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CONVERGENCE OF APPROXIMATE SOLUTIONS TO NONLINEAR CAPUTO NABLA FRACTIONAL DIFFERENCE EQUATIONS WITH BOUNDARY CONDITIONS

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Abstract. This article studies a boundary value problem for a nonlinear Caputo nabla fractional difference equation. We obtain quadratic convergence results for this equation using the generalized quasi-linearization method. Further, we obtain the convergence of the sequences is potentially improved by the Gauss-Seidel method. A numerical example illustrates our main results.

1. Introduction

It is well-known that fractional difference equations can be used to model many problems, such as population models, tumor growth model, and so on. In many situations, fractional difference equations have proved to be better than their counterpart with integer difference. Therefore, research in the theory of fractional difference equations has become very important. Previous studies have mainly focused on the theory of integer-order difference equations, and classical results have been established; see for example the monographs [2, 17]. Recently, there has been a great deal of interest in fractional difference equations. The basic theory of the linear and nonlinear fractional difference equations can be found in [8, 9, 13, 14, 15, 16]. However, we note that the qualitative theory of nonlinear fractional difference equations is not complete and the convergence of approximate solutions plays an important role in the development of qualitative theory.

The generalized quasi-linearization method can be used to construct approximate solutions of nonlinear problems. There are many applications of this method. In [3, 4, 5, 6, 10, 11, 12, 18, 22, 21], the authors used this method to obtain the convergence of the sequences for different types of differential equations. Further, in [7, 20], the authors accelerated the convergence of the sequences by the Gauss-Seidel method. However, there are few applicable results of the above methods to nonlinear fractional difference equations. In [8, 19], the authors only discussed the existence and convergence of solutions for nonlinear fractional difference equations with initial conditions.

In this article, we study the convergence of solutions for a nonlinear Caputo nabla fractional difference equation with boundary conditions. We obtain quadratic convergence by the generalized quasi-linearization method when the forcing function

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is the sum of convex and concave functions. Furthermore, we obtain the convergence of the sequences is potentially improved by the Gauss-Seidel method. Finally, a numerical example illustrates the obtained results.

2. Preliminaries

For the convenience of readers, we will list some relevant results here. We use the notation $\mathbb{N}_a := \{a, a+1, a+2, \dots\}$, $\mathbb{Z}_a := \{\dots, a-2, a-1, a, a+1, a+2, \dots\}$, where a is a real number. For the function $f : \mathbb{Z}_a \to \mathbb{R}$, the backward difference or nabla operator is defined as $\nabla f(t) = f(t) - f(t-1)$ for $t \in \mathbb{Z}_a$ and the higher order differences are defined recursively by $\nabla^n f(t) = \nabla(\nabla^{n-1} f(t))$ for $t \in \mathbb{Z}_a$. In addition, we take ∇^0 as the identity operator. We define the definite nabla integral of $f : \mathbb{Z}_a \to \mathbb{R}$ for $b \in \mathbb{Z}_a$ by

$$\int_a^b f(s)\nabla s = \begin{cases} \sum_{s=a+1}^b f(s), & a < b, \\ 0, & a \geq b. \end{cases} \tag{2.1}$$

Definition 2.1 (See [15, Definition 3.4]). The (generalized) rising function is defined in terms of the gamma function by

$$t^{\overline{r}} := \frac{\Gamma(t+r)}{\Gamma(t)} \tag{2.2}$$

for those values of t and r so that the right-hand side of (2.2) is well defined. Also, we use the convention that if t is a nonpositive integer, or $t+r$ is not a nonpositive integer, then $t^{\overline{r}} = 0$. We then define the ν-th order Taylor monomials based at a (see [15, Definition 3.56]) by

$$H_\nu(t,a) := \frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+1)} = \frac{\Gamma(t-a+\nu)}{\Gamma(t-a)\Gamma(\nu+1)}$$

for $\nu \neq -1, -2, \dots$, and $t \in \mathbb{N}_a$.

In this article, we extend the ν-th order Taylor monomials by defining

$$\widetilde{H}_\nu(t,a) := \lim_{\varepsilon \to 0} \frac{\Gamma(t-a+\nu+\varepsilon)}{\Gamma(t-a+\varepsilon)\Gamma(\nu+1+\varepsilon)}$$

for all $\nu \in \mathbb{R}$, and all $t \in \mathbb{Z}_a$.

Note that when $t > a$ and $\nu > -1$, $\widetilde{H}_\nu(t,a) = H_\nu(t,a)$. The extended Taylor monomials will be used in Lemmas 2.9 and 2.10.

For some important formulas for these Taylor fractional monomials see [15, Theorem 3.57 and Theorem 3.93].

Definition 2.2 (Nabla Fractional Sum [15, Definition 3.58]). Let $f : \mathbb{N}_a \to \mathbb{R}$, $\nu > 0$ be given. Then

$$(\nabla_a^{-\nu} f)(t) = \int_a^t H_{\nu-1}(t, \rho(s)) f(s)\nabla s, \quad t \in \mathbb{N}_a, \tag{2.3}$$

where $\rho(t) := t-1$. Note, by (2.1), $\nabla_a^{-\nu} f(a) = 0$.

Definition 2.3 (Nabla Fractional Difference [15, Definition 3.61]). *Let $f : \mathbb{N}_a \to \mathbb{R}$, $\nu > 0$ be given, and let $N := \lceil \nu \rceil$, where $\lceil \cdot \rceil$ is the ceiling function. Then we define the νth-order nabla fractional difference $(\nabla_a^\nu f)(t)$ by*

$$(\nabla_a^\nu f)(t) = (\nabla^N(\nabla_a^{-(N-\nu)} f))(t), \quad t \in \mathbb{N}_{a+N}. \tag{2.4}$$

Definition 2.4 (Caputo Nabla Fractional Difference [15, Definition 3.117]). *Let $f : \mathbb{N}_{a-N+1} \to \mathbb{R}$, $\nu > 0$ be given, and let $N := \lceil \nu \rceil$. Then we define the νth-order Caputo nabla fractional difference $(\nabla^{\nu}_{a^*} f)(t)$ by*

$$(\nabla^{\nu}_{a^*} f)(t) = (\nabla_a^{-(N-\nu)}(\nabla^N f))(t), \quad t \in \mathbb{N}_{a+1}. \tag{2.5}$$

Since $\nu > 0 \Rightarrow N \geq 1$, it follows from this definition that $\nabla^{\nu}_{a^*} c = 0$ for any constant c.

Lemma 2.5 (See [15, Definition 3.61 and Theorem 3.62]). *Assume $f : \mathbb{N}_a \to \mathbb{R}$, $\nu > 0$, $\nu \notin \mathbb{N}_1$, and choose $N \in \mathbb{N}_1$ such that $N - 1 < \nu < N$. Then*

$$(\nabla^{\nu}_a f)(t) = \int_a^t H_{-\nu-1}(t, \rho(s)) f(s) \nabla s, \quad t \in \mathbb{N}_{a+N}, \tag{2.6}$$

where $\rho(t) := t - 1$. Note, by (2.1), $\nabla^{\nu}_a f(a) = 0$.

Lemma 2.6 (Nabla Leibniz Formula [15, Corollary 3.41]). *Assume that $f : \mathbb{N}_a \times \mathbb{N}_{a+1} \to \mathbb{R}$. Then for $t \in \mathbb{N}_{a+1}$,*

$$\nabla\Big(\int_a^t f(t, \tau) \nabla \tau\Big) = \int_a^t \nabla_t f(t, \tau) \nabla \tau + f(\rho(t), t). \tag{2.7}$$

Also,

$$\nabla\Big(\int_a^t f(t, \tau) \nabla \tau\Big) = \int_a^{t-1} \nabla_t f(t, \tau) \nabla \tau + f(t, t). \tag{2.8}$$

The following corollary appears in Goodrich et al [15, Corollary 3.122, Corollary 3.167].

Corollary 2.7. *For $\nu > 0$, $N = \lceil \nu \rceil$, and $h : \mathbb{N}_{a+1} \to \mathbb{R}$, we have*

$$(\nabla_a^{-(N-\nu)}(\nabla_a^{N-\nu} h))(t) = h(t), \quad t \in \mathbb{N}_{a+1}.$$

Lemma 2.8. *The nabla Taylor monomials satisfy the following:*

(i) $\widetilde{H}_0(t, a) = 1$, $t \in \mathbb{Z}_a$.

(ii) $\nabla \widetilde{H}_r(t, a) = \begin{cases} \widetilde{H}_{r-1}(t, a), & 0 \neq r \in \mathbb{R},\ t \in \mathbb{Z}_a, \\ 0, & r = 0,\ t \in \mathbb{Z}_a. \end{cases}$

Proof. (i) From the definition of $\widetilde{H}_r(t, a)$, we have

$$\widetilde{H}_0(t, a) = \lim_{\varepsilon \to 0} \frac{\Gamma(t - a + \varepsilon)}{\Gamma(t - a + \varepsilon)\Gamma(\varepsilon + 1)} = 1.$$

(ii) For $r = 0$, using (i), we have $\nabla \widetilde{H}_0(t, a) = 0$. For $r \neq 0$, by the definition of $\widetilde{H}_r(t, a)$, we obtain

$$\begin{aligned}
\nabla \widetilde{H}_r(t, a) &= \lim_{\varepsilon \to 0} \frac{\Gamma(t - a + r + \varepsilon)}{\Gamma(t - a + \varepsilon)\Gamma(r + \varepsilon + 1)} - \lim_{\varepsilon \to 0} \frac{\Gamma(t - 1 - a + r + \varepsilon)}{\Gamma(t - 1 - a + \varepsilon)\Gamma(r + \varepsilon + 1)} \\
&= \lim_{\varepsilon \to 0} \Big[\frac{t - 1 - a + r + \varepsilon}{t - 1 - a + \varepsilon} - 1\Big] \frac{\Gamma(t - 1 - a + r + \varepsilon)}{\Gamma(t - 1 - a + \varepsilon)\Gamma(r + \varepsilon + 1)} \\
&= \lim_{\varepsilon \to 0} \frac{r\Gamma(t - 1 - a + r + \varepsilon)}{(r + \varepsilon)\Gamma(t - a + \varepsilon)\Gamma(r + \varepsilon)} \\
&= \lim_{\varepsilon \to 0} \frac{r}{(r + \varepsilon)} \cdot \lim_{\varepsilon \to 0} \frac{\Gamma(t - a + r - 1 + \varepsilon)}{\Gamma(t - a + \varepsilon)\Gamma(r + \varepsilon)}
\end{aligned}$$

$$= \widetilde{H}_{r-1}(t,a).$$

The proof is complete. □

For convenience, in this paper, we define the operator $L_a : \mathcal{D}_{a-N+1} \to \mathcal{D}_{a+1}$ by $(L_a x)(t) := \nabla[(\nabla^{\nu}_{a^*} x)(t+1)]$, $t \in \mathbb{N}_{a+1}$, where $x \in \mathcal{D}_{a-N+1} := \{x : \mathbb{N}_{a-N+1} \to \mathbb{R}\}$.

Lemma 2.9. *Assume $\nu > 0$ and N is a positive integer such that $N-1 < \nu \le N$. Then a general solution of the fractional difference equation $(L_a x)(t) = 0$, $t \in \mathbb{N}_{a+1}$ is given by*

$$x(t) = c_0 \widetilde{H}_0(t,a) + c_1 \widetilde{H}_1(t,a) + c_2 \widetilde{H}_2(t,a) + \cdots + c_{N-1} \widetilde{H}_{N-1}(t,a) + c H_\nu(t,a)$$

for $t \in \mathbb{N}_{a-N+1}$.

Proof. Let $x_k(t) := \widetilde{H}_k(t,a)$, $0 \le k \le N-1$, by Lemma 2.8, we have

$$\begin{aligned}
(L_a x_k)(t) &= \nabla[\nabla_a^{-(N-\nu)} \nabla^N \widetilde{H}_k(t+1,a)] \\
&= \nabla[\nabla_a^{-(N-\nu)} \nabla^{N-k} \nabla^k \widetilde{H}_k(t+1,a)] \\
&= \nabla[\nabla_a^{-(N-\nu)} \nabla^{N-k} \widetilde{H}_0(t+1,a)] \\
&= \nabla[\nabla_a^{-(N-\nu)} 0] = 0
\end{aligned}$$

for $t \in \mathbb{N}_{a+1}$. Let $\bar{x}(t) := H_\nu(t,a)$, by Corollary 2.7, we have

$$\begin{aligned}
(L_a \bar{x})(t) &= \nabla[\nabla_a^{-(N-\nu)} \nabla^N H_\nu(t+1,a)] \\
&= \nabla[\nabla_a^{-(N-\nu)} \nabla^N \nabla_a^{-\nu} 1(t+1)] \\
&= \nabla[\nabla_a^{-(N-\nu)} \nabla_a^{N-\nu} 1(t+1)] \\
&= \nabla[\chi_{(a,\infty)}(t+1)] \\
&= \nabla[\chi_{[a,\infty)}(t)] = 0
\end{aligned}$$

for $t \in \mathbb{N}_{a+1}$, where $\chi_{[a,\infty)}(t) = \begin{cases} 1, & t \in \mathbb{N}_a, \\ 0, & t \notin \mathbb{N}_a. \end{cases}$.

Next, we show that these solutions are linearly independent. We want to show that if

$$c_0 \widetilde{H}_0(t,a) + c_1 \widetilde{H}_1(t,a) + c_2 \widetilde{H}_2(t,a) + \cdots + c_{N-1} \widetilde{H}_{N-1}(t,a) + c H_\nu(t,a) = 0 \quad (2.9)$$

for all $t \in \mathbb{N}_{a-N+1}$, then $c_0 = c_1 = \cdots = c_{N-1} = c = 0$.

Taking $t = a$, $a-1$, $\dots$, $a-(N-1)$, and $a+1$ in (2.9), we obtain

$$\begin{bmatrix}
\widetilde{H}_0(a,a) & \dots & \widetilde{H}_{N-1}(a,a) & H_\nu(a,a) \\
\widetilde{H}_0(a-1,a) & \dots & \widetilde{H}_{N-1}(a-1,a) & H_\nu(a-1,a) \\
\vdots & \ddots & \vdots & \vdots \\
\widetilde{H}_0(a-(N-1),a) & \dots & \widetilde{H}_{N-1}(a-(N-1),a) & H_\nu(a-(N-1),a) \\
\widetilde{H}_0(a+1,a) & \dots & \widetilde{H}_{N-1}(a+1,a) & H_\nu(a+1,a)
\end{bmatrix}
\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \\ c \end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix};$$

that is,

$$\begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ * & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \dots & (-1)^{N-1} & 0 \\ * & * & \dots & * & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

So, we arrive at $c_0 = c_1 = \dots = c_{N-1} = c = 0$. Therefore, these solutions are linearly independent. The proof is complete. □

Lemma 2.10 (See [15, Theorem 3.175]). *Assume $0 < \nu \le 1$, a, $b \in \mathbb{R}$, and $b - a \in \mathbb{N}_2$. Then the Green function for the BVP*

$$\begin{gathered} (L_a x)(t) = 0, \quad t \in \mathbb{N}_{a+1}^{b-1}, \\ x(a) = 0, \quad x(b) = 0 \end{gathered} \tag{2.10}$$

is

$$G(t,s) = \begin{cases} u(t,s), & (t,s) \in \mathbb{N}_a^s \times \mathbb{N}_t^b, \\ v(t,s), & (t,s) \in \mathbb{N}_s^b \times \mathbb{N}_a^t, \end{cases}$$

where

$$\begin{gathered} u(t,s) = -\frac{(b-s)^{\overline{\nu}}(t-a)^{\overline{\nu}}}{\Gamma(\nu+1)(b-a)^{\overline{\nu}}}, \\ v(t,s) = u(t,s) + \frac{(t-s)^{\overline{\nu}}}{\Gamma(\nu+1)} = u(t,s) + x(t,s). \end{gathered}$$

Lemma 2.11 (See [15, Theorem 3.177]). *Assume $0 < \nu \le 1$, a, $b \in \mathbb{R}$, and $b - a \in \mathbb{N}_2$. Then the Green function for the BVP*

$$\begin{gathered} (L_a x)(t) = 0, \quad t \in \mathbb{N}_{a+1}^{b-1}, \\ x(a) = 0, \quad x(b) = 0 \end{gathered}$$

satisfies the following inequalities

- (i) $G(t,s) \le 0$,
- (ii) $G(t,s) \ge -\left(\frac{b-a}{4}\right)\left(\frac{\Gamma(b-a+1)}{\Gamma(\nu+1)\Gamma(b-a+\nu)}\right)$,
- (iii) $\int_a^b |G(t,s)|\nabla s \le \frac{(b-a)^2}{4\Gamma(\nu+2)}$, *for* $t \in \mathbb{N}_a^b$, *and*
- (iv) $\int_a^b |\nabla_t G(t,s)|\nabla s \le \frac{b-a}{\nu+1}$, *for* $t \in \mathbb{N}_{a+1}^b$.

The next corollary is an immediate consequence of [15, Theorem 3.173]. It relates to the nonhomogeneous BVP with homogeneous boundary conditions,

$$\begin{gathered} (L_a y)(t) = h(t), \quad t \in \mathbb{N}_{a+1}^{b-1}, \\ y(a) = 0, \quad y(b) = 0. \end{gathered} \tag{2.11}$$

Corollary 2.12 (See [15, Theorem 3.173]). *Assume $0 < \nu \le 1$, a, $b \in \mathbb{R}$, and $b - a \in \mathbb{N}_2$. The unique solution of (2.11) is*

$$y(t) = \int_a^b G(t,s)h(s)\nabla s = \sum_{s=a+1}^{b} G(t,s)h(s), \quad t \in \mathbb{N}_a^b,$$

where $G(t,s)$ is the Green function of (2.10).

Proof. This is a special case of [15, Theorem 3.173], where $\alpha = \gamma = 1$ and $\beta = \delta = 0$ in equation (3.115) of that theorem. Note that, by [15, Theorem 3.170], $\alpha = \gamma = 1$ and $\beta = \delta = 0$ ensure the hypotheses of [15, Theorem 3.173] are satisfied. □

Lemma 2.13. *Assume $0 < \nu \leq 1$. Then the solution of the BVP*

$$\begin{gathered} (L_a z)(t) = 0, \quad t \in \mathbb{N}_{a+1}^{b-1}, \\ z(a) = A, \quad z(b) = B \end{gathered} \tag{2.12}$$

is

$$z(t) = A + (B - A)\frac{H_\nu(t,a)}{H_\nu(b,a)}, \quad t \in \mathbb{N}_a^b.$$

Proof. Let z be a solution of the fractional difference equation $(L_a z)(t) = 0$. It follows from Lemma 2.9 that

$$z(t) = c_0 \widetilde{H}_0(t,a) + cH_\nu(t,a) = c_0 + cH_\nu(t,a).$$

Using the first boundary condition $z(a) = A$, we obtain

$$A = c_0 + cH_\nu(a,a) = c_0,$$

which implies $c_0 = A$. Using the second boundary condition $z(b) = B$, we obtain

$$B = A + cH_\nu(b,a).$$

Solving for c, we obtain

$$c = \frac{(B-A)}{H_\nu(b,a)}.$$

Thus, we have

$$z(t) = c_0 + cH_\nu(t,a) = A + (B-A)\frac{H_\nu(t,a)}{H_\nu(b,a)}.$$

The proof is complete. □

Lemma 2.14. *Assume $0 < \nu \leq 1$, and $h : \mathbb{N}_{a+1}^{b-1} \to \mathbb{R}$. Then the solution of the nonhomogeneous BVP*

$$\begin{gathered} (L_a x)(t) = h(t), \quad t \in \mathbb{N}_{a+1}^{b-1}, \\ x(a) = A, \quad x(b) = B \end{gathered} \tag{2.13}$$

is

$$x(t) = z(t) + \sum_{s=a+1}^{b} G(t,s)h(s), \quad t \in \mathbb{N}_a^b,$$

where $G(t,s)$ is the Green's function of the BVP (2.10) and $z(t)$ is the unique solution of (2.12).

Proof. Let

$$y(t) = \sum_{s=a+1}^{b} G(t,s)h(s), \quad t \in \mathbb{N}_a^b.$$

By Corollary 2.12, $y(t)$ is the solution of (2.11) on $\mathbb{N}_a^b$. Let $z(t)$ be as in the statement of this theorem. Then

$$\begin{gathered} x(a) = z(a) + y(a) = A + 0 = A, \\ x(b) = z(b) + y(b) = B + 0 = B. \end{gathered}$$

Finally,

$$(L_a x)(t) = (L_a z)(t) + (L_a y)(t) = 0 + h(t) = h(t)$$

for $t \in \mathbb{N}_{a+1}^{b-1}$. The proof is complete. □

Lemma 2.15. *Assume $0 < \nu \le 1$, $x : \mathbb{N}_a^b \to \mathbb{R}$, and put $M := \max\{x(t) : t \in \mathbb{N}_a^b\}$. If $x(t_0) = M$ for some $t_0 \in \mathbb{N}_{a+1}^{b-1}$, then $(L_a x)(t_0) \le 0$.*

Proof. For $\nu = 1$, according to the assumptions, we have

$$\begin{aligned}(L_a x)(t_0) &= (\nabla^2 x)(t)|_{t=t_0+1}\\ &= x(t_0+1) - 2x(t_0) + x(t_0-1)\\ &\le M - 2M + M = 0\end{aligned}$$

For the case when $0 < \nu < 1$, we will use the fact that for any $t \in \mathbb{N}_{a+1}$,

$$(L_a x)(t) = x(t+1) - H_{-\nu-1}(t+1,a)x(a) + \sum_{s=a+1}^{t} H_{-\nu-2}(t+1,\rho(s))x(s). \tag{2.14}$$

To see this note that

$$\begin{aligned}(L_a x)(t) &= \nabla[(\nabla_{a*}^{\nu} x)(t+1)]\\ &= \nabla[\nabla_a^{\nu-1}\nabla x(t+1)]\\ &= \nabla\Big[\sum_{s=a+1}^{t+1} H_{-\nu}(t+1,\rho(s))\nabla x(s)\Big]\\ &= \nabla\Big[\sum_{s=a+1}^{t+1} H_{-\nu}(t+1,\rho(s))x(s) - \sum_{s=a+1}^{t+1} H_{-\nu}(t+1,\rho(s))x(s-1)\Big]\\ &= \nabla\Big[\sum_{s=a+1}^{t+1} H_{-\nu}(t+1,\rho(s))x(s) - \sum_{s=a}^{t} H_{-\nu}(t+1,s)x(s)\Big]\\ &= \nabla\Big[\sum_{s=a+1}^{t} H_{-\nu}(t+1,\rho(s))x(s) + H_{-\nu}(t+1,\rho(t+1))x(t+1)\\ &\quad - \sum_{s=a+1}^{t} H_{-\nu}(t+1,s)x(s) - H_{-\nu}(t+1,a)x(a)\Big]\\ &= \nabla\Big[\sum_{s=a+1}^{t} H_{-\nu-1}(t+1,\rho(s))x(s) + H_{-\nu-1}(t+1,\rho(t+1))x(t+1)\\ &\quad - H_{-\nu}(t+1,a)x(a)\Big]\\ &= \nabla\Big[\sum_{s=a+1}^{t+1} H_{-\nu-1}(t+1,\rho(s))x(s) - H_{-\nu}(t+1,a)x(a)\Big]\\ &= \Big[\sum_{s=a+1}^{t+1} H_{-\nu-1}(t+1,\rho(s))x(s) - H_{-\nu}(t+1,a)x(a)\Big]\\ &\quad - \Big[\sum_{s=a+1}^{t} H_{-\nu-1}(t,\rho(s))x(s) + H_{-\nu}(t,a)x(a)\Big]\end{aligned}$$

$$= x(t+1) + \sum_{s=a+1}^{t} H_{-\nu-2}(t+1,\rho(s))x(s) - H_{-\nu-1}(t+1,a)x(a).$$

Note that for $0 < \nu \le 1$, $-H_{-\nu-1}(t_0+1,a)$ is positive whenever $t_0 \in \mathbb{N}_{a+1}$ and $H_{-\nu-2}(t_0+1,\rho(s))$ is positive whenever $s \in \mathbb{N}_a^{t_0-1}$. Thus, we obtain

$$\begin{aligned}
&(L_a x)(t_0) \\
&= x(t_0+1) - H_{-\nu-1}(t_0+1,a)x(a) + \sum_{s=a+1}^{t_0} H_{-\nu-2}(t_0+1,\rho(s))x(s) \\
&= x(t_0+1) - H_{-\nu-1}(t_0+1,a)x(a) + \sum_{s=a+1}^{t_0-1} H_{-\nu-2}(t_0+1,\rho(s))x(s) \\
&\quad + H_{-\nu-2}(t_0+1,\rho(t_0))x(t_0) \\
&\le M - H_{-\nu-1}(t_0+1,a)M + M\sum_{s=a+1}^{t_0-1} H_{-\nu-2}(t_0+1,\rho(s)) \\
&\quad + H_{-\nu-2}(t_0+1,\rho(t_0))M \\
&= M - H_{-\nu-1}(t_0+1,a)M + M\sum_{s=a+1}^{t_0} H_{-\nu-2}(t_0+1,\rho(s)) \\
&= M[1 - H_{-\nu-1}(t_0+1,a)] + M[(-H_{-\nu-1}(t_0+1,t_0) + H_{-\nu-1}(t_0+1,a)] = 0.
\end{aligned}$$

The proof is complete. □

3. Existence and comparison results

Consider the following BVP for a nonlinear Caputo nabla fractional difference equation

$$\begin{gathered}
(L_a x)(t) = f(t,x(t)), \quad t \in \mathbb{N}_{a+1}^{b-1}, \\
x(a) = A, \quad x(b) = B,
\end{gathered} \tag{3.1}$$

where $f : \mathbb{N}_{a+1}^{b-1} \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to x, $x : \mathbb{N}_a^b \to \mathbb{R}$, and $0 < \nu \le 1$.

In this article, we define the norm of x on $\mathbb{N}_a^b$ by $\|x\| = \max_{s\in\mathbb{N}_a^b} |x(s)|$. Throughout this paper, we use the notation $f^{(k)}(t,x) := \frac{\partial^k f(t,x)}{\partial^k x}$ $(k = 0,1,2\dots)$. For convenience, when $\alpha_0(t)$ and $\beta_0(t)$ are two functions such that $\alpha_0(t) \le \beta_0(t)$ on $\mathbb{N}_a^b$, we use the following sets:

$$\begin{gathered}
\Omega = \Omega(\alpha_0,\beta_0) := \{(t,x) : \alpha_0(t) \le x \le \beta_0(t), \text{ for all } t \in \mathbb{N}_{a+1}^{b-1}\}, \\
\mathcal{S} = \mathcal{S}(\alpha_0,\beta_0) := \{x : \alpha_0(t) \le x(t) \le \beta_0(t), \text{ for all } t \in \mathbb{N}_a^b\}.
\end{gathered}$$

We will simply refer to the sets Ω and $\mathcal{S}$ when it is clear from context what α_0 and β_0 are.

Definition 3.1. A function $\alpha_0(t)$ is said to be a lower solution of (3.1), if

$$\begin{gathered}
(L_a \alpha_0)(t) \ge f(t,\alpha_0(t)), \quad t \in \mathbb{N}_{a+1}^{b-1}, \\
\alpha_0(a) \le A, \quad \alpha_0(b) \le B.
\end{gathered} \tag{3.2}$$

If all three inequalities in (3.2) are reversed, we have an upper solution. Now we present an existence result relative to BVP (3.1), which we will use in our main results. Since the proof is a standard application of Schauder's fixed point theorem we will omit the proof of this lemma.

Lemma 3.2. *Assume that*

(H3.1) *the function* $f : \mathbb{N}_{a+1}^{b-1} \times \mathbb{R} \to \mathbb{R}$ *is continuous with respect to* x*, and for* $M \geq 0$*, define* $C := \max\{|f(t,x)| : t \in \mathbb{N}_{a+1}^{b-1}, |x| \leq 2M\}$*.*

Then the nonlinear BVP (3.1) *has a solution provided there is some* $M \geq \|z\| = \max\{|A|, |B|\}$*, where* z *is the unique solution of* (2.12)*, such that* $C(M) > 0$ *and*

$$(b-a)^2 \leq \frac{4M\Gamma(\nu+2)}{C}.$$

In particular, if $f \not\equiv 0$ and is bounded, then (3.1) has a solution.

Lemma 3.3. *Assume that*

(H3.2) *the function* $f : \mathbb{N}_{a+1}^{b-1} \times \mathbb{R} \to \mathbb{R}$ *is nondecreasing with respect to* x *for each* t*.*

(H3.3) *the functions* α_0*,* $\beta_0 : \mathbb{N}_a^b \to \mathbb{R}$ *are lower and upper solutions respectively of* (3.1)*.*

Then $\alpha_0(t) \leq \beta_0(t)$ *on* $\mathbb{N}_a^b$*.*

Proof. Let us first prove the lemma for strict inequality. That is, suppose

$$\begin{gathered}(L_a\alpha_0)(t) > f(t,\alpha_0(t)), \quad t \in \mathbb{N}_{a+1}^{b-1},\\ \alpha_0(a) \leq A, \quad \alpha_0(b) \leq B.\end{gathered}$$

and

$$\begin{gathered}(L_a\beta_0)(t) \leq f(t,\beta_0(t)), \quad t \in \mathbb{N}_{a+1}^{b-1},\\ \beta_0(a) \geq A, \quad \beta_0(b) \geq B.\end{gathered}$$

The boundary conditions give us $\alpha_0(a) \leq A \leq \beta_0(a)$ and $\alpha_0(b) \leq B \leq \beta_0(b)$. Next, we will show that $\alpha_0(t) < \beta_0(t)$ for $t \in \mathbb{N}_{a+1}^{b-1}$. If this is not true, then there exists $t_0 \in \mathbb{N}_{a+1}^{b-1}$ such that $x(t) := \alpha_0(t) - \beta_0(t)$ has a nonnegative maximum at t_0. That is, for some $t_0 \in \mathbb{N}_{a+1}^{b-1}$,

$$x(t_0) = \alpha_0(t_0) - \beta_0(t_0) = \max\{\alpha_0(t) - \beta_0(t), \quad t \in \mathbb{N}_{a+1}^{b-1}\} \geq 0,$$

But then, $x(t_0) = \max_{t\in\mathbb{N}_a^b} x(t)$, since $x(a) = \alpha_0(a) - \beta_0(a) \leq 0$ and $x(b) = \alpha_0(b) - \beta_0(b) \leq 0$. Therefore, by Lemma 2.15, we obtain

$$(L_a x)(t_0) = (L_a\alpha_0)(t_0) - (L_a\beta_0)(t_0) \leq 0.$$

So, we have

$$f(t_0,\alpha_0(t_0)) < (L_a\alpha_0)(t_0) \leq (L_a\beta_0)(t_0) \leq f(t_0,\beta_0(t_0)),$$

On the other hand, our assumption that $x(t_0) = \alpha_0(t_0) - \beta_0(t_0) \geq 0 \Rightarrow \beta_0(t_0) \leq \alpha_0(t_0)$. By $(H3.2)$, $f(t,x)$ is nondecreasing with respect to x for each t, so this, in turn, implies

$$f(t_0,\beta_0(t_0)) \leq f(t_0,\alpha_0(t_0))$$

which is a contradiction. Hence, $\alpha_0(t) < \beta_0(t)$ on $\mathbb{N}_{a+1}^{b-1}$.

Now, we define $\tilde{\alpha}_0(t) := \alpha_0(t) + \varepsilon(H_{\nu+1}(t,a) - H_{\nu+1}(b,a))$, where $\varepsilon > 0$. Then $\tilde{\alpha}_0(t) < \alpha_0(t)$ for $t \in \mathbb{N}_{a+1}^{b-1}$. Using condition (H3.2), we obtain

$$(L_a\tilde{\alpha}_0)(t) = (L_a\alpha_0)(t) + \varepsilon L_a[H_{\nu+1}(\cdot,a) - H_{\nu+1}(b,a)](t)$$

$$\begin{aligned} &= (L_a\alpha_0)(t) + \varepsilon \\ &\geq f(t, \alpha_0(t)) + \varepsilon \\ &\geq f(t, \tilde{\alpha}_0(t)) + \varepsilon \\ &> f(t, \tilde{\alpha}_0(t)). \end{aligned}$$

Thus $\tilde{\alpha}_0(t)$ is a lower solution for which strict inequality holds. It therefore follows from the previous argument that $\tilde{\alpha}_0(t) < \beta_0(t)$ for all $t \in \mathbb{N}_{a+1}^{b-1}$. Note this holds for all $\varepsilon > 0$. Letting $\varepsilon \to 0$, we obtain $\alpha_0(t) \leq \beta_0(t)$ for all $t \in \mathbb{N}_{a+1}^{b-1}$. Thus, we have $\alpha_0(t) \leq \beta_0(t)$ on $\mathbb{N}_a^b$. The proof is complete. □

Corollary 3.4. *Assume that conditions* (H3.1), (H3.2) *hold. Then BVP* (3.1) *has a unique solution on* $\mathbb{N}_a^b$.

Proof. According to Lemma 3.2, we obtain that BVP (3.1) has a solution on $\mathbb{N}_a^b$. Now, we suppose $x(t)$ and $\bar{x}(t)$ are two solutions of (3.1). Since any solution is both a lower and an upper solution, by Lemma 3.3, we have

$$x(t) \leq \bar{x}(t) \leq x(t) \Rightarrow x(t) = \bar{x}(t) \text{ on } \mathbb{N}_a^b.$$

That is, the solution of (3.1) is unique on $\mathbb{N}_a^b$. The proof is complete. □

Next, we consider BVP (3.1) in the special case where $f(t,x) = C(t)x$ and $A = B = 0$. That is, we consider the BVP

$$(L_a x)(t) = C(t)x(t) \quad \text{for } t \in \mathbb{N}_{a+1}^{b-1},\ x(a) = 0,\ x(b) = 0. \tag{3.3}$$

Corollary 3.5. *Assume that*

(H3.4) *the function* $C(t) \geq 0$ *for* $t \in \mathbb{N}_{a+1}^{b-1}$.

If $x(t)$ *is a lower solution of* (3.3)*, then* $x(t) \leq 0$ *on* $\mathbb{N}_a^b$. *If* $y(t)$ *is an upper solution of* (3.3)*, then* $y(t) \geq 0$ *on* $\mathbb{N}_a^b$.

Proof. By hypothesis, $x(t)$ is a lower solution of (3.3) and, by inspection, $z(t) \equiv 0$ is a (upper) solution to (3.3). So Lemma 3.3 guarantees $x(t) \leq 0$. Similarly, $y(t)$ is an upper solution of (3.3) and $z(t) \equiv 0$ is a (lower) solution to (3.3). So Lemma 3.3 guarantees $y(t) \geq 0$. The proof is complete. □

Lemma 3.6. *Assume that*

(H3.5) *the functions* α_0, $\beta_0 : \mathbb{N}_a^b \to \mathbb{R}$ *are lower and upper solutions respectively of* (3.1) *such that* $\alpha_0(t) \leq \beta_0(t)$ *on* $\mathbb{N}_a^b$.

(H3.6) *the function* $f : \Omega \to \mathbb{R}$ *is continuous in its second variable and* $f \not\equiv 0$ *on* Ω.

Then there exists a solution $x(t)$ *of* (3.1) *satisfying* $\alpha_0(t) \leq x(t) \leq \beta_0(t)$ *on* $\mathbb{N}_a^b$.

Proof. Let $P : \mathbb{N}_{a+1}^{b-1} \times \mathbb{R} \to \mathbb{R}$ be defined by $P(t,x) = \max\{\alpha_0(t), \min\{x, \beta_0(t)\}\}$. Then $f(t, P(t,x))$ defines a extension of f to $\mathbb{N}_{a+1}^{b-1} \times \mathbb{R}$, which is continuous in its second variable, $f \not\equiv 0$ and bounded. Therefore, by Lemma 3.2, the BVP

$$(L_a x)(t) = \bar{f}(t,x) := f(t, P(t,x)) \text{ for } \ t \in \mathbb{N}_{a+1}^{b-1}, \quad x(a) = A,\ x(b) = B. \tag{3.4}$$

has a solution $x(t)$ on $\mathbb{N}_a^b$.

To complete the proof, we just need to show that $\alpha_0(t) \leq x(t) \leq \beta_0(t)$ on $\mathbb{N}_a^b$. Doing so will mean that $\bar{f}(t, x(t)) = f(t, x(t))$, which will imply $x(t)$ is not only a solution of (3.4) but also actually a solution of (3.1). Toward this end, we first show that $\alpha_0(t) \leq x(t)$ on $\mathbb{N}_a^b$. The boundary conditions immediately give us that

$\alpha_0(a) \le x(a)$ and $\alpha_0(b) \le x(b)$. To see that $\alpha_0(t) \le x(t)$ on $\mathbb{N}_{a+1}^{b-1}$, as in the proof of Lemma 3.3, for $\varepsilon > 0$ define $\tilde{\alpha}_\varepsilon(t) := \alpha_0(t) + \varepsilon[H_{\nu+1}(t,a) - H_{\nu+1}(b,a)]$. Then $\tilde{\alpha}_\varepsilon(t) < \alpha_0(t)$ for $t \in \mathbb{N}_{a+1}^{b-1}$, $\tilde{\alpha}_\varepsilon(a) < \alpha_0(a) \le x(a)$, and $\tilde{\alpha}_\varepsilon(b) = \alpha_0(b) \le x(b)$. If we can show $\tilde{\alpha}_\varepsilon(t) < x(t)$ on $\mathbb{N}_{a+1}^{b-1}$, then letting $\varepsilon \to 0$, we obtain $\lim_{\varepsilon\to 0} \tilde{\alpha}_\varepsilon(t) = \alpha_0(t) \le x(t)$ for $t \in \mathbb{N}_{a+1}^{b-1}$ and we will be done. So, toward contradiction, assume that for some fixed $\varepsilon > 0$ this is not true. That is, assume that for some fixed $\varepsilon > 0$ there exists $t_1 \in \mathbb{N}_{a+1}^{b-1}$ such that $d(t) := \tilde{\alpha}_\varepsilon(t) - x(t)$ has a nonnegative maximum at t_1, i.e. for this $t_1 \in \mathbb{N}_{a+1}^{b-1}$,

$$d(t_1) = \tilde{\alpha}_\varepsilon(t_1) - x(t_1) = \max\{\tilde{\alpha}_\varepsilon(t) - x(t), \quad t \in \mathbb{N}_{a+1}^{b-1}\} \ge 0.$$

But then, $d(t_1) = \max_{t\in\mathbb{N}_a^b} d(t)$, since $d(a) = \tilde{\alpha}_\varepsilon(a) - x(a) \le 0$ and $d(b) = \tilde{\alpha}_\varepsilon(b) - x(b) \le 0$. And so, by Lemma 2.15, we must have

$$(L_a d)(t_1) \le 0. \tag{3.5}$$

On the other hand, our assumption that $\tilde{\alpha}_\varepsilon(t_1) - x(t_1) \ge 0$ implies $x(t_1) \le \tilde{\alpha}_\varepsilon(t_1) \le \alpha_0(t_1)$ and so we have $P(t_1, x(t_1)) = \alpha_0(t_1)$. Thus,

$$(L_a\alpha_0)(t_1) \ge f(t_1, \alpha_0(t_1)) = f(t_1, P(t_1, x(t_1))) = (L_a x)(t_1)$$

which implies

$$(L_a\alpha_0)(t_1) - (L_a x)(t_1) \ge 0$$

which, in turn, implies,

$$\begin{aligned}
(L_a d)(t_1) &= (L_a\tilde{\alpha}_\varepsilon)(t_1) - (L_a x)(t_1)\\
&= (L_a\alpha_0)(t_1) + \varepsilon L_a[H_{\nu+1}(\cdot,a) - H_{\nu+1}(b,a)](t_1) - (L_a x)(t_1)\\
&= (L_a\alpha_0)(t_1) + \varepsilon(1-0) - (L_a x)(t_1)\\
&= (L_a\alpha_0)(t_1) - (L_a x)(t_1) + \varepsilon > 0.
\end{aligned}$$

which is in contradiction to (3.5). Thus, for all $\varepsilon > 0$, $\tilde{\alpha}_\varepsilon(t) < x(t)$ on $\mathbb{N}_{a+1}^{b-1}$. And so, as argued above, letting $\varepsilon \to 0$ we obtain our result that $\alpha_0(t) \le x(t)$ on $\mathbb{N}_{a+1}^{b-1}$. Similarly, we can show that $x(t) \le \beta_0(t)$ on $\mathbb{N}_{a+1}^{b-1}$. Therefore, we conclude that $\alpha_0(t) \le x(t) \le \beta_0(t)$, $t \in \mathbb{N}_a^b$. The proof is complete. □

Lemma 3.7. *Assume that the conditions* (H3.2), (H3.5) and (H3.6) *hold. Then* (3.1) *has a unique solution $x(t)$. Furthermore, $\alpha_0(t) \le x(t) \le \beta_0(t)$ on $\mathbb{N}_a^b$.*

Proof. By Lemma 3.6, there exists a solution of (3.1) which lies in $\mathcal{S}$. By Corollary 3.4 this solution is unique. The proof is complete. □

4. Main results

In this section, we consider BVP (3.1) in the special case where $f(t,x)$ is increasing with respect to x and $f(t,x) = f_1(t,x) + f_2(t,x)$, where $f_1(t,x)$ is concave up with respect to x and $f_2(t,x)$ is concave down with respect to x. We obtain sequences of successive approximations by applying the generalized quasi-linearization method to our nonlinear Caputo nabla fractional difference equation and show that the sequences so obtained converge quadratically to the solution. Furthermore, we use the Gauss-Seidel method to improve the rate of convergence.

Theorem 4.1. *Assume that condition* (H3.5) *holds, and*

(A4.1) *$f \not\equiv 0$ and $f = f_1 + f_2$ where the functions f_1, $f_2 : \Omega \to \mathbb{R}$ are such that:*

(i) $f_1^{(i)}(t,x)$, $f_2^{(i)}(t,x)$ $(i=0,1,2)$ *exist and are continuous in the second variable,*
(ii) $f_1^{(1)}(t,x)+f_2^{(1)}(t,y)\ge 0$ *for all* $(t,x),(t,y)\in\Omega$, *and*
(iii) $f_1^{(2)}(t,x)\ge 0$ *and* $f_2^{(2)}(t,x)\le 0$ *on* Ω.

Then there exist two sequences $\{\alpha_n(t)\}$ *and* $\{\beta_n(t)\}$, *which converge monotonically to a function* x *in* $\mathcal{S}(\alpha_0,\beta_0)$ *which is a solution of* (3.1). *Furthermore, the convergence is quadratic.*

As in Agarwal et al [3, 3. Main Results, Theorem 1], we define the quadratic convergence of two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ which both converge to a common point x, by the condition that there exist constants C_1, C_2, C_3, and C_4 such that for all $n\ge 0$,

$$\begin{gathered}\|x-\alpha_{n+1}\|\le C_1\|x-\alpha_n\|^2+C_2\|\beta_n-x\|^2,\\ \|\beta_{n+1}-x\|\le C_3\|\beta_n-x\|^2+C_4\|x-\alpha_n\|^2.\end{gathered}$$

Proof. From condition (A4.1)(iii) and the mean value theorem, we obtain, for all (t,x), $(t,y)\in\Omega$,

$$f_1(t,y)\ge f_1(t,x)+f_1^{(1)}(t,x)(y-x), \tag{4.1}$$

$$f_2(t,y)\ge f_2(t,x)+f_2^{(1)}(t,y)(y-x). \tag{4.2}$$

Next, consider the following two BVPs

$$\begin{gathered}(L_ay)(t)=f_1(t,\alpha)+f_1^{(1)}(t,\beta)(y-\alpha)+f_2(t,\alpha)+f_2^{(1)}(t,\alpha)(y-\alpha)\\ \equiv F(t,\alpha,\beta;y),\quad t\in\mathbb{N}_{a+1}^{b-1},\\ y(a)=A,\quad y(b)=B,\end{gathered}\tag{4.3}$$

and

$$\begin{gathered}(L_az)(t)=f_1(t,\beta)+f_1^{(1)}(t,\beta)(z-\beta)+f_2(t,\beta)+f_2^{(1)}(t,\alpha)(z-\beta)\\ \equiv G(t,\alpha,\beta;z),\quad t\in\mathbb{N}_{a+1}^{b-1},\\ z(a)=A,\quad z(b)=B.\end{gathered}\tag{4.4}$$

Letting $\alpha=\alpha_0$, $\beta=\beta_0$ in BVPs (4.3), (4.4). We first prove that $\alpha_0(t)$ and $\beta_0(t)$ are lower and upper solutions of (4.3), respectively. In fact, from condition (H3.5), we have

$$\begin{gathered}(L_a\alpha_0)(t)\ge f_1(t,\alpha_0)+f_2(t,\alpha_0)=F(t,\alpha_0,\beta_0;\alpha_0),\quad t\in\mathbb{N}_{a+1}^{b-1},\\ \alpha_0(a)\le A,\quad \alpha_0(b)\le B,\end{gathered}$$

and by using inequalities (4.1), (4.2), it follows that

$$\begin{aligned}(L_a\beta_0)(t)&\le f_1(t,\beta_0)+f_2(t,\beta_0)\\ &\le f_1(t,\alpha_0)+f_1^{(1)}(t,\beta_0)(\beta_0-\alpha_0)+f_2(t,\alpha_0)+f_2^{(1)}(t,\alpha_0)(\beta_0-\alpha_0)\\ &=F(t,\alpha_0,\beta_0;\beta_0),\quad t\in\mathbb{N}_{a+1}^{b-1},\\ &\qquad\beta_0(a)\ge A,\quad \beta_0(b)\ge B.\end{aligned}$$

These show that $\alpha_0(t)$ and $\beta_0(t)$ are lower and upper solutions of (4.3). Furthermore, note that $F:\mathbb{N}_{a+1}^{b-1}\times\mathbb{R}\to\mathbb{R}$ is continuous and nondecreasing with respect to y. Thus, by Lemma 3.7, it follows that there exists a unique solution $\alpha_1(t)$ of (4.3) such that $\alpha_0(t)\le\alpha_1(t)\le\beta_0(t)$ on $\mathbb{N}_a^b$.

Similarly, using condition (H3.5), and inequalities (4.1), (4.2), we obtain

$$\begin{aligned}(L_a\alpha_0)(t) &\ge f_1(t,\alpha_0)+f_2(t,\alpha_0)\\ &\ge f_1(t,\beta_0)+f_1^{(1)}(t,\beta_0)(\alpha_0-\beta_0)+f_2(t,\beta_0)+f_2^{(1)}(t,\alpha_0)(\alpha_0-\beta_0)\\ &= G(t,\alpha_0,\beta_0;\alpha_0),\quad t\in\mathbb{N}_{a+1}^{b-1},\\ &\alpha_0(a)\le A,\quad \alpha_0(b)\le B,\end{aligned}$$

and

$$\begin{gathered}(L_a\beta_0)(t)\le f_1(t,\beta_0)+f_2(t,\beta_0)=G(t,\alpha_0,\beta_0;\beta_0),\quad t\in\mathbb{N}_{a+1}^{b-1},\\ \beta_0(a)\ \ge A,\quad \beta_0(b)\ge B.\end{gathered}$$

These show that $\alpha_0(t)$ and $\beta_0(t)$ are lower and upper solutions of (4.4). Furthermore, note that $G:\mathbb{N}_{a+1}^{b-1}\times\mathbb{R}\to\mathbb{R}$ is continuous and nondecreasing with respect to z. Thus, by Lemma 3.7, it follows that there exists a unique solution $\beta_1(t)$ of (4.4) such that $\alpha_0(t)\le\beta_1(t)\le\beta_0(t)$ on $\mathbb{N}_a^b$.

Next, we show that $\alpha_1(t)$ and $\beta_1(t)$ are lower and upper solutions of the original BVP (3.1). Toward this end, using the fact that $\alpha_1(t)$ is the unique solution of (4.3), condition (A4.1)(iii), and the inequalities (4.1), (4.2), we have

$$\begin{aligned}(L_a\alpha_1)(t) &= f_1(t,\alpha_0)+f_1^{(1)}(t,\beta_0)(\alpha_1-\alpha_0)+f_2(t,\alpha_0)+f_2^{(1)}(t,\alpha_0)(\alpha_1-\alpha_0)\\ &\ge f_1(t,\alpha_0)+f_1^{(1)}(t,\alpha_1)(\alpha_1-\alpha_0)+f_2(t,\alpha_0)+f_2^{(1)}(t,\alpha_0)(\alpha_1-\alpha_0)\\ &\ge f_1(t,\alpha_1)+f_2(t,\alpha_1),\quad t\in\mathbb{N}_{a+1}^{b-1},\\ &\alpha_1(a)=A,\quad \alpha_1(b)=B,\end{aligned}$$

which proves $\alpha_1(t)$ is a lower solution of (3.1). Similar arguments show that

$$\begin{gathered}(L_a\beta_1)(t)\le f_1(t,\beta_1)+f_2(t,\beta_1),\quad t\in\mathbb{N}_{a+1}^{b-1},\\ \beta_1(a)=A,\quad \beta_1(b)=B,\end{gathered}$$

which shows that $\beta_1(t)$ is an upper solution of (3.1). So by Lemma 3.3, we have $\alpha_1(t)\le\beta_1(t)$ on $\mathbb{N}_a^b$. Thus, we obtain

$$\alpha_0(t)\le\alpha_1(t)\le\beta_1(t)\le\beta_0(t)\quad\text{on }\mathbb{N}_a^b.$$

Thus, by iteration, we obtain

$$\alpha_0(t)\le\alpha_1(t)\le\dots\le\alpha_n(t)\le\beta_n(t)\le\dots\le\beta_1(t)\le\beta_0(t)\quad\text{on }\mathbb{N}_a^b$$

where we obtain the functions α_{n+1} and β_{n+1} by the iterative schemes:

$$\begin{aligned}&(L_a\alpha_{n+1})(t)\\ &= f_1(t,\alpha_n)+f_1^{(1)}(t,\beta_n)(\alpha_{n+1}-\alpha_n)+f_2(t,\alpha_n)+f_2^{(1)}(t,\alpha_n)(\alpha_{n+1}-\alpha_n)\\ &= F(t,\alpha_n,\beta_n;\alpha_{n+1}),\quad t\in\mathbb{N}_{a+1}^{b-1},\\ &\qquad\alpha_{n+1}(a)=A,\ \alpha_{n+1}(b)=B,\end{aligned}\tag{4.5}$$

and

$$\begin{aligned}&(L_a\beta_{n+1})(t)\\ &= f_1(t,\beta_n)+f_1^{(1)}(t,\beta_n)(\beta_{n+1}-\beta_n)+f_2(t,\beta_n)+f_2^{(1)}(t,\alpha_n)(\beta_{n+1}-\beta_n)\\ &= G(t,\alpha_n,\beta_n;\beta_{n+1}),\quad t\in\mathbb{N}_{a+1}^{b-1},\\ &\qquad\beta_{n+1}(a)=A,\quad \beta_{n+1}(b)=B.\end{aligned}\tag{4.6}$$

In addition, using the fact that $\alpha_n(t)$, $\beta_n(t)$ are lower and upper solutions of (3.1) with $\alpha_n(t) \leq \beta_n(t)$, and the conditions of Lemma 3.7 are satisfied, we conclude that there exists a unique solution $x(t)$ of (3.1) such that $\alpha_n(t) \leq x(t) \leq \beta_n(t)$ on $\mathbb{N}_a^b$. Thus

$$\alpha_0(t) \leq \alpha_1(t) \leq \cdots \leq \alpha_n(t) \leq x(t) \leq \beta_n(t) \leq \cdots \leq \beta_1(t) \leq \beta_0(t) \quad \text{on } \mathbb{N}_a^b$$

For any fixed $t \in \mathbb{N}_a^b$, the monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ are bounded by $\{\alpha_0(t)\}$ and $\{\beta_0(t)\}$. As such, they converge pointwise to some limit functions, ρ and r. That is, the functions ρ, $r : \mathbb{N}_a^b \to \mathbb{R}$ and satisfy

$$\lim_{n\to\infty} \alpha_n(t) = \rho(t) \leq x(t) \leq r(t) = \lim_{n\to\infty} \beta_n(t).$$

By taking the limit as $n \to \infty$ of the difference equation in (4.5) and (4.6) we can show that $\rho(t)$ and $r(t)$ are solutions of (3.1). Since $\rho(t)$ and $r(t)$ also lie in $\mathcal{S}(\alpha_0, \beta_0)$, it must be the case that $\rho(t) = x(t) = r(t)$ on $\mathbb{N}_a^b$. Hence $\alpha_n(t)$ and $\beta_n(t)$ both converge monotonically to $x(t)$.

Finally, we show that the convergence of the sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ is quadratic. For this purpose, set

$$p_{n+1}(t) = x(t) - \alpha_{n+1}(t) \geq 0, \text{ and } q_{n+1}(t) = \beta_{n+1}(t) - x(t) \geq 0, \quad t \in \mathbb{N}_a^b.$$

From condition (A4.1), the mean value theorem, and Lemma 2.11, we obtain

$$\begin{aligned}
p_{n+1}(t) &= \int_a^b G(t,s) f(s,x(s)) \nabla s - \int_a^b G(t,s) F(s,\alpha_n(s),\beta_n(s);\alpha_{n+1}(s)) \nabla s \\
&= \int_a^b G(t,s)\big[f_1(s,x) + f_2(s,x)\big] \nabla s - \int_a^b G(t,s)[f_1(s,\alpha_n) \\
&\quad + f_1^{(1)}(s,\beta_n)(\alpha_{n+1}-\alpha_n) + f_2(s,\alpha_n) + f_2^{(1)}(s,\alpha_n)(\alpha_{n+1}-\alpha_n)] \nabla s \\
&= \int_a^b G(t,s)\big[\big(f_1^{(1)}(s,\beta_n) + f_2^{(1)}(s,\alpha_n)\big)p_{n+1} + \big(f_1^{(1)}(s,\xi_3) - f_1^{(1)}(s,\beta_n)\big)p_n \\
&\quad + \big(f_2^{(1)}(s,\xi_4) - f_2^{(1)}(s,\alpha_n)\big)p_n\big] \nabla s \\
&\leq \int_a^b G(t,s)\big[f_1^{(2)}(s,\eta_1)(\xi_3-\beta_n)p_n + f_2^{(2)}(s,\eta_2)(\xi_4-\alpha_n)p_n\big] \nabla s \\
&\leq \int_a^b |G(t,s)|\big[A_1\|p_n+q_n\|\|p_n\| + B_1\|p_n\|\|p_n\|\big] \nabla s \\
&\leq MA_1\|p_n\|(\|p_n\| + \|q_n\|) + MB_1\|p_n\|^2 \\
&= \Big(\frac{3}{2}MA_1 + MB_1\Big)\|p_n\|^2 + \frac{1}{2}MA_1\|q_n\|^2,
\end{aligned}$$

where

$$\begin{gathered}
M := \max_{t\in\mathbb{N}_a^b} \sum_{s=a+1}^{b} |G(t,s)| = \frac{(b-a)^2}{4\Gamma(\nu+2)}, \\
\alpha_n(t) \leq \xi_3(t), \xi_4(t) \leq x(t), \quad \xi_3(t) \leq \eta_1(t) \leq \beta_n(t), \\
\alpha_n(t) \leq \eta_2(t) \leq \xi_4(t), \quad |f_1^{(2)}(t,x)| \leq A_1, \\
|f_2^{(2)}(t,x)| \leq B_1
\end{gathered}$$

for $t \in \mathbb{N}_a^b$. So, we have

$$\|p_{n+1}\| \le \Big(\frac{3}{2}MA_1 + MB_1\Big)\|p_n\|^2 + \frac{1}{2}MA_1\|q_n\|^2,$$

Similarly, we have

$$\|q_{n+1}\| \le \Big(MA_1 + \frac{3}{2}MB_1\Big)\|q_n\|^2 + \frac{1}{2}MB_1\|p_n\|^2.$$

The proof is complete. □

Next, we apply the Gauss-Seidel method to possibly improve upon the convergence rate of the iterative scheme described in Theorem 4.1.

Theorem 4.2. *Let the hypotheses of Theorem 4.1 hold. Consider the iterative schemes given by*

$$\begin{aligned} (L_a\alpha^*_{n+1})(t) &= f_1(t,\alpha^*_n) + f_1^{(1)}(t,\beta^*_n)(\alpha^*_{n+1}-\alpha^*_n) \\ &\quad + f_2(t,\alpha^*_n) + f_2^{(1)}(t,\alpha^*_n)(\alpha^*_{n+1}-\alpha^*_n) \\ &\equiv F(t,\alpha^*_n,\beta^*_n;\alpha^*_{n+1}), \quad t\in\mathbb{N}_{a+1}^{b-1}, \\ \alpha^*_{n+1}(a) &= A, \quad \alpha^*_{n+1}(b) = B, \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} (L_a\beta^*_{n+1})(t) &= f_1(t,\beta^*_n) + f_1^{(1)}(t,\beta^*_n)(\beta^*_{n+1}-\beta^*_n) \\ &\quad + f_2(t,\beta^*_n) + f_2^{(1)}(t,\alpha^*_{n+1})(\beta^*_{n+1}-\beta^*_n) \\ &\equiv G(t,\alpha^*_{n+1},\beta^*_n;\beta^*_{n+1}), \quad t\in\mathbb{N}_{a+1}^{b-1}, \\ \beta^*_{n+1}(a) &= A, \quad \beta^*_{n+1}(b) = B. \end{aligned} \tag{4.8}$$

starting with $\alpha^*_0 = \alpha_0$ on $\mathbb{N}_a^b$. The two sequences obtained via this iterative scheme $\{\alpha^*_n(t)\}$ and $\{\beta^*_n(t)\}$, $n \ge 0$ converge monotonically to the $x(t)$, the solution of (3.1) *that lies between α_0 and β_0 and the convergence is at least quadratic.*

Proof. Initially, we compute $\alpha_1(t)$ using the following BVPs

$$\begin{aligned} &(L_a\alpha_1)(t) \\ &= f_1(t,\alpha_0) + f_1^{(1)}(t,\beta_0)(\alpha_1-\alpha_0) + f_2(t,\alpha_0) + f_2^{(1)}(t,\alpha_0)(\alpha_1-\alpha_0) \\ &\equiv F(t,\alpha_0,\beta_0;\alpha_1), \quad t\in\mathbb{N}_{a+1}^{b-1}, \\ &\qquad \alpha_1(a) = A, \quad \alpha_1(b) = B. \end{aligned} \tag{4.9}$$

Relabel $\alpha_1(t)$ as $\alpha^*_0(t)$. Now, we compute $\beta_1(t)$ using $\beta_0(t)$ and $\alpha^*_0(t)$, that is, $\beta_1(t)$ is a solution of

$$\begin{aligned} &(L_a\beta_1)(t) \\ &= f_1(t,\beta_0) + f_1^{(1)}(t,\beta_0)(\beta_1-\beta_0) + f_2(t,\beta_0) + f_2^{(1)}(t,\alpha^*_0)(\beta_1-\beta_0) \\ &\equiv G(t,\alpha^*_0,\beta_0;\beta_1), \quad t\in\mathbb{N}_{a+1}^{b-1}, \\ &\qquad \beta_1(a) = A, \quad \beta_1(b) = B. \end{aligned} \tag{4.10}$$

It is clear that $\alpha_0(t) \le \alpha_1(t) \le \alpha^*_0(t)$ and $\beta_1(t) \le \beta_0(t)$ on $\mathbb{N}_a^b$. Put $p(t) := \beta^*_0(t) - \beta_1(t)$. Then $p(a) = p(b) = 0$. Also, we have

$$\begin{aligned} (L_ap)(t) &= [f_1(t,\beta_0) + f_1^{(1)}(t,\beta_0)(\beta^*_0-\beta_0) + f_2(t,\beta_0) + f_2^{(1)}(t,\alpha^*_0)(\beta^*_0-\beta_0)] \\ &\quad - [f_1(t,\beta_0) + f_1^{(1)}(t,\beta_0)(\beta_1-\beta_0) + f_2(t,\beta_0) + f_2^{(1)}(t,\alpha_0)(\beta_1-\beta_0)] \end{aligned}$$

$$\geq f_1^{(1)}(t,\beta_0)(\beta_0^* - \beta_1) + f_2^{(1)}(t,\alpha_0^*)(\beta_0^* - \beta_1)$$
$$= [f_1^{(1)}(t,\beta_0) + f_2^{(1)}(t,\alpha_0^*)]p.$$

So p is a lower solution to a BVP of (3.3). Thus, by Corollary 3.5, we know $p(t) \leq 0$ on $\mathbb{N}_a^b$. That is, $\beta_0^*(t) \leq \beta_1(t)$ on $\mathbb{N}_a^b$. Continuing the process, we will be able to show that the sequences $\{\alpha_n^*(t)\}$ and $\{\beta_n^*(t)\}$ must converge at least as fast as the sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ that were computed using the iterative scheme described in Theorem 4.1. The proof is complete. □

Remark 4.3. When the function $f(t,x)$ is the sum of $(n-1)$-hyperconvex and $(n-1)$-hyperconcave functions (i.e. $f(t,x) = f_1(t,x)+f_2(t,x)$, where $f_1^{(n)}(t,x) \geq 0$, and $f_2^{(n)}(t,x) \leq 0$), we can obtain two monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$, whose convergence is of order n ($n \geq 2$). The proof is similar to that of Theorem 4.1, so we omit it.

5. Examples

Table 1. Three α, β-iterates of (5.1).

t	$\alpha_0(t)$	$\alpha_1(t)$	$\alpha_2(t)$	$\alpha_3(t)$	$\beta_3(t)$	$\beta_2(t)$	$\beta_1(t)$	$\beta_0(t)$
0	0	0	0	0	0	0	0	1
1	0	0.007355	0.021545	0.027694	0.028628	0.067598	0.354038	1
2	0	0.025743	0.058914	0.069876	0.071137	0.115480	0.405802	1
3	0	0.087341	0.152969	0.167201	0.168294	0.205791	0.454208	1
4	0	0.295581	0.391591	0.403170	0.403700	0.424171	0.582092	1
5	0	1	1	1	1	1	1	1

Table 2. Three α^*, β^*-iterates of (5.1).

t	$\alpha_0^*(t)$	$\alpha_1^*(t)$	$\alpha_2^*(t)$	$\alpha_3^*(t)$	$\beta_3^*(t)$	$\beta_2^*(t)$	$\beta_1^*(t)$	$\beta_0^*(t)$
0	0	0	0	0	0	0	0	1
1	0	0.007355	0.021572	0.028417	0.028532	0.066598	0.353610	1
2	0	0.025743	0.058982	0.070027	0.070946	0.112921	0.404336	1
3	0	0.087341	0.153110	0.167290	0.168026	0.200221	0.449635	1
4	0	0.295581	0.391971	0.403220	0.403523	0.418531	0.570861	1
5	0	1	1	1	1	1	1	1

Now, we give an example to illustrate the results established in the previous section.

Example 5.1. Consider the BVP

$$\begin{gathered}(L_a x)(t) = -\frac{1}{3}x^3(t) + \frac{1}{2}x^2(t) + x(t), \quad t \in \mathbb{N}_1^4, \\ x(0) = 0, \quad x(5) = 1,\end{gathered} \tag{5.1}$$

where $a = 0$, $\nu = 1/2$.

Taking $\alpha_0(t) \equiv 0$, $\beta_0(t) \equiv 1$, it is quick to verify that $\alpha_0(t)$, $\beta_0(t)$ are lower and upper solutions of (5.1), respectively. Let $f(t,x(t))$ denote the right-hand side of

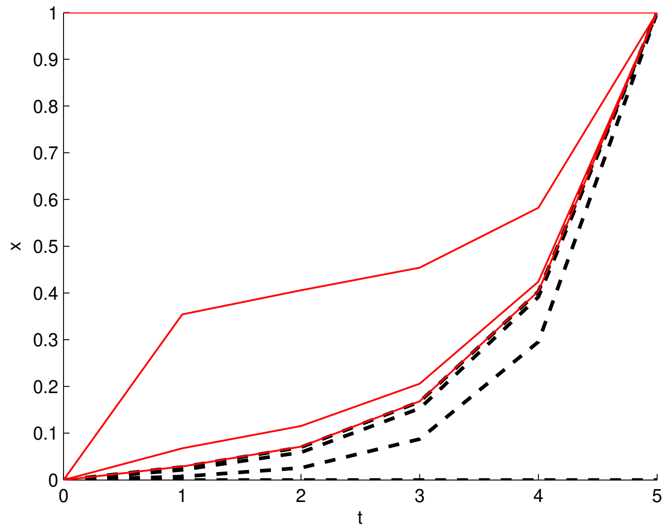

FIGURE 1. α, β-iterates

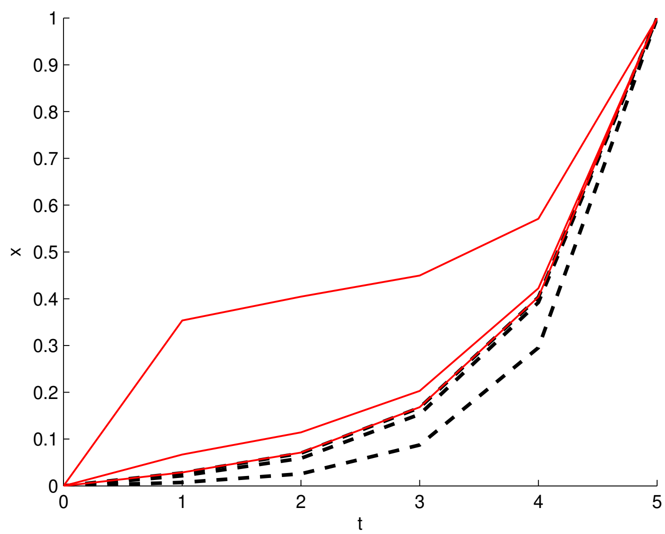

FIGURE 2. α^*, β^*-iterates

(5.1), and split it into two functions as $f(t, x(t)) = f_1(t, x(t)) + f_2(t, x(t))$, where $f_1(t, x(t)) = \frac{1}{2}x^2(t) + x(t)$, $f_2(t, x(t)) = -\frac{1}{3}x^3(t)$. Note that

$$f_1^{(1)}(t, x) = x + 1 > 0, \ f_1^{(2)}(t, x) = 1 > 0 \ \text{ on } \Omega = \mathbb{N}_1^4 \times [0, 1],$$
$$f_2^{(1)}(t, x) = -x^2 \leq 0, \ f_2^{(2)}(t, x) = -2x(t) \leq 0 \ \text{ on } \Omega.$$

Now, we apply the iteration scheme of Theorem 4.1. After three iterations we find the α, β-iterates given in Table 1. The graph in Figure 1 shows the α-iterates (with broken line) and the β-iterates (with unbroken line).

Next, we apply the iteration scheme of Theorem 4.2. After three iterations we find the α^*, β^*-iterates given in Table 2. The graph in Figure 2 shows the α^*-iterates (with broken line) and the β^*-iterates (with unbroken line).

Conclusions. In the above parts, we discussed a nonlinear Caputo nabla fractional difference equation with boundary conditions. By using the generalized quasi-linearization, two monotone sequences are obtained whose rate of convergence is quadratic. Further, by using the Gauss-Seidel method, it is possible that we may improve upon this rate of convergence. Finally, we give a numerical example to illustrate the established results.

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