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ANISOTROPIC LOGARITHMIC SOBOLEV INEQUALITY WITH A GAUSSIAN WEIGHT AND ITS APPLICATIONS

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Abstract. In this article we prove a Logarithmic Sobolev type inequality and a Poincaré type inequality for functions in the anisotropic Gaussian Sobolev space. As an application we study a class of equations, whose anisotropic elliptic condition is given in term of the density of Gauss measure. Finally some extensions of the main results are given for a class of weighted (not Gaussian one) anisotropic Sobolev spaces.

1. Introduction

In the previous years anisotropic problems and spaces have been extensively studied by many authors, motivated by their applications to the mathematical modeling of physical and mechanical processes in anisotropic continuous medium.

Let $N \geq 2$ and $1 \leq p_1, \dots, p_N < +\infty$. Roughly speaking an anisotropic Sobolev space is a space of functions u such that i-th partial derivative of u belongs to the Lebesgue space L^{p_i} with some exponent p_i. If Ω is a bounded open set of $\mathbb{R}^N$ with Lipschitz continuous boundary a Sobolev type inequality (see e.g. [24] for Sobolev type inequalities in the Lebesgue spaces and [23] for similar inequalities in the Lorentz spaces) holds for functions belonging to the anisotropic Sobolev space $W_0^{1,\overrightarrow{p}}(\Omega)$, defined as the closure of $C_0^\infty(\Omega)$ with respect to norm $\sum_{i=1}^N \|\partial_{x_i} u\|_{L^{p_i}(\Omega)}$. Indeed (see e.g. [24]) there exists a constant C_S such that

$$\|u\|_{L^q(\Omega)} \leq C_S \sum_{i=1}^N \|\partial_{x_i} u\|_{L^{p_i}(\Omega)} \quad \forall u \in W_0^{1,\overrightarrow{p}}(\Omega), \tag{1.1}$$

where $q = \overline{p}^* = \frac{N\overline{p}}{N-\overline{p}}$ if $\overline{p} < N$ or $q \in [1, +\infty[$ if $\overline{p} \geq N$ and $\overline{p}$ denote the harmonic mean, i.e. $\frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^N \frac{1}{p_i}$. Moreover an anisotropic Poincaré inequality holds (see e.g. [15]):

$$\|u\|_{L^{p_i}(\Omega)} \leq \frac{p_i}{2} c(\Omega) \|\partial_{x_i} u\|_{L^{p_i}(\Omega)} \quad \forall u \in W_0^{1,\overrightarrow{p}}(\Omega)\ \forall i = 1, \dots, N, \tag{1.2}$$

where $c(\Omega) = \sup_{x,y\in\Omega} (x - y, \mathbf{e_i})$, $\{\mathbf{e}_1, \dots, \mathbf{e}_N\}$ is the canonical basis of $\mathbb{R}^N$ and $(\cdot,\cdot)$ denotes the standard scalar product in $\mathbb{R}^N$. When $\overline{p} < N$, inequality (1.1) implies the continuous embedding of $W_0^{1,\overrightarrow{p}}(\Omega)$ into $L^q(\Omega)$ for every $q \in [1, \overline{p}^*]$. On

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the other hand the continuity of the embedding $W_0^{1,\overrightarrow{p}}(\Omega) \subset L^{p_{\max}}(\Omega)$ with $p_{\max} := \max\{p_1, \dots, p_N\}$ relies on inequality (1.7). Then $p_\infty := \max\{\overline{p}^*, p_{\max}\}$ turns out to be the critical exponent: there is a continuous embedding $W_0^{1,\overrightarrow{p}}(\Omega) \subset L^q(\Omega)$ for $q \in [1, p_\infty]$.

Also there is an increasing interest to Sobolev type inequalities involving weighted Sobolev spaces. In this article we take into account the Gaussian weight. In this contest Gross [17] proved (see [14] for more comments and references) the inequality

$$\int_{\mathbb{R}^N} |u|^p \log |u| d\gamma \leq \frac{p}{2} \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{p-2} \, d\gamma + \|u\|^p_{L^p(\mathbb{R}^N,\gamma)} \log \|u\|_{L^p(\mathbb{R}^N,\gamma)} \tag{1.3}$$

for $u \in W^{1,p}(\mathbb{R}^N, \gamma)$ with $1 < p < +\infty$, where γ states for the Gauss measure. Unlike the classical Sobolev inequality it is independent on dimension and easily extends to the infinite dimensional case. In terms of functional spaces inequality (1.3) implies the imbedding of weighted Sobolev space $W^{1,p}(\mathbb{R}^N, \gamma)$ into the weighted Zygmund space $L^p(\log L)^{1/2}(\mathbb{R}^N, \gamma)$ (see §2.2 for definition). The imbedding holds for $p = 1$ as well and it is related to Gaussian isoperimetric inequality. For $p = 2$ Gross inequality (1.3) entails [9] that

$$\|u - u_\gamma\|^*_{L^2(\log L)^{1/2}(\mathbb{R}^N,\gamma)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^N,\gamma)},$$

where $u_\gamma := \int_{\mathbb{R}^N} u(x) \, \gamma$ and then

$$\|u\|^*_{L^2(\log L)^{1/2}(\mathbb{R}^N,\gamma)} \leq \; C(\|\nabla u\|_{L^2(\mathbb{R}^N,\gamma)} + \|u\|_{L^2(\mathbb{R}^N,\gamma)})$$

for some constant independent on the dimension. Here $\|\cdot\|^*_{L^2(\log L)^{1/2}(\mathbb{R}^N,\gamma)}$ states for the rearrangement invariant quasinorm in the Zygmund space $L^2(\log L)^{1/2}(\mathbb{R}^N, \gamma)$ (see §2.2 for definition). This kind of inequalities hold for $1 \leq p < +\infty$ as well (see e.g. [9, 20]).

A Logarithmic Sobolev-Poincaré inequality (see [13]) is proved in for functions in the weighted Sobolev space $W_0^{1,p}(\Omega, \gamma)$, defined as the closure of $C_0^\infty(\Omega)$ with respect to norm $\|\nabla u\|_{L^p(\Omega,\gamma)}$. Let $1 \leq p < +\infty$ and Ω be an open subset of $\mathbb{R}^N$ (not necessary bounded) with $\gamma(\Omega) < 1$ and $u \in W_0^{1,p}(\Omega, \gamma)$, then $u \in L^p(\log L)^{1/2}(\Omega, \gamma)$ and

$$\|u\|^*_{L^p(\log L)^{1/2}(\Omega,\gamma)} \leq C \|\nabla u\|_{L^p(\Omega,\gamma)} \tag{1.4}$$

for some constant C depending only on p and $\gamma(\Omega)$ (see [13]). Using the continuous embedding between Zygmund spaces the previous inequality yields the following Poincaré inequality:

$$\|u\|_{L^p(\Omega,\gamma)} \leq \; C \|\nabla u\|_{L^p(\Omega,\gamma)} \quad \forall u \in W_0^{1,p}(\Omega, \gamma) \tag{1.5}$$

for some positive constant C(independent of u, but depending on $\gamma(\Omega)$ and p). If $\gamma(\Omega) = 1$ inequality (1.4) holds if in the right hand-side we take into account the norm of u as well. We recall that the analogue inequality when $u \in W^{1,p}(\Omega, \gamma)$ is studied in [13] and, as one can expect, smoothness assumption on $\partial\Omega$ has to be required.

In this article we consider functions such that i-th partial derivative of u belongs to some weighted (with respect to Gauss measure) Lebesgue space L^{p_i} for some exponent p_i. More precisely we prove a Logarithmic Sobolev inequality type inequality and a Poincaré type inequality for functions belonging to the anisotropic

Gaussian Sobolev space $W_0^{1,\overrightarrow{p}}(\Omega,\gamma)$ (see §2 for the definition). As in the Gaussian isotropic case we have to require some additional hypothesis on Ω. Instead of $\gamma(\Omega)<1$ we assume that Ω is an open subset of $\mathbb{R}^N$ such that

(H1) $\gamma_1(a_i,b_i)<1$ with $a_i=\inf_{x\in\Omega}(x,\mathbf{e_i})$ and $b_i=\sup_{x\in\Omega}(x,\mathbf{e_i})$. for a fixed $i=1,\dots,N$.

Theorem 1.1. *Let $N\geq 2$ and Ω be an open subset of $\mathbb{R}^N$ with Lipschitz boundary such that* (H1) *holds for a fixed $i=1,\dots,N$. Then for any $u\in W_0^{1,\overrightarrow{p}}(\Omega,\gamma)$,*

$$\begin{aligned}&\Big(\int_\Omega |u(x)|^{p_i}\log^{p_i/2}(2+|u(x)|)\varphi(x)\,dx\Big)^{1/p_i}\\&\leq c_1(p_i)\Big[\frac{1}{2}\Big(1-\frac{1}{\log(\gamma_1(a_i,b_i))}\Big)\Big]^{1/2}\|\frac{\partial u}{\partial x_i}\|_{L^{p_i}(\Omega,\gamma)},\end{aligned}\tag{1.6}$$

where $c_1(p_i)$ is a constant depending only on p_i.

Moreover an anisotropic Poincaré inequality holds.

Theorem 1.2. *Let $N\geq 2$ and Ω be an open subset of $\mathbb{R}^N$ with Lipschitz boundary such that* (H1) *holds for a fixed $i=1,\dots,N$. Then for any $u\in W_0^{1,\overrightarrow{p}}(\Omega,\gamma)$,*

$$\|u\|_{L^{p_i}(\Omega,\gamma)}\leq c_2(p_i)\big[-\frac{1}{2\log(\gamma_1(a_i,b_i))}\big]^{1/2}\|\partial_{x_i}u\|_{L^{p_i}(\Omega,\gamma)},\tag{1.7}$$

where $c_2(p_i)$ is a constant depending only on p_i.

The constants in (1.6) and (1.7) depend on the 1-dimensional Gauss measure of the diameter of Ω in i-th direction as it happens in (1.2), where the Lebesgue measure of the diameter is involved. In order to emphasize that a 1-dimensional measure is considered we used the notation γ_1.

We stress that hypothesis (H1) guarantees that a_i or b_i is finite. Let us consider examples of sets for which the previous theorem can be applied. If $\Omega=\{x\in\mathbb{R}^N: x_1>\omega\}$ or $\Omega=\{x\in\mathbb{R}^N: x_1<\omega\}$ for some $\omega\in\mathbb{R}$, then the previous theorems hold for $i=1$ but not for $i=2,\dots,N$. Instead if we take into account $\Omega=\{x\in\mathbb{R}^N: x_i>\omega_i \text{ for } i=1,\dots,N\}$ for some $\omega_1,\dots,\omega_N\in\mathbb{R}$, Theorems 1.1 and 1.2 holds for any $i=1,\dots,N$.

Corollary 1.3. *Let $N\geq 2$ and Ω be an open subset of $\mathbb{R}^N$ with Lipschitz boundary such that* (H1) *is in force for any $i=1,\dots,N$. Then for any $u\in W_0^{1,\overrightarrow{p}}(\Omega,\gamma)$*

$$\begin{aligned}&\Big(\int_\Omega |u(x)|^{p_{\max}}\log^{\frac{p_{\max}}{2}}(2+|u(x)|)\varphi(x)\,dx\Big)^{\frac{1}{p_{\max}}}\\&\leq c(p_{\max})\Big[\frac{1}{2}\Big(1-\frac{1}{\log(\gamma_1(a_{\max},b_{\max}))}\Big)\Big]^{1/2}\sum_{i=1}^N\|\frac{\partial u}{\partial x_i}\|_{L^{p_i}(\Omega,\gamma)},\end{aligned}\tag{1.8}$$

where $p_{\max}=\max\{p_1,\dots,p_N\}=p_j$ for some $j\in\{1,\dots,N\}$, $a_{\max}=a_j$, $b_{\max}=b_j$ and $c(p_{\max})$ is a constant depending only on $p_{\max}$.

An anisotropic Sobolev-Poincaré inequality with Orlicz target norm reads as follows:

$$\|u\|_{L^{p_{\max}}(\log L)^{1/2}}\leq c(p_{\max})\Big[\frac{1}{2}\Big(1-\frac{1}{\log(\gamma_1(a_{\max},b_{\max}))}\Big)\Big]^{1/2}\sum_{i=1}^N\|\frac{\partial u}{\partial x_i}\|_{L^{p_i}(\Omega,\gamma)},$$

where $\|u\|_{L^{p_{\max}}(\log L)^{1/2}}$ is the Orlicz norm of u in $L^{p_{\max}}(\log L)^{1/2}(\Omega,\gamma)$ (see (2.5) for a definition). As consequence the continuous embedding of $W_0^{1,\overrightarrow{p}}(\Omega,\gamma)$ into the Zygmund space $L^{p_{\max}}(\log L)^{1/2}(\Omega,\gamma)$ is proved.

This embedding and the previous results are no longer true if the zero trace condition on the boundary is removed as in the isotropic case with weight or not. Moreover we stress that if (H1) is not in force inequality (1.6) holds if in the right hand-side we take into account the norm of u as well.

This article is organized as follows. In Section 2 we recall some definitions and properties. The main results are proved in Section 3. Section 4 is dedicated to the application of these inequalities to study a class of equations, whose anisotropic elliptic condition is given in term of the density of Gauss measure. In particular we prove some uniqueness results. Finally in the last section we show some generalization of our main results to a class of weighted (not Gaussian one) anisotropic Sobolev spaces.

2. Preliminaries

2.1. **Gauss measure and Gaussian rearrangements.** Let γ be the N-dimensional Gauss measure on $\mathbb{R}^N$ defined by

$$d\gamma := \varphi(x)dx := (2\pi)^{-N/2}\exp\Big(-\frac{|x|^2}{2}\Big)\,dx, \quad x\in\mathbb{R}^N$$

normalized by $\gamma(\mathbb{R}^N)=1$.

Let $N\geq 2$ and Ω be an open subset of $\mathbb{R}^N$ not necessary bounded with Lipschitz boundary and let $1\leq p_1,\dots,p_N<\infty$ be N real numbers. A measurable function u belongs to $L^{p_i}(\Omega,\gamma)$ if $\int_\Omega |u|^{p_i}\,d\gamma<+\infty$. The anisotropic Gaussian Sobolev space (see e.g. [24] for the definition without weight) is defined as

$$W^{1,\overrightarrow{p}}(\Omega,\gamma)=\{u\in W^{1,1}(\Omega,\gamma): u_{x_i}\in L^{p_i}(\Omega,\gamma) \text{ for } i=1,\dots,N\}$$

and is a Banach space with respect to the norm

$$\|u\|_{W^{1,\overrightarrow{p}}(\Omega,\gamma)}=\|u\|_{L^1(\Omega,\gamma)}+\sum_{i=1}^N\|u_{x_i}\|_{L^{p_i}(\Omega,\gamma)}. \tag{2.1}$$

As usual the space $W_0^{1,\overrightarrow{p}}(\Omega,\gamma)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.1). When inequality (1.7) is in force, the norm (2.1) is equivalent to the norm that involves only the partial derivatives.

It is well known that an isoperimetric inequality with respect to Gauss measure (see e.g. [10]) holds. For all subsets $E\subset\mathbb{R}^N$ it follows that $P(E)\geq\varphi(\Phi^{-1}(\gamma(E)))$, where $\Phi(\tau)$ is the Gauss measure of the half-space $\{x\in\mathbb{R}^N : x_1>\tau\}$ for every $\tau\in\mathbb{R}\cup\{-\infty\}$ and $P(E)$ is the perimeter with respect to Gauss measure of E. We recall that the isoperimetric function $I_\gamma(t)$ has the following asymptotic behavior

$$I_\gamma(t):=\varphi(\Phi^{-1}(t))\sim t\Big(2\log\frac{1}{t}\Big)^{1/2} \quad \text{for } t\to 0^+;1^-. \tag{2.2}$$

Following [10] we define the one dimensional Gaussian decreasing rearrangement of u by

$$u^\circledast(s)=\inf\{t\geq 0:\gamma(\{x\in\Omega:|u(x)|>t\})\leq s\} \quad s\in]0,1],$$

and the Gaussian rearrangement of u by $u^\star(x)=u^\circledast(\Phi(x_1))$ for $x\in\Omega^\star$, where

$$\Omega^\star=\{x=(x_1,\dots,x_N)\in\mathbb{R}^N : x_1>\omega\}$$

is the half-space such that $\gamma(\Omega^\star) = \gamma(\Omega)$. A Polya-Szëgo type inequality (see [22]) holds for $1 \le p < +\infty$:
$$\|\nabla u^\star\|_{L^p(\Omega^\star,\gamma)} \le \|\nabla u\|_{L^p(\Omega,\gamma)} \quad \forall u \in W_0^{1,p}(\Omega,\gamma). \tag{2.3}$$

2.2. **Zygmund spaces.** We say that a measurable u belongs to the Zygmund space (see e.g. [3]) $L^r(\log L)^\alpha(\Omega,\gamma)$ for $1 \le r < +\infty$ and $\alpha \in \mathbb{R}$ if the quantity
$$\|u\|^*_{L^r(\log L)^\alpha(\Omega,\gamma)} := \Big(\int_0^{\gamma(\Omega)} [(1-\log t)^\alpha u^\circledast(t)]^r\,dt\Big)^{1/r} \tag{2.4}$$
is finite. We emphasize that the Zygmund space $L^r(\log L)^\alpha(\Omega,\gamma)$ coincides with the Lebesgue space $L^r(\Omega,\gamma)$ when $\alpha = 0$. If $1 < r < p < \infty$ and $-\infty < \alpha, \beta < \infty$ we obtain $L^p(\log L)^\alpha(\Omega,\gamma) \hookrightarrow L^r(\log L)^\beta(\Omega,\gamma)$. It is clear from (2.4) that the space $L^r(\log L)^\alpha(\varphi,\Omega)$ decreases as α increases. We remark that (2.4) is a quasinorm and is equivalent to the norm obtained replacing $u^\circledast(t)$ with $u^{\circledast\circledast}(t) := \frac{1}{t}\int_0^t u^\circledast(s)\,ds$ for $p > 1$. Moreover u, $u^\circledast$ and $u^\star$ have the same Zygmund quasinorm.

If we consider the Zygmund space $L^r(\log L)^\alpha(\Omega,\gamma)$ as an Orlicz space, a measurable function u belongs to it if and only if $[|u(x)|\log^\alpha(2+|u(x)|)]^r$ is integrable with respect to γ. Moreover its Orlicz norm is defined as
$$\|u\|_{L^r(\log L)^\alpha(\Omega,\gamma)} := \inf\Big\{\lambda > 0 : \int_\Omega \big[\big|\frac{u(x)}{\lambda}\big|\log^\alpha(2+\big|\frac{u(x)}{\lambda}\big|)\big]^r\,d\gamma \le 1\Big\} \tag{2.5}$$
and it is not in general equivalent to the quasinorm (2.4). The following inequality is useful in working with Zygmund spaces.

Proposition 2.1. *Suppose $r > 0, 1 \le q < +\infty$ and $-\infty < \alpha < +\infty$. Let ψ be a nonnegative measurable function on $(0,b)$ with $0 < b \le 1$, then the following inequality holds:*
$$\Big(\int_0^1 \Big(t^r(1-\log t)^\alpha \int_t^1 \psi(s)\,ds\Big)^q \frac{dt}{t}\Big)^{1/q} \le c\Big(\int_0^1 (t^{1+r}(1-\log t)^\alpha\psi(t))^q\frac{dt}{t}\Big)^{1/q} \tag{2.6}$$
with a constant $c = c(r,q,\alpha)$ are independent on ψ and on b.

When $b = 1$ the previous inequality is proved in [3]. The proof works for $0 < b < 1$ as well and it is easy to check that the constant does not depend on b.

3. Proofs of main results

The idea is to estimate u as a function of x_i variable using the Logarithmic Sobolev-Poincaré inequality with respect to Gauss measure in dimension 1. To do this we need an explicit dependence of the involved constant with respect to the data. For convenience of the reader we detail the dependence of such a constant.

3.1. **Comments on logarithmic Sobolev-Poincaré inequality** (1.4)**.** The main aim of this subsection is to obtain an explicit dependence of the constant with respect to the domain Ω in (1.4). The proof of this inequality is based on properties of rearrangements of functions and on asymptotic behaviour Gaussian isoperimetric function.

Let $u \in W_0^{1,p}(\Omega,\gamma)$. First we observe that
$$(1-\log t) \le \Big(1 - \frac{1}{\log(\gamma(\Omega))}\Big)\log\frac{1}{t} \quad \text{for } 0 < t < \gamma(\Omega). \tag{3.1}$$

Using (2.4), (2.6), (2.2), (3.1) and (2.3) we obtain

$$
\begin{aligned}
&\Big(\|u\|^*_{L^p(\log L)^{1/2}(\Omega,\gamma)}\Big)^p \\
&\le c(p)\int_0^{\gamma(\Omega)} \big[t(1-\log t)^{1/2}\big|\frac{d}{dt}u^{\circledast}(t)\big|\big]^p dt \\
&\le c(p)\Big[\frac{1}{2}\Big(1-\frac{1}{\log(\gamma(\Omega))}\Big)\Big]^{p/2}\int_0^{\gamma(\Omega)} \Big[\big|\frac{d}{dt}u^{\circledast}(t)\big|\varphi_1(\Phi^{-1}(t))\Big]^p dt \\
&= c(p)\Big[\frac{1}{2}\Big(1-\frac{1}{\log(\gamma(\Omega))}\Big)\Big]^{p/2}\|\nabla u^\star\|^p_{L^p(\Omega^\star,\gamma)} \\
&\le c(p)\Big[\frac{1}{2}\Big(1-\frac{1}{\log(\gamma(\Omega))}\Big)\Big]^{p/2}\|\nabla u\|^p_{L^p(\Omega,\gamma)}
\end{aligned}
$$

for some positive constant $c(p)$depending on p, that can be vary from line to line, yielding

$$
\Big(\|u\|^*_{L^p(\log L)^{1/2}(\Omega,\gamma)}\Big)^p \le c(p)\Big[\frac{1}{2}\Big(1-\frac{1}{\log(\gamma(\Omega))}\Big)\Big]^{p/2}\|\nabla u\|^p_{L^p(\Omega,\gamma)}, \tag{3.2}
$$

i.e. (1.4) with an explicit dependence of the constant on the set Ω.

An easy consequence of (3.2) is the inequality

$$
\begin{aligned}
&\int_\Omega |u(x)|^p \log^{p/2}(2+|u(x)|)\varphi(x)\,dx \\
&\le c_3(p)\Big[\frac{1}{2}(1-\frac{1}{\log(\gamma(\Omega))})\Big]^{p/2}\|\nabla u\|^p_{L^p(\Omega,\gamma)}
\end{aligned} \tag{3.3}
$$

for some positive constant $c_3(p)$ depending on p. Indeed observing that by properties of rearrangements of functions it follows that

$$
u^{\circledast}(t) \le u^{\circledast\circledast}(t) := \frac{1}{t}\int_0^t u^{\circledast}(s)\,ds \le \frac{\|u\|_{L^1(\Omega,\gamma)}}{t}
$$

and using (3.2) we obtain

$$
\begin{aligned}
&\int_\Omega |u(x)|^p \log^{p/2}(2+|u(x)|)\varphi(x)\,dx \\
&\le \int_0^{\gamma(\Omega)} \big[u^{\circledast}(t)\log^{1/2}(2+u^{\circledast}(t))\big]^p dt \\
&\le \int_0^{\gamma(\Omega)} [u^{\circledast}(t)\log^{1/2}\Big(2+\frac{\|u\|_{L^1(\Omega,\gamma)}}{t}\Big)]^p dt \\
&\le c(p)\int_0^{\gamma(\Omega)} \big[u^{\circledast}(t)(1-\log t)^{1/2}\big]^p dt \\
&:= c(p)\big(\|u\|^*_{L^p(\log L)^{1/2}(\Omega,\gamma)}\big)^p \\
&\le c(p)\Big[\frac{1}{2}\Big(1-\frac{1}{\log(\gamma(\Omega))}\Big)\Big]^{p/2}\|\nabla u\|^p_{L^p(\Omega,\gamma)}.
\end{aligned}
$$

where $c(p)$ is a positive constant depending on p, that can be vary from line to line.

3.2. **From 1-dimensional to anisotropic logarithmic Sobolev-Poincaré inequality.** By denseness it is sufficient to prove (1.6) for $u \in C_0^1(\Omega)$. Let us fix $i \in \{1,\dots,N\}$ such that $\gamma_1(a_i,b_i) < 1$ with $a_i = \inf_{x\in\Omega}(x,\mathbf{e_i})$ and $b_i = \sup_{x\in\Omega}(x,\mathbf{e_i})$. This hypothesis guarantees that the section of Ω in the direction $\mathbf{e_i}$ is not all the line *a.e.*. We assume without loss of generality that $a_i \in \mathbb{R}$ and $b_i \in \mathbb{R}\cup\{+\infty\}$. It follows that $\Omega \subseteq \{x \in \mathbb{R}^N : a_i < x_i < b_i\}$. We consider u as defined on the whole $\mathbb{R}^N$, setting to 0 outside suppt(u). For for all $x \in \mathbb{R}^N$, we set $x = (x_i, x')$ in order to emphasize its i-th component. Since the Gauss measure is a product measure we can write $\varphi(x) = \varphi_{N-1}(x')\varphi_1(x_i)$, where φ_{N-1} and φ_1 are the density of the $(N-1)$-dimensional Gauss measure γ_{N-1} and 1-dimensional Gauss measure γ_1 respectively. For any fixed $x' \in \mathbb{R}^{N-1}$, by (3.3) and properties of rearrangement of functions we obtain

$$\begin{aligned}
&\int_{a_i}^{b_i} |u(x_i,x')|^{p_i} \log^{p_i/2}(2+|u(x_i,x')|)\varphi_1(x_i)\,dx_i \\
&\le c(p_i)\Big[\frac12\Big(1-\frac{1}{\log(\gamma(a_i,b_i))}\Big)\Big]^{p_i/2}\int_{a_i}^{b_i} |\partial_{x_i}u(x_i,x')|^{p_i}\varphi_1(x_i)\,dx_i
\end{aligned} \tag{3.4}$$

for some constant $c(p_i)$ depending only on p_i. Now multiplying by $\varphi_{N-1}(x')$ and integrating on $\mathbb{R}^{N-1}$, we obtain

$$\begin{aligned}
&\int_\Omega |u(x)|^{p_i}\log^{p_i/2}(2+|u(x)|)\varphi(x)\,dx \\
&\le c(p_i)\Big[\frac12\Big(1-\frac{1}{\log(\gamma(a_i,b_i))}\Big)\Big]^{p_i/2}\int_\Omega \Big|\frac{\partial}{\partial x_i}u(x)\Big|^{p_i}\varphi(x)\,dx,
\end{aligned}$$

which is (1.6).

3.3. **From 1-D logarithmic Sobolev-Poincaré inequality to anisotropic Poincaré inequality.** By denseness it is sufficient to prove (1.7) for $u \in C_0^1(\Omega)$. We consider the same notations and assumptions of the previous subsection. For any fixed $x' \in \mathbb{R}^{N-1}$, by (3.2) and properties of rearrangement of functions we obtain

$$\begin{aligned}
&\int_{a_i}^{b_i} |u(x_i,x')|^{p_i}\varphi_1(x_i)\,dx_i \\
&= \int_0^{\gamma(a_i,b_i)} |u^{\circledast}(t_i,x')|^{p_i}\,dt \\
&\le \sup_{0<t<\gamma_1(a_i,b_i)}(1-\log t)^{-\frac{p_i}{2}}\int_0^{\gamma_1(a_i,b_i)}\big[u^{\circledast}(t,x')(1-\log t)^{1/2}\big]^{p_i}\,dt \\
&:= \sup_{0<t<\gamma_1(a_i,b_i)}(1-\log t)^{-\frac{p_i}{2}}\|u(\cdot,x')\|_{L^{p_i}(\log L)^{1/2}((a_i,b_i),\gamma_1)} \\
&\le c(p_i)\Big[\frac12\Big(1-\frac{1}{\log(\gamma_1(a_i,b_i))}\Big)\Big]^{p_i/2}\big(1-\log(\gamma_1(a_i,b_i))\big)^{-p_i/2} \\
&\quad\times \int_{a_i}^{b_i}\Big|\frac{\partial}{\partial x_i}u(x_i,x')\Big|^{p_i}\varphi_1(x_i)\,dx_i,
\end{aligned}$$

where $u^{\circledast}(t,x')$ is one dimensional Gaussian decreasing rearrangement of u with respect to x_i, for each x' fixed. Now multiplying by $\varphi_{N-1}(x')$ and integrating on

$\mathbb{R}^{N-1}$, we obtain

$$\int_\Omega |u(x)|^{p_i}\varphi(x)\,dx \le c(p_i)\big[-\frac{1}{2\log(\gamma(a_i,b_i))}\big]^{p_i/2}\int_\Omega \big|\frac{\partial}{\partial x_i}u(x)\big|^{p_i}\varphi(x)\,dx_i,$$

for some constant $c(p_i)$ depending only on p_i; thus inequality (1.7) is proved.

Remark 3.1. A similar computation proves that the constant in (1.5) is given by $c(p)[-\frac{1}{2\log(\gamma(\Omega))}]^{1/2}$ for some constant $c(p)$ depending only on p.

4. Application to PDEs

Let us consider the class of nonlinear homogeneous Dirichlet problems

$$\begin{gathered} -\sum_{i=1}^N \partial_{x_i}a_i(x,u,\nabla u) = F \quad \text{in } \Omega \\ u=0 \quad \text{on } \partial\Omega, \end{gathered} \tag{4.1}$$

where $N\ge 2$, Ω is an open subset of $\mathbb{R}^N$ with Lipschitz boundary such that (H1) holds for every $i=1,\dots,N$, $a_i:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a Carathéodory function fulfilling the degenerated anisotropic ellipticity condition

$$\sum_{i=1}^N a_i(x,s,\xi)\xi_i \ge \lambda\sum_{i=1}^N |\xi_i|^{p_i}\varphi(x) \quad \forall s\in\mathbb{R},\xi\in\mathbb{R}^N \text{ a.e. } x\in\Omega, \tag{4.2}$$

with $1<p_i<\infty$ and $\lambda>0$ and the growth condition

$$|a_i(x,s,\xi)| \le [\nu_1|s|^{p_i-1}+\nu_2|\xi_i|^{p_i-1}]\varphi(x) \quad \forall s\in\mathbb{R},\xi\in\mathbb{R}^N \text{ a.e. } x\in\Omega, \tag{4.3}$$

with $\nu_1\ge 0$ and $\nu_2>0$ and F is an element of dual space. For example the datum can be given by $f\varphi-\sum_{i=1}^N(g_i\varphi)_{x_i}$ with $f\in L^{p'_{\max}}(\log L)^{-\frac12}(\Omega,\gamma)$ and $g_i\in L^{p'_i}(\Omega,\gamma)$, where p'_i states for the Hölder conjugate exponent of p_i. We observe that the equation in (4.1) is related to Ornstein-Uhlenbeck operator in the isotropic case.

We take into account weak solutions to problem (4.1). The natural space for searching them is the weighted anisotropic Sobolev space $W_0^{1,\vec{p}}(\Omega,\gamma)$. When the datum F is in the dual space a weak solution to problem (4.1) is a function $u\in W_0^{1,\vec{p}}(\Omega,\gamma)$ such that

$$\int_\Omega a_i(x,u,\nabla u)\frac{\partial}{\partial x_i}\psi\,dx = \langle F,\psi\rangle \quad \forall\psi\in W_0^{1,\vec{p}}(\Omega,\gamma), \tag{4.4}$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing. We stress that under the assumptions (4.2)-(4.3) every term in (4.4) is well-defined and the operator $-\sum_{i=1}^N\partial_{x_i}a_i(x,u,\nabla u)$ is monotone and coercive on the weighted anisotropic Sobolev space $W_0^{1,\vec{p}}(\Omega,\gamma)$. Then there exists (see e.g. [19]) at least a weak solution $u\in W_0^{1,\vec{p}}(\Omega,\gamma)$ to problem (4.1).

In what follows we are interested in some uniqueness results. As in the classical case, to guarantee uniqueness the main hypotheses are a strongly monotonicity and a Lipschitz continuity of the involved operator. More precisely we suppose that every function a_i satisfies the following strongly monotonicity condition

$$(a_i(x,s,\xi)-a_i(x,s,\xi'))(\xi_i-\xi'_i) \ge \alpha(\varepsilon+|\xi_i|+|\xi'_i|)^{p_i-2}|\xi_i-\xi'_i|^2 \tag{4.5}$$

with $\alpha>0$ and $\varepsilon\ge 0$ and the following locally Lipschitz continuity

$$|a_i(x,s,\xi)-a_i(x,s',\xi)| \le \beta|\xi_i|^{p_i-1}|s-s'| \tag{4.6}$$

with $\beta > 0$. We stress Lipschitz continuity condition (4.6) is necessary to get the uniqueness of a solution (for a counterexample see e.g. [4] in the weighted case and [7] in a bounded domain when $\varphi(x) \equiv 1$).

As in no weighted case (see e.g. [2]) we are able to prove an uniqueness result when $\varepsilon = 0$ in (4.5) and at least one $p_i \leq 2$.

Theorem 4.1. *Let us suppose that there exists $j \in \{1, \dots, N\}$ such that $p_j \leq 2$ and that (4.5) with $\varepsilon = 0$ and (4.6) hold. Then problem (4.1) has at most one weak solution in $W_0^{1,\overrightarrow{p}}(\Omega, \gamma)$.*

As in the no weighted case if all $p_i > 2$ for every $i = 1, \dots, N$ we have to take into account only the case $\varepsilon > 0$ in ((4.5). Indeed otherwise the uniqueness is not guaranteed as for the $p-$Laplace operator (for a counterexample see e.g. [4] in the weighted case and [1] in a bounded domain when $\varphi(x) \equiv 1$).

Theorem 4.2. *Let us assume that $p_i > 2$ for every $i = 1, \dots, N$ and that (4.5) with $\varepsilon > 0$ and (4.6) hold. Then problem (4.1) has at most one weak solution in $W_0^{1,\overrightarrow{p}}(\Omega, \gamma)$.*

The proofs of Theorems 4.1 and 4.2 follow the idea in [1] (see also [4] for isotropic weighted case).

Let us consider the case when the datum is $f\varphi$ with $f \in L^{p'_{\max}}(\log L)^{-\frac{1}{2}}(\Omega, \gamma)$. The restriction $\varepsilon > 0$ when all $p_i > 2$ can be avoided if f does not change sign (see [8] and [12] in the no weighted case and [4] for isotropic weighted case). We obtain the following uniqueness result holding for the model operator $\sum_{i=1}^{N}(|\partial_{x_i}u|^{p_i-2}\partial_{x_i}u\,\varphi)_{x_i}$ with all $p_i > 2$, which does not fulfill (4.5) with $\varepsilon > 0$.

Theorem 4.3. *Let us assume that $p_i > 2$ for every $i = 1, \dots, N$ and that (4.3) with $\nu_1 = 0$, (4.5) with $\varepsilon = 0$ and (4.6) hold and that the sign of f is constant on Ω. Then problem (4.1) has at most one weak solution in $W_0^{1,\overrightarrow{p}}(\Omega, \gamma)$.*

When $\varphi(x) \equiv 1$ and Ω is bounded, uniqueness results for elliptic problems are proved for example in [1, 5, 8, 12, 16, 21] (see also the bibliography therein).

4.1. **Proof of Theorem 4.1.** Let u and v be two weak solutions to problem (4.1). Let $(u-v)^+ := \max\{0, u-v\}$, $D = \{x \in \Omega : (u-v)^+ > 0\}$, $D_t = \{x \in D : (u-v)^+ < t\}$ for $t \in [0, \sup(u-v)^+[$ and let us suppose that D has positive measure. Let $T_t(s)$ be the truncation function at level t, i.e.

$$T_t(s) = \min\{t, \max\{s, -t\}\}. \tag{4.7}$$

Taking $\frac{T_t((u-v)^+)}{t}$ as test function in (4.4) written for u and v, making the difference of the two equations, we obtain

$$\sum_{i=1}^{N} \int_{D_t} [a_i(x, u, \nabla u) - a_i(x, v, \nabla v)]\partial_{x_i}\psi \, dx \leq 0.$$

Hypothesis (4.5) with $\varepsilon = 0$ and (4.6) yield

$$\sum_{i=1}^{N} \int_{D_t} |\partial_{x_i}\psi|^2(|\partial_{x_i}u| + |\partial_{x_i}v|)^{p_i-2} \, d\gamma \leq \frac{\beta}{\alpha} \sum_{i=1}^{N} \int_{D_t} |\partial_{x_i}v|^{p_i-1}|\partial_{x_i}\psi| \, d\gamma. \tag{4.8}$$

By Young inequality with some $\delta > 0$ we obtain

$$\begin{aligned}&\int_{D_t} |\partial_{x_i} v|^{p_i-1} |\partial_{x_i}\psi|\, d\gamma \\ &\le \frac{\delta}{2}\int_{D_t} |\partial_{x_i}\psi|^2 (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i-2}\, d\gamma + \frac{1}{4\delta}\int_{D_t} (|\partial_{x_i} v| + |\partial_{x_i} v|)^{p_i}\, d\gamma.\end{aligned} \tag{4.9}$$

Putting (4.9) in (4.8) and choosing δ small enough we obtain

$$\sum_{i=1}^N \int_{D_t} |\partial_{x_i}\psi|^2 (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i-2}\, d\gamma \le c_1 \sum_{i=1}^N \int_{D_t} (|\partial_{x_i} v| + |\partial_{x_i} v|)^{p_i}\, d\gamma \tag{4.10}$$

for some positive constant c_1 independent on t. Let $p_j \le 2$. Using Poincaré inequality (1.7), Young inequality and (4.10) we obtain

$$\begin{aligned}\gamma(D\backslash D_t) &= \int_{D_t} |\psi|\, d\gamma \le C\int_{D_t} |\partial_{x_i}\psi|\ d\gamma \\ &\le \frac{C}{2}\Big[\int_{D_t} \tfrac{|\partial_{x_j}\psi|^2}{(|\partial_{x_j} u|\, d\gamma + |\partial_{x_j} v|)^{2-p_j}}\, d\gamma + \int_{D_t} (|\partial_{x_j} v| + |\partial_{x_j} v|)^{2-p_j}\, d\gamma\Big] \\ &\le \frac{c_1 C}{2}\sum_{i=1}^N \int_{D_t} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i}\, d\gamma + \frac{C}{2}\int_{D_t} (|\partial_{x_j} u| + |\partial_{x_j} v|)^{2-p_j}\, d\gamma \\ &=: \Lambda(t),\end{aligned}$$

where C is the constant in Poincaré inequality (1.7). Since $\lim_{t\to 0}\Lambda(t) = 0$ we obtain

$$\gamma(D) = \lim_{t\to 0}\gamma(D\backslash D_t) = 0 \tag{4.11}$$

and the conclusion follows.

Remark 4.4. In the last step of proof of Theorem (4.1) we only need that Poincaré inequality (1.7) holds for $j \in \{1, \dots, N\}$ such that $p_j < 2$. Then that we can relax the hypothesis on Ω, requiring that (H1) is fulfilled for such a index j.

4.2. **Proof of Theorem 4.2.** Arguing as in the proof of Theorem 4.1 we obtain

$$\sum_{i=1}^N \int_{D_t} (\varepsilon + |\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i-2} |\partial_{x_i}\psi|^2\, d\gamma \le \frac{\beta}{\alpha}\sum_{i=1}^N \int_{D_t} |\partial_{x_i} v|^{p_i-1} |\partial_{x_i}\psi|\, d\gamma \tag{4.12}$$

Let us estimate the right hand side. By Young inequality with some $\delta > 0$ we have

$$\int_{D_t} |\partial_{x_i} v|^{p-1} |\partial_{x_i}\psi|\, d\gamma \le \frac{1}{4\delta}\int_{D_t} |\partial_{x_i} v|^{p_i}\, d\gamma + \frac{\delta}{2}\int_{D_t} |\partial_{x_i} v|^{p_i-2} |\partial_{x_i}\psi|^2\, d\gamma. \tag{4.13}$$

Choosing δ small enough inequalities (4.13) and (4.12) yield

$$\sum_{i=1}^N \int_{D_t} (\varepsilon + |\partial_{x_i} u| + |\partial_{x_i} v|)^{p-2} |\partial_{x_i}\psi|^2\, d\gamma \le c_1 \sum_{i=1}^N \int_{D_t} |\partial_{x_i} v|^{p_i}\, d\gamma := \Lambda(t) \tag{4.14}$$

for some constant c_1 independent of t. Moreover Young inequality and (4.14) imply

$$\begin{aligned}\int_D |\partial_{x_i}\psi|\, d\gamma &= \int_{D_t} |\partial_{x_i}\psi|\, d\gamma \\ &\le \frac{1}{2}\gamma(D_t) + \frac{1}{2}\int_{D_t} |\partial_{x_i}\psi|^2\, d\gamma \le \frac{\gamma(D_t)}{2} + \frac{\Lambda(t)}{2\varepsilon^{p_i-2}}.\end{aligned} \tag{4.15}$$

On the other hand Poincaré inequality (1.7) gives

$$\gamma(D\backslash D_t)=\int_{D\backslash D_t}\psi\,d\gamma\le\int_D\psi\,d\gamma\le C\int_D|\partial_{x_i}\psi|\,d\gamma.$$

Observing that $\lim_{t\to 0}\Lambda(t)=0$, by (4.15) we obtain (4.11) and we have the conclusion.

4.3. **Proof of Theorem 4.3.** Let u and v be two weak solutions to problem (4.1) and let $(u-v)^+:=\max\{0,u-v\}$, $D=\{x\in\Omega:(u-v)^+>0\}$, $D_t=\{x\in D:(u-v)^+<t\}$ for $t>0$ and let us suppose that D has positive measure. We proceed by steps.

Step 1. We prove that

$$\lim_{t\to 0}\frac{1}{t^2}\sum_{i=1}^N\int_{D_t}(|\partial_{x_i}u|+|\partial_{x_i}v|)^{p_i-2}|\partial_{x_i}(u-v)|^2\,d\gamma=0. \tag{4.16}$$

For $t>0$, denoting by T_t the function defined as in (4.7), putting as test function $T_t[(u-v)^+]$ in (4.4) we obtain

$$\sum_{i=1}^N\int_\Omega[a_i(x,u,\nabla u)-a_i(x,v,\nabla v)]\partial_{x_i}T_t[(u-v)^+]\,dx=0.$$

Denoting $D_i^1=\{x\in D_t,|\partial_{x_i}u|\le|\partial_{x_i}v|\}$ and $D_i^2=\{x\in D_t,|\partial_{x_i}v|\le|\partial_{x_i}u|\}$, we obtain

$$\begin{aligned}
&\sum_{i=1}^N\int_{D_i^1}[a_i(x,v,\nabla u)-a_i(x,v,\nabla v)]\partial_{x_i}(u-v)\,dx\\
&+\sum_{i=1}^N\int_{D_i^2}[a_i(x,u,\nabla u)-a_i(x,u,\nabla v)]\partial_{x_i}(u-v)\,dx\\
&\le-\sum_{i=1}^N\int_{D_i^1}[a_i(x,u,\nabla u)-a_i(x,v,\nabla u)]\partial_{x_i}(u-v)\,dx\\
&\quad-\sum_{i=1}^N\int_{D_i^2}[a_i(x,u,\nabla v)-a_i(x,v,\nabla v)]\partial_{x_i}(u-v)\,dx
\end{aligned}$$

for every $t>0$. By (4.5) and (4.6) we have

$$\begin{aligned}
&\alpha\sum_{i=1}^N\int_{D_t}(|\partial_{x_i}u|+|\partial_{x_i}v|)^{p_i-2}|\partial_{x_i}(u-v)|^2\,d\gamma\\
&\le\beta\sum_{i=1}^N\int_{D_i^1}|\partial_{x_i}u|^{p_i-1}|\partial_{x_i}(u-v)|\,|u-v|d\gamma\\
&\quad+\beta\sum_{i=1}^N\int_{D_i^2}|\partial_{x_i}v|^{p_i-1}|\partial_{x_i}(u-v)|\,|u-v|\,d\gamma\\
&\le\beta t\sum_{i=1}^N\int_{D_t}[\min\{|\partial_{x_i}u|,|\partial_{x_i}v|\}]^{p_i-1}|\partial_{x_i}(u-v)|\,d\gamma.
\end{aligned} \tag{4.17}$$

Using Young's inequality we obtain

$$
\begin{aligned}
&\alpha \sum_{i=1}^{N} \int_{D_t} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i-2} |\partial_{x_i}(u-v)|^2 \, d\gamma \\
&\le \beta t \sum_{i=1}^{N} \int_{D_t} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i-1} |\partial_{x_i}(u-v)| \, d\gamma \\
&\le \frac{\alpha}{2} \sum_{i=1}^{N} \int_{D_t} (|\partial_{x_i} u| + |\partial_{x_i}|)^{p_i-2} |\partial_{x_i}(u-v)|^2 \, d\gamma \\
&\quad + \frac{\beta^2 t^2}{2\alpha} \sum_{i=1}^{N} \int_{D_t} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i} \, d\gamma
\end{aligned}
$$

and then

$$
\begin{aligned}
&\frac{\alpha}{2t^2} \sum_{i=1}^{N} \int_{D_t} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i-2} |\partial_{x_i}(u-v)|^2 \, d\gamma \\
&\le \frac{\beta^2}{2\alpha} \sum_{i=1}^{N} \int_{D_t} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i} \, d\gamma.
\end{aligned}
$$

Since the second term in the previous estimate approaches zero as $t \to 0$, (4.16) follows.

Step 2. We prove that

$$
\begin{gathered}
\sum_{i=1}^{N} \int_{D} a_i(x, u, \nabla u) \partial_{x_i} \Psi \, dx = \lim_{t \to 0} \int_{\Omega} f \frac{T_t[(u-v)^+]}{t} \Psi \, d\gamma \\
\sum_{i=1}^{N} \int_{D} a_i(x, v, \nabla v) \partial_{x_i} \Psi \, dx = \lim_{t \to 0} \int_{\Omega} f \frac{T_t[(u-v)^+]}{t} \Psi \, d\gamma
\end{gathered}
\tag{4.18}
$$

for every $\Psi \in L^\infty(\Omega) \cap W^{1,\overrightarrow{p}}(\Omega, \gamma)$.

Taking $\frac{T_t[(u-v)^+]}{t} \Psi$ as test function in (4.4), we obtain

$$
\begin{aligned}
&\sum_{i=1}^{N} \int_{\Omega} a_i(x, u, \nabla u) \partial_{x_i} \Psi \frac{T_t[(u-v)^+]}{t} dx + \frac{1}{t} \sum_{i=1}^{N} \int_{D_t} a_i(x, u, \nabla u) \partial_{x_i}(u-v) \Psi \, dx \\
&= \int_{\Omega} f \frac{T_t[(u-v)^+]}{t} \Psi \, d\gamma.
\end{aligned}
$$

We easily pass to the limit in the first term by using Lebesgue dominated convergence theorem. For the second term using (4.3) and Hölder inequality we obtain

$$
\begin{aligned}
&\frac{1}{t} \int_{D_t} a_i(x, u, \nabla u) \partial_{x_i}(u-v) \Psi \, dx \\
&\le \nu_2 \|\Psi\|_{L^\infty(\Omega)} \Big(\frac{1}{t^2} \int_{D_t} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i-2} |\partial_{x_i}(u-v)|^2 \, d\gamma \Big)^{1/2} \\
&\quad \times \Big(\int_{D_t} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i} \, d\gamma \Big)^{1/2},
\end{aligned}
$$

which tends to zero by (4.16). Then we obtain (4.18).

Step 3. We prove that D has zero measure. Taking $\Psi = 1$ in (4.18) we obtain

$$\lim_{t\to 0}\int_\Omega f\frac{T_t[(u-v)^+]}{t}\,d\gamma = 0.$$

Since the sign of f is constant we obtain $f\chi_D = 0$ a.e. in Ω and the right-hand side of (4.18) is zero.

Now taking $\psi = T_k(u)$ in (4.18) and passing to the limit as $k \to \infty$, we obtain

$$\sum_{i=1}^N \int_D a_i(x,u,\nabla u)\partial_{x_i} u\,dx = 0.$$

By (4.5) with $\varepsilon = 0$ and (4.3) with $\nu_1 = 0$ we obtain

$$\sum_{i=1}^N \int_D |\partial_{x_i} u|^{p_i}\,d\gamma = 0.$$

Then

$$\partial_{x_i} u = 0 \quad \text{a.e. on } D \text{ for every } i \in \{1,\dots,N\}. \tag{4.19}$$

By (4.17) and (4.19) it follows that $\partial_{x_i} v = 0$ a.e. on D_t for every $t > 0$ and for every $i \in \{1,\dots,N\}$ and then in D. Then $\partial_{x_i}(u-v) = 0$ a.e. on D for every $i \in \{1,\dots,N\}$. Since $u = v = 0$ on $\partial\Omega$, by Poincaré inequality (1.7)

$$\int_D |u-v|^{p_i}\,d\gamma = \int_\Omega |(u-v)^+|^{p_i}\,d\gamma \le C\int_D |\partial_{x_i}(u-v)|^{p_i}\,d\gamma = 0.$$

Then the conclusion follows.

Remark 4.5. In the last step of proofs of Theorems (4.2) and (4.3) we only need that Poincaré inequality (1.7) holds for some $j \in \{1,\dots,N\}$, that means that we can relax the hypothesis on Ω, requiring that (H1) is fulfilled for some $j \in \{1,\dots,N\}$.

5. An extension to Boltzmann measures

Let us consider the weight

$$Z^{-1}\exp(-W(x)) \in L^1(\mathbb{R}^n), \tag{5.1}$$

where $W(x)$ is a positive smooth function. We can see (5.1) as the density of the following measure on $\mathbb{R}^N$, called Boltzmann measure, defined by

$$d\mu = Z^{-1}\exp(-W(x))\,dx \quad x \in \mathbb{R}^N$$

and normalized by $\mu(\mathbb{R}^N) = 1$. Let us suppose that measure μ satisfies an isoperimetric inequality and its isoperimetric function $I_\mu(t)$ is estimated by the Gaussian isoperimetric function:

$$I_\mu(t) \le c_\mu I_\gamma(t) \tag{5.2}$$

for a suitable positive constant $c_\mu > 0$. For example (5.2) is satisfied if the Hessian matrix satisfies

$$D^2W(x) \ge c_\mu^2 Id \tag{5.3}$$

as symmetric matrix uniformly in x (see [18]). We remark that the Gauss measureγ is such a measure.

Under the previous assumption arguing as in subsection 3.1 it is possible to prove that

$$\Big(\|u\|^\circ_{L^p(\log L)^{1/2}(\Omega,\mu)}\Big)^p \le c_\mu c(p)\Big[\frac{1}{2}\Big(1-\frac{1}{\log(\mu(\Omega))}\Big)\Big]^{p/2}\|\nabla u\|^p_{L^p(\Omega,\mu)} \tag{5.4}$$

for some constant $c(p)$ depending only on p, and then the analogue of our main results. The main tools are a suitable definition of rearrangement and Polya-Szëgo inequality for it. If u is a measurable function in Ω, we define

$$u^{\circ}(s) = \inf\{t \geq 0 : \mu(\{x \in \Omega : |u| > t\}) \leq s\} \quad \text{for } s \in]0,1]$$

the one dimensional rearrangement of u with respect to Boltzmann measure and $u^{\blacklozenge}(x) = u^{\circ}(\Phi(x_1))$ for $x \in \Omega^{\blacklozenge}$ the rearrangement with respect to Boltzmann, where $\Omega^{\blacklozenge}$ is the half-space such that $\gamma(\Omega^{\blacklozenge}) = \mu(\Omega)$. In [11] is proved the Polya-Szëgo inequality

$$\|\nabla u^{\blacklozenge}\|_{L^p(\Omega^{\blacklozenge},\gamma)} \leq \sqrt{c_\mu}\|\nabla u\|_{L^p(\Omega,\mu)}. \tag{5.5}$$

In this framework u°, $u^{\blacklozenge}$ and (5.5) play the role of $u^{\circledast}, u^{\star}$ and (2.3) respectively. Moreover $\|u\|^{\circ}_{L^p(\log L)^{1/2}(\Omega,\mu)}$ is defined as in (2.4) with $u^{\circledast}$ replaced by u°.

Stating from (5.4) arguing as in §3.2 and §3.3 we can prove the analogue of Theorem 1.1, Theorem 1.2, and Corollary 1.3. Moreover using such inequalities it is possible to study the analogue of problem (4.1), where the density of Gauss measure is replaced by the (5.1) following the ideas of §4.

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