11}u + E_{12}v \\ E_{21}u + E_{22}v \end{bmatrix} \Big\|_{X^{1/2}\times X} = \|A^{1/2}[E_{11}u + E_{12}v]\|_X + \|E_{21}u + E_{22}v\|_X.$$

In the following discussion we develop estimates for $\|A^{1/2}[E_{11}u + E_{12}v]\|_X$ and $\|E_{21}u + E_{22}v\|_X$ with the functions $a^{\alpha/2}(t)$, $\alpha \in (0,1)$. Note that

$$\begin{aligned} &\|E_{11}A^{1/2}u\|_X \\ &\leqslant \cos^2\frac{\pi\alpha}{2}|a(t)^{\alpha/2} - a(\tau)^{\alpha/2}|a(s)^{-\alpha/2}\|u\|_{X^{1/2}} \\ &\quad + \sin^2\frac{\pi\alpha}{2}|a(t)^{\frac{-1+\alpha}{2}} - a(\tau)^{\frac{-1+\alpha}{2}}|a(s)^{\frac{1-\alpha}{2}}\|u\|_{X^{1/2}} \\ &\leqslant \max\{1, a_{\max}^{1/2}\}a_{\min}^{-\alpha/2}\big[|a(t)^{\alpha/2} - a(\tau)^{\alpha/2}| + |a(t)^{\frac{-1+\alpha}{2}} - a(\tau)^{\frac{-1+\alpha}{2}}|\big]\|u\|_{X^{1/2}}, \end{aligned} \tag{3.11}$$

$$\begin{aligned} &\|E_{12}A^{1/2}v\|_X \\ &\leqslant \cos\frac{\pi\alpha}{2}\sin\frac{\pi\alpha}{2}|a(t)^{\alpha/2} - a(\tau)^{\alpha/2}|a(s)^{\frac{-1-\alpha}{2}}\|v\|_X \\ &+ \cos\frac{\pi\alpha}{2}\sin\frac{\pi\alpha}{2}|a(t)^{\frac{-1+\alpha}{2}} - a(\tau)^{\frac{-1+\alpha}{2}}|a(s)^{-\alpha/2}\|v\|_X \\ &\leqslant \max\{1, a_{\max}^{-1/2}\}a_{\min}^{-\alpha/2}\big[|a(t)^{\alpha/2} - a(\tau)^{\alpha/2}| + |a(t)^{\frac{-1+\alpha}{2}} - a(\tau)^{\frac{-1+\alpha}{2}}|\big]\|v\|_X, \end{aligned} \tag{3.12}$$

$$\begin{aligned} \|E_{21}u\|_X &\leqslant \cos\frac{\pi\alpha}{2}\sin\frac{\pi\alpha}{2}|a(t)^{\frac{1+\alpha}{2}} - a(\tau)^{\frac{1+\alpha}{2}}|a(s)^{-\alpha/2}\|u\|_{X^{1/2}} \\ &\quad + \cos\frac{\pi\alpha}{2}\sin\frac{\pi\alpha}{2}|a(t)^{\alpha/2} - a(\tau)^{\alpha/2}|a(s)^{-\alpha/2}\|u\|_{X^{1/2}} \\ &\leqslant a_{\min}^{-\alpha/2}\Big[|a(t)^{\frac{1+\alpha}{2}} - a(\tau)^{\frac{1+\alpha}{2}}| + |a(t)^{\alpha/2} - a(\tau)^{\alpha/2}|\Big]\|u\|_{X^{1/2}}, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &\|E_{22}v\|_X \\ &\leqslant \sin^2\frac{\pi\alpha}{2}|a(t)^{\frac{1+\alpha}{2}} - a(\tau)^{\frac{1+\alpha}{2}}|a(s)^{\frac{-1-\alpha}{2}}\|v\|_X \\ &\quad + \cos^2\frac{\pi\alpha}{2}|a(t)^{\alpha/2} - a(\tau)^{\alpha/2}|a(s)^{-\alpha/2}\|v\|_X \\ &\leqslant \max\{1, a_{\min}^{-1/2}\}a_{\min}^{-\alpha/2}\big[|a(t)^{\frac{1+\alpha}{2}} - a(\tau)^{\frac{1+\alpha}{2}}| + |a(t)^{\alpha/2} - a(\tau)^{\alpha/2}|\big]\|v\|_X. \end{aligned} \tag{3.14}$$

Hence, we can estimate $\|A^{1/2}[E_{11}u + E_{12}v]\|_X$ using (3.11) with (3.12) as follows

$$\begin{aligned} \|A^{1/2}[E_{11}u + E_{12}v]\|_X &\leqslant 2a_{\min}^{-\alpha/2}\max\{a_{\max}^{-1/2}, a_{\max}^{1/2}\}\big[|a(t)^{\alpha/2} - a(\tau)^{\alpha/2}| \\ &\quad + |a(t)^{\frac{-1+\alpha}{2}} - a(\tau)^{\frac{-1+\alpha}{2}}|\big]\Big\|\begin{bmatrix} u \\ v \end{bmatrix}\Big\|_{X^{1/2}\times X}. \end{aligned}$$

Since for all $t, \tau \in \mathbb{R}$,

$$\begin{aligned} &|a(t)^{\frac{-1+\alpha}{2}} - a(\tau)^{\frac{-1+\alpha}{2}}| \\ &\leqslant a(t)^{-1/2}|a(t)^{\alpha/2} - a(\tau)^{\alpha/2}| + a(\tau)^{\alpha/2}|a(t)^{-1/2} - a(\tau)^{-1/2}| \end{aligned}$$

$$
\begin{aligned}
&\leqslant a_{\min}^{-1/2}|a(t)^{\alpha/2}-a(\tau)^{\alpha/2}|+\max\{1,a_{\max}^{1/2}\}|a(t)^{-1/2}-a(\tau)^{-1/2}|\\
&\leqslant a_{\min}^{-1/2}|a(t)^{\alpha/2}-a(\tau)^{\alpha/2}|+\max\{1,a_{\max}^{1/2}\}|a(t)^{1/2}-a(\tau)^{1/2}|,
\end{aligned}
$$

it follows that there exists a positive constant C' dependent of $a_{\min}$ and $a_{\max}$, but independent of α, such that

$$
\begin{aligned}
&\|A^{1/2}[E_{11}u+E_{12}v]\|_X\\
&\leqslant C'\big[|a(t)^{\alpha/2}-a(\tau)^{\alpha/2}|+|a(t)^{1/2}-a(\tau)^{1/2}|\big]\Big\|\begin{bmatrix}u\\ v\end{bmatrix}\Big\|_{X^{1/2}\times X}.
\end{aligned}
\tag{3.15}
$$

Also we can estimate $\|E_{21}u+E_{22}v\|_X$ using (3.13) and (3.14) as follows

$$
\begin{aligned}
&\|E_{21}u+E_{22}v\|_X\\
&\leqslant 2a_{\min}^{-\alpha/2}\max\{1,a_{\min}^{-1/2}\}\big[|a(t)^{\frac{1+\alpha}{2}}-a(\tau)^{\frac{1+\alpha}{2}}|+|a(t)^{\alpha/2}-a(\tau)^{\alpha/2}|\big]\Big\|\begin{bmatrix}u\\ v\end{bmatrix}\Big\|_{X^{1/2}\times X}.
\end{aligned}
$$

Since for all $t,\tau\in\mathbb{R}$,

$$
\begin{aligned}
|a(t)^{\frac{1+\alpha}{2}}-a(\tau)^{\frac{1+\alpha}{2}}|&\leqslant a(t)^{1/2}|a(t)^{\alpha/2}-a(\tau)^{\alpha/2}|+a(\tau)^{\alpha/2}|a(t)^{1/2}-a(\tau)^{1/2}|\\
&\leqslant a_{\max}^{1/2}|a(t)^{\alpha/2}-a(\tau)^{\alpha/2}|+\max\{1,a_{\max}^{1/2}\}|a(t)^{1/2}-a(\tau)^{1/2}|,
\end{aligned}
$$

it follows that there exists a positive constant C'' dependent of $a_{\min}$ and $a_{\max}$, but independent of α, such that

$$
\begin{aligned}
&\|E_{21}u+E_{22}v\|_X\\
&\leqslant C''\big[|a(t)^{\alpha/2}-a(\tau)^{\alpha/2}|+|a(t)^{1/2}-a(\tau)^{1/2}|\big]\Big\|\begin{bmatrix}u\\ v\end{bmatrix}\Big\|_{X^{1/2}\times X}.
\end{aligned}
\tag{3.16}
$$

Combining (3.10), (3.15) and (3.16) we concluded that there exists a positive constant $C'''=2(C'+C'')$ such that

$$
\begin{aligned}
&\Big\|[\Lambda(t)^{\alpha}-\Lambda(\tau)^{\alpha}]\Lambda(s)^{-\alpha}\begin{bmatrix}u\\ v\end{bmatrix}\Big\|_{X^{1/2}\times X}\\
&\leqslant C'''\big[|a(t)^{\alpha/2}-a(\tau)^{\alpha/2}|+|a(t)^{1/2}-a(\tau)^{1/2}|\big]\Big\|\begin{bmatrix}u\\ v\end{bmatrix}\Big\|_{X^{1/2}\times X}.
\end{aligned}
$$

From Lemma 3.6 there exists a positive constant C dependent of $a_{\min},a_{\max},\kappa$, but independent of α such that

$$
\begin{aligned}
&\Big\|[\Lambda(t)^{\alpha}-\Lambda(\tau)^{\alpha}]\Lambda(s)^{-\alpha}\begin{bmatrix}u\\ v\end{bmatrix}\Big\|_{X^{1/2}\times X}\\
&\leqslant C'''\big[|a(t)^{\alpha/2}-a(\tau)^{\alpha/2}|+|a(t)^{1/2}-a(\tau)^{1/2}|\big]\Big\|\begin{bmatrix}u\\ v\end{bmatrix}\Big\|_{X^{1/2}\times X}\\
&\leqslant C|t-\tau|^{1/4}\Big\|\begin{bmatrix}u\\ v\end{bmatrix}\Big\|_{X^{1/2}\times X},
\end{aligned}
$$

for any $t,\tau\in\mathbb{R}$. From this the proof of the Theorem 1.1(i) it follows from [13, 26].

For $\alpha=1$ it is easily seen that $0\in\rho(\Lambda(t))$ for any $t\in\mathbb{R}$, and its inverse is given by

$$
\Lambda(t)^{-1}=\begin{bmatrix}0 & a(t)^{-1}A^{-1}\\ -I & 0\end{bmatrix},\quad \text{for all } t\in\mathbb{R}.
$$

Observe that, the adjoint of $\Lambda(t)$, is

$$\Lambda(t)^* = \begin{bmatrix} 0 & I \\ -a(t)A & 0 \end{bmatrix}, \quad \text{for all } t \in \mathbb{R},$$

and $\Lambda(t) = -\Lambda(t)^*$ for all $t \in \mathbb{R}$; that is, for every $t \in \mathbb{R}$ the operator $\Lambda(t)$ is skew-adjoint. It follows that $i\Lambda(t)$ is self-adjoint and, from Stone's theorem, $\Lambda(t)$ is the infinitesimal generator of a C_0-group of unitary operators on $X^{1/2} \times X$ (see Pazy [25, Theorem 1.10.8, pg. 41]).

For $\alpha \in (0,1)$ it follows from results of [13, Subsection 2.1.2] and [26] that (1.8) is valid. From the above analysis we have proved the Theorem 1.1(ii).

Finally, let us consider the linear problem (1.6). Thanks to boundedness of η, it is easy to see that $F(t,\cdot) : X^{\frac{1+\alpha}{2}} \times X^{\alpha/2} \to X^{1/2} \times X$ is Lipschitz continuous in bounded subsets of $X^{\frac{1+\alpha}{2}} \times X^{\alpha/2}$ and by the [9, Theorem 2.3] (see also [13, Theorem 1.1 and Theorem 3.1] for a more general version that includes de critical growth case) we have proved the Theorem 1.1(iii). □

3.3. **Energy functionals associated with perturbed problems.** In this subsection, we will consider the function a equal to 1 and η be a decreasing function on $\mathbb{R}$ in (1.1). Let $\begin{bmatrix} u^\alpha(t) \\ v^\alpha(t) \end{bmatrix}$ be the local solution of (1.6). In this case $\begin{bmatrix} u^\alpha(t) \\ v^\alpha(t) \end{bmatrix}$ satisfies the following system

$$\begin{aligned} u_t^\alpha + \cos\frac{\pi\alpha}{2} A^{\alpha/2} u^\alpha - \sin\frac{\pi\alpha}{2} A^{\frac{-1+\alpha}{2}} v^\alpha = 0 \\ v_t^\alpha + \sin\frac{\pi\alpha}{2} A^{\frac{1+\alpha}{2}} u^\alpha + \cos\frac{\pi\alpha}{2} A^{\alpha/2} v^\alpha + \eta(t) v^\alpha = 0. \end{aligned} \tag{3.17}$$

From the first equation we obtain

$$\sin\frac{\pi\alpha}{2} v^\alpha = A^{\frac{1-\alpha}{2}} \Big(u_t^\alpha + \cos\frac{\pi\alpha}{2} A^{\alpha/2} u^\alpha \Big)$$

and then

$$\sin\frac{\pi\alpha}{2} v_t^\alpha = A^{\frac{1-\alpha}{2}} \Big(u_{tt}^\alpha + \cos\frac{\pi\alpha}{2} A^{\alpha/2} u_t^\alpha \Big). \tag{3.18}$$

It is not difficult to see (second equation of (3.17)) that

$$\sin\frac{\pi\alpha}{2} v_t^\alpha + \sin^2\frac{\pi\alpha}{2} A^{\frac{1+\alpha}{2}} u^\alpha + \sin\frac{\pi\alpha}{2}\cos\frac{\pi\alpha}{2} A^{\alpha/2} v^\alpha + \eta(t)\sin\frac{\pi\alpha}{2} v^\alpha = 0. \tag{3.19}$$

Combining (3.18) with (3.19), we obtain

$$A^{\frac{1-\alpha}{2}} u_{tt}^\alpha + 2\cos\frac{\pi\alpha}{2} A^{1/2} u_t^\alpha + A^{\frac{1+\alpha}{2}} u^\alpha + \eta(t) A^{\frac{1-\alpha}{2}} u_t^\alpha + \eta(t)\cos\frac{\pi\alpha}{2} A^{1/2} u^\alpha = 0. \tag{3.20}$$

Multiplying (3.20) by u_t^α and integrating, we obtain a function $\mathcal{V}_\alpha$ given by

$$\mathcal{V}_\alpha(u^\alpha, u_t^\alpha) = \frac{1}{2}\|u_t^\alpha\|^2_{X^{\frac{1-\alpha}{4}}} + \frac{1}{2}\|u^\alpha\|^2_{X^{\frac{1+\alpha}{4}}} + \frac{\eta(t)}{2}\cos\frac{\pi\alpha}{2}\|u^\alpha\|^2_{X^{1/4}}.$$

This function satisfies the differential equation

$$\frac{d}{dt}\left(\mathcal{V}_\alpha(u^\alpha, u_t^\alpha)\right) = -2\cos\frac{\pi\alpha}{2}\|u_t^\alpha\|^2_{X^{1/4}} - \eta(t)\|u_t^\alpha\|^2_{X^{\frac{1-\alpha}{4}}} + \eta'(t)\cos\frac{\pi\alpha}{2}\|u^\alpha\|^2_{X^{1/4}}.$$

Since $u_t^\alpha = \sin\frac{\pi\alpha}{2} A^{\frac{-1+\alpha}{2}} v^\alpha - \cos\frac{\pi\alpha}{2} A^{\alpha/2} u^\alpha$ (see (3.17)), we can consider a functional $\mathcal{L}_\alpha : X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}} \to \mathbb{R}$, given by

$$\mathcal{L}_\alpha\left(\begin{bmatrix} w \\ z \end{bmatrix}\right)$$

$$= \frac{1}{2}\|w\|^2_{X^{\frac{1+\alpha}{4}}} + \frac{1}{2}\|\sin\frac{\pi\alpha}{2}A^{\frac{-1+\alpha}{2}}z - \cos\frac{\pi\alpha}{2}A^{\alpha/2}w\|^2_{X^{\frac{1-\alpha}{4}}} + \frac{\eta(t)}{2}\cos\frac{\pi\alpha}{2}\|w\|^2_{X^{1/4}}$$
$$= \frac{1}{2}\|w\|^2_{X^{\frac{1+\alpha}{4}}} + \frac{1}{2}\|\sin\frac{\pi\alpha}{2}A^{\frac{-1+\alpha}{4}}z - \cos\frac{\pi\alpha}{2}A^{\frac{1+\alpha}{4}}w\|^2_{X} + \frac{\eta(t)}{2}\cos\frac{\pi\alpha}{2}\|w\|^2_{X^{1/4}}.$$

Remark 3.7. For positive times and as long as the solutions exist we have
$$\mathcal{V}_\alpha(u^\alpha, u^\alpha_t) = \mathcal{L}_\alpha\Big(\begin{bmatrix} u^\alpha \\ u^\alpha_t \end{bmatrix}\Big)$$
and hence since $\eta'(t) \leqslant 0$ for all t (since η is decreasing) it follows that
$$\frac{d}{dt}\Big(\mathcal{V}_\alpha(u^\alpha, u^\alpha_t)\Big) = \frac{d}{dt}\Big(\mathcal{L}_\alpha\Big(\begin{bmatrix} u^\alpha \\ u^\alpha_t \end{bmatrix}\Big)\Big) \leqslant 0.$$

Remark 3.8. For $\mathbb{A} = A^\alpha$, we can rewrite (3.20) as
$$u^\alpha_{tt} + \eta(t)\cos\frac{\pi\alpha}{2}\mathbb{A}^{1/2}u^\alpha + \mathbb{A}u^\alpha + \eta(t)u^\alpha_t + 2\cos\frac{\pi\alpha}{2}\mathbb{A}^{1/2}u^\alpha_t = 0.$$
this latter equation can be viewed as an fractional approximation of the PDE in (1.1) (see Bezerra, Carvalho, Cholewa and Nascimento [3], Caraballo, Carvalho, Langa and Rivero [9, 10], Carvalho, Cholewa and Dłotko [11], Carvalho, Langa and Robinson [12], Chen and Russell [15], Chen and Triggiani [16, 17, 18, 19], Sun, Cao and Duan [27] and references therein for the extensive studies of the strongly damped wave equations).

4. Spectral properties

In this section we will study the spectral properties of the operators $-\Lambda(t)^\alpha$. We characterize the functions $\lambda_\alpha = \lambda_\alpha(t)$ such that
$$-\Lambda(t)^\alpha \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \lambda_\alpha(t)\begin{bmatrix} \varphi \\ \psi \end{bmatrix}, \tag{4.1}$$
for some not null vector $\begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in D(\Lambda(t)^\alpha)$, in terms of the eigenvalues of A. Moreover, we will prove the convergence with rate of the functions λ_α at $\alpha = 1$.

Let $\sigma(-\Lambda(t)^\alpha)$ be the spectrum of $-\Lambda(t)^\alpha$ for any $\alpha \in [0,1]$, the next result shows the convergence of the spectrum of $-\Lambda(t)^\alpha$ at $\alpha = 1$. The proof follows the same ideas of the [5, Lemmas 2.3 and 2.6], and we will omit the proof.

Proposition 4.1. *The following statements hold:*

(i) *If $\mu_0 \in \sigma(-\Lambda(t))$, then exists a sequence $\alpha_n \to 1$ and $\{\mu_n\}$, with $\mu_n \in \sigma(-\Lambda(t)^{\alpha_n})$, $n \in \mathbb{N}$ such that $\mu_n \to \mu_0$ as $n \to \infty$;*

(ii) *If for some sequences $\alpha_n \to 1$ and $\mu_n \to \mu_0$ as $n \to \infty$, with $\mu_n \in \sigma(-\Lambda(t)^{\alpha_n})$, $n \in \mathbb{N}$, then $\mu_0 \in \sigma(-\Lambda(t))$.*

Lemma 4.2. *If $\lambda \in \rho(-\Lambda(t)) \cap \rho(-\Lambda(t)^\alpha)$, then*
$$\begin{aligned}&(\lambda I + \Lambda(t)^\alpha)^{-1} - (\lambda I + \Lambda(t))^{-1}\\ &= \Lambda(t)^\alpha(\lambda I + \Lambda(t)^\alpha)^{-1}[\Lambda(t)^{-\alpha} - \Lambda(t)^{-1}]\Lambda(t)(\lambda I + \Lambda(t))^{-1},\end{aligned} \tag{4.2}$$
and
$$\begin{aligned}&\Lambda(t)^\alpha(\lambda I + \Lambda(t)^\alpha)^{-1} - \Lambda(t)(\lambda I + \Lambda(t))^{-1}\\ &= (\lambda I + \Lambda(t)^\alpha)^{-1}\Lambda(t)[\Lambda(t)^{-1} - \Lambda(t)^{-\alpha}]\lambda\Lambda(t)^\alpha(\lambda I + \Lambda(t))^{-1}.\end{aligned} \tag{4.3}$$

Proposition 4.3. *Let $\alpha \in (1/2, 1)$ and $\lambda \in \rho(-\Lambda(t)) \cap \rho(-\Lambda(t)^\alpha)$. There exists a constant $M > 0$ such that*

$$\|(\lambda I + \Lambda(t)^\alpha)^{-1} - (\lambda I + \Lambda(t))^{-1}\|_{\mathcal{L}(X^{1/2} \times X)} \leqslant M(1-\alpha), \tag{4.4}$$

where

$$M = C\|\Lambda(t)(\lambda I + \Lambda(t))^{-1}\|_{\mathcal{L}(X^{1/2} \times X)} \sup_{\alpha \in (0,1)} \|\Lambda(t)^\alpha(\lambda I + \Lambda(t)^\alpha)^{-1}\|_{\mathcal{L}(X^{1/2} \times X)}$$

and C is given by Theorem 3.3 and it is independent of α.

Proof. Using the Theorem 3.3 we obtain that (4.4) is a direct consequence of (4.2). $\square$

Proposition 4.4. *Let $\alpha \in (1/2, 1)$ and $\lambda \in \rho(-\Lambda(t)) \cap \rho(-\Lambda(t)^\alpha)$. There exists a constant $M' > 0$ such that*

$$\|\Lambda(t)^\alpha(\lambda I + \Lambda(t)^\alpha)^{-1} - \Lambda(t)(\lambda I + \Lambda(t))^{-1}\|_{\mathcal{L}(X^{1/2} \times X)} \leqslant M'(1-\alpha)|\lambda|, \tag{4.5}$$

where

$$M' = C\|(\lambda I + \Lambda(t)^\alpha)^{-1}\Lambda(t)\|_{\mathcal{L}(X^{1/2} \times X)} \sup_{\alpha \in (0,1)} \|\Lambda(t)^\alpha(\lambda I + \Lambda(t))^{-1}\|_{\mathcal{L}(X^{1/2} \times X)}$$

and C_α is given by Theorem 3.3 and it is independent of α.

Proof. Using Theorem 3.3 we obtain that (4.5) is a direct consequence of (4.3). $\square$

Let $\mathcal{C}$ be a compact oriented counterclockwise contour in $\rho(-\Lambda(t))$. From Proposition 4.1, we have that $\mathcal{C} \subset \rho(-\Lambda(t)^\alpha)$ for all $\alpha \in [\alpha_{\mathcal{C}}, 1]$, for some $\alpha_{\mathcal{C}} > 0$. Define the spectral projections on X by

$$Q(t)_\alpha(\mu_0) = \frac{1}{2\pi i}\int_{\mathcal{C}} (\lambda I + \Lambda(t)^\alpha)^{-1} d\lambda,$$

for any $\mu_0 \in \mathbb{C}$ such that $\frac{1}{2\pi i}\int_{\mathcal{C}}(\lambda - \mu_0)^{-1} d\lambda = 1$.

Proposition 4.5. *Let $\alpha \in (1/2, 1)$. There exists a constant $M > 0$ such that*

$$\|Q(t)_\alpha(\mu_0) - Q(t)(\mu_0)\|_{\mathcal{L}(X^{1/2} \times X)} \leqslant \delta M(1-\alpha),$$

where

$$M = C\|\Lambda(t)(\lambda I - \Lambda(t))^{-1}\|_{\mathcal{L}(X^{1/2} \times X)} \sup_{\alpha \in (1/2,1)} \|\Lambda(t)^\alpha(\lambda I - \Lambda(t)^\alpha)^{-1}\|_{\mathcal{L}(X^{1/2} \times X)},$$

C is given by Theorem 3.3, and is independent of α, and $\delta > 0$ is such that $\mathcal{C}$ is contained in the ball centered at the origin of radius $\sqrt{2\delta}$, $B(0, \sqrt{2\delta})$.

Proof. The proof it follows from the same arguments used to proof the Proposition 4.3; namely

$$\begin{aligned}
&\|Q(t)_\alpha(\mu_0) - Q(t)(\mu_0)\|_{\mathcal{L}(X^{1/2} \times X)} \\
&\leqslant \frac{1}{2\pi}\int_{\mathcal{C}} \|(\lambda I + \Lambda(t)^\alpha)^{-1} - (\lambda I + \Lambda(t))^{-1}\|_{\mathcal{L}(X^{1/2} \times X)} d\lambda.
\end{aligned}$$

Since

$$\|(\lambda I + \Lambda(t)^\alpha)^{-1} - (\lambda I + \Lambda(t))^{-1}\|_{\mathcal{L}(X^{1/2} \times X)} \leqslant M(1-\alpha),$$

where M is defined in the Proposition 4.3, and $\mathcal{C} \subset B(0, \sqrt{2\delta})$, it follows that

$$\|Q(t)_\alpha(\mu_0) - Q(t)(\mu_0)\|_{\mathcal{L}(X^{1/2} \times X)} \leqslant \delta M(1-\alpha),$$

since $\int_{B(0,\sqrt{2\delta})} d\lambda = 2\delta\pi$. $\square$

Proposition 4.6. *Let $\alpha \in (1/2, 1)$. There exists a constant $C > 0$ independent of α such that*

$$\|\Lambda(t)^\alpha Q(t)_\alpha(\mu_0) - \Lambda(t)Q(t)(\mu_0)\|_{\mathcal{L}(X^{1/2}\times X)} \leqslant CM'(1-\alpha),$$

where M' is defined in the Proposition 4.4 and it is independent of α.

Proof. We have

$$\begin{aligned}
&\|\Lambda(t)^\alpha Q(t)_\alpha(\mu_0) - \Lambda(t)Q(t)(\mu_0)\|_{\mathcal{L}(X^{1/2}\times X)} \\
&\leqslant \frac{1}{2\pi}\int_{\mathcal{C}} \|\Lambda(t)^\alpha(\lambda I + \Lambda(t)^\alpha)^{-1} - \Lambda(t)(\lambda I + \Lambda(t))^{-1}\|_{\mathcal{L}(X^{1/2}\times X)} d\lambda.
\end{aligned}$$

From (4.3) we obtain

$$\|\Lambda(t)^\alpha(\lambda I + \Lambda(t)^\alpha)^{-1} - \Lambda(t)(\lambda I + \Lambda(t))^{-1}\|_{\mathcal{L}(X^{1/2}\times X)} \leqslant M'(1-\alpha)|\lambda|,$$

where M' is defined by Proposition 4.4, By the compactness of the contour $\mathcal{C}$, it follows that

$$\|\Lambda(t)^\alpha Q(t)_\alpha(\mu_0) - \Lambda(t)Q(t)(\mu_0)\|_{\mathcal{L}(X^{1/2}\times X)} \leqslant CM'(1-\alpha),$$

where $C = \int_{\mathcal{C}} |\lambda| d\lambda > 0$ is independent of α. □

Now prove the convergence with rate of $\{\lambda_\alpha\}_{\alpha\in(0,1]}$ at $\alpha = 1$, where the functions $\lambda_\alpha = \lambda_\alpha(t)$ are defined by (4.1).

Theorem 4.7 (Spectral properties of $\Lambda(t)$)**.** *Let the skew-adjoint operator $\Lambda(t)$, and the family $\{\lambda_n^\pm(t)\}_{n\in\mathbb{N}}$, where for every $t \in \mathbb{R}$*

$$\Lambda(t)\begin{bmatrix}\varphi\\ \psi\end{bmatrix} = \lambda_n^\pm(t)\begin{bmatrix}\varphi\\ \psi\end{bmatrix},$$

for some no-null vector $\begin{bmatrix}\varphi\\ \psi\end{bmatrix} \in D(\Lambda(t))$. Then

$$\lambda_n^\pm(t) = \pm i a(t)^{1/2}\mu_n^{1/2},\ n \in \mathbb{N},$$

where $\{\mu_n\}_{n\in\mathbb{N}}$ denote the eigenvalues of the operator $A = -\Delta_D$ with zero Dirichlet boundary conditions.

Proof. Since $\Lambda(t)$ has compact resolvent, all points in the spectrum $\sigma(\Lambda(t))$ of $\Lambda(t)$ are eigenvalues. The eigenvalue problem for $\Lambda(t)$ is

$$\begin{bmatrix}0 & -I\\ a(t)A & 0\end{bmatrix}\begin{bmatrix}\varphi\\ \psi\end{bmatrix} = \lambda\begin{bmatrix}\varphi\\ \psi\end{bmatrix}, \quad \begin{bmatrix}\varphi\\ \psi\end{bmatrix} \in D(\Lambda(t)),$$

i.e.

$$a(t)A\varphi = -\lambda^2\varphi, \quad \varphi \in D(A).$$

Recall that $A = -\Delta_D$ is a positive self-adjoint operator with compact resolvent. Denote by $\{\mu_n\}_{n\in\mathbb{N}}$ the eigenvalues of A ordered increasingly and repeated according to multiplicity. Hence, the eigenvalues of $\Lambda(t)$ are solutions of the equation $\lambda^2 = -a(t)\mu_n,\ n \in \mathbb{N}$, and therefore

$$\lambda = \lambda_n^\pm(t) = \pm i a(t)^{1/2}\mu_n^{1/2},\ n \in \mathbb{N}.$$

□

Theorem 4.8 (Spectral properties of $-\varLambda(t)^\alpha, 0<\alpha\leqslant 1$)**.** *Let the operator $-\varLambda(t)^\alpha$. For every $t\in\mathbb{R}$ the spectrum of $-\varLambda(t)^\alpha$ consists of functions $\lambda^{\pm}_{\alpha,n}(t)$ only, where*

$$-\varLambda(t)^\alpha \begin{bmatrix}\varphi\\ \psi\end{bmatrix} = \lambda^{\pm}_{\alpha,n}(t)\begin{bmatrix}\varphi\\ \psi\end{bmatrix},$$

for some non-null vector $\begin{bmatrix}\varphi\\ \psi\end{bmatrix}\in D(\varLambda(t)^\alpha)$; namely, they are given by

$$\lambda^{\pm}_{\alpha,n}(t) = e^{\pm i\frac{\pi(2-\alpha)}{2}} a(t)^{\alpha/2}\mu_n^{\alpha/2}, \quad n\in\mathbb{N},$$

where $\{\mu_n\}_{n\in\mathbb{N}}$ denotes the ordered sequence of eigenvalues of the operator A repeated according to multiplicity.

Proof. For each $t\in\mathbb{R}$, the eigenvalue problem for $-\varLambda(t)^\alpha$ is

$$-\begin{bmatrix}\cos\frac{\pi\alpha}{2}a(t)^{\alpha/2}A^{\alpha/2} & -\sin\frac{\pi\alpha}{2}a(t)^{\frac{-1+\alpha}{2}}A^{\frac{-1+\alpha}{2}}\\ \sin\frac{\pi\alpha}{2}a(t)^{\frac{1+\alpha}{2}}A^{\frac{1+\alpha}{2}} & \cos\frac{\pi\alpha}{2}a(t)^{\alpha/2}A^{\alpha/2}\end{bmatrix}\begin{bmatrix}\varphi\\ \psi\end{bmatrix} = \lambda\begin{bmatrix}\varphi\\ \psi\end{bmatrix},\ \begin{bmatrix}\varphi\\ \psi\end{bmatrix}\in D(\varLambda(t)^\alpha);$$

that is, $\lambda\in\mathbb{C}$ is an eigenvalue for $-\varLambda(t)^\alpha$ if and only if there is a $0\neq\begin{bmatrix}\varphi\\ \psi\end{bmatrix}\in X^{\frac{1+\alpha}{2}}\times X^{\alpha/2}$ such that

$$\begin{gathered}-\cos\frac{\pi\alpha}{2}a(t)^{\alpha/2}A^{\alpha/2}\varphi + \sin\frac{\pi\alpha}{2}a(t)^{\frac{-1+\alpha}{2}}A^{\frac{-1+\alpha}{2}}\psi = \lambda\varphi\\ -\sin\frac{\pi\alpha}{2}a(t)^{\frac{1+\alpha}{2}}A^{\frac{1+\alpha}{2}}\varphi - \cos\frac{\pi\alpha}{2}a(t)^{\alpha/2}A^{\alpha/2}\psi = \lambda\psi.\end{gathered}$$

With this, we obtain $\lambda\in\mathbb{C}$ is an eigenvalue for $-\varLambda(t)^\alpha$ if and only if

$$\begin{aligned}0 &= \lambda^2 + 2\lambda\cos\frac{\pi\alpha}{2}a(t)^{\alpha/2}A^{\alpha/2} + a(t)^\alpha A^\alpha\\ &= (\lambda - e^{i\frac{\pi(2-\alpha)}{2}}a(t)^{\alpha/2}A^{\alpha/2})(\lambda - e^{-i\frac{\pi(2-\alpha)}{2}}a(t)^{\alpha/2}A^{\alpha/2})\end{aligned}$$

is not injective. Then, the eigenvalues λ of $-\varLambda(t)^\alpha$ are solutions of the equation

$$(\lambda - e^{i\pi(2-\alpha)/2}a(t)^{\alpha/2}\mu_n^{\alpha/2})(\lambda - e^{-i\pi(2-\alpha)/2}a(t)^{\alpha/2}\mu_n^{\alpha/2}) = 0;$$

that is, $\lambda^{\pm}_{\alpha,n}(t) = e^{\pm i\frac{\pi(2-\alpha)}{2}}a(t)^{\alpha/2}\mu_n^{\alpha/2}$, $n\in\mathbb{N}$, and this concludes the proof. $\square$

Remark 4.9. We can see that the eigenvalues $-\varLambda(t)^\alpha$ lie in the semi-axes

$$\{re^{\pm i\frac{\pi(2-\alpha)}{2}} : r\geqslant 0\}.$$

These semi-axes form the edges of a sector of angle $\frac{\pi(2-\alpha)}{2}$ in the complex plane that, as α tends to 1 approaches the semi-plane $\{\lambda\in\mathbb{C} : \mathrm{Re}\lambda\geqslant 0\}$. This behavior reflects the loss of sectoriallity that the operator $\varLambda(t)^\alpha$ experiences as α tends to 1.

Moreover, the eigenvalues $\lambda^{\pm}_n(t)$ and $\lambda^{\pm}_{\alpha,n}(t)$ of the operators $-\varLambda(t)$ and $-\varLambda^\alpha(t)$, respectively, have the same regularity of the functional coefficient $a(t)$, and the following estimates are hold

$$\begin{gathered}|\lambda^{\pm}_n(t)|\leqslant a_{\max}^{1/2}\mu_n^{1/2}, \quad \text{for all } t\in\mathbb{R},\\ |\lambda^{\pm}_{\alpha,n}(t)|\leqslant a_{\max}^{\alpha/2}\mu_n^{\alpha/2}, \quad \text{for all } t\in\mathbb{R}.\end{gathered}$$

Remark 4.10. The analysis made in the previous sections can be applied on other examples, e.g., singularly non-autonomous plate equation with structural damping and non-autonomous Schrödinger equations. In each example a careful study of the fractional power of the operator that governs the problem it is necessary.

Acknowledgments. M. J. D. Nascimento was partially supported by FAPESP #2017/06582-2, Brazil

References

[1] H. Amann; *Linear and Quasilinear Parabolic Problems. Volume I: Abstract Linear Theory*, Birkhäuser Verlag, Basel, 1995.

[2] D. A. Benson, S. W. Wheatcraft, M. M. Meerschaert; *Application of a fractional advection-dispersion equation*, Water Resour. Res., **36** (2000), 1403-1412.

[3] F. D. M. Bezerra, A. N. Carvalho, J. W. Cholewa, M. J. D. Nascimento; *Parabolic approximation of damped wave equations via fractional powers: Fast growing nonlinearities and continuity of the dynamics*. J. Math. Anal. Appl., **450** (2017), 377-405.

[4] F. D. M. Bezerra, A. N. Carvalho, T. Dłotko, M. J. D. Nascimento; *Fractional Schrödinger equation; solvability, asymptotic behaviour and connection with classical Schrödinger equation*, J. Math. Anal. Appl., **457** (1) (2018), 336-360.

[5] S. M. Bruschi, A. N. Carvalho, J. W. Cholewa, T. Dłotko; *Uniform exponential dichotomy and continuity of attractors for singularly perturbed damped wave equations*, J. Dynam. Differential Equations **18** (2006), 767-814.

[6] J. W. Cholewa, T. Dłotko; *Fractional Navier-Stokes equation*, Discrete Contin. Dyn. Syst. Ser. B, **23** (8) (2018), 2967-2988.

[7] J. W. Cholewa, T. Dłotko; *Remarks on the powers of elliptic operators*, Rev. Mat. Complut., **13** (2000), 1-12.

[8] T. Caraballo, A. N. Carvalho, J. A. Langa, F. Rivero; *Existence of pullback attractors for pullback asymptotically compact processes*, Nonlinear Anal., **72** (2010), 1967-1976.

[9] T. Caraballo, A. N. Carvalho, J. A. Langa, F. Rivero; *A non-autonomous strongly damped wave equation: Existence and continuity of the pullback attractor*, Nonlinear Anal., **74** (2011), 2272-2283.

[10] T. Caraballo, A. N. Carvalho, J. A. Langa, F. Rivero; *Some gradient-like non-autonomous evolution processes*, Internat. J. Bifur. Chaos, **20** (9) (2010), 2751-2760.

[11] A. N. Carvalho, J. W. Cholewa, T. Dłotko; *Damped wave equations with fast growing dissipative nonlinearities*, Discrete Contin. Dyn. Syst., **24** (4) (2009), 1147-1165.

[12] A. N. Carvalho, J. A. Langa, J. C. Robinson; *Attractors for Infinite-dimensional Non-autonomous Dynamical Systems*, Applied Mathematical Sciences 182, Springer-Verlag, 2012.

[13] A. N. Carvalho, M. J. D. Nascimento; *Singularly non-autonomous semilinear parabolic problems with critical exponents and applications*, Discrete Contin. Dyn. Syst. Ser. S, **2** (3) (2009), 449-471.

[14] A. N. Carvalho, S. Piskarev; *A general approximation scheme for attractors of abstract parabolic problems*, Numer. Funct. Anal. Optim., **27** (7-8) (2006), 785-829.

[15] G. Chen, D. L. Russell; *A mathematical model for linear elastic systems with structural damping*, Quart. Appl. Math., (1982), 433-454.

[16] S. Chen, R. Triggiani; *Proof of two conjectures of G. Chen and D. L. Russell on structural damping for elastic systems*, Approximation and Optimization (Havana, 1987), 234-256, Lecture Notes in Math., 1354, Springer, Berlin, 1988.

[17] S. Chen, R. Triggiani; *Proof of extension of two conjectures on structural damping for elastic systems: the case* $1 \leqslant \alpha \leqslant \frac{1}{2}$. Pacific J. Math., **136** (1989), 15-55.

[18] S. Chen, R. Triggiani; *Gevrey class semigroups arising from elastic systems with gentle dissipation: the case* $0 < \alpha < \frac{1}{2}$. Proc. Amer. Math. Soc., **110** (1990), 401-415.

[19] S. Chen, R. Triggiani; *Characterization of domains of fractional powers of certain operators arising in elastic systems*, and applications J. Differential Equations, **88** (1990), 279-293.

[20] T. Dłotko; *Navier-Stokes equation and its fractional approximations*, Appl. Math. Optim., **77** (2018), 99-128.

[21] H. Fazli, F. Bahrami; *On the steady solutions of fractional reaction-diffusion equations*, Published by Faculty of Sciences and Mathematics, University of Niš, Serbia, Filomat 31:6 (2017), 1655–1664.

[22] D. Henry; *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics 840, Springer-Verlag, Berlin, 1981.

[23] T. Kato; *Note on fractional powers of linear operators*, Proc. Japan Acad., **36** (1960), 94-96.

[24] M. Pan, L. Zheng, F. Liu, C. Liu, X. Chen; *A spatial-fractional thermal transport model for nanofluid in porous media*, Appl. Math. Model., **53** (2018), 622–634.
[25] A. Pazy; *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.
[26] P. E. Sobolevskiĭ; *Equations of parabolic type in a Banach space*, Amer. Math. Soc. Transl., **49** (1966), 1-62.
[27] C. Sun, D. Cao, J. Duan; *Non-autonomous dynamics of wave equations with nonlinear damping and critical nonlinearity*, Nonlinearity, **19** (11) (2006), 2645-2665.
[28] H. Triebel; *Interpolation Theory, Function Spaces, Differential Operators*, Veb Deutscher Verlag, Berlin 1978.

Marcelo J. D. Nascimento
Universidade Federal de São Carlos, Departamento de Matemática, 13565-905 São Carlos SP, Brazil
Email address: `marcelo@dm.ufscar.br`

Flank D. M. Bezerra
Departamento de Matemática, Universidade Federal da Paraíba, 58051-900 João Pessoa PB, Brazil
Email address: `flank@mat.ufpb.br`