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PASSING TO THE LIMIT ON SMALL PARAMETERS FOR GENERALIZED VISCOUS CAHN-HILLIARD TYPE EQUATIONS WITH NONLINEAR SOURCE

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Abstract. We study the well-posedness of the generalized viscous Cahn-Hilliard equation with nonlinear source term. Then, we analyze the singular limits when the relaxed terms vanish. In the sense of Young measures, we obtain the measure-valued solution of a forward-backward parabolic type equation.

1. Introduction

We study the forward-backward parabolic problem

$$\begin{gathered} u_t = \Delta\varphi(u) + f(u) \quad \text{in } \Omega \times (0,T) =: Q_T \\ u = 0 \quad \text{on } \partial\Omega \times (0,T) \\ u = u_0 \quad \text{in } \Omega \times \{0\}\,, \end{gathered} \tag{1.1}$$

by considering the limit of solutions of the generalized viscous Cahn-Hilliard problems

$$\begin{gathered} u_t = \Delta[\varphi(u) - \varepsilon\Delta u + \delta u_t] + f(u) \quad \text{in } \Omega \times (0,T) =: Q_T \\ u = \Delta u = 0 \quad \text{on } \partial\Omega \times (0,T) \\ u = u_0 \quad \text{in } \Omega \times \{0\} \end{gathered} \tag{1.2}$$

where Ω is a smooth bounded subset of $\mathbb{R}^N$ $(N \le 3)$, $\varepsilon > 0$, and $\delta > 0$. We use the the following assumptions:

(H1) $\varphi \in C^2(\mathbb{R})$, $\varphi(0) = 0$, $f \in C^1(\mathbb{R})$, $f(0) = 0$;
(H2) $\varphi'(s) \ge -C_0$, $C_0 \ge 0$;
(H3) $\varphi(s)s \ge C_1\Phi(s) - C_2 \ge -C_3$ where $C_1, C_2, C_3 \ge 0$ and

$$\Phi(s) = \int_0^s \varphi(r)dr; \tag{1.3}$$

(H4) There exist $C_1 > 0$, $C_2 \in \mathbb{R}$ such that

$$f^2(s) \le C_1\Phi(s) + C_2;$$

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(H5) There exist a $\delta > 0$ such that for all $u \in L^2(\Omega)$,
$$\|f(u)\|\|u\| \le \delta \int_\Omega \Phi(u)dx + C_\delta;$$
(H6) There exist $C_1, C_2 > 0$ such that
$$\|f'(u)\|^2 \le C_1 \int_\Omega \Phi(u)dx + C_2,$$
where $\|\cdot\|$ denotes the usual norm in $L^2(\Omega)$.

Note that when $\varepsilon = 0$ and $\delta = 0$, problem (1.2) becomes (1.1). Letting $\varepsilon > 0$ and $\delta = 0$ in (1.2) leads to the generalized Cahn-Hilliard equation
$$u_t = \Delta[\varphi(u) - \varepsilon\Delta u] + f(u). \tag{1.4}$$

Depending on the choice of f, we get the corresponding equation which was widely investigated in the literature. For example, in the case $f(s) = -cs$ with $c > 0$, equation (1.4) is known as the Cahn-Hilliard-Oono equation which is an application in the phase separation process (see [14]). If $f(s) = \alpha s(1-s)$ and $\alpha > 0$, then (1.4) has an application in biology, in particular, in models wound healing and tumor growth (see [10]). The well-posedness of equation (1.4) was studied in [12] with the Dirichlet boundary condition, and in [6] with the Neumann boundary condition. In these articles, they also gave the asymptotic behavior of solution in terms of finite-dimensional attractors.

The case $\varepsilon = 0$ and $\delta > 0$ leads to the equation
$$u_t = \Delta[\varphi(u) + \delta u_t] + f(u). \tag{1.5}$$
This equation arises as a model for populations with the tendency to form groups which was studied by Padron (see [15]). The model of aggregating population with a migration rate determined by φ, and total birth and mortality rates characterized by f. He showed that the aggregating mechanism induced by φ allows the survival of a species in danger of extinction. For more information on the application of equation (1.5), we refer to [15] and the references therein.

Now taking into account $\varepsilon = 0$ in equation (1.4) or $\delta = 0$ in equation (1.5), we obtain the forward-backward parabolic type equation
$$u_t = \Delta\varphi(u) + f(u). \tag{1.6}$$
This equation has a variety of applications in biology such as aggregating populations (see [8, 11, 7] and references therein). In aggregation of population models, the nonlinearity φ may be increasing or decreasing therefore, the standard initial boundary value problems for (1.6) are in general ill-posed. That is the reason for studying the regularized problem of equation (1.6) by adding some regular terms. It is worth to mention that in the case of vanishing source term, the forward-backward parabolic equation
$$u_t = \Delta\varphi(u) \tag{1.7}$$
has no weak solution if the general initial data is considered. Often a higher order term is added to the right-hand side to regularize the equation. There are mainly two classes of additional terms which can be found in the mathematical literature, which, *e.g.* in case of equation (1.7) reduce to:

(i) $\varepsilon\Delta[\psi(u)]_t$, with $\psi' > 0$, leading to third order *pseudo-parabolic equations* ($\varepsilon > 0$ being a small parameter; see for example [1, 13, 17]);

(ii) $-\varepsilon\Delta^2 u$, leading to fourth-order Cahn-Hilliard type equations (see for example [16, 18] and references therein).

It is remarkable, taking advantage of the cubic-like growth of φ at infinity, which gives rise to better estimates of the family $\{u_\varepsilon\}$ of solutions of the regularized problem, they proved the existence of solutions in the sense of Young measures. Moreover, it is worthy to mention that the Cahn-Hilliard equation with a logarithmic nonlinear term has been investigated by many authors (see [5, 4, 3] and references therein). In all these references, the logarithmic potential is approximated by regular ones. Then, when passing to the limit in the approximated problems, it is difficult to prove that the limit of the order parameter remains in $(-1, 1)$. We refer readers to the survey [2] for more applications and other aspects of the Cahn-Hiliard equation.

In light of the above considerations, we first prove the existence and uniqueness of solution of problem (1.2) by Galerkin approximation and compactness method which are the same approaches as in [12, 6]. Secondly, we give the rigorous analysis of the convergence of a family $\{u_{\varepsilon,\delta}\}$ of solutions of (1.2) as $\delta \to 0$ to obtain the existence of solution of (1.1). Finally, we investigate the convergence of a family $\{u_\varepsilon\}$ of solutions of (1.1). Because of lacking of the compactness, we prove the appearance of measure-valued solution of (1.1). Actually, taking of the advantage of the growth of nonlinearities φ, f, we only have the L^2 uniform bounded estimate on solutions. Our approach is almost the same as in [18, 16].

Remark 1.1. If we choose $\varphi(u) = u^3 - u$ and $f(u) = \alpha u(1-u)$ with $\alpha > 0$, then φ, f will satisfy the assumptions (H1)–(H6). The choice of φ, f is widely used in the literature.

This article is organized as follows: we introduce our problem and some assumptions in Section 1, we study the well-posedness of problem (1.2) in Section 2 and give the rigorous convergence of the solutions of (1.2) as δ vanishing in Section 3. Finally, we investigate the existence of measure-valued solution of (1.1) in Section 4.

2. Well-posedness of problem (1.2)

In what follows, the symbols c, c', c'', c_i ($i \geq 0$) will denote positive constants and may vary from line to line. $Q : \mathbb{R}^+ \to \mathbb{R}^+$ will be a positive increasing monotone function and may also vary from line to line or even in a same line.

2.1. Mathematical formulation and results.

In this section, we study the well-posedness of problem (1.2). By setting $A := -\Delta$, the first equation of problem (1.2) is written in the form

$$u_t + A\big(\varphi(u) + \varepsilon A u + \delta u_t\big) - f(u) = 0. \tag{2.1}$$

Operator $A : D(A) \to L^2(\Omega)$ is a strictly positive self-adjoint linear with compact inverse on $L^2(\Omega)$, and domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. In this article, we denote $\big(\cdot,\cdot\big)$ as an usual scalar product in $L^2(\Omega)$ and set $\|\cdot\|_{-1} = \|A^{-1/2}\cdot\|$. In general, we introduce the family of Hilbert spaces

$$H^{2s} = D(A^s), \quad \forall s \in \mathbb{R}$$

with scalar products

$$((u,v))_{2s} := (A^s u, A^s v), \quad \forall u, v \in H^s.$$

Definition 2.1. For any interval $(0,T)$, a function $u_{\varepsilon,\delta}(x,t)=u(x,t)$ is called a solution of problem (1.2) if $(u,u_t)\in L^\infty(0,T;D(A)\times L^2(\Omega))$ and

$$u_t+A\left(\varphi(u)+\varepsilon Au+\delta u_t\right)-f(u)=0 \quad \text{in } D(A^{-1}),\ \text{a.e.}\quad t\in(0,T),$$

and

$$u(0)=u_0(x)\in D(A),\qquad \text{for a.e. } x\in\Omega.$$

Theorem 2.2. *Let assumptions* (H1)–(H6) *hold and* $u_0\in H^2(\Omega)\cap H_0^1(\Omega)$. *Then problem* (1.2) *admits a unique global solution as in definition 2.1. Moreover, let* u_1, u_2 *be two solutions of* (1.2) *with initial data* $u_{0,1}$ *and* $u_{0,2}$ *respectively, then there exists a constant* $c\ge 0$ *such that*

$$\begin{aligned}
&\|u_1(t)-u_2(t)\|_{-1}^2+\delta\|u_1(t)-u_2(t)\|^2\\
&\le e^{ct}Q(\|u_{0,1}\|_{H^2},\|u_{0,2}\|_{H^2})(\|u_{0,1}-u_{0,2}\|_{-1}^2+\delta\|u_{0,1}-u_{0,2}\|^2)
\end{aligned}$$

for any $t\ge 0$.

Proof of Theorem 2.2. We first prove the uniqueness of solution of (1.2). Let u_1,u_2 be two solutions of (1.2) with initial data $u_{0,1}$ and $u_{0,2}$ respectively. We set $u=u_1-u_2$ and $u_0=u_{0,1}-u_{0,2}$ and then u satisfies

$$\begin{gathered}
u_t+A\big(\varphi(u_1)-\varphi(u_2)+\varepsilon Au+\delta u_t\big)-f(u_1)+f(u_2)=0,\\
u=\Delta u=0\quad \text{on } \partial\Omega,\\
u(x,0)=u_0(x).
\end{gathered}\tag{2.2}$$

We multiply (2.2) by $A^{-1}u$ and we have

$$\begin{aligned}
&\frac12\frac{d}{dt}\big(\|A^{-1/2}u\|^2+\delta\|u\|^2\big)+\varepsilon\|A^{1/2}u\|^2+\big(\varphi(u_1)-\varphi(u_2),u\big)\\
&-\big(f(u_1)-f(u_2),A^{-1}u\big)=0.
\end{aligned}\tag{2.3}$$

Note that from (H2),

$$(\varphi(u_1)-\varphi(u_2),u)\ge -C_0\|u\|^2.$$

Furthermore,

$$\begin{aligned}
\left|\big(f(u_1)-f(u_2),A^{-1}u\big)\right|&\le\int_\Omega |A^{-1}u||u|\int_0^1|f'(su_1+(1-s)u_2)|\,ds\,dx\\
&\le Q(\|u_{0,1}\|_{H^2},\|u_{0,2}\|_{H^2})\|A^{-1}u\|_\infty\|u\|\\
&\le Q(\|u_{0,1}\|_{H^2},\|u_{0,2}\|_{H^2})\|u\|^2
\end{aligned}$$

thank to the continuous embedding $H^2\subset C(\bar\Omega)$. Therefore,

$$\frac{d}{dt}\big(\|A^{-1/2}u\|^2+\delta\|u\|^2\big)+\varepsilon\|A^{1/2}u\|^2\le Q(\|u_{0,1}\|_{H^2},\|u_{0,2}\|_{H^2})\|u\|^2.\tag{2.4}$$

From $\|u\|^2=\big(A^{-1/2}u,A^{1/2}u\big)$ and Young's inequality, we have

$$\begin{aligned}
&\frac{d}{dt}\big(\|A^{-1/2}u\|^2+\delta\|u\|^2\big)+\varepsilon\|A^{1/2}u\|^2\\
&\le Q(\|u_{0,1}\|_{H^2},\|u_{0,2}\|_{H^2})(\|A^{-1/2}u\|^2+\delta\|u\|^2).
\end{aligned}\tag{2.5}$$

By Gronwall's lemma we obtain

$$\begin{aligned}
&\|u_1(t)-u_2(t)\|_{-1}^2+\delta\|u_1(t)-u_2(t)\|^2\\
&\le e^{ct}Q(\|u_{0,1}\|_{H^2},\|u_{0,2}\|_{H^2})(\|u_{0,1}-u_{0,2}\|_{-1}^2+\delta\|u_{0,1}-u_{0,2}\|^2)
\end{aligned}$$

for any $t \geq 0$, $c \geq 0$. We have the uniqueness and the continuous dependence with respect to initial data. $\square$

The existence result relies on a standard approximation - a priori estimates and passage to the limit procedure. The Faedo-Galerkin scheme is as follows. Since A^{-1} is compact and self-adjoint operator on $L^2(\Omega)$, there exists an orthonormal basis of $L^2(\Omega)$ consisting of eigenvectors $\{e_i\}$ of A and the corresponding eigenvalues λ_i with Dirichlet boundary condition; that is,

$$Ae_i = \lambda_i e_i, \quad \text{for } i = 1, 2, \dots$$

and $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \to \infty$. It is easy to check that $\{\lambda_i^{-1/2} e_i\}$ is an orthonormal basis of $H_0^1(\Omega)$. For any integer number $n \geq 1$, let

$$V_n := \operatorname{span}\{e_1, \dots, e_n\}.$$

We state the approximating problem as follows.

Problem P_n**:** Find $t_n > 0$ and $u_i \in C^2([0, t_n])$ for $i = 1, \cdots, n$ such that

$$u^n := \sum_{i=1}^{N} u_i(t) e_i(x),$$

belongs to $C^2([0, t_n], D(A))$ and satisfies

$$\langle u_t^n, v\rangle + \langle \varphi(u^n) + \varepsilon A u^n + \delta u_t^n, Av\rangle - \langle f(u^n), v\rangle = 0, \quad \forall v \in V_n, \tag{2.6}$$

$$u^n(0) = u_0^n, \tag{2.7}$$

where $u_0^n \in V_n$ such that $u_0^n \to u_0$ in $D(A)$.

Problem P_n consist of a n-dimensional system of nonlinear ordinary differential equations. By the Cauchy-Lipschitz Theorem, there exists a unique local in time solution u^n in the maximal interval $[0, T^*)$. We now derive some a priori estimates that will permit us to prove the existence result by passage the limit as $n \to \infty$. The procedure is standard and so we only give a priori estimates in the next subsection.

2.2. **A priori estimates.** We multiply the first equation of (1.2) (2.1) by $A^{-1}u$, and integrate over Ω and by parts to obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{-1}^2 + \delta\|u\|^2\right) + \big(\varphi(u), u\big) + \varepsilon\|A^{1/2}u\|^2 - \big(f(u), A^{-1}u\big) = 0. \tag{2.8}$$

Thank to assumptions (H3) and (H5), we have

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\left(\|u\|_{-1}^2 + \delta\|u\|^2\right) + \varepsilon\|A^{1/2}u\|^2 + c_1\int_\Omega \Phi(u)dx \leq \|f(u)\|\|u\| + c_2,\\ \frac{d}{dt}\left(\|u\|_{-1}^2 + \delta\|u\|^2\right) + c\Big(\varepsilon\|u\|_{H^1}^2 + \int_\Omega \Phi(u)dx\Big) \leq c', \quad c > 0.\end{gathered} \tag{2.9}$$

Multiplying (2.1) by u and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\big(\|u\|^2 + \delta\|A^{1/2}u\|^2\big) + \varepsilon\|Au\|^2 + (\varphi'(u)\nabla u, \nabla u) - (f(u), u) = 0. \tag{2.10}$$

Thank to (H2) and (H5), we have

$$\frac{1}{2}\frac{d}{dt}\Big(\|u\|^2 + \delta\|A^{1/2}u\|^2\Big) + \varepsilon\|Au\|^2 \leq c_0\|A^{1/2}u\| + \|f(u)\|\|u\|. \tag{2.11}$$

Therefore,
$$\frac{d}{dt}\big(\|u\|^2+\delta\|A^{1/2}u\|^2\big)+c\varepsilon\|u\|^2_{H^2}\le c_1\|A^{1/2}u\|+c_2\int_\Omega\Phi(u)dx+c_3, \tag{2.12}$$
with $c>0$. Finally, we take sum of (2.9) and (2.12) times $\delta_1>0$, where δ_1 is small enough, to obtain
$$\begin{aligned}&\frac{d}{dt}\Big(\|u\|^2_{-1}+\delta\|u\|^2+\delta_1\|u\|^2+\delta_1\delta\|A^{1/2}u\|^2\Big)\\&+c\Big(\varepsilon\|u\|^2_{H^2}+\int_\Omega\Phi(u)dx\Big)\le c',\quad c>0.\end{aligned} \tag{2.13}$$
Note that from (2.13) and Gronwall's lemma, we have
$$\|u(t)\|^2_{-1}+\delta\|u(t)\|^2+\delta_1\|u(t)\|^2+\delta_1\delta\|A^{1/2}u(t)\|^2\le e^{-ct}Q(\|u_o\|_{H^1_0})+c', \tag{2.14}$$
for $t>0$ with $c>0$. Multiplying equation (2.1) by Au and integrating over Ω, we have
$$(u_t+\delta Au_t,Au)+\varepsilon\|A^{3/2}u\|^2+(A\varphi(u)-f(u),Au)=0. \tag{2.15}$$
By Holder's inequality,
$$\frac12\frac{d}{dt}\Big(\|A^{1/2}u\|^2+\delta\|Au\|^2\Big)+\varepsilon\|A^{3/2}\|^2\le c\left(\|A\varphi(u)\|^2+\|f(u)\|^2\right). \tag{2.16}$$
Since $H^2(\Omega)\subset C(\bar\Omega)$ with continuous embedding as $N\le 3$ and $\varphi,f\in C^2(\mathbb{R})$,
$$\|A\varphi(u)\|^2+\|f(u)\|^2\le Q(\|u\|_{H^2}).$$
Thus,
$$\frac{d}{dt}\Big(\|A^{1/2}u\|^2+\delta\|Au\|^2\Big)\le Q(\|A^{1/2}u\|^2+\delta\|Au\|^2). \tag{2.17}$$
Let y be the solution to the ordinary differential equation
$$y'=Q(y),\quad y(0)=\|A^{1/2}u_0\|^2+\delta\|Au_0\|^2.$$
Then by the comparison principle, there exists a time $T^*=T^*(\|u_0\|_{H^2})>0$ such that
$$\|A^{1/2}u(t)\|^2+\delta\|Au(t)\|^2\le y(t),\quad t\le T^*.$$
In summary we have
$$\delta\|u(t)\|_{H^2}\le Q(\|u_0\|_{H_2}),\quad \forall t\le T^*. \tag{2.18}$$
Multiplying (2.1) by $A^{-1}u_t$ and have
$$\frac{\varepsilon}{2}\frac{d}{dt}\|A^{1/2}u\|^2+\|u_t\|^2_{-1}+\delta\|u_t\|^2+(\varphi(u),u_t)-\big(f(u),A^{-1}u_t\big)=0, \tag{2.19}$$
which, by (2.18) and Holder's inequality, for $t\le T^*$ yields
$$\begin{aligned}\frac{\varepsilon}{2}\frac{d}{dt}\|A^{1/2}u\|^2+\|u_t\|^2_{-1}+\delta\|u_t\|^2&\le c\left(\|\varphi(u)\|^2_{H^1}+\|f(u)\|^2\right)\\&\le Q(\|u\|_{H^2})\le Q(\|u_0\|_{H^2}).\end{aligned} \tag{2.20}$$
Therefore,
$$\varepsilon\|A^{1/2}u(t)\|^2+\int_0^{T^*}\|u_t\|^2_{-1}+\delta\|u_t\|^2ds\le Q(\|u_0\|_{H^2}),\quad t\le T^*. \tag{2.21}$$
Differentiating (2.1) with respect to time and setting $v=u_t$ we have
$$A^{-1}v_t+\delta v_t+\varepsilon Av+\varphi'(u)v-A^{-1}(f'(u)v)=0. \tag{2.22}$$

Multiplying (2.22) by tv, for $t \le T^*$ we obtain

$$\frac{d}{dt}\left(t\|v\|_{-1}^2+\delta t\|v\|^2\right)+\varepsilon t\|A^{1/2}v\|^2 \le Q(\|u_0\|_{H^2})(t\|v\|^2)+\|v\|_{-1}^2+\delta\|v\|^2. \tag{2.23}$$

Noting that $\|v\|^2 \le c\|v\|_{-1}\|A^{1/2}v\|$, we have

$$\frac{d}{dt}\left(t\|v\|_{-1}^2+\delta t\|v\|^2\right) \le Q(\|u_0\|_{H^2})(t\|v\|^2+\delta t\|v\|^2)+\|v\|_{-1}^2+\delta\|v\|^2. \tag{2.24}$$

Integrate (2.24) over $(0,t)$ with $t \le T^*$; then thanks to (2.21) we have

$$\begin{aligned} t\|v\|_{-1}^2+\delta t\|v\|^2 &\le c\int_0^t (s\|v\|^2+\delta s\|v\|^2)ds+\int_0^t \|v\|_{-1}^2+\delta\|v\|^2 ds && (2.25)\\ &\le c\int_0^t (s\|v\|^2+\delta s\|v\|^2)ds+Q(\|u_0\|_{H^2}). && (2.26)\end{aligned}$$

By Gronwall's inequality,

$$\|v\|_{-1}^2+\delta\|v\|^2 \le \frac{1}{t}Q(\|u_0\|_{H^2}), \quad 0<t\le T^*. \tag{2.27}$$

We now multiply (2.22) by v to have

$$\frac{d}{dt}\left(\|v\|_{-1}^2+\delta\|v\|^2\right)+2\varepsilon\|A^{1/2}v\|+2\left(\varphi'(u)v,v\right) \le 2|\left(A^{-1}(f'(u)v),v\right)|. \tag{2.28}$$

Thank to (H2), Young's inequality and $\|f'(u)\| \le c_1\int\Phi(u)dx+c_2$, we have

$$\begin{aligned} \frac{d}{dt}\left(\|v\|_{-1}^2+\delta\|v\|^2\right)+2\varepsilon\|A^{1/2}v\| &\le c_0\|v\|^2+c\|f'(u)\|\|v\|_{-1}\|A^{1/2}v\| \\ &\le \left(c_1\int\Phi(u)dx+c_2\right)\left(\|v\|_{-1}^2+\delta\|v\|^2\right). \end{aligned} \tag{2.29}$$

From (2.9), we have

$$\int_0^t\int_\Omega \Phi(u)\,dx\,ds \le c\|u_0\|^2+c't+c''. \tag{2.30}$$

Using Gronwall's inequality and (2.29) we have

$$\|v\|_{-1}^2+\delta\|v\|^2 \le \left(\|v(T^*)\|_{-1}^2+\delta\|v(T^*)\|^2\right)e^{\int_0^t dt\int_\Omega c_1\Phi(u)dx+c_2}. \tag{2.31}$$

Thanks to (2.27), we have

$$\|v\|_{-1}^2+\delta\|v\|^2 \le e^{ct}Q(\|u_0\|_{H^2}),\ \ t\ge T^*. \tag{2.32}$$

Again rewritten our equation (2.1) in the following form

$$\varepsilon Au+\varphi(u)-A^{-1}f(u)=-A^{-1}v-\delta v:=h. \tag{2.33}$$

It is clearly that from (2.32)

$$\|h\| \le e^{ct}Q(\|u_0\|_{H^2}),\ \ \forall t\ge T^*.$$

Multiplying (2.33) by u and integrating over Ω, we have

$$\varepsilon\|A^{1/2}u\|^2+(\varphi(u),u) \le \|h\|\|u\|+\|f(u)\|\|u\|. \tag{2.34}$$

This implies

$$\varepsilon\|A^{1/2}u\|^2+c\int_\Omega \Phi(u)dx \le e^{ct}Q(\|u_0\|_{H_2})+c''. \tag{2.35}$$

Multiplying (2.33) by Au and integrating over Ω, we have

$$\varepsilon\|Au\|^2+(\varphi'(u)\nabla u,\nabla u) \le \|h\|\|Au\|+\|f(u)\|\|u\|. \tag{2.36}$$

This implies

$$\varepsilon\|Au\|^2 \le \|h\|^2 + c_0\|A^{1/2}u\|^2 + c_1\int_\Omega \Phi(u)dx + c_2. \tag{2.37}$$

Adding (2.29) and δ_2 times (2.37), where δ_2 is small enough, yields

$$\varepsilon\|u(t)\|^2_{H^2} \le e^{ct}Q(\|u_0\|_{H_2}) + c', \quad \forall t \ge T^*,\ c \ge 0. \tag{2.38}$$

Combining this with (2.18), we obtain

$$\varepsilon\|u(t)\|^2_{H^2} \le e^{ct}Q(\|u_0\|_{H_2}) + c', \quad \forall t \ge 0,\ c \ge 0. \tag{2.39}$$

From (2.13), we have

$$\varepsilon\int_0^1 \|u(t)\|_{H^2}dt \le c\|u_0\|^2_{H^1} + c', \tag{2.40}$$

so that there exists a $T \in (0,1)$ such that

$$\|u(T)\|_{H^2} \le c\|u_0\|^2_{H^1} + c'. \tag{2.41}$$

If we start from time $t = T$ instead of $t = 0$, inequality (2.41) holds for $T = 1$, that is,

$$\|u(1)\|_{H^2} \le c\|u_0\|^2_{H^1} + c', \quad c \ge 0. \tag{2.42}$$

Again from (2.13) and Gronwall's lemma, we can prove that for any $t \ge 0$,

$$\int_t^{t+1} \|u(s)\|_{H^2}ds \le e^{-ct}Q(\|u_0\|_{H^1}) + c', \quad c \ge 0.$$

Hence for every $t \ge 1$, there exists a $t_1 \in [t-1, t]$ such that

$$\|u(t_1)\|_{H^2} \le e^{-ct}Q(\|u_0\|^2_{H^1}) + c', \tag{2.43}$$

which implies, for $t_2 \in [0,1]$, that $t = t_1 + t_2$. Thanks to (2.39) and (2.43),

$$\begin{aligned}
\|u(t)\|^2_{H^2} &= \|u(t_1+t_2)\|^2_{H^2} \\
&\le e^{ct_2}Q(\|u(t_1)\|_{H^2}) + c' \\
&\le c_1e^{-c_2t_2}Q(\|u(t_1)\|_{H^2}) + c_3 \\
&\le c_1e^{-c_2t_2}Q\Big(e^{-c't}Q'(\|u_0\|^2_{H^2}) + c''\Big) + c_3 \\
&\le e^{-ct}Q'(\|u_0\|^2_{H^1}) + c',
\end{aligned}$$

that is,

$$\|u(t)\|_{H^2} \le e^{ct}Q(\|u_0\|_{H^2}) + c', \quad c > 0,\ t \ge 0. \tag{2.44}$$

3. Convergence of solutions of problem (1.2) as $\delta \to 0$

In this section, we study the well-posedness of problem (1.2) in the sense of convergence of a family of solutions $\{u_{\varepsilon,\delta}\}$ of (1.2) as $\delta \to 0$.

Definition 3.1. Let $T > 0$, $u_0 \in H^1_0(\Omega) \cap H^2(\Omega)$, $\varepsilon > 0$, by a solution u_ε of (1.2), we mean a function $u_\varepsilon \in L^\infty(0,T; H^1_0(\Omega)) \cap L^2(0,T; H^2(\Omega))$, $u_{\varepsilon t} \in L^\infty(0,T; H^{-1}(\Omega))$, $\varphi(u_\varepsilon)$, $f(u_\varepsilon) \in L^2(Q_T)$ such that

$$\int_0^T \langle u_{\varepsilon t}, \eta\rangle ds + \int_0^T \langle \varphi(u_\varepsilon) + \varepsilon A u_\varepsilon, A\eta\rangle ds - \int_0^T \langle f(u_\varepsilon), \eta\rangle ds = 0, \tag{3.1}$$

for any test function $\eta \in C^1(0,T; H^1_0(\Omega) \cap H^2(\Omega))$, and $u_\varepsilon(x,0) = u_0(x)$ for a.e. $x \in \Omega$.

Theorem 3.2. *Let* (H1)–(H6) *hold, and* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. *Then problem* (1.1) *admits a unique solution as in Definition 3.1.*

Proof. We focus mainly on proving the existence result. Thanks to estimates (2.13), (2.14), (2.21), (2.32) and (2.44), there exists a positive constant C independent of δ such that

$$\|u_{\varepsilon,\delta}\|_{L^\infty(0,T:H_0^1(\Omega))} \le C, \tag{3.2}$$

$$\|u_{\varepsilon,\delta}\|_{L^2(0,T:H^2(\Omega))} \le C, \tag{3.3}$$

$$\|u_{\varepsilon,\delta t}\|_{L^\infty(0,T:H^{-1}(\Omega))} \le C, \tag{3.4}$$

$$\sqrt{\delta}\|u_{\varepsilon,\delta t}\|_{L^\infty(0,T:L^2(\Omega))} \le C, \tag{3.5}$$

By standard argument of compactness and Aubin-Lions Lemma, there exist a function $u_\varepsilon \in L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$, $u_{\varepsilon t} \in L^\infty(0,T;H^{-1}(\Omega))$ and a subsequence of $\{u_{\varepsilon,\delta}\}$ (still denote $\{u_{\varepsilon,\delta}\}$) such that

- $u_{\varepsilon,\delta}$ converges weakly-star to u_ε in $L^\infty(0,T;H_0^1(\Omega))$,
- $u_{\varepsilon,\delta}$ converges weakly to u_ε in $L^2(0,T;H^2(\Omega))$,
- $u_{\varepsilon,\delta}$ converges strongly to u_ε in $L^2(Q_T)$ and a.e. in Q_T,
- $u_{\varepsilon,\delta t}$ converges weakly-star to $u_{\varepsilon t}$ in $L^\infty(0,T;H^{-1}(\Omega))$.

Now we are ready to take limit $\delta \to 0$ in the weak formulation of solution $u_{\varepsilon,\delta}$ of problem (1.2). For any $\eta \in C^1(0,T;H_0^1 \cap H^2)$, $T > 0$, we have

$$\int_0^T \langle u_{\varepsilon,\delta t}, \eta\rangle ds + \int_0^T \langle \varphi(u_{\varepsilon,\delta}) + \varepsilon A u_{\varepsilon,\delta} + \delta u_{\varepsilon,\delta t}, A\eta\rangle ds - \int_0^T \langle f(u_{\varepsilon,\delta}), \eta\rangle ds = 0. \tag{3.6}$$

Taking $\delta \to 0$ in (3.6), using above convergences of $\{u_{\varepsilon,\delta}\}$ and noting that

$$\begin{aligned}
\Big|\int_0^T \int_\Omega \delta u_{\varepsilon,\delta t}\eta \,dx\,ds\Big| &= \Big|\int_0^T \int_\Omega \delta^{1/2}\delta^{1/2} u_{\varepsilon\delta t}\eta \,dx\,ds\Big| \\
&\le \delta^{1/2}\|\delta^{1/2}u_{\varepsilon,\delta t}\|_{L^2(Q_T)}\|\eta\|_{L^2(Q_T)} \\
&\le C\delta^{1/2} \to 0 \quad \text{as } \delta \to 0,
\end{aligned}$$

which yields the weak formulation (3.1) as in Definition 3.1. It is also easy to prove that $u_\varepsilon(x,0) = u_0(x)$ a.e. in Ω. Concerning the uniqueness, we first observe that for a.e. $t \in (0,T)$,

$$\langle u_{\varepsilon t}, \eta\rangle + \langle \varphi(u_\varepsilon) + \varepsilon A u_\varepsilon, A\eta\rangle - \langle f(u_\varepsilon), \eta\rangle = 0 \tag{3.7}$$

for any $\eta \in H_0^1(\Omega) \cap H^2(\Omega)$. Let u_1, u_2 be two solutions of (1.1) with initial data $u_{0,1}$ and $u_{0,2}$ respectively. We set $u = u_1 - u_2$ and $u_0 = u_{0,1} = u_{0,2}$ and then u_1, u_2 satisfy (3.7). We choose $\eta = A^{-1}u$ and subtract equations of u_1 and u_2 to obtain

$$\frac{1}{2}\frac{d}{dt}\|A^{-1/2}u\|^2 + \varepsilon\|A^{1/2}u\|^2 + (\varphi(u_1) - \varphi(u_2), u) - \big(f(u_1) - f(u_2), A^{-1}u\big) = 0. \tag{3.8}$$

Note that from (H2)

$$(\varphi(u_1) - \varphi(u_2), u) \ge -C_0\|u\|^2.$$

Furthermore,

$$\begin{aligned}
\big|\big(f(u_1) - f(u_2), A^{-1}u\big)\big| &\le \int_\Omega |A^{-1}u||u| \int_0^1 |f'(su_1 + (1-s)u_2)|\,ds\,dx \\
&\le Q(\|u_{0,1}\|_{H^2}, \|u_{0,2}\|_{H^2})\|A^{-1}u\|_\infty\|u\|
\end{aligned}$$

$$\leq Q(\|u_{0,1}\|_{H^2}, \|u_{0,2}\|_{H^2})\|u\|^2$$

by the continuous embedding $H^2 \subset C(\bar{\Omega})$. And thus, we have

$$\frac{d}{dt}\|A^{-1/2}u\|^2 + \varepsilon\|A^{1/2}u\|^2 \leq Q(\|u_{0,1}\|_{H^2}, \|u_{0,2}\|_{H^2})\|u\|^2. \tag{3.9}$$

From $\|u\|^2 = \big(A^{-1/2}u, A^{1/2}u\big)$ and Young's inequality we have

$$\frac{d}{dt}\big(\|A^{-1/2}u\|^2\big) + \varepsilon\|A^{1/2}u\|^2 \leq Q(\|u_{0,1}\|_{H^2}, \|u_{0,2}\|_{H^2})\|A^{-1/2}u\|^2. \tag{3.10}$$

By Gronwall's lemma we obtain

$$\|u_1(t) - u_2(t)\|_{-1}^2 \leq e^{ct}Q(\|u_{0,1}\|_{H^2}, \|u_{0,2}\|_{H^2})\|u_{0,1} - u_{0,2}\|_{-1}^2$$

for any $t \geq 0$, some $c \geq 0$. We complete the proof of Theorem 3.2. □

4. Existence of measure-valued solution of problem (1.1)

Concerning the well-posedness of problem (1.2), we refer to Section 3 or to Alain Miranville [12].

Set $v_\varepsilon(x,t) = \varphi(u_\varepsilon(x,t)) - \varepsilon\Delta u_\varepsilon(x,t)$, then $u_t = \Delta v + f(u)$. We state the equivalence of problem (1.2), finding $(u_\varepsilon,\, v_\varepsilon)$ of

$$\begin{gathered}
u_{\varepsilon t} = \Delta v_\varepsilon + f(u_\varepsilon) \quad \text{in } \Omega \times (0,T) =: Q_T \\
u_\varepsilon = v_\varepsilon = 0 \quad \text{on } \partial\Omega \times (0,T) \\
u = u_0 \quad \text{in } \Omega \times \{0\}
\end{gathered} \tag{4.1}$$

Theorem 4.1 (Well-posedness of (4.1))**.** *Let* (H1)–(H6) *hold and* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. *Then problem* (4.1) *admits a unique global solution* $u_\varepsilon(\cdot,t) \in H^2(\Omega) \cap H_0^1(\Omega)$, $v_\varepsilon \in L^2((0,T), H_0^1(\Omega) \cap H^2(\Omega))$, $u_{\varepsilon t} \in L^2(Q_T)$ *for all* $t \geq 0$ *in strong sense.*

Proposition 4.2 (A priori estimates)**.** *Let* (H1)–(H6) *hold and* $\int_\Omega \Phi(u_0)dx < \infty$. *Then the family of solutions* $\{u_\varepsilon,\, v_\varepsilon\}_{\varepsilon>0}$ *which are guaranteed by Theorem 4.1 satisfy the following inequalities*

$$\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \leq C, \quad \forall t \in [0,T], \tag{4.2}$$

$$\int_\Omega \varepsilon|\nabla u_\varepsilon(x,t)|^2 dx \leq C, \ \forall t \in [0,T], \tag{4.3}$$

$$\|v_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \leq C \tag{4.4}$$

where C *is a positive constant independent of* ε.

Next we prove the existence of Young measure solutions of problem (1.1) in the sense of the following definition. For the definition and properties of Young measure, we refer to [19].

Definition 4.3. Let $N \leq 3$, $u_0 \in H_0^1(\Omega)$. By a Young measure solution of problem (1.1) in Q_T we mean a triplet (u, v, τ) such that:

(i) $u \in L^2(Q_T)$, $u_t \in L^2((0,T), H^{-1}(\Omega))$;
(ii) $v \in L^2((0,T); H_0^1(\Omega))$, $\tau \in \mathcal{Y}(Q_T; \mathcal{P}(\mathbb{R}))$;
(iii) for almost every $(x,t) \in Q_T$ it holds

$$u(x,t) = \langle \tau_{(x,t)}, id \rangle_{\mathbb{R}} = \int_{\mathbb{R}} \xi \, d\tau_{(x,t)}(\xi)\,, \tag{4.5}$$

where $id(\xi) := \xi$ $(\xi \in \mathbb{R})$ and $\tau_{(x,t)} \in \mathcal{P}(\mathbb{R})$ denotes the disintegration of τ;

(iv) for any $\zeta \in C^1([0,T); C_c^1(\Omega))$ and $t \in (0,T)$

$$\begin{aligned} &\int_0^t\int_\Omega \big[u\,\zeta_s - \nabla v\cdot\nabla\zeta + f^*\zeta\big](x,s)\,dx\,ds \\ &= \int_\Omega u(x,t)\zeta(x,t)\,dx - \int_\Omega u_0(x)\,\zeta(x,0)\,dx\,, \end{aligned} \tag{4.6}$$

where $v(x,t)$ and f^* satisfy

$$v(x,t) = \varphi^*(x,t) := \langle \tau_{(x,t)}, \varphi\rangle_{\mathbb{R}} = \int_{\mathbb{R}} \varphi(\xi)\,d\tau_{(x,t)}(\xi), \tag{4.7}$$

$$f^*(x,t) := \langle \tau_{(x,t)}, f\rangle_{\mathbb{R}} = \int_{\mathbb{R}} f(\xi)\,d\tau_{(x,t)}(\xi) \tag{4.8}$$

for almost every $(x,t) \in Q_T$.

A Young measure solution of problem (1.1) in Q_∞, which exists in Q_T for any $T \in (0,\infty)$, is said to be global.

Proposition 4.4. *Let assumptions in Theorem 4.1 hold. Then there exist functions $(u,v) \in L^2(Q_T)\times L^2(0,T;H_0^1(\Omega))$ and subsequences $\{u_{\varepsilon_k}\}, \{v_{\varepsilon_k}\}$ of $\{u_\varepsilon\}$, $\{v_\varepsilon\}$ (still denote $\{u_\varepsilon\}, \{v_\varepsilon\}$ for convenience) such that:*

(i) *u_ε converges weakly to u in $L^2(Q_T)$;*

(ii) *v_ε converges weakly to v in $L^2(0,T;H_0^1(\Omega))$;*

(iii) *The sequence of Young measure $\{\tau_k\}$ associated with the sequence $\{u_{\varepsilon_k}\}$. There exists a Young measure τ such that*

$$\tau_k \to \tau \text{ narrowly in the sense of Definition 5.5};$$

(iv) *We have*

$$u(x,t) = \langle \tau_{(x,t)}, id\rangle_{\mathbb{R}} = \int_{\mathbb{R}} \xi\,d\tau_{(x,t)}(\xi)\,. \tag{4.9}$$

Moreover for any $\phi \in C^1(\mathbb{R})$ there exists a function ϕ^* such that

$$\phi^*(x,t) := \langle \tau_{(x,t)}, \phi\rangle_{\mathbb{R}} = \int_{\mathbb{R}} \phi(\xi)\,d\tau_{(x,t)}(\xi)\,. \tag{4.10}$$

Our main result is as follows.

Theorem 4.5. *Let assumptions in Theorem 4.1 hold. Then problem* (1.1) *admits a global Young measure solution as in Definition 4.3.*

Proof of Proposition 4.2. Multiplying the two-sides of the first equation of (1.1) by v_ε and integrating over Ω, yields

$$\begin{gathered} \int_\Omega u_{\varepsilon t}[\varphi(u_\varepsilon) - \varepsilon\Delta u_\varepsilon]dx = \int_\Omega \Delta v_\varepsilon v_\varepsilon + f(u_\varepsilon)v_\varepsilon dx, \\ \frac{d}{dt}\Big(\int_\Omega \Phi(u_\varepsilon) + \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 dx\Big) + \int_\Omega |\nabla v_\varepsilon|^2 dx = \int_\Omega f(u_\varepsilon)v_\varepsilon dx. \end{gathered}$$

Using Poincare' inequality for v_ε and Hölder's inequality for the integral in the right-hand side, we obtain

$$\begin{gathered} \int_\Omega \Phi(u_\varepsilon) + \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 dx + \int_0^t\int_\Omega |\nabla v_\varepsilon|^2 dx = \int_0^t\int_\Omega f(u_\varepsilon)v_\varepsilon dx + C(\|u_0\|_{H_0^1(\Omega)}), \\ \int_\Omega \Phi(u_\varepsilon) + \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 dx + \int_0^t\int_\Omega |\nabla v_\varepsilon|^2 dx \le C_1\int_0^t\int_\Omega f^2(u_\varepsilon)dx + C_2. \end{gathered}$$

Now using assumption (H4),

$$\int_\Omega \Phi(u_\varepsilon) + \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 dx + \int_0^t \int_\Omega |\nabla v_\varepsilon|^2 dx \le C \int_0^t \int_\Omega \Phi(u_\varepsilon) dx + M.$$

By Gronwall's inequality, we obtain

$$\int_\Omega \Phi(u_\varepsilon) dx \le C(u_0, T, \Omega).$$

Then the assumption on the growth of Φ implies inequality (4.2),

$$\|u_\varepsilon\|_{L^2(\Omega)} \le C.$$

From this, we can easily obtain the remaining estimates in Proposition 4.2 $\square$

Proof of Proposition 4.4. The statements in this Proposition follow directly from estimates (4.2), (4.4) and the Fundamental Theorem of Young measure. $\square$

Proof of Theorem 4.5. Firstly, we prove the statement (4.7). Indeed, for any $\eta \in C_c^\infty(Q_T)$,

$$\begin{aligned}
\Big|\int_{Q_T} [\varphi(u_{\varepsilon_k}) - v]\eta \, dx\, dt\Big| &= \Big|\int_{Q_T} [\varphi(u_{\varepsilon_k}) - v_{\varepsilon_k} + v_{\varepsilon_k} - v]\eta \, dx\, dt\Big| \\
&\le \Big|\int_{Q_T} [\varphi(u_{\varepsilon_k}) - v_{\varepsilon_k}]\eta \, dx\, dt\Big| + \Big|\int_{Q_T} [v_{\varepsilon_k} - v]\eta \, dx\, dt\Big| \\
&= \Big|\int_{Q_T} \varepsilon_k \Delta u_{\varepsilon_k} \eta \, dx\, dt\Big| + \Big|\int_{Q_T} [v_{\varepsilon_k} - v]\eta \, dx\, dt\Big| \\
&= \Big|\int_{Q_T} \varepsilon_k \nabla u_{\varepsilon_k} \nabla\eta \, dx\, dt\Big| + \Big|\int_{Q_T} [v_{\varepsilon_k} - v]\eta \, dx\, dt\Big| \\
&\to 0 \text{ as } k \to \infty,
\end{aligned}$$

$$\Big|\int_{Q_T} [v_{\varepsilon_k} - v]\eta \, dx\, dt\Big| \to 0 \quad \text{by Proposition 4.4 (ii)},$$

$$\Big|\int_{Q_T} \varepsilon_k \nabla u_{\varepsilon_k} \nabla\eta \, dx\, dt\Big| = \sqrt{\varepsilon_k}\|\sqrt{\varepsilon_k}\nabla u_{\varepsilon_k}\|_{L^2(Q_T)}\|\nabla\eta\|_{L^2(Q_T)} \to 0$$

by the uniform bounded estimate (4.3) of Proposition 4.2

Secondly, by the well-posedness Theorem 4.1 for problem (1.2), with ε_k instead of ε, and with the initial datum $u_{0\varepsilon_k} := u_0$. For any function $\zeta \in C^1([0,T); C_c^1(\Omega))$ and $t \in (0,T)$ we have

$$\int_0^t \int_\Omega \big[(u_{\varepsilon_k})_s \, \zeta - \Delta v \zeta\big](x,s)\, dx\, ds = \int_\Omega f(u_{\varepsilon_k}(x,t))\zeta(x,t)\, dx.$$

Integration by parts yields

$$\int_0^t \int_\Omega \big[u_{\varepsilon_k} \, \zeta_s - \nabla v \cdot \nabla\zeta + f(u_{\varepsilon_k})\zeta\big](x,s)\, dx\, ds = \int_\Omega [u_{\varepsilon_k}\zeta](x,t) dx - \int_\Omega u_0(x)\zeta(x,0).$$

By Proposition 4.4 and the above statement of weak convergence of $\{v_{\varepsilon_k}\}$, we send $k \to \infty$ to get the weak formulation (4.6),

$$\int_0^t \int_\Omega \big[u \, \zeta_s - \nabla v \cdot \nabla\zeta + f^*\zeta\big](x,s)\, dx\, ds$$

$$= \int_\Omega u(x,t)\zeta(x,t)\,dx - \int_\Omega u_0(x)\,\zeta(x,0)\,dx\,,$$

where f^* is barycenter of the nonlinearity f which is defined as in (4.8). This completes the proof. □

5. Appendix

Concerning Young measures on $Q \times \mathbb{R}$ (*e.g.*, see [9, 19] and references therein).

Definition 5.1. By a Young measure on $Q \times \mathbb{R}$ we mean any positive Radon measure τ such that

$$\tau(E \times \mathbb{R}) = |E| \tag{5.1}$$

for any Lebesgue measurable set $E \subseteq Q$. The set of Young measures on $Q \times \mathbb{R}$ will be denoted by $\mathcal{Y}(Q;\mathbb{R})$.

If $f : Q \to \mathbb{R}$ is Lebesgue measurable, the Young measure associated to f is the measure $\tau \in \mathcal{Y}(Q;\mathbb{R})$ such that

$$\tau(E \times F) = |E \cap f^{-1}(F)| \tag{5.2}$$

for any Lebesgue measurable set $E \subseteq Q$ and any Borel set $F \subseteq \mathbb{R}$.

Remark 5.2. In view of (5.2), if τ is the Young measure associated to a Lebesgue measurable function $f : Q \to \mathbb{R}$, for any τ-integrable function $\psi : Q \times \mathbb{R} \to \bar{\mathbb{R}}$ we have

$$\int_{Q\times\mathbb{R}} \psi\,d\tau = \iint_Q \psi(x,t,f(x,t))\,dx\,dt\,. \tag{5.3}$$

Proposition 5.3. *Let $\tau \in \mathcal{Y}(Q;\mathbb{R})$. Then for almost every $(x,t) \in Q$ there exists a measure $\tau_{(x,t)} \in \mathcal{P}(\mathbb{R})$, such that for any function $\psi : Q \times \mathbb{R} \to \mathbb{R}$ bounded and continuous:*

(i) *the map*

$$(x,t) \to \langle \tau_{(x,t)}, \psi(x,t,\cdot)\rangle_{\mathbb{R}} = \int_{\mathbb{R}} \psi(x,t,\xi)\,d\tau_{(x,t)}(\xi)$$

is Lebesgue measurable;

(ii) *it holds*

$$\begin{aligned}\langle \tau, \psi\rangle_{Q\times\mathbb{R}} := \int_{Q\times\mathbb{R}} \psi\,d\tau &= \iint_Q \langle \tau_{(x,t)}, \psi(x,t,\cdot)\rangle_{\mathbb{R}}\,dx\,dt \\ &= \iint_Q dx\,dt \int_{\mathbb{R}} \psi(x,t,\xi)\,d\tau_{(x,t)}(\xi)\,.\end{aligned} \tag{5.4}$$

Therefore, every $\tau \in \mathcal{Y}(Q \times \mathbb{R})$ can be identified with the associated family $\{\tau_{(x,t)} : (x,t) \in Q\}$, which is called the *disintegration of τ.*

Remark 5.4. If τ is the Young measure associated to a Lebesgue measurable function $f : Q \to \mathbb{R}$, equalities (5.3)-(5.4) imply

$$\psi(x,t,f(x,t)) = \langle \tau_{(x,t)}, \psi(x,t,\cdot)\rangle_{\mathbb{R}} = \int_{\mathbb{R}} \psi(x,t,\xi)\,d\tau_{(x,t)}(\xi) \tag{5.5}$$

for almost every $(x,t) \in Q$; where $\psi \in BC(Q \times \mathbb{R})$ and $\{\tau_{(x,t)}\}$ is the disintegration of τ. In this case

$$\tau_{(x,t)} = \delta_{f(x,t)} \quad \text{for almost every } (x,t) \in Q\,,$$

where δ_P denotes the Dirac mass concentrated in $P \in \mathbb{R}$.

Definition 5.5. Let $\{\tau^n\} \subseteq \mathcal{Y}(Q;\mathbb{R})$, $\tau \in \mathcal{Y}(Q;\mathbb{R})$ $(n \in \mathbb{N})$. We say that $\tau^n \to \tau$ narrowly in $Q \times \mathbb{R}$, if

$$\int_{Q\times\mathbb{R}} \psi \, d\tau^n \to \int_{Q\times\mathbb{R}} \psi \, d\tau \tag{5.6}$$

for any function $\psi : Q \times \mathbb{R} \to \mathbb{R}$ bounded and measurable, such that $\psi(x,t,\cdot)$ is continuous for almost every $(x,t) \in Q$.

Theorem 5.6. *Let $\{f_n\}$ be a bounded sequence in $L^1(Q)$, and $\{\tau^n\}$ the sequence of associated Young measures. Then:*

(i) *there exist subsequences $\{f_k\} \equiv \{f_{n_k}\} \subseteq \{f_n\}$, $\{\tau^k\} \equiv \{\tau^{n_k}\} \subseteq \{\tau^n\}$ and a Young measure τ on $Q \times \mathbb{R}$ such that $\tau^k \to \tau$ narrowly in $Q \times \mathbb{R}$;*

(ii) *for any $\rho \in C(\mathbb{R})$ such that the sequence $\{\rho \circ f_n\} \subseteq L^1(Q)$ is uniformly integrable, it holds*

$$\rho \circ f_k \equiv \rho \circ f_{n_k} \rightharpoonup \rho^* \quad \text{in } L^1(Q)\,, \tag{5.7}$$

where

$$\rho^*(x,t) := \langle \tau_{(x,t)}, \rho \rangle_{\mathbb{R}} = \int_{\mathbb{R}} \rho(\xi) \, d\tau_{(x,t)}(\xi) \quad \text{a.e. } (x,t) \in Q \tag{5.8}$$

and $\{\tau_{(x,t)}\}$ is the disintegration of τ.

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