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# MULTIPLICITY OF POSITIVE SOLUTIONS FOR A GRADIENT TYPE COOPERATIVE/COMPETITIVE ELLIPTIC SYSTEM

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Abstract. We study the existence of positive solutions for gradient type cooperative, competitive elliptic systems, which depends on real parameters $\lambda, \mu$. Our analysis is purely variational and depends on finer estimates with respect to the Nehari sets, in fact, we determine the extremal parameter $\lambda^*(\mu)$ for which the Nehari set is a manifold and hence standard variational techniques can be applied. We also discuss the cases where the Nehari set is not a manifold.

## 1. Introduction

In this work we study the gradient type cooperative or competitive elliptic system

$$\begin{gathered} -\Delta u = \mu u + \lambda v + f(x)|u|^{p-2}u \quad \text{in } \Omega, \\ -\Delta v = \lambda u + \mu v - g(x)|v|^{q-2}v \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\lambda, \mu$ are real parameters, $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$, $2 < q < p < 2^*$, $2^* = 2N/(N-2)$. We look for weak solutions in $\mathcal{X} := H_0^1(\Omega) \times H_0^1(\Omega)$.

Gradient type systems and cooperative or competitive type systems have been studied by many authors: see for example the works of deFigueiredo [8], Clément et al. [5], Alves et al. [1], Bozhkov and Mitidieri [3], Wenming [14], Costa and Magalhães [6], da Silva [7] and the references therein. Such systems appear in many phenomena in Physics, Chemistry, Biology, etc. (see for example Brown [4] and the references therein), in particular they are related to reaction-diffusion systems that appear in chemical and biological phenomena.

Our plan is to study system (1.1) with respect to the parameters $\lambda, \mu$ only by a variational method. We will provide a relation between the parameters $\lambda, \mu$ with some topological properties of the Nehari set and the existence of solutions to problem (1.1). From now on, a solution to (1.1) is a critical point to the energy functional $\Phi_{\lambda,\mu} : \mathcal{X} \to \mathbb{R}$ which is defined by

$$\Phi_{\lambda,\mu}(u,v) = \frac{1}{2}\Big(\int |\nabla u|^2 + \int |\nabla v|^2\Big) - \frac{\mu}{2}\Big(\int |u|^2 + \int |v|^2\Big)$$

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$$- \lambda \int uv + \frac{1}{q} \int g|v|^q - \frac{1}{p} \int f|u|^p.$$

Depending on the values of the parameters $\lambda, \mu$, the energy functional $\Phi_{\lambda,\mu}$ is unbounded from below and from above. This kind of behavior is similar to that of indefinite problems (see Berestycki et al. [2]) and we should expect multiplicity of solutions (see Ouyang [11]). In fact, by analyzing the Nehari set associated to $\Phi_{\lambda,\mu}$:

$$\mathcal{N}_{\lambda,\mu} = \{(u,v) \in \mathcal{X} : \Phi'_{\lambda,\mu}(u,v)(u,v) = 0\},$$

one is lead to the conclusion that for some parameters, the Nehari set $\mathcal{N}_{\lambda,\mu}$ is in fact a manifold which split in two disjoint sets $\mathcal{N}^+_{\lambda,\mu}, \mathcal{N}^-_{\lambda,\mu}$ satisfying $\overline{\mathcal{N}^+}_{\lambda,\mu} \cap \mathcal{N}^-_{\lambda,\mu} = \emptyset$ and $\mathcal{N}^+_{\lambda,\mu} \cap \overline{\mathcal{N}^-}_{\lambda,\mu} = \emptyset$, where

$$\begin{aligned}\mathcal{N}^+_{\lambda,\mu} &= \{(u,v) \in \mathcal{X} : \Phi''_{\lambda,\mu}(u,v)(u,v)^2 > 0\},\\ \mathcal{N}^-_{\lambda,\mu} &= \{(u,v) \in \mathcal{X} : \Phi''_{\lambda,\mu}(u,v)(u,v)^2 < 0\}.\end{aligned}$$

This suggest multiplicity of solutions, more precisely, the existence of critical points to $\Phi_{\lambda,\mu}$ in each manifold $\mathcal{N}^+_{\lambda,\mu}$ and $\mathcal{N}^-_{\lambda,\mu}$. In fact, let $(\lambda_1, \phi_1)$ be the first eigenpair of $-\Delta$ in $\Omega$ with Dirichlet boundary conditions. We consider the following hypothesis on $f$ and $g$:

(H1) $f, g \in L^\infty(\Omega)$ and $g(x) > 0$ a.e. $x \in \Omega$, $f(x) \geq 0$ a.e. $x \in \Omega$ and $f \neq 0$.

Our main result reads as follows.

**Theorem 1.1.** *Assume* (H1) *and that* $\mu < \lambda_1$. *Then there exists* $0 < \lambda_1(\mu) < \lambda^*(\mu) < \infty$ *such that*

(1) *For each* $\lambda \in (-\infty, \lambda^*(\mu)]$ *problem* (1.1) *has at least one positive solution* $(\bar{u}_{\lambda,\mu}, \bar{v}_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$.
(2) *For each* $\lambda \in (\lambda_1(\mu), \lambda^*(\mu)]$ *problem* (1.1) *has at least one positive solution* $(u_{\lambda,\mu}, v_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$.

By a positive solution we mean that both coordinates are positive functions. The parameters $\lambda_1(\mu), \lambda^*(\mu)$ which appears in Theorem 1.1 are the so-called extremal parameters (see Il'yasov [9]) and they describe the topological changes on the Nehari set with respect to $\lambda, \mu$. In fact, if $\lambda < \lambda_1(\mu)$ we have that $\mathcal{N}^+_{\lambda,\mu} = \emptyset$, while if $\lambda \geq \lambda^*(\mu)$, then $\mathcal{N}_{\lambda,\mu}$ is no longer a $C^1$ manifold. They can be found through the study of the so-called nonlinear Rayleigh quotient

$$R_\mu(u,v) := \frac{\int(|\nabla u|^2 + |\nabla v|^2) - \mu \int(|u|^2 + |v|^2)}{\int uv} + \frac{\int g|v|^q - \int f|u|^p}{\int uv}.$$

Since $\mathcal{N}_{\lambda,\mu}$ is no longer a manifold when $\lambda \geq \lambda^*(\mu)$, the technique used to prove Theorem 1.1 can not be used to prove existence of solutions in this case, therefore, we need a finer analysis over the Nehari sets. In this work we deal only with the case where $\mathcal{N}_{\lambda,\mu}$ is a manifold (and its limiting case), although some recent works of Il'yasov and Silva [10], Silva and Macedo [13] suggests multiplicity of solutions for $\lambda > \lambda^*(\mu)$.

This article is organized as follows: in Section 2 we study the fiber maps associated to $\Phi_{\lambda,\mu}$ and the extremal parameters. In Section 3 analyze some topological properties of the energy functional. In Section 4 we show existence of two positive solutions to equation (1.1).

## 2. Non-linear generalized Rayleigh quotient and extremal parameters

In this Section we establish some notation and technical results which will be used throughout the paper. In particular we study the Nehari set and its decomposition and we analyze the values of the parameters $\lambda, \mu$ for which $\mathcal{N}_{\lambda,\mu}$ is a manifold.

From now on we denote $w := (u, v) \in \mathcal{X}$. We equip $\mathcal{X}$ with the norm

$$\|w\| = \Big( \int |\nabla u|^2 + \int |\nabla v|^2 \Big)^{1/2}.$$

If $w \in \mathcal{X}$, we denote

$$\|w\|_2^2 = \|u\|_2^2 + \|v\|_2^2.$$

For $(\lambda, \mu) \in \mathbb{R}^2$, we recall the definition of the Nehari set

$$\mathcal{N}_{\lambda,\mu} = \{w \in \mathcal{X} \setminus \{0\} : \Phi'_{\lambda,\mu}(w)w = 0\}.$$

Note that the Nehari set can be written as

$$\mathcal{N}_{\lambda,\mu} = \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu},$$

where

$$\begin{aligned}
\mathcal{N}^+_{\lambda,\mu} &= \{w \in \mathcal{X} \setminus \{0\} : \Phi'_{\lambda,\mu}(w)w = 0,\ \Phi''_{\lambda,\mu}(w)w^2 > 0\},\\
\mathcal{N}^0_{\lambda,\mu} &= \{w \in \mathcal{X} \setminus \{0\} : \Phi'_{\lambda,\mu}(w)w = 0,\ \Phi''_{\lambda,\mu}(w)w^2 = 0\},\\
\mathcal{N}^-_{\lambda,\mu} &= \{w \in \mathcal{X} \setminus \{0\} : \Phi'_{\lambda,\mu}(w)w = 0,\ \Phi''_{\lambda,\mu}(w)w^2 < 0\}.
\end{aligned}$$

**Lemma 2.1.** *If $\mathcal{N}^+_{\lambda,\mu}, \mathcal{N}^-_{\lambda,\mu}$ are nonempty sets then $\mathcal{N}^+_{\lambda,\mu}, \mathcal{N}^-_{\lambda,\mu}$ are $C^1$ manifolds of codimension 1 in $\mathcal{X}$. Moreover, $w \in \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu}$ is a critical point of $(\Phi_{\lambda,\mu})_{|\mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu}}$ if and only if $w$ is a critical of $\Phi_{\lambda,\mu}$.*

Since all critical points of $\Phi_{\lambda,\mu}$ belongs to $\mathcal{N}_{\lambda,\mu}$, in order to find critical points to $\Phi_{\lambda,\mu}$ in $\mathcal{X}$, we restrict our attention to critical points of $\Phi_{\lambda,\mu}$ over $\mathcal{N}_{\lambda,\mu}$, however, to apply Lemma 2.1 we need to understand the Nehari sets $\mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu}$ and $\mathcal{N}^0_{\lambda,\mu}$. In fact, when $\mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu} \neq \emptyset$ and $\mathcal{N}^0_{\lambda,\mu} = \emptyset$ it is easy to show existence of solutions to problem (1.1), however, when $\mathcal{N}^0_{\lambda,\mu}$ we have to provide a more finer analysis over the Nehari sets.

For $\lambda, \mu \in \mathbb{R}$ and $w \in \mathcal{X}$ we introduce

$$H_{\lambda,\mu}(w) = \|w\|^2 - \mu \|w\|_2^2 - 2\lambda \int uv.$$

First, let us characterize the Nehari set by using the Fibering Method of Pohozaev (see [12]): for each $w \in \mathcal{X} \setminus \{0\}$, define $\psi_{\lambda,\mu,w} : [0, \infty) \to \mathbb{R}$ by $\psi_{\lambda,\mu,w}(t) = \Phi_{\lambda,\mu}(tw)$.

**Proposition 2.2.** *For each $\lambda, \mu \in \mathbb{R}$ and $w \in \mathcal{X} \setminus \{0\}$, the function $\psi_{\lambda,\mu,w}$ is of class $C^\infty$ on $(0, \infty)$. Moreover, the only three cases where $\psi_{\lambda,\mu,w}$ has a critical point are:*

***Case 1:*** *$H_{\lambda,\mu}(w) > 0$.*

- (i) *There is only one critical point at $t^-_\lambda(w) \in (0, \infty)$, and this point satisfies $\psi''_{\lambda,w}(t^-_{\lambda,\mu}(w)) < 0$ if only if $\int f|u|^p > 0$;*

***Case 2:*** *$H_{\lambda,\mu}(w) = 0$.*

- (ii) *$\psi_{\lambda,\mu,w}$ is constant equal to zero if and only if $\int f|u|^p, \int g|v|^q = 0$;*

(iii) *There is only one critical point at* $t_\lambda^-(w) \in (0,\infty)$ *and this point satisfies* $\psi''_{\lambda,w}(t^-_{\lambda,\mu}(w)) < 0$ *if only if* $\int f|u|^p, \int g|v|^q > 0$*;*

***Case 3:*** $H_{\lambda,\mu}(w) < 0$.

(I) *if* $\int f|u|^p = 0$ *and* $\int g|v|^q > 0$ *then there is only one critical point at* $t^+_{\lambda,\mu}(w) \in (0,\infty)$ *which satisfies* $\psi''_{\lambda,\mu,w}(t^-_{\lambda,\mu}(w)) > 0$*;*

*If* $\int f|u|^p > 0$ *and* $\int g|v|^q > 0$ *there are two possibilities:*

(II) *There are only two critical points for* $\psi_{\lambda,w}$*. One critical point at* $t^-_{\lambda,\mu}(w)$ *with* $\psi''_{\lambda,\mu,w}(t^-_{\lambda,\mu}(w)) < 0$ *and the other one at* $t^+_{\lambda,\mu}(w)$ *with* $\psi''_{\lambda,\mu,w}(t^+_{\lambda,\mu}(w)) > 0$*. Moreover* $\psi_{\lambda,\mu,w}$ *is decreasing over the intervals* $[0, t^+_{\lambda,\mu}(w)], [t^-_\lambda(w),\infty]$ *and increasing over the interval* $[t^+_{\lambda\mu}(w), t^-_{\lambda,\mu}(w)]$*;*

(III) *The function* $\psi_{\lambda,\mu,w}$ *has only one critical point which is an inflection point at* $t^0_{\lambda,\mu}(w)$*. Moreover,* $\psi_{\lambda,\mu,w}$ *is decreasing;*

We start with the study of $\mathcal{N}^+_{\lambda,\mu}$. Observe from Proposition 2.2 that if $\mathcal{N}^+_{\lambda,\mu} \neq \emptyset$ then there exist $(\lambda,\mu) \in \mathbb{R}^2$ and $w \in \mathcal{X}$ such that $H_\lambda(w) < 0$ or equivalently

$$\frac{\int |\nabla u|^2 + \int |\nabla v|^2 - \mu\big(\int |u|^2 + \int |v|^2\big)}{2\int uv} < \lambda,$$

therefore we are led to the study of the function

$$\lambda_{\min}(\mu; w) := \frac{\|w\|^2 - \mu\|w\|_2^2}{2\int uv}, \quad w \in \mathcal{X},\ \int uv > 0. \tag{2.1}$$

Now we turn our attention to the Nehari set $\mathcal{N}^0_{\lambda,\mu}$. From Proposition 2.2 we have that if $w \in \mathcal{N}^0_{\lambda,\mu}$, then

$$\int f|u|^p > 0, \quad \text{and} \quad \int uv > 0.$$

Let us introduce the set (the open subset of $\mathcal{X}$)

$$\mathcal{X}_+ := \big\{w \in \mathcal{X} : \int f|u|^p > 0,\ \int uv > 0\big\},$$

so $\mathcal{N}^0_{\lambda,\mu} \subset \mathcal{X}_+$. For each $w \in \mathcal{X}_+$, consider the corresponding so-called scalar fibered Rayleigh quotient (see Il'yasov [9])

$$\begin{aligned} R_\mu(tw) = &\frac{\int (|\nabla u|^2 + |\nabla v|^2)\,dx - \mu\int (|u|^2 + |v|^2)\,dx}{\int uv\,dx} \\ &+ \frac{t^{q-2}\int g|v|^q\,dx - t^{p-2}\int f|u|^p\,dx}{\int uv\,dx}. \end{aligned}$$

As was shown in [9] we have that

$$w \in \mathcal{N}^0_{\lambda,\mu}, \quad \text{if and only if} \quad R_\mu(w) = \lambda,\ \frac{d}{dt}R(tw)_{|t=1} = 0.$$

and hence the extremal values of $[0,\infty) \ni t \mapsto R_\mu(tw)$ provide regions of parameters where $\mathcal{N}^0_{\lambda,\mu} = \emptyset$. Assume that $\mu < \lambda_1$ and observe that $[0,\infty) \ni t \mapsto R_\mu(tw)$ has two extremal values. The first one is a local minimum attained at $t = 0$, indeed, it corresponds to $\lambda_{\min}(\mu; w)$ as defined in (2.1): one can easily see that

$$\lambda_{\min}(\mu; w) \geq 1 - \frac{\mu}{\lambda_1} > 0, \quad \forall \mu < \lambda_1,\ \forall w \in \mathcal{X}_+.$$

The second one corresponds to a local maximum which can be computed by using standard calculus in the following way:

$$\frac{d}{dt}R_\mu(tw) = \frac{(q-2)t^{q-3}\int g|v|^q dx - (p-2)t^{p-3}\int f|u|^p dx}{\int uv\, dx} = 0, t > 0,$$

if and only if

$$(q-2)\int g|v|^q dx - (p-2)t^{p-q}\int f|u|^\gamma dx = 0,$$

and hence

$$t_0(w) = \Big(\frac{(q-2)\int g|v|^q dx}{(p-2)\int f|u|^p dx}\Big)^{\frac{1}{p-q}} \tag{2.2}$$

is the critical point of $[0,\infty) \ni t \mapsto R_\mu(tw)$ which corresponds to a global maximum. Therefore we have the nonlinear generalized Rayleigh quotient

$$\lambda_{\max}(\mu; w) := \max_{t\geq 0} R_\mu(tw) = \frac{1}{2\int uv}\bigg(\|w\|^2 - \mu\|w\|_2^2 + C_{p,q}\frac{\big(\int g|v|^q dx\big)^{\frac{p-2}{p-q}}}{\big(\int f|u|^p dx\big)^{\frac{q-2}{p-q}}}\bigg),$$

where $C_{p,q} > 0$ is given by

$$C_{p,q} = \Big(\frac{q-2}{p-2}\Big)^{\frac{q-2}{p-q}} - \Big(\frac{q-2}{p-2}\Big)^{\frac{p-2}{p-q}}.$$

**Remark 2.3.** We observe here that to study the scalar fibered Rayleigh quotient, there is no need to assume that $w \in \mathcal{X}_+$; however, $[0,\infty) \ni t \mapsto R_\mu(tw)$ has a global maximum if, and only if $w \in \mathcal{X}_+$. Furthermore, note that if $\mu < \lambda_1$ and $\int uv > 0$, then $\lambda_{\min}(\mu; w)$ is just the local minimum of $[0,\infty) \ni t \mapsto R_\mu(tw)$ which is attained at $t = 0$.

The functions $\lambda_{\min}(\mu; w)$ and $\lambda_{\max}(\mu; w)$ have the following geometrical interpretation, with respect to the fiber maps, which can be proved from Proposition 2.2 and their definitions.

**Proposition 2.4.** *The following holds:*

1. *For each $\mu < \lambda_1$ and $\lambda \in \mathbb{R}$ we have that $\mathcal{N}^-_{\lambda,\mu} \neq \emptyset$. Moreover $\mathcal{N}^+_{\lambda,\mu} \neq \emptyset$ if, and only if $\lambda > \lambda_{\min}(\mu, w)$ and $\mu < \lambda_1$.*
2. *For each $\mu < \lambda_1$ and $w \in \mathcal{X}_+$ we have that: $\lambda_{\max}(\mu; w)$ is the unique parameter $\lambda > 0$ for which the fiber map $\psi_{\lambda,w}$ has a critical point with second derivative zero at $t(w)$. If $\lambda_{\min}(\mu, w) < \lambda < \lambda_{\max}(\mu; w)$, then $\psi_{\lambda,\mu,w}$ satisfies II) of the Proposition 2.2 while if $\lambda > \lambda_{\max}(\mu; w)$, then $\psi_{\lambda,w}$ is decreasing and has no critical points.*

Let us consider the following critical values:

$$\lambda_1^*(\mu) = \inf\big\{\lambda_{\min}(\mu; w) : w \in \mathcal{X} : \int uv\, dx > 0\big\}, \tag{2.3}$$

$$\lambda^*(\mu) = \inf\big\{\lambda_{\max}(\mu; w) : w \in \mathcal{X}_+\big\}. \tag{2.4}$$

**Lemma 2.5.** *For each $\mu < \lambda_1$ it holds $0 < \lambda_1^*(\mu) < \lambda^*(\mu) < +\infty$. Moreover,*

- (i) $\lambda_1^*(\mu) = \lambda_1 - \mu$.
- (ii) *There exists a minimizer $w^* := (u^*, v^*) \in \mathcal{X}_+$ of (2.4), which means $\lambda_{\max}(\mu; w^*) = \lambda^*(\mu)$.*

*Proof.* $(i)$ Indeed, we have

$$\lambda_{\min}(\mu; w) \geq \Big(1 - \frac{\mu}{\lambda_1}\Big)\frac{\|w\|^2}{2\int uv}, \tag{2.5}$$

and

$$\begin{aligned}\int uv &\leq \|u\|_2\|v\|_2 \leq \Big(\frac{1}{\sqrt{\lambda_1}}\|u\|\Big)\Big(\frac{1}{\sqrt{\lambda_1}}\|v\|\Big)\\ &= \frac{1}{\lambda_1}\|u\|\|v\| \leq \frac{1}{\lambda_1}\Big(\frac{\int |\nabla u|^2 + |\nabla v|^2}{2}\Big) = \frac{1}{\lambda_1}\frac{\|w\|^2}{2}\,.\end{aligned}$$

Then we obtain

$$\lambda_1 \leq \inf \frac{\|(u,v)\|^2}{2\int uv}. \tag{2.6}$$

It follows from (2.6) and (2.5) that

$$\lambda_1^*(\mu) \geq \lambda_1 - \mu.$$

Since $\int \phi_1^2 > 0$ and $\lambda_{min}(\mu, \phi_1, \phi_1) = \lambda_1 - \mu$ it follows that $\lambda_1^*(\mu) = \lambda_1 - \mu$.

(ii) Now let us prove there exists $w^* \in \mathcal{X}_+$ such that $\lambda^*(\mu) = \lambda_{max}(\mu; w^*)$. Choose a sequence $w_n := (u_n, v_n) \in \mathcal{X}_+$ such that $\lambda_{max}(\mu, w_n) \to \lambda^*(\mu)$ as $n \to \infty$ and since $\lambda_{\max}(\mu, tw) = \lambda_{\max}(\mu; w)$ for $t > 0$, we can assume without loss of generality that $\|w_n\| = 1$ and therefore $w_n \rightharpoonup w$ in $\mathcal{X}$ and $w_n \to w$ in $L^p(\Omega)\times L^q(\Omega)$. Note that $\int uv > 0$ because

$$\lambda_{\max}(\mu; w_n) \geq \Big(1 - \frac{\mu}{\lambda_1}\Big)\frac{\|w_n\|^2}{2\int u_n v_n} = \Big(1 - \frac{\mu}{\lambda_1}\Big)\frac{1}{2\int u_n v_n}, \quad \forall n,$$

and on the contrary, we would have $\lambda_{\max}(\mu; w_n) \to +\infty$ which is clearly a contradiction. It follows that $u, v \neq 0$. We claim that $\int f|u|^p > 0$, indeed suppose on the contrary that $\int f|u|^p = 0$. Since

$$\lambda_{\max}^*(\mu; w_n) \geq C_{p,q}\frac{\big(\int g|v_n|^q dx\big)^{\frac{p-2}{p-q}}}{\big(\int f|u_n|^p dx\big)^{\frac{q-2}{p-q}}}, \quad \forall n, \tag{2.7}$$

and $\int g|v|^q > 0$ we conclude that $\lambda_{\max}(\mu; w_m) \to +\infty$ which is a contradiction, therefore $\int f|u|^p > 0$. We denote by $\bar{u} = \frac{u}{\|w\|}$, $\bar{v} = \frac{v}{\|w\|}$, then $\bar{w} = (\bar{u}, \bar{v})$ satisfies $\|\bar{w}\| = 1$, $\int \bar{u}\bar{v} > 0$ and $\int f|\bar{u}|^p, \int g|\bar{v}|^q > 0$. We claim that $w_n \to w$ in $\mathcal{X}$, indeed if not, by the weak lower semi-continuity of the norm, we have

$$\lambda_{\max}(\mu; \bar{w}) < \liminf_{n\to\infty} \lambda_{\max}(w_n) = \lambda^*(\mu) \tag{2.8}$$

which is an absurd and hence $\lambda^*(\mu) = \lambda_{\max}(\mu; \bar{w})$. By defining $w^* = \bar{w}$ the proof is complete. $\square$

**Proposition 2.6.** *Let $\mu < \lambda_1$, then $\mathcal{N}^0_{\lambda^*(\mu),\mu} \neq \emptyset$. Moreover, each $w \in \mathcal{N}^0_{\lambda^*(\mu),\mu}$ satisfies*

$$\begin{gathered}2\big(-\Delta u - \mu u - \lambda^*(\mu)v\big) - pf(x)|u|^{p-2}u = 0,\\ 2\big(-\Delta v - \mu v - \lambda^*(\mu)u\big) + qg(x)|v|^{q-2}v = 0,\end{gathered} \tag{2.9}$$

*Proof.* From Lemma 2.5 there exists $w \in \mathcal{X} \setminus \{0\}$ such that $\lambda_{\max}(\mu; w) = \lambda^*(\mu)$. From the definition of $\lambda_{\max}(\mu; w)$ it follows that $\mathcal{N}^0_{\lambda^*(\mu),\mu} \neq \emptyset$. To prove that each

$w \in \mathcal{N}^0_{\lambda^*(\mu),\mu}$ satisfies (2.9), we observe that $\lambda'_{\max}(\mu; w)\bar{w} = 0$ for all $\bar{w} = (\bar{u}, \bar{v}) \in \mathcal{X}$, hence we obtain

$$\begin{aligned} 0 =&2\Big(\int \nabla u \nabla \bar{u} - \mu \int u\bar{u} - \lambda^*(\mu) \int v\bar{u}\Big) \\ &- p\Big(\frac{q-2}{p-2}\Big)^{\frac{p-2}{p-q}} \Big(\frac{\int g|v|^q}{\int f|u|^p}\Big)^{\frac{p-2}{q-2}} \int f|u|^{p-2}u\bar{u}, \\ 0 =&2\Big(\int \nabla v \nabla \bar{v} - \mu \int v\bar{v} - \lambda^*(\mu) \int u\bar{v}\Big) \\ &- q\Big(\frac{q-2}{p-2}\Big)^{\frac{p-2}{p-q}} \Big(\frac{\int g|v|^q}{\int f|u|^p}\Big)^{\frac{p-2}{q-2}} \int g|v|^{q-2}v\bar{v}. \end{aligned} \tag{2.10}$$

For $w \in \mathcal{N}^0_{\lambda^*(\mu),\mu}$ we have

$$\Big(\frac{q-2}{p-2}\Big)^{\frac{p-2}{p-q}} \Big(\frac{\int g|v|^q}{\int f|u|^p}\Big)^{\frac{p-2}{q-2}} = 1\,. \tag{2.11}$$

Then, from (2.10) and (2.11) we conclude the proof. □

**Corollary 2.7.** *Let $\mu < \lambda_1$. Then*

(i) *For each $\lambda \in \mathbb{R}$ we have that $\mathcal{N}^-_{\lambda,\mu} \neq \emptyset$.*
(ii) *$\mathcal{N}^+_{\lambda,\mu} \neq \emptyset$ if, and only if $\lambda > \lambda_1^*(\mu)$.*
(iii) *$\mathcal{N}^0_{\lambda,\mu} \neq \emptyset$ if, and only if $\lambda \geq \lambda^*(\mu)$.*

*Proof.* (i) Given $\lambda \in \mathbb{R}$ there exists $w_n := (u_n, v_n) \in \mathcal{X}_+$ such that $\|u_n\| = 1$, $v_n \neq 0$ and $v_n \to 0$ in $H^1_0(\Omega)$, so

$$\begin{aligned} \lim_{n\to\infty} H_{\lambda,\mu}(w_n) &\geq \lim_{n\to\infty} \Big(1 - \frac{\mu}{\lambda_1}\Big) + \|v_n\|^2 - \mu\|v_n\|_2^2 - 2\lambda \int u_n v_n \\ &= \Big(1 - \frac{\mu}{\lambda_1}\Big) > 0, \end{aligned}$$

then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we obtain $H_{\lambda,\mu}(w_n) > 0$ and from Proposition 2.2 we conclude that $\mathcal{N}^-_{\lambda,\mu} \neq \emptyset$.

(ii) Suppose $\mathcal{N}^+_{\lambda,\mu} \neq \emptyset$ and take $w \in \mathcal{N}^+_{\lambda,\mu}$. By Proposition 2.2 we conclude that $H_{\lambda,\mu}(w) < 0$ which implies $\frac{\|w\|^2 - \mu\|w\|_2^2}{2\int uv} < \lambda$, therefore $\lambda_1^*(\mu) < \lambda$.

Now suppose that $\lambda_1^*(\mu) < \lambda$ and take $w = (\phi_1, \phi_1)$. It follows that

$$\lambda_1^*(\mu) = \frac{\|w\|^2 - \mu\|w\|_2^2}{2\int \phi_1^2} < \lambda,$$

hence $H_{\lambda,\mu}(w) < 0$. Since $\int g|\phi_1|^q > 0$, from Proposition 2.2 we conclude that $t^+_{\lambda,\mu}(w)w \in \mathcal{N}^+_{\lambda,\mu}$.

(iii) Suppose $\mathcal{N}^0_{\lambda,\mu} \neq \emptyset$. We know that $w \in \mathcal{N}^0_{\lambda,\mu}$ if, and only if

$$R_\mu(w) = \lambda, \quad \frac{d}{dt} R(tw)_{|t=1} = 0,$$

and therefore by definition of $\lambda^*(\mu)$, we conclude that $\lambda^*(\mu) \leq \lambda$.

Now observe from Lemma 2.5 that there exists $w^* \in \mathcal{X}_+$ such that $w^* \in \mathcal{N}^0_{\lambda^*(\mu),\mu}$. Moreover there exists $w_n := (u_n, v_n) \in \mathcal{X}_+$ such that $\|u_n\| = 1$, $v_n \neq 0$ and $v_n \to 0$

in $H_0^1(\Omega)$, then

$$\lim_{n\to\infty}\lambda_{\max}(\mu;w_n)\geq\lim_{n\to\infty}\frac{1}{2\int u_nv_n}\Big(\Big(1-\frac{\mu}{\lambda_1}\Big)+\|v_n\|^2-\mu\|v_n\|_2^2\Big)=\infty,$$

therefore, from the continuity of $\lambda_{\max}(\mu;w)$ with respect to $w$, given $\lambda\geq\lambda^*(\mu)$ we there exists $w\in\mathcal{X}_+$ such that $\lambda_{\max}(\mu;w)=\lambda$ and from Proposition 2.4 we conclude that $\mathcal{N}^0_{\lambda,\mu}\neq\emptyset$. $\square$

## 3. Topological properties of the energy functional

In this Section we study the energy functional $\Phi_{\lambda,\mu}$, in particular, we show that $\Phi_{\lambda,\mu}$ has some well know topological properties when restricted to the Nehari set, as for example coerciveness, which allow us to minimize over the Nehari manifolds $\mathcal{N}^-_{\lambda,\mu}$ and $\mathcal{N}^+_{\lambda,\mu}$.

For $\lambda>0$ we define

$$\hat{\mathcal{N}}^-_{\lambda,\mu}=\big\{w\in\mathcal{N}^-_{\lambda,\mu}:H_{\lambda,\mu}(w)\leq 0\big\}.$$

**Proposition 3.1.** *For each $\mu<\lambda_1$ and $\lambda\in\mathbb{R}$, we have the following:*

(i) *There exists a constant $C>0$ such that $\|w\|\leq C$ for all $w\in\mathcal{N}^+_{\lambda,\mu}\cup\hat{\mathcal{N}}^-_{\lambda,\mu}$.*

(ii) *The functional $\Phi_{\lambda,\mu}$ restricted to $\mathcal{N}^+_{\lambda,\mu}\cup\mathcal{N}^-_{\lambda,\mu}$ is coercive that is if $w_n\in\mathcal{N}^-_{\lambda,\mu}$ is such that $\|w_n\|\to\infty$ as $n\to\infty$, then $\Phi_{\lambda,\mu}(w_n)\to\infty$ as $n\to\infty$.*

*Proof.* Assume that $w_n=(u_n,v_n)\in\mathcal{N}^+_{\lambda,\mu}\cup\mathcal{N}^-_{\lambda,\mu}$ satisfies $\|w_n\|\to\infty$. We claim that

$$\int\Big|\frac{u_n}{\|w_n\|}\Big|^p\to 0,\quad\text{as } n\to\infty. \tag{3.1}$$

If not, then there exists $\bar{C}>0$ such that, up to a subsequence, $\int\Big|\frac{u_n}{\|w_n\|}\Big|^p>\bar{C}$. Denote by $\bar{u}_n=\frac{u_n}{\|w_n\|}$ and $\bar{v}_n=\frac{v_n}{\|w_n\|}$. Since $w_n\in\mathcal{N}^+_{\lambda,\mu}\cup\mathcal{N}^-_{\lambda,\mu}$, we have

$$0=1-\mu\Big(\int|\bar{u}_n|^2+|\bar{v}_n|^2\Big)-2\lambda\int\bar{u}_n\bar{v}_n+\|w_n\|^{q-2}\int g|\bar{v}_n|^q-\|w_n\|^{p-2}\int f|\bar{u}_n|^p. \tag{3.2}$$

By Sobolev embedding and Poincare's inequality there exist constants $C_1,C_2,C_3>0$ such that $\int g|\bar{v}_n|^q\leq C_1$, $\int\big(|\bar{u}_n|^2+|\bar{v}_n|^2\big)\leq C_2$ and $\int\bar{u}_n\bar{v}_n\leq C_3$. It follows from (3.2) that

$$\begin{aligned}0&=1-\mu\Big(\int|\bar{u}_n|^2+|\bar{v}_n|^2\Big)-2\lambda\int\bar{u}_n\bar{v}_n+\|w_n\|^{q-2}\int g|\bar{v}_n|^q\\&\quad-\|w_n\|^{p-2}\int f|\bar{u}_n|^p\\&\leq 1+|\mu|C_2+2|\lambda|C_3+C_1\|w_n\|^{q-2}-\bar{C}\|w_n\|^{p-2},\forall n,\end{aligned}$$

which is a contradiction since $p>q$ and therefore (3.1) is true.

Let us prove (i). Suppose on the contrary that there exists a sequence $w_n\in\mathcal{N}^+_{\lambda,\mu}\cup\hat{\mathcal{N}}^-_{\lambda,\mu}$ such that $\|w_n\|\to\infty$ as $n\to\infty$. From (3.1) we obtain that $\int|\bar{u}_n|^2\to 0$ and since $H_{\lambda,\mu}(w_n)\leq 0$ and $\mu<\lambda_1$ we conclude that

$$\begin{aligned}0&\geq 1-\mu\Big(\int|\bar{u}_n|^2+|\bar{v}_n|^2\Big)-2\lambda\int\bar{u}_n\bar{v}_n\\&\geq\Big(1-\frac{\mu}{\lambda_1}\Big)-2\lambda\int\bar{u}_n\bar{v}_n\end{aligned}$$

$$\to \Big(1-\frac{\mu}{\lambda_1}\Big) > 0, \quad n\to\infty,$$

which is a contradiction and therefore there exists a constant $C > 0$ such that $\|w\| \le C$ for all $w \in \mathcal{N}^+_{\lambda,\mu} \cup \hat{\mathcal{N}}^-_{\lambda,\mu}$.

Let us prove (ii): Assume that $\|w_n\| \to \infty$ as $n \to \infty$. From (3.1) and (3.2) we conclude that

$$\|w_n\|^{q-2}\int g|\bar v_n|^q - \|w_n\|^{p-2}\int f|\bar u_n|^p - \mu\int |\bar v|^2 = o(1) - 1. \tag{3.3}$$

Now observe that

$$\begin{aligned} \Phi_{\lambda,\mu}(w_n) &= \|w_n\|^2\Big(\frac12 - \mu\frac12\int |\bar u_n|^2 + |\bar v_n|^2 - \lambda\int \bar u_n \bar v_n\Big) \\ &\quad + \|w_n\|^2\Big(\frac{\|w_n\|^{q-2}}{q}\int g|\bar v_n|^q - \frac{\|w_n\|^{p-2}}{p}\int f|\bar u_n|^p\Big). \end{aligned}\tag{3.4}$$

Observe that $\Phi_{\lambda,\mu}(w) > 0$ for all $w \in \mathcal{N}^-_{\lambda,\mu}$. If we assume on the contrary that $\Phi_{\lambda,\mu}(w_n)$ does not converge to $\infty$ then from (3.4) we are forced to assume that

$$\frac{\|w_n\|^{q-2}}{q}\int g|\bar v_n|^q - \frac{\|w_n\|^{p-2}}{p}\int f|\bar u_n|^p - \frac12\mu\int |\bar v|^2 = o(1) - \frac12. \tag{3.5}$$

from (3.3) and (3.5) we obtain

$$\|w_n\|^{p-2}\int f|\bar u_n|^p = o(1) + \frac2p\Big(\frac{q-2}{q-p}\Big)\Big(1-\mu\int |\bar v_n|^2\Big), \tag{3.6}$$

$$\|w_n\|^{q-2}\int g|\bar v_n|^q = o(1) + \frac2p\Big(\frac{q-2}{q-p}\Big)\Big(1-\mu\int |\bar v_n|^2\Big). \tag{3.7}$$

Once $\mu < \lambda_1$ and $q < p$ it follows from (3.6), (3.7) that

$$\begin{gathered} \|w_n\|^{p-2}\int f|\bar u_n|^p \le o(1) + \frac2p\Big(\frac{q-2}{q-p}\Big)\Big(1-\frac{\mu}{\lambda_1}\Big), \\ \|w_n\|^{q-2}\int g|\bar v_n|^q \le o(1) + \frac2p\Big(\frac{q-2}{q-p}\Big)\Big(1-\frac{\mu}{\lambda_1}\Big), \end{gathered}$$

which is a contradiction and therefore $\Phi_{\lambda,\mu}(w_n) \to \infty$ as $n \to \infty$. □

From Proposition 3.1 we have the following result.

**Corollary 3.2.** *Suppose that $\mu < \lambda_1$ and $\lambda \in \mathbb{R}$. Then there exists a constant $C > 0$ such that $\Phi_\lambda(w) \ge -C$, for all $w \in \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu}$.*

**Lemma 3.3.** *For each $\mu < \lambda_1$ and $\lambda \in \mathbb{R}$ there exists a constant $C > 0$ such that $\|w\| \ge C$, for all $w \in \mathcal{N}^-_{\lambda,\mu}$. Moreover, if $A \subset \mathcal{N}^-_{\lambda,\mu}$ is a bounded set, then $\|u\|_p \ge C$ for each $(u,v) \in A$.*

*Proof.* Indeed, suppose on the contrary that there exists $w_n = (u_n, v_n) \in \mathcal{N}^-_{\lambda,\mu}$ such that $\|w_n\| \to 0$. If $v_n = 0$ for all $n$ the proof is immediate, therefore there is no loss of generality in assuming that $v_n \ne 0$ for all $n$. Moreover from Proposition 2.2 we also have that $u_n \ne 0$ for all $n$. Define $\bar u_n = \frac{u_n}{\|w_n\|}$ and $\bar v_n = \frac{v_n}{\|w_n\|}$ and $\bar w_n = (\bar u_n, \bar v_n)$. It follows that $\bar w_n \rightharpoonup (u_0, v_0)$ in $\mathcal{X}$ and $\bar w_n \to (u_0, v_0)$ in $L^p(\Omega) \times L^q(\Omega)$. Once $w_n \in \mathcal{N}^-_{\lambda,\mu}$ we know that

$$1 - \mu\|\bar w_n\|_2^2 - 2\lambda\int \bar u_n \bar v_n = \|w_n\|^{p-2}\int f|\bar u_n|^p - \|w_n\|^{q-2}\int g|\bar v_n|^q, \ \forall n, \tag{3.8}$$

and

$$1-\|\bar{w}_n\|_2^2-2\lambda\int\bar{u}_n\bar{v}_n+(q-1)\|w_n\|^{q-2}\int g|\bar{v}_n|^q-(p-1)\|w_n\|^{p-2}\int f|\bar{u}_n|^p<0,$$

and hence

$$(q-2)\|w_n\|^{q-2}\int g|\bar{v}_n|^q-(p-2)\|w_n\|^{p-2}\int f|\bar{u}_n|^p<0,\quad \forall n,$$

which implies

$$\frac{1}{\|w_n\|^{p-q}}<\frac{p-2}{q-2}\frac{\int f|\bar{u}_n|^p}{\int g|\bar{v}_n|^q},\quad \forall n,$$

Hence $\int g|\bar{v}_n|^q\to 0$ as $n\to\infty$ which combined with (3.8) gives us an absurd since $\mu<\lambda_1$ and therefore $\mathcal{N}_\lambda^-$ is bounded always from the origin.

Now assume that $A\subset\mathcal{N}_{\lambda,\mu}^-$ is a bounded set. For each $w\in A$ we have that

$$\|w\|^2-\mu\|w\|_2^2-2\lambda\int uv+\int g|v|^q-\int f|u|^p=0. \tag{3.9}$$

If on the contrary we can find $w_n\in A$ such that $u_n\to 0$ in $L^p(\Omega)$, then since $A$ is bounded, from (3.9) we obtain $\|w_n\|^2-\mu\|v_n\|_2^2+\int g|v_n|^q=o(1)$ and once $\mu<\lambda_1$ we conclude that $\|w_n\|=o(1)$ that is a contradiction and therefore, there exists $C>0$ such that $\|u\|_p\geq C$ for each $w\in A$. □

## 4. Existence of solutions in $(-\infty,\lambda^*(\mu)]$

In this section, by using the properties of the fiber maps, we prove existence of positive solutions to the problem (1.1) for $\lambda\in(-\infty,\lambda^*(\mu)]$ and $\mu<\lambda_1$.

**Remark 4.1.** We claim that there is no non-negative solution of (1.1) for $\mu>\lambda_1$ and $\lambda>0$. Indeed, take $\phi_1\in H_0^1(\Omega)$ and let $w:=(u,v)\in\mathcal{X}$ be a non-negative solution for (1.1), then

$$\int\nabla u\nabla\phi_1=\lambda_1\int u\phi_1=\mu\int u\phi_1+\lambda\int v\phi_1+\int f|u|^{p-2}u\phi_1\geq\mu\int u\phi_1$$

we obtain

$$(\lambda_1-\mu)\int u\phi_1\geq 0$$

which implies that $u=v=0$, since $\mu>\lambda_1$. Therefore there is no non-negative solution of (1.1) for $\mu>\lambda_1$ and $\lambda>0$. If $w$ is a positive solution, then the same argument holds for all $\mu\geq\lambda_1$ and $\lambda>0$.

For $\lambda\in\mathbb{R}$ define

$$\hat{\mathcal{M}}_{\lambda,\mu}:=\{w\in\mathcal{X}:\psi_{\lambda,\mu,w}\text{ satisfies (I) or (II) of Proposition 2.2}\},$$

and

$$\hat{\mathcal{M}}_{\lambda\,\mu}^-:=\Big\{w\in\mathcal{X}\setminus\{0\}:H_{\lambda,\mu}(w)\geq 0,\ \int f|u|^p>0\Big\}.$$

For $\lambda\in\mathbb{R}$, let $J_{\lambda,\mu}^-:\hat{\mathcal{M}}_{\lambda,\mu}\cup\hat{\mathcal{M}}_{\lambda,\mu}^-\to\mathbb{R}$ and $J_{\lambda,\mu}^+:\hat{\mathcal{M}}_{\lambda,\mu}\to\mathbb{R}$ be defined by

$$J_{\lambda,\mu}^-(w)=\Phi_{\lambda,\mu}(t_{\lambda,\mu}^-(w)w),\quad\text{and}\quad J_{\lambda,\mu}^+(w)=\Phi_{\lambda,\mu}(t_{\lambda,\mu}^+(w)w).$$

**Remark 4.2.** Observe from Proposition 2.2 that $\mathcal{N}_{\lambda,\mu}^+\cup\mathcal{N}_{\lambda,\mu}^-\subset\hat{\mathcal{M}}_{\lambda,\mu}\cup\hat{\mathcal{M}}_{\lambda,\mu}^-$ and from Corollary 2.7 we have that $\mathcal{N}_{\lambda,\mu}^+\neq\emptyset$ if $\lambda>\lambda_1(\mu)$ and $\mathcal{N}_{\lambda,\mu}^-\neq\emptyset$ if $\lambda\in\mathbb{R}$. Moreover $J_{\lambda,\mu}^-,J_{\lambda,\mu}^+$ are the restrictions of $\Phi_{\lambda,\mu}$ to $\mathcal{N}_{\lambda,\mu}^-$ and $\mathcal{N}_{\lambda,\mu}^+$ respectively.

We consider the following constrained minimization problems

$$\hat{J}^{-}_{\lambda,\mu} := \inf\{J^{-}_{\lambda,\mu}(w) : w \in \mathcal{N}^{-}_{\lambda,\mu}\}, \quad \forall \lambda \in \mathbb{R},$$

and

$$\hat{J}^{+}_{\lambda,\mu} := \inf\{J^{+}_{\lambda,\mu}(w) : w \in \mathcal{N}^{+}_{\lambda,\mu}\}, \quad \forall \lambda > \lambda_1(\mu).$$

**Proposition 4.3.** *It holds:*

- *For each $\lambda \in (\lambda_1(\mu), \lambda^*(\mu))$ there there exists $w_\lambda := (u_\lambda, v_\lambda) \in \mathcal{N}^{+}_{\lambda,\mu}$ such that $\hat{J}^{+}_{\lambda,\mu} = J^{+}_{\lambda,\mu}(w_\lambda)$.*
- *For each $\lambda \in (-\infty, \lambda^*(\mu))$ there there exists $\bar{w}_\lambda := (\bar{u}_\lambda, \bar{v}_\lambda) \in \mathcal{N}^{-}_{\lambda,\mu}$ such that $\hat{J}^{-}_{\lambda,\mu} = J^{-}_{\lambda,\mu}(\bar{w}_\lambda)$.*

*Proof.* Firstly, we start with $\hat{J}^{+}_{\lambda,\mu}$. We may suppose that there exists $w_n := (u_n, v_n) \in \mathcal{N}^{+}_{\lambda,\mu}$ such that $J^{+}_{\lambda,\mu}(w_n) \to \hat{J}^{+}_{\lambda,\mu}$. From Proposition 3.1 we have $w_n \rightharpoonup w := (u, v)$ in $\mathcal{X}$ and $w_n \to w$ in $L^p(\Omega) \times L^q(\Omega)$. Since

$$\Phi_{\lambda,\mu}(w) \le \liminf \Phi_{\lambda,\mu}(w_n) = \hat{J}^{+}_{\lambda,\mu}, \tag{4.1}$$

and by Proposition 2.2 we have that $\hat{J}^{+}_{\lambda,\mu} < 0$, we conclude that $w \neq 0$. We claim that $w_n \to w$ in $\mathcal{X}$. Indeed suppose on the contrary that it is false. By one hand note from Proposition 2.2 that $H_{\lambda,\mu}(w) < \liminf_{n\to\infty} H_{\lambda,\mu}(w_n) \le 0$ and since $\lambda \in (\lambda_1, \lambda^*(\mu))$ we conclude that $w \in \hat{\mathcal{M}}_{\lambda,\mu}$. On the other hand

$$0 = \psi'_{\lambda,\mu,w}(t^{+}_{\lambda,\mu}(w)) < \liminf_{n\to\infty} \psi'_{\lambda,\mu,w_n}(t^{+}_{\lambda,\mu}(w)),$$

and hence $t^{+}_{\lambda,\mu}(w) > 1$ which implies that

$$J^{+}_{\lambda,\mu}(w) < \liminf_{n\to\infty} \Phi_{\lambda,\mu}(w) < \liminf_{n\to\infty} \Phi_{\lambda,\mu}(w_n) = \hat{J}^{+}_{\lambda,\mu},$$

which is a contradiction. Therefore $w_n \to w$ in $\mathcal{X}$, $w \in \mathcal{N}^{+}_{\lambda,\mu}$ and $\hat{J}^{+}_{\lambda,\mu} = J^{+}_{\lambda,\mu}(w)$.

Now we consider $w_n := (u_n, v_n) \in \mathcal{N}^{-}_{\lambda,\mu}$ such that $J^{-}_{\lambda,\mu}(w_n) \to \hat{J}^{-}_{\lambda,\mu}$. From Proposition 3.1 we have $w_n \rightharpoonup w := (u, v)$ in $\mathcal{X}$ and $w_n \to w$ in $L^p(\Omega) \times L^q(\Omega)$. Then from Lemma 3.3 we have that $u \neq 0$ and hence from Proposition 2.2, $t^{-}_{\lambda,\mu}(w)$ is well defined. We claim that $w_n \to w$ in $\mathcal{X}$, so suppose that is not true. Observe that

$$0 = \psi'_{\lambda,\mu,w}(t^{-}_{\lambda,\mu}(w)) < \liminf_{n\to\infty} \psi'_{\lambda,\mu,w_n}(t^{-}_{\lambda,\mu}(w)),$$

and hence $t^{+}_{\lambda,\mu}(w_n) < t^{-}_{\lambda,\mu}(w) < 1$ for sufficiently large $n$ in case $t^{+}_{\lambda,\mu}(w_n)$ is well defined and $t_{\lambda,\mu}(w) < 1$ in case $t^{+}_{\lambda,\mu}(w_n)$ is not defined. In both cases we have

$$J^{-}_{\lambda,\mu}(w) < \liminf_{n\to\infty} \Phi_{\lambda,\mu}(t^{-}_{\lambda,\mu}(w)w_n) \le \liminf_{n\to\infty} \Phi_{\lambda,\mu}(w_n) = \hat{J}^{-}_{\lambda,\mu},$$

that is an absurd. Therefore $w_n \to w$ in $\mathcal{X}$, $w \in \mathcal{N}^{-}_{\lambda,\mu}$ and $\hat{J}^{-}_{\lambda,\mu} = J^{-}_{\lambda,\mu}(w)$. $\square$

The next Proposition will be useful in order to prove existence of solutions when $\lambda \ge \lambda^*(\mu)$.

**Proposition 4.4.** *Fix $\mu < \lambda_1$ and take $w \in \mathcal{X} \setminus 0$ such that $\int uv > 0$. Let $I \subset \mathbb{R}$ be an open interval such that $t^{\mp}_{\lambda,\mu}(w)$ are well defined for all $\lambda \in I$. It holds:*

(i) *The functions $I \ni \lambda \mapsto t^{\mp}_{\lambda,\mu}(w)$ are $C^1$. Moreover, $I \ni \lambda \mapsto t^{-}_{\lambda,\mu}(w)$ is decreasing while $I \ni \lambda \mapsto t^{+}_{\lambda,\mu}(w)$ is increasing.*

(ii) *The functions* $I \ni \lambda \mapsto J^{\mp}_{\lambda,\mu}(w)$ *are continuous and decreasing.*

*Proof.* (i) For each $w \in \mathcal{X} \setminus 0$ fixed we define

$$F(\lambda, t) = H_{\lambda,\mu}(tu, tv) + G(v) - F(u).$$

Since $t^{\mp}_{\lambda,\mu}(w)w \in \mathcal{N}^{\mp}_{\lambda,\mu}$, it follows that

$$\begin{gathered}F(\lambda, t^{\mp}_{\lambda,\mu}(w)w) = 0,\\ \frac{\partial}{\partial t}F(\lambda, t^{\mp}_{\lambda,\mu}(w)) \neq 0,\end{gathered}$$

which implies from the implicit function theorem that $t^{\mp}_{\lambda,\mu}(w)$ is $C^1$ and

$$\frac{\partial}{\partial \lambda} t^{\mp}_{\lambda,\mu}(w) = \frac{2\int uv}{\psi''_{\lambda,\mu,w}(t^{\mp}_{\lambda,\mu}(w))},$$

therefore, $\frac{\partial}{\partial\lambda} t^{+}_{\lambda,\mu}(w) > 0$ and $\frac{\partial}{\partial\lambda} t^{-}_{\lambda,\mu}(w) < 0$.

(ii) Indeed,

$$\frac{\partial}{\partial \lambda} J^{\mp}_{\lambda,\mu}(w) = -\int uv;$$

therefore, $J^{\mp}_{\lambda,\mu}$ is decreasing. $\square$

**Proposition 4.5.** *For each* $\mu < \lambda_1$*, there exists* $w \in \mathcal{N}^{+}_{\lambda^*(\mu),\mu}$ *and* $\bar{w} \in \mathcal{N}^{-}_{\lambda^*(\mu),\mu}$ *such that* $\hat{J}^{+}_{\lambda^*(\mu),\mu} = J^{+}_{\lambda^*(\mu),\mu}(w)$ *and* $\hat{J}^{-}_{\lambda^*(\mu),\mu} = J^{-}_{\lambda^*(\mu),\mu}(\bar{w})$.

*Proof.* Take $\lambda_n \uparrow \lambda^*(\mu)$ and $w_n := (u_n, v_n) \in \mathcal{N}^{-}_{\lambda_n,\mu}$ with $\hat{J}^{-}_{\lambda_n,\mu} = J_{\lambda_n,\mu}(w_n)$. From Lemma 2.1 we have

$$\begin{gathered}-\Delta u_n - \mu u_n - \lambda^*(\mu) v_n - f(x)|u_n|^{p-2}u_n = 0,\\ -\Delta v_n - \mu v_n - \lambda^*(\mu) u_n + g(x)|v_n|^{q-2}v_n = 0,\end{gathered}$$

for each $n$. Using similar arguments to those in Proposition 3.1 and Lemma 3.3, we can show that there exist constants $C, c > 0$ such that $c \leq \|w_n\| \leq C$. We can suppose without loss generality that $w_n \rightharpoonup w := (u, v)$ in $\mathcal{X}$ and $w_n \to w$ in $L^p(\Omega) \times L^q(\Omega)$. Hence $w_n \to w \neq 0$ in $\mathcal{X}$ and we conclude that

$$\begin{gathered}-\Delta u - \mu u - \lambda^*(\mu) v - f(x)|u|^{p-2}u = 0,\\ -\Delta v - \mu v - \lambda^*(\mu) u + g(x)|v|^{q-2}v = 0,\end{gathered} \tag{4.2}$$

We claim that $w \in \mathcal{N}^{-}_{\lambda^*(\mu),\mu}$. If not, then $w \in \mathcal{N}^{0}_{\lambda^*(\mu),\mu}$ and from Proposition 2.6,

$$\begin{gathered}2\left(-\Delta u - \mu u - \lambda^*(\mu) v\right) - pf(x)|u|^{p-2}u = 0,\\ 2\left(-\Delta v - \mu v - \lambda^*(\mu) u\right) + q|v|^{q-2}g(x)v = 0.\end{gathered} \tag{4.3}$$

From (4.2) and (4.3) we have

$$\begin{gathered}(2-p)f(x)|u|^{p-2}u = 0,\\ (2+q)g(x)|v|^{q-2}v = 0,\end{gathered} \tag{4.4}$$

which implies $w = 0$, an absurd. Therefore $w \in \mathcal{N}^{-}_{\lambda^*(\mu),\mu}$ and hence $J^{-}_{\lambda^*(\mu),\mu}(w) \geq \hat{J}^{-}_{\lambda^*(\mu),\mu}$. To conclude the proof we need to show that $J^{-}_{\lambda^*(\mu),\mu}(w) = \hat{J}^{-}_{\lambda^*(\mu),\mu}$ so suppose on the contrary that $J^-_{\lambda^*(\mu),\mu}(w) > \hat{J}^-_{\lambda^*(\mu),\mu}$. Given $\varepsilon > 0$ there exists $z \in \mathcal{N}^-_{\lambda^*(\mu),\mu}$ such that

$$0 < J^-_{\lambda^*(\mu),\mu}(z) - \hat{J}^-_{\lambda^*(\mu),\mu} < \varepsilon. \tag{4.5}$$

From Proposition 4.4 we can also find $N > 0$ such that

$$0 < J^-_{\lambda_n,\mu}(z) - J^-_{\lambda^*(\mu),\mu}(z) < \varepsilon, \quad \forall\ n > N. \tag{4.6}$$

From (4.5) and (4.6) we conclude that

$$\begin{aligned} \hat{J}^-_{\lambda_n,\mu} &= J^-_{\lambda^*(\mu),\mu}(w) + o(1) > \hat{J}^-_{\lambda^*(\mu),\mu} + o(1) \\ &> J^-_{\lambda_n,\mu}(z) - 2\varepsilon + o(1) \ge \hat{J}^-_{\lambda_n,\mu} - 2\varepsilon + o(1), \end{aligned}$$

which is a contradiction and hence $J^-_{\lambda^*(\mu),\mu}(w) = \hat{J}^-_{\lambda^*(\mu),\mu}$. A similar proof can be carried out for $\hat{J}^+_{\lambda^*}$. $\square$

*Proof of Theorem 1.1.* From Propositions 4.3 and 4.5, it follows that there exist $w_{\lambda,\mu} := (u_{\lambda,\mu}, v_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$ and $\bar{w}_{\lambda,\mu} := (\bar{u}_{\lambda,\mu}, \bar{v}_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$, such that $\hat{J}^+_{\lambda,\mu} = J^+_{\lambda,\mu}(w_{\lambda,\mu})$ and $\hat{J}^-_{\lambda,\mu} = J^-_{\lambda,\mu}(\bar{w}_{\lambda,\mu})$. For simplicity we define $w := w_{\lambda,\mu}$ and $\bar{w} := \bar{w}_{\lambda,\mu}$, then from Lemma 2.1 we have that $w$ and $\bar{w}$ are solutions of problem (1.1).

Let us prove now that $w$ and $\bar{w}$ can be chosen as positive functions. We do it only to $\bar{w}$ since for $w$ the calculations are similar. First, observe that $H_{\lambda,\mu}(|\bar{w}|) \le H_{\lambda,\mu}(\bar{w})$, where $|\bar{w}| := (|\bar{u}|, |\bar{v}|)$. We claim that $H_{\lambda,\mu}(|\bar{w}|) = H_{\lambda,\mu}(\bar{w})$. Suppose on the contrary that $H_{\lambda,\mu}(|\bar{w}|) < H_{\lambda,\mu}(\bar{w})$.

**Case 1:** $\lambda \in (-\infty, \lambda^*(\mu))$. From Proposition 2.2 and since $|\bar{u}| \neq 0$, there exist $t^- := t^-_{\lambda,\mu}(|\bar{w}|) > 0$ such that $t^-|\bar{w}| \in \mathcal{N}^-_{\lambda,\mu}$. Once $H_{\lambda,\mu}(|\bar{w}|) < H_{\lambda,\mu}(\bar{w})$, we have

$$0 = \psi'_{\lambda,\mu,|\bar{w}|}(t^-) < \psi'_{\lambda,\mu,\bar{w}}(t^-),$$

which from Proposition 2.2 implies $t^- < 1$ and in this case $t^+_{\lambda,\mu}(\bar{w})$ is defined; we also have that $t^+_{\lambda,\mu}(\bar{w}) < t^- < 1$. It follows that

$$\begin{aligned} \Phi_{\lambda,\mu}(t^-|\bar{w}|) &= \frac{(t^-)^2}{2} H_{\lambda,\mu}(|\bar{w}|) + \frac{(t^-)^q}{q} \int g|\bar{v}|^q - \frac{(t^-)^p}{p} \int f|\bar{u}|^p \\ &< \frac{(t^-)^2}{2} H_{\lambda,\mu}(\bar{w}) + \frac{(t^-)^q}{q} \int g|\bar{v}|^q - \frac{(t^-)^p}{p} \int f|\bar{u}|^p \\ &= \Phi_{\lambda,\mu}(t^-\bar{w}) < \Phi_{\lambda,\mu}(\bar{w}) = \hat{J}^-_{\lambda,\mu} \end{aligned}$$

which is a contradiction and therefore $H_{\lambda,\mu}(|\bar{w}|) = H_{\lambda,\mu}(\bar{w})$.

**Case 2:** $\lambda = \lambda^*(\mu)$. Indeed, we claim that $\hat{J}^-_{\lambda,\mu} = \Phi_{\lambda,\mu}(\bar{w}) < 0$ so $H_{\lambda,\mu}(\bar{w}) < 0$. If not, then $H_{\lambda,\mu}(\bar{w}) \ge 0$ and by Proposition 2.2 we obtain that $\hat{J}^-_{\lambda,\mu} \ge 0$ which is an absurd. By the definition of $\lambda^*(\mu)$ and Propositions 2.2 and 2.4, there exists $t := t_{\lambda,\mu}(|\bar{w}_{\lambda,\mu}|) > 0$ such that $t|\bar{w}| \in \mathcal{N}^-_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu}$ and hence

$$0 = \psi'_{\lambda,\mu,|\bar{w}|}(t) < \psi'_{\lambda,\mu,\bar{w}}(t).$$

From Proposition 2.2 it follows that $t < 1$. Then

$$\Phi_{\lambda,\mu}(t|\bar{w}|) < \Phi_{\lambda,\mu}(t\bar{w}) < \Phi_{\lambda,\mu}(\bar{w}) = \hat{J}^-_{\lambda,\mu} \tag{4.7}$$

which is a contradiction. Therefore $H_{\lambda,\mu}(|\bar{w}|) = H_{\lambda,\mu}(\bar{w})$ which implies that

$$\begin{aligned} \psi'_{\lambda,\mu,|\bar{w}|}(1) &= \psi'_{\lambda,\mu,\bar{w}}(1) = 0, \\ \psi''_{\lambda\mu,|\bar{w}|}(1) &= \psi''_{\lambda,\mu,\bar{w}}(1) < 0. \end{aligned} \tag{4.8}$$

Therefore we can assume that $w, \bar{w} \geq 0$. Moreover, one can easily see from (1.1) that the functions $u, v, \bar{u}, \bar{v}$ are non-zero. From standard regularity theory we conclude that $u, v, \bar{u}, \bar{v} \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and they are positive everywhere in $\Omega$. □

## References


[1] C. O. Alves, D. C. de Morais Filho, M. A. S. Souto; *On systems of elliptic equations involving subcritical or critical Sobolev exponents*, Nonlinear Anal. **42** (2000), no. 5, Ser. A: Theory Methods, 771–787. MR 1776922.

[2] Henri Berestycki, Italo Capuzzo-Dolcetta, Louis Nirenberg; *Variational methods for indefinite superlinear homogeneous elliptic problems*, NoDEA Nonlinear Differential Equations Appl., **2** (1995), no. 4, 553–572. MR 1356874.

[3] Yuri Bozhkov, Enzo Mitidieri; *Existence of multiple solutions for quasilinear systems via fibering method*, J. Differential Equations, **190** (2003), no. 1, 239–267. MR 1970963.

[4] K. J. Brown; *Spatially inhomogeneous steady state solutions for systems of equations describing interacting populations*, J. Math. Anal. Appl., **95** (1983), no. 1, 251–264. MR 710432.

[5] Ph. Clément, D. G. de Figueiredo, E. Mitidieri; *Positive solutions of semilinear elliptic systems*, Comm. Partial Differential Equations **17** (1992), no. 5-6, 923–940. MR 1177298.

[6] D. G. Costa, C. A. Magalhães; *A variational approach to subquadratic perturbations of elliptic systems*, J. Differential Equations, **111** (1994), no. 1, 103–122. MR 1280617.

[7] Edcarlos D. da Silva; *Multiplicity of solutions for gradient systems using Landesman-Lazer conditions*, Abstr. Appl. Anal., (2010), Art. ID 237826, 22. MR 2720025.

[8] Djairo G. deFigueiredo; *Nonlinear elliptic systems*, An. Acad. Brasil. Ciênc. **72** (2000), no. 4, 453–469. MR 1800060.

[9] Yavdat Ilyasov; *On extreme values of Nehari manifold method via nonlinear Rayleigh's quotient*, Topol. Methods Nonlinear Anal., **49** (2017), no. 2, 683–714. MR 3670482.

[10] Yavdat Ilyasov, Kaye Silva; *On branches of positive solutions for p-Laplacian problems at the extreme value of the Nehari manifold method*, Proc. Amer. Math. Soc., **146** (2018), no. 7, 2925–2935. MR 3787354.

[11] Tiancheng Ouyang; *On the positive solutions of semilinear equations* $\Delta u + \lambda u + hu^p = 0$ *on compact manifolds. II*, Indiana Univ. Math. J., **40** (1991), no. 3, 1083–1141. MR 1129343.

[12] S. I. Pokhozhaev; *The fibration method for solving nonlinear boundary value problems*, Trudy Mat. Inst. Steklov., **192** (1990), 146–163, Translated in Proc. Steklov Inst. Math., **1** 992, no. 3, 157–173, Differential equations and function spaces (Russian). MR 1097896.

[13] Kaye Silva, Abiel Macedo; *Local minimizers over the Nehari manifold for a class of concave-convex problems with sign changing nonlinearity*, J. Differential Equations, **265** (2018), no. 5, 1894–1921. MR 3800105.

[14] Wenming Zou; *Multiple solutions for asymptotically linear elliptic systems*, J. Math. Anal. Appl., **255** (2001), no. 1, 213–229. MR 1813819.



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