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EXPONENTIAL STABILITY FOR SOLUTIONS OF CONTINUOUS AND DISCRETE ABSTRACT CAUCHY PROBLEMS IN BANACH SPACES

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Abstract. Let $\mathbf{T}$ be a strongly continuous semigroup acting on a complex Banach space X and let A be its infinitesimal generator. It is well-known [29, 33] that the uniform spectral bound $s_0(A)$ of the semigroup $\mathbf{T}$ is negative provided that all solutions to the Cauchy problems

$$\dot{u}(t) = Au(t) + e^{i\mu t}x, \quad t \geq 0, \quad u(0) = 0,$$

are bounded (uniformly with respect to the parameter $\mu \in \mathbb{R}$). In particular, if X is a Hilbert space, then this yields all trajectories of the semigroup $\mathbf{T}$ are exponentially stable, but if X is an arbitrary Banach space this result is no longer valid. Let $\mathcal{X}$ denote the space of all continuous and 1-periodic functions $f : \mathbb{B} \to X$ whose sequence of Fourier-Bohr coefficients $(c_m(f))_{m\in\mathbb{Z}}$ belongs to $\ell^1(\mathbb{Z}, X)$. Endowed with the norm $\|f\|_1 := \|(c_m(f))_{m\in\mathbb{Z}}\|_1$ it becomes a non-reflexive Banach space [15]. A subspace $\mathcal{A}_{\mathbf{T}}$ of X (related to the pair $(\mathbf{T}, \mathcal{X})$) is introduced in the third section of this paper. We prove that the semigroup $\mathbf{T}$ is uniformly exponentially stable provided that in addition to the above-mentioned boundedness condition, $\mathcal{A}_{\mathbf{T}} = X$. An example of a strongly continuous semigroup (which is not uniformly continuous) and fulfills the second assumption above is also provided. Moreover an extension of the above result from semigroups to 1-periodic and strongly continuous evolution families acting in a Banach space is given. We also prove that the evolution semigroup $\mathcal{T}$ associated with $\mathbf{T}$ on $\mathcal{X}$ does not verify the spectral determined growth condition. More precisely, an example of such a semigroup with uniform spectral bound negative and uniformly growth bound non-negative is given. Finally we prove that the assumption $\mathcal{A}_{\mathbf{T}} = X$ is not needed in the discrete case.

1. Introduction

The following conditions are of a great interest in linear systems theory in infinite dimensional spaces,

$$(0, \infty) \subseteq \rho(A) \text{ and } \|R(\lambda, A)\| \leq \frac{1}{\lambda}, \quad \forall \lambda > 0, \tag{1.1}$$

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$$i\mathbb{R} \subseteq \rho(A) \text{ and } \lim_{|\theta|\to\infty} \|R(i\theta, A)\| = 0, \tag{1.2}$$

and the "spectrum determined growth condition" (SDGC on short) that states that

$$s(A) = \omega_0(\mathbf{T}). \tag{1.3}$$

For further details, concrete examples, and related issues see for example the monographs [19, 39].

In (1.1), $A : D(A) \subset X \to X$ is assumed to be a densely defined closed linear operator, and (1.1) is a special case of the Hille-Yoshida assumption and ensures the existence and uniqueness of solutions in many concrete problems. In fact any operator A verifying (1.1) generates a strongly continuous semigroup of contractions. In (1.2), $A : D(A) \subset X \to X$ is assumed to be the generator of a strongly continuous semigroup $\mathbf{T}$ acting on a complex Hilbert space X. In fact, (1.2) yields boundedness of the resolvent operator-valued map $z \mapsto R(z, A)$ on the imaginary axis; that is,

$$i\mathbb{R} \subseteq \rho(A) \text{ and } \sup_{\mu\in\mathbb{R}} \|R(i\mu, A)\| < \infty. \tag{1.4}$$

As is well-known, (1.4) yields the uniform exponential stability of $\mathbf{T}$; that is, the negativeness of its uniform growth bound $\omega_0(\mathbf{T})$. This result is known as the Gearhart-Prüss Theorem (see [21, 30, 42]) (it was settled independently in many other places). The reader can find further details and other references in the monographs [1, 33, 20].

Prior to discussing assumptions (1.2), (1.4) and (1.3), let us recall briefly the definitions of the two growth bounds and that of the two spectral bounds associated with a strongly continuous semigroup $\mathbf{T}$ and its generator A acting on a Banach space X. Further details and many other equivalent definitions can be found in [1, Chapter 5]. In the case when the spectrum of A (denoted by $\sigma(A)$) is a nonempty set, then the spectral bound of A is defined by

$$s(A) = \sup\big\{\Re(z) : z \in \sigma(A)\big\} \tag{1.5}$$

and (by convention) $s(A) = -\infty$ whenever the spectrum of A is the empty set.

The uniform spectral bound of A is given by

$$s_0(A) = \inf\big\{\omega > s(A) : \quad \sup_{b\in\mathbb{R}} \|R(a + ib, A)\| < \infty, a \geq \omega\big\} \tag{1.6}$$

and the growth bound of the trajectory $t \mapsto u_x(t) := T(t)x$ is given by

$$\omega(x) := \inf\big\{\omega \in \mathbb{R} : \lim_{t\to\infty} \|e^{-\omega t}u_x(t)\| = 0\big\}. \tag{1.7}$$

We mention that the condition $\lim_{t\to\infty} \|e^{-\omega t}u_x(t)\| = 0$ in (1.6) can be replaced by a weaker one, namely that the map $t \mapsto \|e^{-\omega t}u_x(t)\|$ is bounded on $\mathbb{R}_+ := [0, \infty)$.

The growth bound of $\mathbf{T}$ (denoted by $\omega_1(\mathbf{T})$) is the supremum of the set $\{\omega(x) : x \in D(A)\}$ and the uniform growth bound of $\mathbf{T}$ (denoted by $\omega_0(\mathbf{T})$) is the supremum of the set $\{\omega(x) : x \in X\}$.

Using the definitions we deduce the inequalities $s(A) \leq s_0(A)$ and $\omega_1(\mathbf{T}) \leq \omega_0(\mathbf{T})$. Actually it is known that

$$s(A) \leq \omega_1(\mathbf{T}) \leq s_0(A) \leq \omega_0(\mathbf{T}). \tag{1.8}$$

The most difficult part in the proof of (1.8) is the inequality $\omega_1(\mathbf{T}) \leq s_0(A)$ and that was first obtained by Weis and Wrobel [41] using tools such as interpolation theory and a previous result in [31] and independently (using an elementary method) obtained by van Neerven [32].

A result of Neubrander [26] (which leads to the first inequality in (1.8)) asserts that $\omega_1(\mathbf{T})$ is equal with the infimum of all real numbers ω for which the limit $\lim_{t\to\infty}\int_0^t e^{-\omega s}T(s)x\,ds$ exists in X for all $x \in X$. Combining this with the well-known fact that if for some $z \in \mathbb{C}$ we have that $R_z := \lim_{t\to\infty}\int_0^t e^{zs}T(s)x\,ds$ exists in X, for all $x \in X$, then $z \in \rho(A), R(z,A) = R_z$, and the first inequality in (1.8) follows easily. On the other hand from a result of Pazy [28] it follows that if $\int_0^\infty \|e^{zs}T(s)x\,ds\| < \infty$ for some $x \in X$ and $z \in \mathbb{C}$ then $\lim_{t\to\infty}\|e^{zs}T(s)x\| = 0$. Combining this with the previous result and the definitions above the final inequality in (1.8) becomes clear.

Going back to (1.4), let us mention that in the general case when the state space is a complex Banach space, the absence of the spectrum of A on the imaginary axis is a consequence of the following piecewise boundedness condition

$$\sup_{t\geq 0}\|\int_0^t e^{i\omega s}T(s)x\,ds\| := K(\omega,x) < \infty \quad \forall \omega \in \mathbb{R},\ \forall x \in X;$$

see for example [36, Proposition 3.3] and [29]. Moreover, the uniform boundedness condition

$$\sup_{\omega\in\mathbb{R}} K(\omega,x) < \infty \quad x \in X \tag{1.9}$$

does ensure that (1.4) is fulfilled [33, 29] and then $s_0(A)$ is negative using an analytical continuation argument (see for example [33]).

However, except in the particular case when X is a complex Hilbert space, (1.9) does not guarantee that the uniform growth bound of $\mathbf{T}$ is negative. To make this possible we add to (1.9) a new assumption as in Theorem 3.1 or in Theorem 3.5 below.

Howland, [23] introduced evolution semigroups in 1974. He was interested in scattering theory for Schrödinger operators with potentials that were periodic functions of time and created a way to do this as an autonomous problem by working in the space $L^2(\mathbb{R},H)$ rather than in the Hilbert space H. Some of the history and earlier references can be found in the monograph [17]. Later, this was generalized by Rau [34] and many others, with semigroups on Banach spaces replacing unitary groups on Hilbert spaces.

Weis [40] obtained a result that states that positive semigroups on $L^p(\Omega,m)$ verify SDGC. The proof of the Weis theorem (in the context of [40]) is based on the fact that (noticed earlier in [24]), the evolution semigroup (associated to an arbitrary strongly continuous semigroup acting on a Banach space X) on the Lebesgue-Bochner space of functions $L^p(\mathbb{R},X)$ satisfies SDGC.

Generally speaking an evolution semigroup and its infinitesimal generator verify SDGC and the Spectral Mapping Theorem; see for example [6, 5, 8, 9, 17, 18, 24, 27, 37] and the references therein.

Applications of evolution semigroups in the theory of inequalities were highlighted (for example) in [7, 10, 14, 13].

We also mention the papers [3, 12, 25] where the evolution semigroups theory was applied to the study of nonuniform asymptotic behavior of evolutionary families of operators acting in Banach spaces.

In the third section of this paper (Corollary 4.5) we highlight an example of an evolution semigroup for which the spectral determined growth condition is not fulfilled. This raises the question as to whether the first and the second inequalities in (1.8) could be strict for evolution semigroups.

2. Background and previous results

Let X be a complex Banach space and let X' be its (topological) dual. Let $\mathcal{L}(X)$ stand for the Banach algebra of all bounded linear operators on X. The norms on X, X' and $\mathcal{L}(X)$ are denoted by the same symbol, namely $\|\cdot\|$. The duality pair between X and X' is denoted by $\langle x, x'\rangle$. Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup that acts on X and let $(A, D(A))$ be its infinitesimal generator. In a slightly different form, the following result was originally obtained by van Neerven [32].

Theorem 2.1. *If for all $b \in X$ the solutions of the Cauchy problems associated to the generator A of $\mathbf{T}$*

$$\begin{gathered} u'(t) = Au(t) + e^{i\mu t}b, \quad t > 0, \quad \mu \in \mathbb{R} \\ u(0) = 0 \end{gathered} \tag{2.1}$$

are bounded on $\mathbb{R}_+$ (uniformly with respect to the parameter μ) then for each $x \in D(A)$ the solution of the abstract Cauchy Problem

$$u'(t) = Au(t), \quad u(0) = x \tag{2.2}$$

is uniformly exponentially stable, that is there exists $\nu > 0$ such that the map

$$t \mapsto e^{-\nu t}T(t)x \text{ is bounded on } \mathbb{R}_+ := [0, \infty). \tag{2.3}$$

Obviously, the boundedness assumption of Theorem 2.1 can be written as

$$\sup_{\mu\in\mathbb{R}} \sup_{t\geq 0} \Big\| \int_0^t e^{i\mu s} T(t-s) b\, ds \Big\| := M(b) < \infty, \quad \text{for all } b \in X. \tag{2.4}$$

In the next example, $\mathbf{T}$, is a strongly continuous semigroup that is weakly L^1-integrable (that is (2.7) is fulfilled) and is not uniformly exponentially stable (that is its uniform growth bound is nonnegative) and was originally introduced in [22]. We shall use this example in the next section (Corollary 4.5).

Example 2.2. Let $X_1 := C_0(\mathbb{R}_+, \mathbb{C})$ be the set of all continuous functions $x : \mathbb{R}_+ \to \mathbb{C}$ which vanish at infinity. This is a Banach space when it is equipped with the uniform norm $\|\cdot\|_\infty$. Let $X_2 := L_1(\mathbb{R}_+, \mathbb{C}, e^s ds)$ be the set of all measurable functions $x : \mathbb{R}_+ \to \mathbb{C}$ for which

$$\|x\|_{X_2} := \int_0^\infty e^t |x(t)| dt < \infty. \tag{2.5}$$

Let $X := X_1 \cap X_2$. The space X becomes a Banach space when it is equipped with the norm

$$\|x\|_X := \|x\|_\infty + \|x\|_{X_2}, \quad x \in X. \tag{2.6}$$

For each $t \geq 0$, the map $(S(t)x)(\tau) = x(t + \tau)$, defined for $\tau \geq 0$, acts on X and the family $\mathbf{S} = \{S(t)\}_{t\geq 0}$ is a strongly continuous semigroup. Moreover, for each $t > 0$ the spectrum of $S(t)$ consists by all complex numbers z with $|z| \leq 1$, [22]. Let $\mathbf{T}(t) = e^{\frac{t}{2}} S(t)$, $t \geq 0$. The semigroup $\mathbf{T}$ satisfies condition (2.4) (see for instance [4]) although $\omega_0(\mathbf{T}) = \mathbf{1/2}$. In fact, as is shown in [22], the semigroup $\mathbf{T}$ is weakly-L^1-stable; that is

$$\int_0^\infty |\langle \mathbf{T}(t)x, x' \rangle| dt < \infty \quad \text{for all } x \in X \text{ and } x' \in X'. \tag{2.7}$$

The Fourier-Bohr coefficients associated to any 1-periodic continuous function $f : \mathbb{B} \to X$ is given by

$$c_n(f) := \int_0^1 e^{2int\pi} f(t) dt, \quad n \in \mathbb{Z}. \tag{2.8}$$

Let us denote (ad-hoc) by $\mathcal{X}$ the space consisting of all 1-periodic continuous functions $f : \mathbb{B} \to X$ for which

$$\|f\|_1 := \sum_{n \in \mathbb{Z}} \|c_n(f)\| < \infty. \tag{2.9}$$

Note that (see for example [15]) the space $\mathcal{X}$ endowed with the norm $\| \cdot \|_1$ is a nonreflexive Banach space.

For every $f \in \mathcal{X}$, consider its Bohr-Fourier sum (with respect to the uniform norm) defined by

$$s_f(t) = \sum_{n \in \mathbb{Z}} e^{-2int\pi} c_n(f), \quad t \in \mathbb{R}. \tag{2.10}$$

Clearly $s_f \in \mathcal{X}$ and thus $f = s_f$.

Let $\mathcal{H}$ be the set of all scalar valued functions defined on $[0, 1]$, of class C^1, with $h(0) = h(1) = 0$ and having the following property

$$I(\mu, h) := \int_0^1 e^{i\mu t} h(t) dt \neq 0 \quad \text{for all } \mu \in \mathbb{R}. \tag{2.11}$$

It is easy to show that the map $t \mapsto h_0(t) := \sin(\pi t) : [0, 1] \to \mathbb{R}$ belongs to $\mathcal{H}$.

A family $\mathcal{U} = \{U(t, s) : (t, s) \in \mathbb{R}^2, t \geq s\}$ of bounded linear operators acting on X is called a 1-periodic strongly continuous evolution family if $U(t, t) = I$- the identity operator on X, $U(t, s)U(s, r) = U(t, r)$ for all reals $t \geq s \geq r$, the map $(t, s) \mapsto U(t, s)$ is strongly continuous on the set $\Delta := \{(t, s) \in \mathbb{R}^2 : t \geq s\}$ and $U(t, s) = U(t + 1, s + 1)$ for every pair $(t, s) \in \Delta$. The evolution family $\mathcal{U}$ is called exponentially bounded if there exists a real number ω such that the map $(t, s) \mapsto e^{-\omega(t-s)} U(t, s)$ is bounded in $\mathcal{L}(X)$ and uniformly exponentially stable if there exists a negative ω with that property.

Throughout this article we assume that the evolution families are exponentially bounded. Clearly, if $\mathcal{U}$ satisfies the convolution condition $U(t, s) = U(t - s, 0)$ for every $(t, s) \in \Delta$ then the family $\mathbf{T} = \{U(t, 0) : t \geq 0\}$ is a strongly continuous semigroup that acts on X.

3. Stability result and its natural consequences

For $h \in \mathcal{H}$ and $x \in X$ denote

$$H_{xh}(t) := h(t)U(t, 0)x, \quad t \in [0, 1]. \tag{3.1}$$

Obviously, $H_{xh}(0) = H_{xh}(1)$. Let $\tilde{H}_{xh}$ be the extension by periodicity to $\mathbb{R}$ of H_{xh}.

Theorem 3.1. *Let $\mathcal{U} = \{U(t,s) : (t,s) \in \mathbb{R}^2, t \geq s\}$ be a strongly continuous 1-periodic evolution family acting on the Banach space X. With the above notation assume that for a given $x \in X$ and some $h \in \mathcal{H}$ the map $\tilde{H}_{xh}$ belongs to $\mathcal{X}$; that is*

$$\|\tilde{H}_{xh}\|_1 := \sum_{n=1}^{\infty} \|c_n(\tilde{H}_{xh})\| < \infty, \tag{3.2}$$

and that there exists a positive constant M (depending only of the family $\mathcal{U}$) such that for all $\mu \in \mathbb{R}$ and all $t \geq 0$, one has

$$\| \int_0^t e^{i\mu s} U(t,s) y ds\| \leq M\|y\|, \quad \text{for all } y \in X. \tag{3.3}$$

Then for each $\mu \in \mathbb{R}$ and all $n \in \mathbb{Z}, n \geq 1$, one has

$$\| \sum_{j=1}^{n} e^{-i\mu j} U(1,0)^j x\| \leq \frac{M}{|I(\mu,h)|} \|\tilde{H}_{xh}\|_1. \tag{3.4}$$

Proof. The proof is contained in the proof of [9, Theorem 2.2] and so we omit it. Note however that the result in [9] is stated under the stronger assumptions that X is a Hilbert space and that the map $\tilde{H}_{xh}$ is α-Hölder continuous for some $\alpha > 1/2$. □

Remark 3.2.
(i) If (3.2) holds for all $x \in X$, then (3.4) holds for all $x \in X$ and in turn it implies the exponential stability of the evolution family $\mathcal{U}$; see for example [11].
(ii) We mention that the constant M in (3.3) is independent of the real parameter μ, and moreover (under assumption (i) of this Remark) it cannot be dropped; see [4] or [29] for examples in the semigroup case.

Corollary 3.3. *Let $t \mapsto a(t) : \mathbb{R} \to \mathbb{C}$ be a continuous and 1-periodic function. The following three statements are equivalent.*

(1) *For all $z \in \mathbb{C}$ and all $s \in \mathbb{R}$ the solution of the homogeneous Cauchy Problem*

$$u'(t) + a(t)u(t) = 0, \quad u(s) = z, \quad t \geq s \tag{3.5}$$

is uniformly exponentially stable; that is, there exist absolute constants N and ν such that

$$|U(t,s)| \leq N \exp[-N(t-s)], \quad \forall t \geq s, \tag{3.6}$$

where

$$U(t,s) := \exp\Big(-\int_s^t a(r)dr\Big). \tag{3.7}$$

(2) *All solutions of the non-homogeneous Cauchy Problems*

$$v'(t) + a(t)v(t) = e^{i\mu t}, \quad u(0) = 0, \quad t \geq 0 \tag{3.8}$$

are bounded on $[0,\infty)$ uniformly with respect to the parameter $\mu \in \mathbb{R}$.

(3) *One has*

$$\int_0^1 \Re[a(r)]dr > 0. \tag{3.9}$$

Proof. (1) $\Rightarrow$ (2). The solution of (3.8) is

$$v(t) = \int_0^t U(t,s)e^{i\mu s}ds$$

and its boundedness on $[0,\infty)$ follows from (3.6).

(2) $\Rightarrow$ (3). It is sufficient to see that the continuation by periodicity $(\tilde{H})$ of the function $t \mapsto H(t) := h_0^2(t)U(t,0)$ is continuously differentiable on $\mathbb{R}$ and its derivative $\tilde{H}'$ is bounded. Thus, $\tilde{H}$ belongs to $\mathcal{X}$ (with $X = \mathbb{C}$) and we can apply Remark 3.2.

(3) $\Rightarrow$ (1). It is sufficient to see that (3.9) implies the inequality

$$|U(1,0)| = \Big|\exp\Big(-\int_0^1 a(r)\Big)dr\Big| < 1$$

and then (3.6) becomes clear. □

Let $n \geq 1$ be an integer. Let $\mathbb{C}^{n\times n}$ be the vector space of all $n \times n$ square matrices with complex entries. As is well-known, $\mathbb{C}^{n\times n}$ becomes a Banach algebra when it is endowed with the usual matrix norm. For $t \in \mathbb{R}$ consider the matrix $A(t) := (a_{ij}(t))_{i,j\in\{1,2,\cdots,n\}} \in \mathbb{C}^{n\times n}$ and assume that the map $t \mapsto A(t) : \mathbb{R} \to \mathbb{C}^{n\times n}$ is 1-periodic and continuous on $\mathbb{R}$. With $P(t)$ we denote the solution of the matrix Cauchy Problem

$$X'(t) = A(t)X(t), \quad t \in \mathbb{R}, \quad X(0) = I_n;$$

here I_n denotes the identity matrix in $\mathbb{C}^{n\times n}$. Let

$$\operatorname{tr}(A(t)) := \sum_{i=1^n} a_{ii}(t), \quad t \in \mathbb{R},$$

be the trace of $A(t)$. The next Corollary uses the Liouville Theorem in ODE's, asserting that

$$\det[P(t)] = \det[P(s)] \exp\Big(\int_s^t \operatorname{tr}(A(r))dr\Big), \quad t \geq s \in \mathbb{R}; \tag{3.10}$$

see [16, pp.152-153].

Corollary 3.4. *With the above notation and assumptions the following two statements are equivalent:*

(i) *There exist absolute constants N and ν such that*

$$|\det[P(t)]|e^{\nu t} \leq N|\det[P(s)]|e^{\nu s}, \quad \forall t \geq s. \tag{3.11}$$

(ii) *One has*

$$\int_0^1 \Re[\operatorname{tr}(A(r))]dr > 0. \tag{3.12}$$

Proof. Let $a(t) := \operatorname{tr}(A(t)), t \in \mathbb{R}$. Thus, via (3.7) and (3.10), the evolution family associated to the differential equation $u'(t) = a(t)u(t)$ satisfies

$$U(t,s) = \exp\Big(-\int_s^t a(r)dr\Big) = \frac{\det[P(s)]}{\det[P(t)]}.$$

Now we can apply Corollary 3.3. □

Theorem 3.5. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup acting on the Banach space X and $A : D(A) \subseteq X \to X$ its infinitesimal generator. With the above notation assume that the following two statements are fulfilled.*

(i) *For all $x \in X$ and some $h \in \mathcal{H}$ the map $\tilde{H}_{xh}$ in (3.1) (with $U(t,0)x$ replaced by $T(t)x$) belongs to $\mathcal{X}$.*

(ii) *All solutions of the abstract Cauchy problems*
$$\dot{u}(t) = A(u(t)) + e^{i\mu t}x, \quad t \in \mathbb{R}_+, x \in X, \quad u(0) = 0 \tag{3.13}$$
are bounded on $\mathbb{R}_+$ uniformly with respect to the real parameter μ, or equivalently, for all $\mu \in \mathbb{R}$ and all $t \geq 0$, one has
$$\sup_{t\geq 0, \mu\in\mathbb{R}} \| \int_0^t e^{i\mu s} T(t-s) y ds\| := M(y) < \infty, \quad \text{for all } y \in X. \tag{3.14}$$

Then $\mathbf{T}$ is uniformly exponentially stable, that is its uniform growth bound is negative.

Proof. Clearly the family $\{U(t,s) : t \geq s \geq 0\}$ with $U(t,s) := T(t-s)$ is a strongly continuous and 1-periodic evolution family on X and (3.3) is an easy consequence of (3.14) and the uniform boundedness principle. The assertion follows via Theorem 3.1 and Remark 3.2(i). □

With any strongly continuous semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ and any $h \in \mathcal{H}$ we associate the set $\mathcal{A}_{\mathbf{T}h}$ consisting of all $x \in X$ with the property that the map $\tilde{H}_{xh}$ belongs to $\mathcal{X}$. We denote by $\mathcal{A}_{\mathbf{T}}$ the smallest linear subspace of X containing the set $\cup_{h\in\mathcal{H}} \mathcal{A}_{\mathbf{T}h}$.

When X is a Hilbert space, $x \in D(A)$ and $h \in \mathcal{H}$ it is easy to see that $\tilde{H}_{xh}$ is a Lipschitz continuous function and then it belongs to $\mathcal{X}$, (see [9, Lemma 1.3]). As is shown below, a similar result holds for semigroups acting on Banach spaces but only for $x \in D(A^2)$ and for certain functions $h \in \mathcal{H}$. More precisely the following result holds.

Proposition 3.6. *Let $\mathbf{T}$ be a strongly continuous semigroup acting on complex Banach space X and let A be its infinitesimal generator. Then $\mathcal{A}_{\mathbf{T}}$ contains $D(A^2)$.*

Proof. First we prove that $h_0^2 \in \mathcal{H}$. Indeed when $e^{i\mu} = 1$ obviously one has that $I(\mu, h_0^2) \neq 0$ and when $e^{i\mu} \neq 1$ (after elementary calculus whose details are omitted) we obtain
$$I(\mu, h_0^2) = \frac{2\pi^2(e^{i\mu} - 1)}{i\mu(4\pi^2 - \mu^2)} \neq 0.$$
Now for $m \in \mathbb{Z}, m \neq 0$ and $x \in D(A^2)$, integrating by parts two times, we obtain
$$\begin{aligned} c_m(\tilde{H}_{xh_0^2}) = \frac{1}{(2\pi i m)^2} \int_0^1 & e^{2i\pi m t}[(2h_0'(t)^2 + 2h_0(t)h_0''(t))T(t)x \\ & + 4h_0(t)h_0'(t)T(t)Ax + h_0^2 T(t)A^2x]dt. \end{aligned} \tag{3.15}$$
Finally by passing to the norms in both sides of (3.15) we obtain
$$\|c_m(\tilde{H}_{xh_0^2})\| \leq \frac{K}{4\pi^2 m^2}(4\pi^2\|x\| + 4\pi\|Ax\| + \|A^2x\|),$$
where $K := \sup_{t\in[0,1]} \|T(t)\|$. □

It seems that Theorem 3.5 and Proposition 3.6 can produce a new proof for Theorem 2.1 and other results in this area. This will be analyzed in a separate paper.

We recall that condition (2.7) does not imply the exponential stability of the semigroup $\mathbf{T}$ when it acts on an (arbitrary) Banach space. However, a nice result of Storozhuk [38] asserts that if for a given nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ that is positive on $(0, \infty)$ one has

$$\int_0^\infty \phi(|\langle T(t)x, x'\rangle|)dt \quad \forall x \in X, \quad x' \in X' \tag{3.16}$$

then the pseudo-spectral bound of its generator (i.e. $s_0(A)$) is negative. In turn, via the well-known Gearhart-Prüss Theorem, this implies that the semigroup is uniformly exponentially stable when X is a Hilbert space.

Corollary 3.7. *A semigroup $\mathbf{T} = \{T(t) : t \geq 0\}$ acting on a complex Banach space X, with $\mathcal{A}_{\mathbf{T}} = X$, is uniformly exponentially stable if and only if for all $x \in X$ and $x' \in X'$, one has*

$$\sup_{\|x'\|\leq 1} \int_0^\infty |\langle T(t)x, x'\rangle|dt < \infty \tag{3.17}$$

Proof. It is sufficient to see that for all $\mu \in \mathbb{R}$ and $x \in X$ one has

$$\|\int_0^t e^{i\mu s}T(s)x\| \leq \sup_{\|x'\|\leq 1} \int_0^\infty |\langle T(t)x, x'\rangle|dt$$

and apply the uniform boundedness principle and Theorem 3.5. □

Corollary 3.8. *Let $\mathbf{T} = \{T(t) : t \geq 0\}$ acting on a complex Banach space X, with $\mathcal{A}_{\mathbf{T}} = X$, and let $t \mapsto a(t) : \mathbb{R} \to [1, \infty)$ be a 1-periodic and continuous map. The 1-periodic evolution family defined by*

$$U(t,s)x = T\Big(\int_s^t a(r)dr\Big)x, \quad t \geq s, \ x \in X \tag{3.18}$$

is uniformly exponentially stable if and only if for all $x \in X$ and $x' \in X'$ the map $t \mapsto |\langle U(t,0)x, x'\rangle|$ belongs to $L^1(\mathbb{R}_+)$.

Proof. For $t \geq 0$ let $G(t) := \int_0^t a(r)dr$. By a change of variables $G(t) = u$ we obtain

$$\||\langle U(\cdot,0)x, x'\rangle|\|_{L_1} = \int_0^\infty |\langle T(u)x, x'\rangle| \frac{1}{a(G^{-1}(u))} du \geq \frac{1}{K}\||\langle T(\cdot)x, x'\rangle|\|_{L_1}, \tag{3.19}$$

where $K := \sup_{t\in\mathbb{R}} a(t)$. Now, from the uniform boundedness principle and Corollary 3.7, there exist absolute positive constants N and ν such that for all $t \geq s \in \mathbb{R}$ one has

$$\|U(t,s)\| = \|T(G(t)) - T(G(s))\| \leq Ne^{-\nu(F(t)-F(s))} \leq Ne^{-\nu(t-s)}.$$

□

Keeping in mind the assumptions of the above theorems it is important to highlight classes of strongly continuous semigroups $\mathbf{T}$ for which $\mathcal{A}_{\mathbf{T}} = X$. First we mention that for the semigroup $\mathbf{T}$ in Example 2.2, we have that $\mathcal{A}_{\mathbf{T}}$ is a proper subset of X. On the other hand, for every uniformly continuous semigroup $\mathbf{T}$ acting on a Banach space X, one has $\mathcal{A}_{\mathbf{T}} = X$. The next example shows that there exists strongly continuous semigroups which are not uniformly continuous with $\mathcal{A}_{\mathbf{T}} = X$.

Example 3.9. Let $(H, \langle , \rangle)$ be a separable complex Hilbert space and let $B = \{e_n, n = 1, 2, \cdots\}$ be an orthonormal basis of it, i.e. $\langle e_n, e_m\rangle = 0$ when $m \neq n$, $\|e_n\|^2 := \langle e_n, e_n\rangle = 1$ and the linear span of B is dense in H. Thus any $x \in H$ can be represented uniquely as $x = \sum_{n=1}^{\infty}\langle x, e_n\rangle e_n$. Let (λ_n) be a sequence of negative real numbers. For every $t \geq 0$ and every $x \in H$ let

$$T(t)x := \sum_{n=1}^{\infty} e^{\lambda_n t}\langle x, e_n\rangle e_n. \tag{3.20}$$

It is well-known and is easy to prove that:

(1) The family $\mathbf{T} := \{T(t) : t \geq 0\}$ is a strongly continuous semigroup that acts on H;
(2) $\omega_0(\mathbf{T}) = \sup_{n\geq 1}\lambda_n$;
(3) $D(A) = \{x \in H : \sum_{n=1}^{\infty}|\lambda_n|^2|\langle x, e_n\rangle|^2 < \infty\}$;
(4) $Ax := \sum_{n=1}^{\infty}\lambda_n\langle x, e_n\rangle e_n$ for all $x \in D(A)$.

See [43].

Proposition 3.10. *Let $\mathbf{T}$ be the semigroup in the previous Example and let $m \in \mathbb{Z}$ with $|m| \geq 1$. Then for every $x \in H$ one has*

$$\|c_m(\tilde{H}_{xh_0})\| \leq \frac{2}{\pi^2(4m^2-1)}\|x\|. \tag{3.21}$$

In particular, $\mathcal{A}_{\mathbf{T}} = H$.

Proof. By using the well-known Euler formula $\sin(\pi t) = \frac{1}{2i}(e^{i\pi t} - e^{i\pi t})$ we obtain

$$c_m(\tilde{H}_{xh_0}) = \pi\sum_{n=1}^{\infty}\frac{e^{\lambda_n}+1}{\lambda_n^2 - \lambda_n(4m\pi i) - \pi^2(4m^2-1)}\langle x, e_n\rangle e_n. \tag{3.22}$$

Since $e^{\lambda_n} + 1 \leq 2$ and $|\lambda_n^2 - \lambda_n(4m\pi i) - \pi^2(4m^2-1)|^2 \geq \pi^4(4m^2-1)^2$, (3.22) and Bessel's inequality yield (3.21). $\square$

Corollary 3.11. *Let $\mathbf{T}$ be the strongly continuous semigroup in Example 3.9. The evolution family $\mathcal{U} = \{U(t,s)\}$ defined in (3.18) is uniformly exponentially stable provided the series*

$$\Big(\sum_{n\geq 1} -\frac{1}{\lambda_n}\Big)$$

converges.

Proof. Since $a(t) \geq 1$ for every $t \geq 0$ and taking into account (3.19) it follows that the integral $\int_0^{\infty}|\langle U(t,0)x, y\rangle|dt$ is convergent if and only if the integral $\int_0^{\infty}|\langle T(t)x, y\rangle|dt$ has the same property. On the other hand a simple calculation gives

$$\langle T(t)x, y\rangle = \sum_{n=1}^{\infty} e^{\lambda_n t}\langle x, e_n\rangle\overline{\langle y, e_n\rangle}$$

and this yields

$$\int_0^{\infty}|\langle T(t)x, y\rangle|dt \leq \sum_{n=1}^{\infty} -\frac{1}{\lambda_n}|\langle x, e_n\rangle\overline{\langle y, e_n\rangle}| \leq \sum_{n=1}^{\infty} -\frac{1}{\lambda_n}\|x\|\|y\|\,.$$

Then we obtain the conclusion. $\square$

4. Determined spectral growth condition for evolution semigroups

Let X be a complex Banach space and let $\mathbf{T} := \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup acting on X.

Lemma 4.1. *The operator $\mathcal{T}(t)$ given by*
$$(\mathcal{T}(t)f)(s) = T(t)f(s-t), s \in \mathbb{R}, t \geq 0, f \in \mathcal{X} \tag{4.1}$$
is well defined and acts on $\mathcal{X}$. Moreover, the family $\{\mathcal{T}(t), t \geq 0\}$ is a strongly continuous semigroup on $\mathcal{X}$, called the evolution semigroup associated to $\mathbf{T}$ on $\mathcal{X}$.

Proof. Let $f \in \mathcal{X}$ and $m \in \mathbb{Z}$. Then the map $s \mapsto e^{-2i\pi ms}f(s) : \mathbb{R} \to X$ belongs to $\mathcal{X}$ and for each $t \geq 0$ one has
$$\begin{aligned} c_m(\mathcal{T}(t)f) &= \int_0^1 e^{-2i\pi ms}T(t)f(s-t)ds \\ &= \int_{-t}^{1-t} e^{-2i\pi m(t+\tau)}T(t)f(\tau)d\tau \\ &= e^{-2i\pi mt}T(t)\int_0^1 e^{-2i\pi ms}f(s)ds \\ &= e^{-2i\pi mt}T(t)c_m(f). \end{aligned} \tag{4.2}$$
Now, since $\mathbf{T}$ is exponentially bounded there exist positive constants M and ω such that
$$\|\mathcal{T}(t)f\|_1 \leq \sum_{m\in\mathbb{Z}} \|T(t)\|\|c_m(f)\| \leq Me^{\omega t}\|f\|_1 < \infty. \tag{4.3}$$
Moreover, for $0 < t \leq 1$ we have $\|\mathcal{T}(t)f\|_1 \leq S_1(t) + S_2(t)$, where
$$\begin{gathered} S_1(t) = \sum_{m\in\mathbb{Z}} \|T(t)c_m(f) - c_m(f)\|, \\ S_2(t) = \sum_{m\in\mathbb{Z}} \|c_m(f)\||1 - e^{2i\pi mt}|. \end{gathered}$$
Since,
$$\begin{gathered} \|T(t)c_m(f) - c_m(f)\| \leq \sup_{t\in[0,1]} \|T(t) - I\|\|c_m(f)\|, \\ \|c_m(f)\||1 - e^{2i\pi mt}| \leq 2\|c_m(f)\|, \end{gathered}$$
the Dominated Convergence Theorem assures that $S_1(t)$ and $S_2(t)$ converge to 0 as $t \to 0+$; that is, the semigroup $\{\mathcal{T}(t)\}$ is strongly continuous. $\square$

Proposition 4.2. *Let $\mathbf{T}$ and $\mathcal{T}$ be the semigroups from the above Lemma 4.1. Then $\omega_0(\mathbf{T}) = \omega_0(\mathcal{T})$.*

Proof. Using (4.2) we obtain
$$\|\mathcal{T}(t)f\|_1 \leq \|T(t)\|\|f\|_1, \quad \forall f \in \mathcal{X}. \tag{4.4}$$
Thus $\omega_0(\mathbf{T}) \leq \omega_0(\mathcal{T})$.

To establish the converse inequality let $f_x(t) = (h_0 \otimes x)(t) := h_0(t)x$. An easy argument shows that $\tilde{f}_x$ (the continuation by periodicity of f_x to the real axis) belongs to $\mathcal{X}$ for each $x \in X$ and moreover
$$c_m(\tilde{f}_x) = \frac{2}{\pi(1-4m^2)}x. \quad \forall m \in \mathbb{Z}. \tag{4.5}$$

Then (4.2) and (4.5) yield $\|\mathcal{T}(t)f_x\|_1 = \frac{4}{\pi}\|T(t)x\|$ and $\|f_x\|_1 = \frac{4}{\pi}\|x\|$. Thus

$$\|\mathcal{T}(t)\|_{\mathcal{L}(\mathcal{X})} \geq \sup_{x\neq 0} \frac{\frac{4}{\pi}\|T(t)x\|}{\frac{4}{\pi}\|x\|} = \|T(t)\|. \tag{4.6}$$

□

Theorem 4.3. *With the above notation if*

$$\sup_{\mu\in\mathbb{R}} \sup_{t\geq 0} \| \int_0^t e^{i\mu s} T(t-s)x\, ds\| := M(x) < \infty \quad \forall x \in X, \tag{4.7}$$

then

$$\sup_{\mu\in\mathbb{R}} \sup_{t\geq 0} \| \int_0^t e^{i\mu s} \mathcal{T}(t-s)f ds\|_{\mathcal{X}} := N(f) < \infty \quad \forall f \in \mathcal{X}. \tag{4.8}$$

Proof. Let $f \in \mathcal{X}$ and $m \in \mathbb{Z}$. Using [35, Lemma 2.2], one has

$$\begin{aligned}
&c_m\Big(\int_0^t e^{i\mu s}\mathcal{T}(t-s)f ds\Big)\\
&= \int_0^1 \Big(\int_0^t e^{i\mu s - 2im\pi\tau} T(t-s)f(\tau - t + s)ds\Big)\, d\tau\\
&= \int_0^t \int_0^1 e^{i\mu s - 2im\pi\tau} T(t-s)f(\tau - t + s)d\tau ds\\
&= \int_0^t \int_{s-t}^{s-t+1} e^{i\mu s - 2im\pi\tau} T(t-s)f(\tau - t + s)d\tau ds\\
&= \int_0^t e^{i\mu s - 2im\pi(t-s)} T(t-s)\Big(\int_{s-t}^{s-t+1} e^{-2im\pi\rho} f(\rho)d\rho\Big)ds\\
&= e^{i\mu t}\int_0^t e^{-(i\mu+2im\pi)r} T(r)c_m(f)dr.
\end{aligned}$$

Now (4.8) is an easy consequence of (4.7) and the uniform boundedness principle; indeed, let $M > 0$ be such that

$$\| \int_0^t e^{i\mu s} T(t-s)x\, ds\| \leq M\|x\| \quad \forall t \geq 0, \forall x \in X,$$

and then

$$\begin{aligned}
\| \int_0^t e^{i\mu s} \mathcal{T}(t-s)f ds\|_{\mathcal{X}} &= \sum_{m\in\mathbb{Z}} \| \int_0^t e^{-(i\mu+2im\pi)r} T(r)c_m(f)dr\|\\
&\leq M \sum_{m\in\mathbb{Z}} \|c_m(f)\|\\
&= M\|f\|_{\mathcal{X}} := N_f.
\end{aligned}$$

□

Remark 4.4. We know [29, Theorem 2] that if X is a Hilbert space then (4.7) implies the negativeness of $\omega_0(\mathbf{T})$ and then, via Proposition 4.2, the evolution semigroup $\mathcal{T}$ is uniformly exponentially stable and (4.8) follows automatically. It could be of some interest if the converse statement in Theorem 4.3 is true or not.

Corollary 4.5. *Let X and $\mathbf{T}$ as in Example 2.2 and let $\mathcal{T}$ the evolution semigroup associated to $\mathbf{T}$ on the Banach space of functions $\mathcal{X}$. Denote by G the generator of $\mathcal{T}$. Then*

$$s_0(G) < 0 \le \omega_0(\mathcal{T}). \tag{4.9}$$

Proof. Via Proposition 4.2 one has

$$\omega_0(\mathcal{T}) = \omega_0(\mathbf{T}) = \frac{1}{2}.$$

Since $\mathbf{T}$ satisfies (4.7), $\mathcal{T}$ satisfies (4.8) (via Theorem 4.3) and from Theorem 2.1 the uniform spectral bound $s_0(G)$ is negative. □

5. The discrete case revisited

Let $q \ge 2$ be a integer number, X be a complex Banach space and let $\mathcal{U} = \{U(n,m) : n \ge m \ge 0\} \subset \mathcal{L}(X)$ be a q-periodic discrete evolution family on X, that is $U(n,n)x = x, U(m,n)U(n,r) = U(m,r)$ and $U(n+q, m+q) = U(n,m)$ hold for all nonnegative integers m,n,r with $m \ge n \ge r$ and all $x \in X$. We denote by $T_q := U(q,0)$ the monodromy operator associated with the evolution family $\mathcal{U}$. It is well known that the family $\mathcal{U}$ is uniformly exponentially stable, that is, there exists the positive constants N and ν such that

$$\|U(n,m)\|_{\mathcal{L}(X)} \le Ne^{-\nu(n-m)} \text{ for all } n \ge m$$

if and only if the spectral radius of T_q,

$$r(T_q) := \lim_{k\to\infty} \|T_q^k\|^{1/k},$$

is less than 1.

With any X-valued continuous and q-periodic function defined on $\mathbb{R}$ we associate the Fourier-Bohr coefficients given by

$$c_n(f) := \frac{1}{q}\int_0^q e^{-2i\pi nt/q} f(t)dt, \tag{5.1}$$

and its norm

$$\sum_{n\in\mathbb{Z}} \|c_n(f)\| := \|f\|_1.$$

Let $P_q(\mathbb{Z}, X)$ denote the space of all X-valued and q-periodic sequences defined on $\mathbb{Z}$.

Lemma 5.1. *Let $w = (w_m)_{m\in\mathbb{Z}}$ be a sequence in $P_q(\mathbb{Z}, X)$ with $w_0 = 0$. There exists a X-valued continuous and q-periodic function defined on $\mathbb{R}$ with $\|f\|_1 < \infty$ and such that $f(m) = w_m$ for all integers m.*

Proof. We highlight a function f whose Range is included in the space $Y := \operatorname{span}[\operatorname{Range}(w)]$ which is finite dimensional so it is a Hilbert space with an equivalent norm. Therefore the continuous and q-periodic function, defined on $[0,q]$ by

$$f(t) = \begin{cases} w(t), & \text{if } t \in [0,q] \cap \mathbb{Z} \\ \text{linear}, & \text{if } t \in (k, k+1),\ k = 0, 1, \dots, q-1, \end{cases} \tag{5.2}$$

is Y-valued and Lipschitz continuous. Taking into account that $c_m(f) \in Y$ for every $m \in \mathbb{Z}$, we have that $\|f\|_1 < \infty$ (cf. [9, Lemma 1.3]). □

Theorem 5.2. *Let $q \geq 2$ be an integer, X be a complex Banach space and let $\mathcal{U} = \{U(n,m) : n \geq m\}$ be a q-periodic evolution family of bounded linear operators acting on X. The discrete evolution semigroup associated to the evolution family $\mathcal{U}$ on the space $P_q(\mathbb{Z}, X))$ is defined by*

$$(\mathcal{T}(n)w)(m) := U(m, m-n)w(m-n), \quad n \in \mathbb{Z}_+, m \in \mathbb{Z}. \tag{5.3}$$

The following three statements are equivalent:

1. $\sup_{\mu\in\mathbb{R}} \sup_{n\in\mathbb{Z}_+} \|\sum_{k=0}^{n} e^{i\mu k} U(n,k)x\| := M_1(x) < \infty$, *for all* $x \in X$.
2. $\sup_{\mu\in\mathbb{R}} \sup_{n\in\mathbb{Z}_+} \|\big(\sum_{k=0}^{n} e^{i\mu k}\mathcal{T}(n-k)w\big)(n)\| := M_2(w) < \infty$, *for all* $w \in P_q(\mathbb{Z}, X)$ *with* $w_0 = 0$.
3. *The family $\mathcal{U}$ is uniformly exponentially stable.*

Proof. (1) $\Rightarrow$ (2). Let f be a function as in the previous Lemma 5.1. Thus for each integer m one has

$$w_m = \sum_{j\in\mathbb{Z}} e^{2\pi ij\frac{m}{q}} c_j(f), \tag{5.4}$$

which yields

$$\begin{aligned}
\Big(\sum_{k=0}^{n} e^{i\mu k}\mathcal{T}(n-k)w\Big)(n) &= \sum_{k=0}^{n} e^{i\mu k} U(n,k)w_k \\
&= \sum_{k=0}^{n}\sum_{j\in\mathbb{Z}} e^{i(\mu k + 2\pi j\frac{k}{q})} U(n,k)c_j(f) \\
&= \sum_{j\in\mathbb{Z}}\sum_{k=0}^{n} e^{i(\mu k + 2\pi j\frac{k}{q})} U(n,k)c_j(f).
\end{aligned} \tag{5.5}$$

Now, the conclusion is an easy consequence of the uniformly boundedness principle.

(2) $\Rightarrow$ (3). Is an easy consequence of the first identity in (5.5) and [11, Theorem 2].

(3) $\Rightarrow$ (1). This is clear. $\square$

Remark 5.3. The equivalence between the first statement and the third one in Theorem 5.2 was stated earlier in [2, Theorem 1] using a different approach.

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