Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 101, pp. 1–21.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

MULTIPLE POSITIVE SOLUTIONS TO THE FRACTIONAL KIRCHHOFF PROBLEM WITH CRITICAL INDEFINITE NONLINEARITIES

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Abstract. This article concerns the existence and multiplicity of positive solutions to the fractional Kirchhoff equation with critical indefinite nonlinearities by applying the Nehari manifold approach and fibering maps.

1. Introduction and statement of results

In this paper, we study the existence and multiplicity of positive solutions to the fractional Kirchhoff type problem

$$\begin{gathered} M\Big(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)(-\Delta)^s u = f_\lambda(x)|u|^{q-2}u + g(x)|u|^{2_s^*-2}u, \quad \text{in } \Omega, \\ u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with the Lipschitz boundary $\partial\Omega$, dimension $N > 2s$ with $s \in (0,1)$, $2_s^* = \frac{2N}{N-2s}$ is the fractional critical Sobolev exponent and $0 < s < 1 < q < \min\{2, \frac{N}{N-2s}\} < \infty$. Here, $M(t) = a + bt^{m-1}$ with $m > 1$, $a, b > 0$, $f_\lambda \in L^{q^*}(\Omega)$, $q^* = \frac{2_s^*}{2_s^*-q}$, $f_\lambda = \lambda f_+ - f_-$ with $\lambda > 0$, and $f_\pm = \max\{\pm f, 0\}$ and $g \in L^\infty(\Omega)$. Furthermore, g satisfies the condition

(A1) $g(x) = \max_{x\in\bar\Omega} g(x) \equiv 1$ in $B_\rho(0)$ for some $\rho > 0$.

We denote by $(-\Delta)^s$ the usual fractional Laplacian operator which is defined (up to normalization factors) as follows (see for instance [18] and the references therein for further details on the fractional Laplacian) by

$$(-\Delta)^s u(x) = 2\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy, \tag{1.2}$$

where P.V. stands for the principle value.

When $M(t) \equiv 1$, $\lambda = 1$ and $s = 1$, equation (1.1) can be reduced to the semilinear elliptic problem

$$\begin{gathered} -\Delta u = f(x)|u|^{q-2} + g(x)|u|^{2^*-2}u, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.3}$$

2010 *Mathematics Subject Classification.* 35A15, 35B33, 35R11.
Key words and phrases. Fractional Kirchhoff equation; Nehari manifold; fibering maps.

Submitted June 12, 2020. Published September 28, 2020.

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, $1 < q < 2$, and the weight functions f, g are continuous and sign-changing. By using the Nehari manifold, fibering maps and Ljusternik-Schnirelmann category, Wu [25] proved that there existed at least three positive solutions of (1.3). Xie-Chen [26] presented a multiplicity result on the Kirchhoff-type problems in the bounded domain by using a similar strategy. A number of works dealt with the fractional differential equations [3, 6, 7, 11, 21] and some recent results on problem (1.3) can be seen in [4, 5, 10, 12, 13, 14, 15, 16, 22, 23, 27] and the references therein.

As we know, the variational problems involving fractional and nonlocal operators are much more complicated and challenging. In the last decade, considerable attention focused on the fractional Laplacian operator and nonlocal operator. We refer to [19] for the Brezis-Nirenberg type results for the following elliptic equation involving the fractional Laplacian $(-\Delta)^s (0 < s < 1)$ in a bounded domain,

$$\begin{gathered}(-\Delta)^s u = \lambda u + |u|^{2_s^*-2}u, \quad x \in \Omega,\\ u = 0, \quad x \in \partial\Omega,\end{gathered}$$

where $\lambda > 0$, $s \in (0,1)$ is fixed, $2_s^* = \frac{2N}{N-2s}$, $\Omega \subset \mathbb{R}^N (N > 2s)$ is open, bounded and with the Lipschitz boundary, and $(-\Delta)^s$ is the fractional Laplace operator. The classical Brezis-Nirenberg result was generalized to the case of nonlocal fractional operators through variational techniques. The existence of multiple solutions to the fractional Laplacian equations of Kirchhoff type was considered in [17] and two positive solutions for proper selection of positive parameter λ was obtained.

The main purpose of this article is to establish the existence and multiplicity of positive solutions to problem (1.1) with the critical growth and sign-changing weight functions. Our results encompass and improve the corresponding results presented in [26] for the fractional Kirchhoff type equations involving the critical growth.

The energy functional associated with problem (1.1) is

$$\begin{aligned}I_\lambda(u) = \frac{a}{2}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \frac{b}{2m}\Big(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^m\\ - \frac{1}{q}\int_\Omega f_\lambda(x)|u|^q dx - \frac{1}{2_s^*}\int_\Omega g(x)|u|^{2_s^*} dx\end{aligned}$$

for $u \in H_0^s(\Omega)$. We can prove that $I_\lambda \in C^1(H_0^s(\Omega), \mathbb{R})$ and a critical point of I_λ in $H_0^s(\Omega)$ corresponds to a weak solution of problem (1.1). We summarize our main results as follows.

Theorem 1.1. *Assume that $m < \frac{N}{N-2s}$, $f_\pm \not\equiv 0$ and condition* (A1) *holds. Then there exist $0 < \Lambda_* \leq \Lambda_0$ and $\bar{b} > 0$ such that*

- (i) *for any $\lambda \in (0, \Lambda_0)$, problem* (1.1) *admits at least one positive solution u_1 with $I_\lambda(u_1) < 0$, and u_1 is a ground state solution;*
- (ii) *for any $\lambda \in (0, \Lambda_*)$ and $b \in (0, \bar{b})$, problem* (1.1) *admits at least two positive solutions u_1 and u_2 satisfying $I_\lambda(u_1) < 0 < I_\lambda(u_2)$, and u_1 is a ground state solution.*

Theorem 1.2. *Assume that $m = \frac{N}{N-2s}$, $f_\pm \not\equiv 0$ and condition* (A1) *holds. Then the following two statements hold:*

- (i) *For $b \geq 1/S^m$ and any $\lambda > 0$, problem* (1.1) *admits at least one positive solution.*

(ii) *For $b < 1/S^m$, there exist $0 < \tilde{\Lambda}_* \le \Lambda_0$ and $\tilde{b} > 0$ such that*
   (1) *for any $\lambda \in (0, \Lambda_0)$, problem (1.1) admits at least one positive solution;*
   (2) *for any $\lambda \in (0, \tilde{\Lambda}_*)$ and $b \in (0, \tilde{b})$, problem (1.1) admits at least two positive solutions u_1 and u_2 satisfying $I_\lambda(u_1) < 0 < I_\lambda(u_2)$, and u_1 is a ground state solution.*

Theorem 1.3. *Assume that $m > \frac{N}{N-2s}$, $f_- \equiv 0$, and condition (A1) holds. Then there exist $b^*, \Lambda^* > 0$ such that for any $b \in (0, b^*)$ and $\lambda \in (0, \Lambda^*)$, problem (1.1) admits at least three positive solutions $u_b, u_\lambda, u_{\lambda,b}$ with*

$$I_\lambda(u_\lambda) < I_\lambda(u_b) < 0 < I_\lambda(u_{\lambda,b}),$$

and u_λ is a ground state solution.

Note that the corresponding results in [26] are generalized to the nonlocal fractional Kirchhoff problem and the existence results are extended in the sense that the restriction on the Kirchhoff coefficient M is eliminated.

When $g(x) \equiv 1$, by Theorems 1.1 and 1.2, we obtain the existence and multiplicity of positive solutions to the problem

$$M\Big(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)(-\Delta)^s u = f_\lambda(x)|u|^{q-2}u + |u|^{2_s^*-2}u, \quad \text{in } \Omega,$$
$$u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

where $M(t) = a + bt^{m-1}$ with $a, b > 0$ for $t \ge 0$ and $m \in [1, 2_s^*/2]$, which generalizes [17, Theorem 1.1].

In view of [2, 5], problem (1.1) appears more complicated because of the lack of compactness and the nonlocal nature of the fractional Laplacian. Theorems 1.1–1.3 can be regarded as generalizations of [26] for fractional Laplacian operators.

The rest of this paper is organized as follows. In Section 2, we present mathematical notation and technical lemmas. We prove Theorems 1.1 and 1.2 in Section 3, and prove Theorem 1.3 in Section 4.

2. Preliminary results

In this section, we introduce some notation, definitions and useful lemmas which will be used in the proofs of main results. We define the Hilbert space $H^s(\mathbb{R}^N)$ by

$$H^s(\mathbb{R}^N) := \big\{u \in L^2(\mathbb{R}^N) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N+2s}{2}}} \in L^2\left(\mathbb{R}^N \times \mathbb{R}^N\right)\big\}$$

endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{1/2}, \tag{2.1}$$

where the term

$$[u]_s = \Big(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{1/2}$$

is the so-called Gagliardo semi-norm of u. In view of (1.2) and [18, Proposition 3.6], we have

$$\|(-\Delta)^{s/2}u\|_2^2 = \frac{1}{C_s}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy,$$

where C_s is a positive constant depending on s. We define $\mathcal{D}^{s,2}(\mathbb{R}^N)$ as the closure of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with the norm

$$\|u\|_{\mathcal{D}^{s,2}} = \Big(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 dx\Big)^{1/2}.$$

Then $\mathcal{D}^{s,2}(\mathbb{R}^N)$ is continuously embedded into $L^{2_s^*}(\mathbb{R}^N)$. As in [7, Theorem 1.1], let S be the best constant of the fractional Sobolev embedding $\mathcal{D}^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ defined by

$$S = \inf_{u\in\mathcal{D}^{s,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy}{\big(\int_{\mathbb{R}^N} |u|^{2_s^*} dx\big)^{2/2_s^*}}, \tag{2.2}$$

which is well-defined and strictly positive.

We define

$$E_0 = \{u \in H^s(\mathbb{R}^N) : u = 0 \quad \text{a.e. in } \mathbb{R}^N \setminus \Omega\}$$

with the norm

$$\|u\|_{E_0} = \Big(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{1/2},$$

which is equivalent to (2.1) [19, 20]. The embedding $E_0 \hookrightarrow L^r(\Omega)$ is continuous for any $r \in [1, 2_s^*]$ and compact whenever $r \in [1, 2_s^*)$. We recall that $(E_0, \|\cdot\|_{E_0})$ is a Hilbert space with the inner product defined by

$$\langle u, v\rangle = \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy.$$

For simplicity, we will just denote $\|\cdot\|_{E_0}$ and $\|\cdot\|_{L^p(\Omega)}$ by $\|\cdot\|$ and $|\cdot|_p$, respectively. Throughout this paper, the letters $C, C_i,\ i = 1, 2, \dots$ denote positive constants which may vary from line to line but independent of the associated terms and parameters.

As we see, I_λ is of class C^1 in E_0 and for any $v \in E_0$ it holds

$$\begin{aligned}\langle I_\lambda'(u), v\rangle =& M(\|u\|^2)\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy\\ &- \int_\Omega f_\lambda(x)|u|^{q-2}uv dx - \int_\Omega g(x)|u|^{2_s^*-2}uv dx.\end{aligned}$$

Define the Nehari manifold associated with I_λ by

$$N_\lambda = \{u \in E_0 \setminus \{0\} : \langle I_\lambda'(u), u\rangle = 0\}.$$

It is well-known that the Nehari manifold is closely related to the behavior of the fibering map $\phi_u : t \in \mathbb{R}^+ \to I_\lambda(tu)$ [2, 8]. Thus, we have

$$\begin{aligned}\phi_u'(t) &= at\|u\|^2 + bt^{2m-1}\|u\|^{2m} - t^{q-1}\int_\Omega f_\lambda(x)|u|^q dx - t^{2_s^*-1}\int_\Omega g(x)|u|^{2_s^*} dx,\\ \phi_u''(t) &= a\|u\|^2 + (2m-1)bt^{2m-2}\|u\|^{2m}\\ &\quad - (q-1)t^{q-2}\int_\Omega f_\lambda(x)|u|^q dx - (2_s^*-1)t^{2_s^*-2}\int_\Omega g(x)|u|^{2_s^*} dx.\end{aligned}$$

Then $u \in N_\lambda$ if and only if $\phi_u'(1) = 0$. Moreover, for $u \in N_\lambda$ we have

$$\phi_u''(1) = a(2-q)\|u\|^2 + b(2m-q)\|u\|^{2m} - (2_s^*-q)\int_\Omega g(x)|u|^{2_s^*} dx, \tag{2.3}$$

or

$$\phi_u''(1) = a(2-2_s^*)\|u\|^2 + b(2m-2_s^*)\|u\|^{2m} - (q-2_s^*)\int_\Omega f_\lambda(x)|u|^q dx. \tag{2.4}$$

We split N_λ into three parts:

$$N_\lambda^+ = \{u \in N_\lambda | \phi_u''(1) > 0\},$$
$$N_\lambda^- = \{u \in N_\lambda | \phi_u''(1) < 0\},$$
$$N_\lambda^0 = \{u \in N_\lambda | \phi_u''(1) = 0\},$$

and define

$$H^+ = \{u \in E_0 | \int_\Omega f_\lambda(x)|u|^q dx > 0\}, \quad H^- = \{u \in E_0 | \int_\Omega f_\lambda(x)|u|^q dx \le 0\},$$
$$G^+ = \{u \in E_0 | \int_\Omega g(x)|u|^{2_s^*} dx > 0\}, \quad G^- = \{u \in E_0 | \int_\Omega g(x)|u|^{2_s^*} dx \le 0\}.$$

In view of $m \le \frac{N}{N-2s}$ and following [17, Lemma 3.2], we can derive the following lemma immediately.

Lemma 2.1. *If u is a minimizer of I_λ on N_λ such that $u \notin N_\lambda^0$, then $I_\lambda'(u) = 0$ in E_0^{-1}.*

Lemma 2.2. *For any $\lambda > 0$, the functional I_λ is coercive and bounded below on N_λ.*

Proof. For $u \in N_\lambda$, from (2.2) and Hölder's inequality, we have

$$\begin{aligned} I_\lambda(u) &= I_\lambda(u) - \frac{1}{2_s^*}\langle I_\lambda'(u), u\rangle \\ &= \Big(\frac{1}{2} - \frac{1}{2_s^*}\Big)a\|u\|^2 + \Big(\frac{1}{2m} - \frac{1}{2_s^*}\Big)b\|u\|^{2m} - \Big(\frac{1}{q} - \frac{1}{2_s^*}\Big)\int_\Omega f_\lambda(x)|u|^q dx \\ &\ge \Big(\frac{1}{2} - \frac{1}{2_s^*}\Big)a\|u\|^2 - \Big(\frac{1}{q} - \frac{1}{2_s^*}\Big)\lambda|f_+|_{q^*} S^{-q/2}\|u\|^q. \end{aligned}$$

Recalling that $1 < q < 2$, we obtain that I_λ is coercive and bounded below on N_λ. $\square$

Let

$$\lambda_1 = \Big[\frac{a(2-q)}{2_s^*-q}\Big]^{\frac{2-q}{2_s^*-2}} \frac{a(2_s^*-2)S^{\frac{2_s^*-q}{2_s^*-2}}}{(2_s^*-q)|f_+|_{q^*}}. \tag{2.5}$$

Lemma 2.3. *There exists $\lambda_1 > 0$ such that $N_\lambda^0 = \emptyset$ for $\lambda \in (0, \lambda_1)$.*

Proof. By contradiction assume that for some $\lambda \in (0, \lambda_1)$, there is a function $u \in N_\lambda^0$. Then from (2.3) and (2.4), we have

$$a(2-q)\|u\|^2 + b(2m-q)\|u\|^{2m} - (2_s^*-q)\int_\Omega g(x)|u|^{2_s^*} dx = 0, \tag{2.6}$$

$$a(2-2_s^*)\|u\|^2 + b(2m-2_s^*)\|u\|^{2m} - (q-2_s^*)\int_\Omega f_\lambda(x)|u|^q dx = 0. \tag{2.7}$$

It follows from (A1), (2.6) and (2.2) that

$$\|u\|^2 \le \frac{2_s^*-q}{a(2-q)}|u|_{2_s^*}^{2_s^*} \le \frac{2_s^*-q}{a(2-q)} S^{-\frac{2_s^*}{2}}\|u\|^{2_s^*}. \tag{2.8}$$

Similarly, from (2.2), (2.7) and Hölder's inequality, we can deduce that

$$\|u\|^2 \le \frac{2_s^*-q}{a(2_s^*-2)}\lambda\int_\Omega f_+|u|^q dx \le \frac{2_s^*-q}{a(2_s^*-2)}\lambda|f_+|_{q^*}S^{-q/2}\|u\|^q. \tag{2.9}$$

Combining (2.8) and (2.9) yields

$$\Big[\frac{a(2-q)}{2_s^*-q}S^{\frac{2_s^*}{2}}\Big]^{\frac{1}{2_s^*-2}} \le \|u\| \le \Big[\frac{(2_s^*-q)\lambda|f_+|_{q^*}S^{-q/2}}{a(2_s^*-2)}\Big]^{\frac{1}{2-q}}.$$

Therefore,

$$\lambda \ge \Big[\frac{a(2-q)}{2_s^*-q}\Big]^{\frac{2-q}{2_s^*-2}}\frac{a(2_s^*-2)S^{\frac{2_s^*-q}{2_s^*-2}}}{(2_s^*-q)|f_+|_{q^*}} = \lambda_1.$$

This is a contradiction. $\square$

We define

$$\lambda_2 = \begin{cases}\lambda_1, & m < \frac{N}{N-2s},\\ \big(\frac{1}{1-bS^m}\big)^{\frac{2-q}{2_s^*-2}}\lambda_1, & m = \frac{N}{N-2s},\ b < 1/S^m.\end{cases} \tag{2.10}$$

The lemma below shows that the component sets N_λ^+ and N_λ^- are nonempty.

Lemma 2.4. *Assume $m < \frac{N}{N-2s}$. Then the following two statements are true.*

(i) *For any $u \in G^+ \cap H^+$ and $\lambda \in (0, \lambda_2)$, there exist $0 < t^+ = t^+(u) < t_{\max} < t^- = t^-(u)$ such that $t^+u \in N_\lambda^+$, $t^-u \in N_\lambda^-$ and*

$$I_\lambda(t^+u) = \inf_{0\le t\le t^-} I_\lambda(tu), \quad I_\lambda(t^-u) = \sup_{t\ge t_{\max}} I_\lambda(tu).$$

(ii) *For any $u \in G^+ \cap H^-$ and $\lambda > 0$, there exists a unique $t^- = t^-(u) > t_{\max}$ such that $t^-u \in N_\lambda^-$ and*

$$I_\lambda(t^-u) = \sup_{t\ge 0} I_\lambda(tu).$$

Proof. Fix $u \in E_0 \setminus \{0\}$ and define $\psi_u(t) : \mathbb{R}^+ \to \mathbb{R}$ by

$$\psi_u(t) = at^{2-q}\|u\|^2 + bt^{2m-q}\|u\|^{2m} - t^{2_s^*-q}\int_\Omega g(x)|u|^{2_s^*}dx. \tag{2.11}$$

We remark that $tu \in N_\lambda$ if and only if $\psi_u(t) = \int_\Omega f_\lambda|u|^q dx$.

(i) Let $u \in G^+ \cap H^+$. From (2.11), it is easy to check that

$$\psi_u(0) = 0,\ \lim_{t\to\infty}\psi_u(t) = -\infty,\ \lim_{t\to 0^+}\psi_u'(t) > 0 \text{ and } \lim_{t\to\infty}\psi_u'(t) < 0.$$

Define $\psi_u'(t) = t^{1-q}h_u(t)$, where

$$h_u(t) = a(2-q)\|u\|^2 + (2m-q)bt^{2m-2}\|u\|^{2m} - (2_s^*-q)t^{2_s^*-2}\int_\Omega g(x)|u|^{2_s^*}dx.$$

Then, there exists a unique $t_0 > 0$ such that $h_u'(t_0) = 0$, where

$$t_0 = \Big(\frac{(2m-q)(2m-2)b\|u\|^{2m}}{(2_s^*-q)(2_s^*-2)\int_\Omega g(x)|u|^{2_s^*}dx}\Big)^{\frac{1}{2_s^*-2m}}.$$

From $m < \frac{N}{N-2s}$ it follows that $\lim_{t\to 0^+} h_u(t) > 0$ and $\lim_{t\to\infty} h_u(t) = -\infty$. This implies that there is a unique $t_{\max} > t_0$ such that $h_u(t_{\max}) = 0$. Hence, $\psi_u'(t) >$

0 for $t \in (0, t_{\max})$, $\psi_u'(t) < 0$ for $t \in (t_{\max}, \infty)$ and $\psi_u'(t_{\max}) = 0$. Moreover, $\psi_u(t_{\max}) = \max_{t>0} \psi_u(t) \geq \max_{t>0} \bar{\psi}_u(t)$, where

$$\bar{\psi}_u(t) = at^{2-q}\|u\|^2 - t^{2_s^*-q} \int_\Omega g(x)|u|^{2_s^*} dx.$$

From (2.2) it follows that

$$\begin{aligned}
\max_{t>0} \bar{\psi}_u(t) &= \|u\|^q \frac{a(2_s^*-2)}{2_s^*-q} \Big(\frac{(2-q)a\|u\|^{2_s^*}}{(2_s^*-q)\int_\Omega g(x)|u|^{2_s^*} dx} \Big)^{\frac{2-q}{2_s^*-2}} \\
&\geq \|u\|^q \frac{a(2_s^*-2)}{2_s^*-q} \Big(\frac{(2-q)aS^{\frac{2_s^*}{2}}}{2_s^*-q} \Big)^{\frac{2-q}{2_s^*-2}}.
\end{aligned}$$

For $u \in H^+$, it holds

$$\psi_u(0) = 0 < \int_\Omega f_\lambda(x)|u|^q dx \leq \lambda \int_\Omega f_+|u|^q dx \leq \lambda |f_+|_{q^*} S^{-q/2}\|u\|^q.$$

So, if

$$\lambda < \lambda_1 = \frac{a(2_s^*-2)S^{\frac{2_s^*-q}{2_s^*-2}}}{(2_s^*-q)|f_+|_{q^*}} \Big(\frac{(2-q)a}{2_s^*-q} \Big)^{\frac{2-q}{2_s^*-2}},$$

there exist unique $t^+ = t^+(u) < t_{\max}$ and $t^- = t^-(u) > t_{\max}$ such that

$$\psi_u(t^+) = \int_\Omega f_\lambda(x)|u|^q dx = \psi_u(t^-), \quad \psi_u'(t^+) > 0, \quad \psi_u'(t^-) < 0,$$

which implies t^+u, $t^-u \in N_\lambda$. According to $\phi_u''(1) = t^{q+1}\psi_u'(t)$, we can deduce that $t^+u \in N_\lambda^+$ and $t^-u \in N_\lambda^-$. Since $\phi_u'(t) = t^{q-1}\big(\psi_u(t) - \int_\Omega f_\lambda(x)|u|^q dx\big)$, it is clear that $\phi_u'(t) < 0$ for $t \in [0, t^+)$ and $\phi_u'(t) > 0$ for $t \in (t^+, t^-)$. This indicates that $I_\lambda(t^+u) = \inf_{0\leq t\leq t^-} I_\lambda(tu)$.

Similarly, from $\phi_u'(t) > 0$ for $t \in (t^+, t^-)$ and $\phi_u'(t) < 0$ for $t \in (t^-, \infty)$, we can obtain $I_\lambda(t^-u) = \sup_{t\geq t_{\max}} I_\lambda(tu)$.

(ii) The proof is essentially the same as that in Part (i), so we omit it. $\square$

As in Lemma 2.4, we can deduce the following two lemmas.

Lemma 2.5. *Assume that $m = \frac{N}{N-2s}$ and $b \geq 1/S^m$. Then for any $u \in H^+$, there exists a unique $0 < t^+ < t_{\max}$ such that $t^+u \in N_\lambda$ and $I_\lambda(t^+u) = \inf_{t\geq 0} I_\lambda(tu)$.*

Lemma 2.6. *Assume that $m = \frac{N}{N-2s}$ and $b < 1/S^m$. Then the following two statements are true.*

(i) *For any $u \in H^+$ and $\lambda \in (0, \lambda_2)$, there exist $0 < t^+ = t^+(u) < t_{\max} < t^- = t^-(u)$ such that $t^+u \in N_\lambda^+$, $t^-u \in N_\lambda^-$ and*
$$I_\lambda(t^+u) = \inf_{0\leq t\leq t^-} I_\lambda(tu), \quad I_\lambda(t^-u) = \sup_{t\geq t_{\max}} I_\lambda(tu).$$

(ii) *For any $u \in H^-$ and $\lambda > 0$, there exists a unique $t^- = t^-(u) > t_{\max}$ such that $t^-u \in N_\lambda^-$ and*
$$I_\lambda(t^-u) = \sup_{t\geq 0} I_\lambda(tu).$$

Lemma 2.7. *Assume $\lambda \in (0, \lambda_1)$. Then for any $u \in N_\lambda^+$ and $v \in N_\lambda^-$, there exist $B_0 > B_\lambda > 0$ such that $\|v\| > B_0 > B_\lambda > \|u\|$.*

Proof. Let $u \in N_\lambda^+ \subset N_\lambda$. In view of (2.2) and (2.4), it follows from Hölder's inequality that

$$a(2_s^*-2)\|u\|^2 < (2_s^*-q)\int_\Omega f_\lambda(x)|u|^q dx \le (2_s^*-q)\lambda S^{-q/2}|f_+|_{q^*}\|u\|^q.$$

Then

$$\|u\| < \Big(\frac{(2_s^*-q)\lambda S^{-q/2}|f_+|_{q^*}}{a(2_s^*-2)}\Big)^{\frac{1}{2-q}} = B_\lambda.$$

Similarly, if $v \in N_\lambda^- \subset N_\lambda$, from (2.3) and (A1) we have

$$a(2-q)\|v\|^2 < (2_s^*-q)\int_\Omega g(x)|v|^{2_s^*}dx \le (2_s^*-q)S^{-2_s^*/2}\|v\|^{2_s^*}.$$

Hence, we have

$$\|v\| > \Big(\frac{a(2-q)S^{\frac{2_s^*}{2}}}{2_s^*-q}\Big)^{\frac{1}{2_s^*-2}} = B_0.$$

By a direct calculation, we can verify that $B_0 > B_\lambda$ for $\lambda \in (0,\lambda_1)$, where λ_1 is given in (2.5). ☐

Corollary 2.8 ([11])**.** *For any $\lambda \in (0,\lambda_1)$, N_λ^- is a closed set in E_0 topology.*

3. Proof of Theorems 1.1 and 1.2

In this section, we discuss the existence and multiplicity of solutions to problem (1.1) when $m \le \frac{N}{N-2s}$. From Lemmas 2.3, 2.4 and 2.6, if $m < \frac{N}{N-2s}$ or $m = \frac{N}{N-2s}$, and $b < 1/S^m$ holds for any $\lambda \in (0,\lambda_1)$, then $N_\lambda = N_\lambda^+ \cup N_\lambda^-$. Now, we study the infimum of I_λ on the $N_\lambda^\pm$ by defining $c_\lambda^\pm = \inf_{N_\lambda^\pm} I_\lambda(u)$ and $\lambda_3 = \frac{q}{2}\lambda_1$.

Lemma 3.1. *Assume that $m < \frac{N}{N-2s}$ or $m = \frac{N}{N-2s}$, and $b < 1/S^m$. Then*

(i) *for any $\lambda \in (0,\lambda_1)$, we have $c_\lambda^+ = \inf_{u\in N_\lambda^+} I_\lambda(u) < 0$;*

(ii) *for any $\lambda \in (0,\lambda_3)$, we have $c_\lambda^- \ge \alpha > 0$. In particular, if $\lambda \in (0,\lambda_1)$, then*

$$c_\lambda^+ = \inf_{u\in N_\lambda} I_\lambda(u).$$

Proof. (i) For $u \in N_\lambda^+$, it follows from (2.4) that

$$\int_\Omega f_\lambda(x)|u|^q dx \ge \Big(\frac{2_s^*-2}{2_s^*-q}\Big)a\|u\|^2 + \Big(\frac{2_s^*-2m}{2_s^*-q}\Big)b\|u\|^{2m}. \tag{3.1}$$

By (3.1), we obtain

$$\begin{aligned}
c_\lambda^+ &\le I_\lambda(u) - \frac{1}{2_s^*}\langle I_\lambda'(u),u\rangle\\
&= \Big(\frac12 - \frac{1}{2_s^*}\Big)a\|u\|^2 + \Big(\frac{1}{2m} - \frac{1}{2_s^*}\Big)b\|u\|^{2m} - \Big(\frac1q - \frac{1}{2_s^*}\Big)\int_\Omega f_\lambda(x)|u|^q dx\\
&\le -\Big(\frac1q - \frac12\Big)\Big(1 - \frac{2}{2_s^*}\Big)a\|u\|^2 - \Big(\frac1q - \frac{1}{2m}\Big)\Big(1 - \frac{2m}{2_s^*}\Big)b\|u\|^{2m}\\
&< 0.
\end{aligned}$$

(ii) For $u \in N_\lambda^-$, applying Lemma 2.7 and $\lambda \in (0,\lambda_3)$, we deduce

$$I_\lambda(u) = \Big(\frac12 - \frac{1}{2_s^*}\Big)a\|u\|^2 + \Big(\frac{1}{2m} - \frac{1}{2_s^*}\Big)b\|u\|^{2m} - \Big(\frac1q - \frac{1}{2_s^*}\Big)\int_\Omega f_\lambda(x)|u|^q dx$$

$$
\begin{aligned}
&\geq \|u\|^q \Big[\frac{as}{N}\Big(\frac{a(2-q)}{2_s^*-q}S^{\frac{2_s^*}{2}}\Big)^{\frac{2-q}{2_s^*-2}} - \lambda\Big(\frac{1}{q}-\frac{1}{2_s^*}\Big)|f_+|_{q^*}S^{-q/2}\Big] \\
&\geq \frac{(2_s^*-q)|f_+|_{q^*}\|u\|^q}{2_s^* q S^{\frac{q}{2}}}(\lambda_3-\lambda) \\
&\geq \alpha > 0.
\end{aligned}
$$

□

Lemma 3.2. *For each $u \in N_\lambda^{\pm}$ and $\lambda \in (0,\lambda_1)$, there is a number ϵ and a differentiable function $\zeta : B(0,\epsilon) \subseteq E \to \mathbb{R}$ such that $\zeta(0)=1$, the function $\zeta(v)(u-v) \in N_\lambda^{\pm}$, and*

$$
\begin{aligned}
&\langle \zeta'(0), v\rangle \\
&= \frac{2a\langle u,v\rangle + 2mb\|u\|^{2(m-1)}\langle u,v\rangle - q\int_\Omega f_\lambda|u|^{q-2}uv dx - 2_s^*\int_\Omega g|u|^{2_s^*-2}uv dx}{(2-q)a\|u\|^2 + (2m-q)b\|u\|^{2m} - (2_s^*-q)\int_\Omega g|u|^{2_s^*}dx},
\end{aligned}
$$

where

$$
\langle u,v\rangle = \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy
$$

for $v \in B_\epsilon(0) = \{v \in E_0 : \|v\| \leq \epsilon\}$.

The proof of the above lemma is similar to that of [11, Lemma 3.4], we omit it here.

Lemma 3.3. *Assume that $\lambda \in (0,\lambda_1)$. Then there exists a minimizing sequence $\{u_k\} \subset N_\lambda$ such that*

$$
I_\lambda(u_k) \to c_\lambda \quad \text{and} \quad \|I'_\lambda(u_k)\|_{E_0^{-1}} \to 0 \quad \text{as } k \to \infty \tag{3.2}
$$

with $c_\lambda = \inf_{u\in N_\lambda} I_\lambda(u)$.

Proof. It follows form Lemma 2.2 and the Ekeland's variational principle [9] that there exists a minimizing sequence $\{u_k\} \subset N_\lambda$ such that

$$
c_\lambda < I_\lambda(u_k) < c_\lambda + \frac{1}{k}, \tag{3.3}
$$

$$
I_\lambda(u_k) < I_\lambda(u) + \frac{1}{k}\|u-u_k\|, \quad u \in N_\lambda. \tag{3.4}
$$

From (3.3) and Lemma 2.2, we have $\sup_k \|u_k\| < \infty$. Now, we claim that $\|I'_\lambda(u_k)\|_{E_0^{-1}} \to 0$ as $k \to \infty$. From Lemma 3.2, we know the differentiable functions $\zeta_k : B_{\epsilon_k}(0) \to \mathbb{R}$ for some $\epsilon_k > 0$ such that $\zeta_k(v)(u_k - v) \in N_\lambda$ for $v \in B_{\epsilon_k}(0)$. For a fixed k, we take $0 < \varrho < \epsilon_k$ and define $v_\varrho = \varrho u/\|u\|$ with $u \in E_0, u \not\equiv 0$ and $\omega_\varrho = \zeta_k(v_\varrho)(u_k - v_\varrho)$. Then it is easy to see that $\omega_\varrho \in N_\lambda$. By (3.4), we can deduce that

$$
I_\lambda(\omega_\varrho) - I_\lambda(u_k) \geq -\frac{1}{k}\|\omega_\varrho - u_k\|,
$$

which implies

$$
\langle I'_\lambda(u_k), \omega_\varrho - u_k\rangle + o_k\left(\|\omega_\varrho - u_k\|\right) \geq -\frac{1}{k}\|\omega_\varrho - u_k\|.
$$

Therefore,

$$
-\langle I'_\lambda(u_k), v_\varrho\rangle + (\zeta_k(v_\varrho) - 1)\langle I'_\lambda(u_k), u_k - v_\varrho\rangle \geq -\frac{1}{k}\|\omega_\varrho - u_k\| + o_k\left(\|\omega_\varrho - u_k\|\right).
$$

Then $\langle I'_\lambda(\omega_\varrho), u_k - v_\varrho\rangle = 0$ yields

$$\begin{aligned}
&-\varrho\langle I'_\lambda(u_k), \frac{u}{\|u\|}\rangle + (\zeta_k(v_\varrho)-1)\langle I'_\lambda(u_k) - I'_\lambda(\omega_\varrho), u_k - v_\varrho\rangle \\
&\geq -\frac{1}{k}\|\omega_\varrho - u_k\| + o_k(\|\omega_\varrho - u_k\|).
\end{aligned}$$

That is,

$$\begin{aligned}
\langle I'_\lambda(u_k), \frac{u}{\|u\|}\rangle &\leq \frac{1}{k\varrho}\|\omega_\varrho - u_k\| + \frac{o_k(\|\omega_\varrho - u_k\|)}{\varrho} \\
&\quad + \frac{(\zeta_k(v_\varrho)-1)}{\varrho}\langle I'_\lambda(u_k) - I'_\lambda(\omega_\varrho), u_k - v_\varrho\rangle.
\end{aligned} \tag{3.5}$$

Since $\|\omega_\varrho - u_k\| \leq \rho|\zeta_k(v_\varrho)| + |\zeta_k(v_\varrho)-1|\|u_k\|$ and $\lim_{\varrho\to 0}\frac{|\zeta_k(v_\varrho)-1|}{\varrho} \leq \|\zeta'_k(0)\|$, taking the limit $\varrho \to 0^+$ in (3.5), we obtain

$$\langle I'_\lambda(u_k), \frac{u}{\|u\|}\rangle \leq \frac{C}{k}\big(1 + \|\zeta'_k(0)\|\big)$$

for some $C > 0$ independent of u.

It suffices to show that $\|\zeta'_k(0)\|$ is bounded. Assume by contradiction that $\langle\zeta'(0), v\rangle = \infty$. It follows from Lemma 3.2 and Hölder's inequality that

$$\langle\zeta'_k(0), v\rangle = \frac{C\|v\|}{(2-q)a\|u_k\|^p + (2m-q)b\|u_k\|^{2m} - (2_s^*-q)\int_\Omega g(x)|u_k|^{2_s^*}dx}$$

for some $C > 0$, which implies that there exists a subsequence $\{u_k\}$ such that

$$(2-q)a\|u_k\|^2 + (2m-q)b\|u_k\|^{2m} - (2_s^*-q)\int_\Omega g(x)|u_k|^{2_s^*}dx = o_k(1). \tag{3.6}$$

Analogously, we can obtain

$$a(2-2_s^*)\|u_k\|^2 + b(2m-2_s^*)\|u_k\|^{2m} - (q-2_s^*)\int_\Omega f_\lambda(x)|u_k|^q dx = o_k(1). \tag{3.7}$$

From (3.6) and (3.7), as in the proof of Lemma 2.3, we can see that $\lambda \geq \lambda_1$, which is impossible. □

We define

$$c_\lambda^* := \frac{s}{N}(aS)^{\frac{N}{2s}} - D\lambda^{\frac{2}{2-q}}, \tag{3.8}$$

where

$$D = \frac{(2-q)(2_s^*-q)|f_+|_{q^*}^{\frac{2}{2-q}}}{2q2_s^*}\Big(\frac{2_s^*-q}{(2_s^*-2)S}\Big)^{\frac{q}{2-q}}.$$

Lemma 3.4. *Assume that $m \leq \frac{N}{N-2s}$. Then I_λ satisfies the (PS) condition at the level $c_\lambda < c_\lambda^*$, where c_λ^* is given in* (3.8).

Proof. Let $\{u_n\}$ be a $(PS)_{c_\lambda}$ sequence satisfying (3.2). It follows from Lemma 2.2 that $\{u_n\}$ is bounded in E_0. Hence, we may assume that, up to a subsequence, there exists $u \in E_0$ such that

$$\begin{gathered}
u_n \to u, \quad \text{a. e. in } \Omega, \\
u_n \rightharpoonup u, \quad \text{weakly in } E_0, \\
u_n \to u, \quad \text{strongly in } L^r(\Omega),\ 1 \leq r < 2_s^*.
\end{gathered} \tag{3.9}$$

Meanwhile, there exists $\bar{h} \in L^2(\Omega)$ such that $|u_n(x)| \leq \bar{h}(x)$ a.e. in Ω. Note that $\lim_{n\to\infty} \|u_n\| = \beta$ and M is continuous. We derive that $M(\|u_n\|^2) \to M(\beta^2)$ as $n \to \infty$. Set $v_n = u_n - u$. We can assume that $\lim_{n\to\infty} \|v_n\| = d_1 > 0$. Otherwise, the conclusion follows. From [1, Lemma 2.7], (3.9) and condition (A1), we have

$$\begin{gathered} \|u_n\|^2 = \|u_n - u\|^2 + \|u\|^2 + o_n(1), \\ \int_\Omega g(x)|u_n|^{2_s^*} dx = \int_\Omega g(x)|u_n - u|^{2_s^*} dx + \int_\Omega g(x)|u|^{2_s^*} dx + o_n(1), \end{gathered} \tag{3.10}$$

as $n \to \infty$. By (3.9)-(3.10), we obtain

$$\begin{aligned} o_n(1) &= \langle I_\lambda'(u_n), u_n \rangle \\ &= M(\|u_n\|^2)\|u_n\|^2 - \int_\Omega f_\lambda(x)|u|^q dx - \int_\Omega g(x)|u|^{2_s^*} dx \\ &\quad - \int_\Omega g(x)|v_n|^{2_s^*} dx, \end{aligned} \tag{3.11}$$

and

$$o_n(1) = \langle I_\lambda'(u_n), u \rangle = M\left(\|u_n\|^2\right)\|u\|^2 - \int_\Omega f_\lambda(x)|u|^q dx - \int_\Omega g(x)|u|^{2_s^*} dx. \tag{3.12}$$

As a consequence of (3.11) and (3.12), we obtain

$$M\left(\|u_n\|^2\right)\|v_n\|^2 - \int_\Omega g(x)|v_n|^{2_s^*} dx = o_n(1).$$

Let $\lim_{n\to\infty} \int_\Omega g(x)|v_n|^{2_s^*} dx = d_2$. We derive

$$\left(a + b\beta^{2(m-1)}\right)d_1^2 = d_2, \tag{3.13}$$

which implies $d_2 > 0$. Moreover, from the definition of S in (2.2), we have

$$d_1^2 \geq S d_2^{2/2_s^*}. \tag{3.14}$$

Combining (3.13) and (3.14), we obtain

$$d_1^2 \geq a^{\frac{N-2s}{2s}} S^{\frac{N}{2s}}. \tag{3.15}$$

It follows from Hölder's inequality that

$$\begin{aligned} c_\lambda &= \lim_{n\to\infty} \Big(I_\lambda(u_n) - \frac{1}{2_s^*}\langle I_\lambda'(u_n), u_n\rangle \Big) \\ &= \lim_{n\to\infty} \Big\{ \Big(\frac{1}{2} - \frac{1}{2_s^*}\Big) a\|u_n\|^2 + \Big(\frac{1}{2m} - \frac{1}{2_s^*}\Big) b\|u_n\|^{2m} - \Big(\frac{1}{q} - \frac{1}{2_s^*}\Big)\int_\Omega f_\lambda |u_n|^q dx \Big\} \\ &\geq \Big(\frac{1}{2} - \frac{1}{2_s^*}\Big) a d_1^2 + \Big(\frac{1}{2} - \frac{1}{2_s^*}\Big) a\|u\|^2 - \Big(\frac{1}{q} - \frac{1}{2_s^*}\Big)\lambda |f_+|_{q^*} S^{-q/2}\|u\|^q. \end{aligned}$$

Setting

$$F_\lambda(t) = \Big(\frac{1}{2} - \frac{1}{2_s^*}\Big) a t^2 - \Big(\frac{1}{q} - \frac{1}{2_s^*}\Big)\lambda |f_+|_{q^*} S^{-q/2} t^q,$$

we deduce that $F_\lambda(t)$ attains its minimum as

$$\min_{t\geq 0} F_\lambda(t) = -\frac{(2-q)(2_s^* - q)(\lambda|f_+|_{q^*})^{\frac{2}{2-q}}}{22_s^* q}\Big(\frac{2_s^* - q}{(2_s^* - 2)S}\Big)^{\frac{q}{2-q}} = -D\lambda^{\frac{2}{2-q}},$$

where

$$D=\frac{(2-q)(2_s^*-q)|f_+|_{q^*}^{\frac{2}{2-q}}}{2q2_s^*}\Big(\frac{2_s^*-q}{(2_s^*-2)S}\Big)^{\frac{q}{2-q}}.$$

By applying (3.15), we obtain

$$c_\lambda\geq\frac{s}{N}\,(aS)^{\frac{N}{2s}}-D\lambda^{\frac{2}{2-q}}=c_\lambda^*,$$

which yields a contradiction with the hypothesis $c_\lambda<c_\lambda^*$. □

We define

$$\lambda_4:=\Big(\frac{s}{N}(aS)^{\frac{N}{2s}}/D\Big)^{\frac{2-q}{2}}$$

and $\Lambda_0=\min\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$, where λ_1,λ_2 and λ_3 are given in (2.5), (2.10) and Lemma 3.1, respectively.

Proposition 3.5. *Assume that $m<\frac{N}{N-2s}$ or $m=\frac{N}{N-2s}$ and $b<1/S^m$. Then for $\lambda\in(0,\Lambda_0)$, I_λ has a minimizer u_1 in N_λ, which is a positive solution to problem (1.1) with $I_\lambda(u_1)=c_\lambda^+$ and $\|u_1\|\to 0$ as $\lambda\to 0$.*

Proof. For $\lambda\in(0,\Lambda_0)$, combining the definition of c_λ^* and Lemma 3.1 gives

$$c_\lambda^+<0<c_\lambda^*.$$

In view of the Ekeland's variational principle [9], there exists a $(PS)_{c_\lambda^+}$ sequence $\{u_n\}\subset N_\lambda^+$ satisfying (3.2). It follows from Lemma 3.4 that there exists $u_1\in N_\lambda$ such that

$$I'_\lambda(u_1)=0,\quad I_\lambda(u_1)=c_\lambda^+<0,$$

We now show that $u_1\in N_\lambda^+$. Consider the case $m<\frac{N}{N-2s}$, while the case $m=\frac{N}{N-2s}$ and $b<1/S^m$ follows similarly. Suppose by contradiction that $u_1\in N_\lambda^-$. Combining this with (2.3), we have $u_1\in G^+$. On the other hand, from $u_1\in N_\lambda$ and $I_\lambda(u_1)=c_\lambda^+<0$, we can see that $u_1\in H^+$. Hence, from Lemma 2.4, we can infer that there exist $t^-(u_1)>t^+(u_1)>0$ such that $t^-u_1\in N_\lambda^-$ and $t^+u_1\in N_\lambda^+$. This implies $t^-=1$ and $t^+<1$. Therefore, there exists $\tilde t\in(t^+,t^-)$ such that

$$I_\lambda(t^+u_1)=\min_{0\leq t\leq t^-}I_\lambda(tu_1)<I_\lambda(\tilde t u_1)<I_\lambda(t^-u_1)=I_\lambda(u_1)=c_\lambda^+,$$

which yields a contradiction. This implies $u_1\in N_\lambda^+$.

Furthermore, we show that u_1 is positive. Note that $I_\lambda(u)\neq I_\lambda(|u|)$ and $\|u\|\neq\||u|\|$ in E_0. We consider the positive part of problem (1.1) by defining

$$I_\lambda^+(u)=\frac{a}{2}\|u\|^2+\frac{b}{2m}\|u\|^{2m}-\frac{1}{q}\int_\Omega f_\lambda(x)|u^+|^q dx-\frac{1}{2_s^*}\int_\Omega g(x)|u^+|^{2_s^*}dx.$$

Then there exists a critical point $u_1\in N_\lambda^+$ for I_λ^+. That is, for any $v\in E_0$ it holds

$$\begin{aligned} &M\left(\|u_1\|^2\right)\int_{\mathbb{R}^{2N}}\frac{(u_1(x)-u_1(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy\\ &=\int_\Omega f_\lambda(x)|u_1^+|^{q-1}v dx-\int_\Omega g(x)|u_1^+|^{2_s^*-1}v dx. \end{aligned}\tag{3.16}$$

Taking $v=u_1^-=\min\{u_1,0\}$ as a test function in (3.16) and applying the inequality

$$\begin{aligned}(u_1(x)-u_1(y))(u_1^-(x))-u_1^-(y))&=-u_1^+(x)u_1^-(y)-u_1^-(x)u_1^+(y)-[u_1^-(x)-u_1^-(y)]^2\\ &\leq-[u_1^-(x)-u_1^-(y)]^2,\end{aligned}$$

we obtain

$$(a+b\|u_1^-\|^{2(m-1)})\int_{\mathbb{R}^{2N}}\frac{|u_1^-(x)-u_1^-(y)|^2}{|x-y|^{N+2s}}\,dx\,dy=o(1),$$

which implies $\|u_1^-\|=0$, i.e. $u_1\geq 0$ in $\mathbb{R}^N$. Moreover, by the strong maximum principle [3], we know that u_1 is positive.

Then, we prove that u_1 is a local minimizer of I_λ in E_0. From Lemmas 2.4 and 2.6, we have $t^+(u_1)=1<t_{\max}(u_1)$. From continuity of $u\mapsto t_{\max}(u)$, for the fixed $\epsilon>0$, there exists $\delta_1=\delta_1(\epsilon)>0$ such that $t_{\max}(u_1-u)>1+\epsilon$ for all $\|u\|<\delta_1$. Meanwhile, by Lemma 3.2, we can see that for a given $\delta_2>0$, there exists a C^1 map $\zeta: B_{\delta_2}(0)\to\mathbb{R}^+$ such that $\zeta(u)(u_1-u)\in N_\lambda^+$ and $\zeta(0)=1$. Hence, taking into account $0<\delta=\min\{\delta_1,\delta_2\}$ and the uniqueness of zeros of fibering map, we have $t^+(u_1-u)=\zeta(u)<1+\epsilon<t_{\max}(u_1-u)$ for all $\|u\|<\delta$. By $t_{\max}(u_1-u)>1$, we obtain $I_\lambda(u_1)\leq I_\lambda(t^+(u_1-u)(u_1-u))\leq I_\lambda(u_1-u)$, which implies that u_1 is a local minimizer of I_λ in E_0.

By Lemma 2.1, we obtain that u_1 is a positive solution to problem (1.1) . By Lemma 2.7, we arrive at the desired result. □

In [19], it is shown that the infimum in (2.2) is attained by

$$u_\epsilon(x)=\frac{\epsilon^{(N-2s)/2}}{(\epsilon^2+|x|^2)^{(N-2s)/2}},\quad \epsilon>0, \tag{3.17}$$

which satisfies

$$\int_{\mathbb{R}^{2N}}\frac{|u_\epsilon(x)-u_\epsilon(y)|^2}{|x-y|^{N+2s}}\,dx\,dy=S|u_\epsilon|_{2_s^*}^{2_s^*}.$$

We define

$$u_{\epsilon,\eta}(x)=\eta(x)u_\epsilon(x), \tag{3.18}$$

where $\eta(x)\in\mathcal{C}_0^\infty(B_\rho(0))$ satisfies $0\leq\eta\leq 1$ in $B_\rho(0)$, $\eta\equiv 1$ in $B_{\rho/2}(0)$ and $\eta\equiv 0$ in $\mathbb{R}^N\setminus B_\rho(0)$, for some $\rho>0$ sufficiently small as given in condition (A1). From [19], we have

$$\|u_{\epsilon,\eta}\|^2\leq S^{N/(2s)}+O\left(\epsilon^{N-2s}\right)\quad\text{and}\quad |u_{\epsilon,\eta}|_{2_s^*}^{2_s^*}=S^{N/(2s)}+O\left(\epsilon^N\right). \tag{3.19}$$

It follows from (3.17) and (3.18) that

$$\begin{aligned}
\int_{B_\rho(0)}|u_{\epsilon,\eta}|^q dx&\leq C\Big(\int_{B_\epsilon(0)}\frac{1}{\epsilon^{q(N-2s)/2}}dx+\int_{B_\rho(0)\setminus B_\epsilon(0)}\frac{\epsilon^{q(N-2s)/2}}{|x|^{q(N-2s)}}dx\Big)\\
&=C\omega_N\Big(\epsilon^{N-\frac{(N-2s)q}{2}}+\epsilon^{\frac{q(N-2s)}{2}}\int_\epsilon^\rho r^{N-1-q(N-2s)}dr\Big)\\
&=O\big(\epsilon^{N-\frac{(N-2s)q}{2}}\big)+O\big(\epsilon^{\frac{q(N-2s)}{2}}\big)\\
&=O\big(\epsilon^{\frac{q(N-2s)}{2}}\big),
\end{aligned}$$

where $1<q<N/(N-2s)$, and ω_N denotes the unit sphere in $\mathbb{R}^N$.

In view of condition (A1) and the definition of η, we have the following lemma.

Lemma 3.6 ([11, 19]). *For small $\epsilon>0$, the following statements are true.*

(i) $\int_{B_\rho(0)}|u_{\epsilon,\eta}|^q dx=O\left(\epsilon^{q(N-2s)/2}\right)$;

(ii) $\int_{B_\rho(0)}|u_{\epsilon,\eta}|^{2_s^*-1}dx\geq C\epsilon^{\frac{N-2s}{2}}$;

(iii) $\int_{B_\rho(0)}g(x)|u_{\epsilon,\eta}|^{2_s^*}dx=S^{\frac{N}{2s}}+O(\epsilon^N)$.

We consider the following two sets

$$A_1 = \{u \in E \setminus \{0\} : \frac{1}{\|u\|} t^- \big(\frac{u}{\|u\|}\big) > 1\} \cup \{0\},$$
$$A_2 = \{u \in E \setminus \{0\} : \frac{1}{\|u\|} t^- \big(\frac{u}{\|u\|}\big) < 1\},$$

where t^- is given in Lemma 2.4. It follows from [25] that $t^-(u)$ is continuous for $u \in E_0 \setminus \{0\}$ and $N_\lambda^- = \{u \in E_0 \setminus \{0\} : \frac{1}{\|u\|} t^- \big(\frac{u}{\|u\|}\big) = 1\}$ splits E_0 into two connected parts A_1 and A_2. It follows from Lemmas 2.4 and 2.6 that for any $u \in N_\lambda^+$ and $\lambda \in (0, \lambda_2)$, we have $1 < t_{\max}(u) < t^-(u)$. Then $N_\lambda^+ \subset A_1$. Particularly, $u_\lambda^+ \in A_1$.

Lemma 3.7. *Assume that $m < \frac{N}{N-2s}$ or $m = \frac{N}{N-2s}$ and $b < 1/S^m$. Then for any $\epsilon > 0$, there exists $t_1 > 0$ such that $u_1 + t_1 u_{\epsilon,\eta} \in A_2$.*

Proof. We just prove the case of $m = \frac{N}{N-2s}$ and $b < 1/S^m$, since the case of $m < \frac{N}{N-2s}$ can be processed in a similar manner.

We claim that there exists a constant $\tilde{c} > 0$ such that $0 < t^- \big(\frac{u_1 + t u_{\epsilon,\eta}}{\|u_1 + t u_{\epsilon,\eta}\|}\big) < \tilde{c}$ for $m = \frac{N}{N-2s}$ and $b < 1/S^m$. Otherwise, there is a sequence $\{t_n\} \subset \mathbb{R}^+$ such that $t_n \to \infty$ and $t^- \Big(\frac{u_1 + t_n u_{\epsilon,\eta}}{\|u_1 + t_n u_{\epsilon,\eta}\|}\Big) \to \infty$ as $n \to \infty$. Let $v_n = \frac{u_1 + t_n u_{\epsilon,\eta}}{\|u_1 + t_n u_{\epsilon,\eta}\|}$. From (3.19), we deduce that

$$\begin{aligned}
\int_{B_\rho(0)} g|v_n|^{2_s^*} dx - b\|v_n\|^{2_s^*} &= \frac{\int_{B_\rho(0)} g|u_1 + t_n u_{\epsilon,\eta}|^{2_s^*} dx}{\|u_1 + t_n u_{\epsilon,\eta}\|^{2_s^*}} - b \\
&\to \frac{\int_{B_\rho(0)} g|u_{\epsilon,\eta}|^{2_s^*} dx}{\|u_{\epsilon,\eta}\|^{2_s^*}} - b \\
&\geq 1/S^m - b + O\big(\epsilon^N\big) > 0
\end{aligned}$$

for $0 < \epsilon < \epsilon_1$ with some $\epsilon_1 > 0$, as $n \to \infty$. Thus, $I_\lambda(t^-(v_n)v_n) \to -\infty$ as $n \to \infty$ for $m = \frac{N}{N-2s}$ and $\epsilon \in (0, \epsilon_1)$, which contradicts Lemma 2.2. According to [25, Lemma 3.6], we obtain $u_1 + t_1 u_{\epsilon,\eta} \in A_2$ immediately. $\square$

Lemma 3.8. *Assume that $m < \frac{N}{N-2s}$ or $m = \frac{N}{N-2s}$ and $b < 1/S^m$. Then there exist $\Lambda_* \in (0, \Lambda_0]$ and $\bar{b} > 0$ such that for any $\lambda \in (0, \Lambda_*)$ and $b \in (0, \bar{b})$ it holds*

$$\sup_{t \geq 0} I_\lambda(u_1 + t u_{\epsilon,\eta}) < c_\lambda^*,$$

where c_λ^* is given in (3.8).

Proof. For any $\alpha, \beta \geq 0$ and $m \geq 1$, we recall the inequality

$$(\alpha + \beta)^m \leq \alpha^m + C_m(\alpha^m + \beta^m) + m\alpha^{m-1}\beta,$$

where $C_m > 0$ is a constant depending on m. It follows from Young's inequality that

$$\begin{aligned}
&\frac{b}{2m}\|u_1 + t u_{\epsilon,\eta}\|^{2m} \\
&\leq \frac{b}{2m}\|u_1\|^{2m} + b\|u_1\|^{2(m-1)} t \int_{\mathbb{R}^{2N}} \frac{(u_1(x) - u_1(y))(u_{\epsilon,\eta}(x) - u_{\epsilon,\eta}(y))}{|x-y|^{N+2s}} \\
&\quad + bC_m\|u_1\|^{2m} + bD_m t^{2m}\|u_{\epsilon,\eta}\|^{2m},
\end{aligned} \tag{3.20}$$

where $D_m > 0$. Since u_1 is a critical point of I_λ, we obtain

$$\langle I'_\lambda(u_1), tu_{\epsilon,\eta}\rangle = 0. \tag{3.21}$$

In view of the inequality

$$(\alpha+\beta)^p - \alpha^p - \beta^p - p\alpha^{p-1}\beta \geq C\alpha\beta^{p-1}, \quad \alpha,\beta \geq 0,\ p > 2,$$

it follows from the definition of η, (3.20), (3.21) and condition (A1) that

$$\begin{aligned}
& I_\lambda(u_1 + tu_{\epsilon,\eta}) \\
&= \frac{a}{2}\|u_1 + tu_{\epsilon,\eta}\|^2 + \frac{b}{2m}\|u_1 + tu_{\epsilon,\eta}\|^{2m} \\
&\quad - \frac{1}{q}\int_{B_\rho(0)} f_\lambda |u_1 + tu_{\epsilon,\eta}|^q dx - \frac{1}{2_s^*}\int_{B_\rho(0)} g|u_1 + tu_{\epsilon,\eta}|^{2_s^*} dx \\
&\leq I_\lambda(u_1) + \frac{a}{2}\|tu_{\epsilon,\eta}\|^2 + bC_m\|u_1\|^{2m} + bD_m\|tu_{\epsilon,\eta}\|^{2m} \\
&\quad - \frac{1}{q}\int_{B_\rho(0)} f_\lambda\Big(\int_0^{tu_{\epsilon,\eta}} [|u_1 + s|^{q-1} - |u_1|^{q-1}] ds\Big) dx \\
&\quad - \frac{1}{2_s^*}\int_{B_\rho(0)} g\big[|u_1 + tu_{\epsilon,\eta}|^{2_s^*} - |u_1|^{2_s^*} - 2_s^* tu_{\epsilon,\eta}|u_1|^{2_s^*-1}\big] dx \\
&\leq c_\lambda^+ + \frac{a}{2}\|tu_{\epsilon,\eta}\|^2 + bC_m\|u_1\|^{2m} + bD_m\|tu_{\epsilon,\eta}\|^{2m} + C|f_-|_\infty |tu_{\epsilon,\eta}|_q^q \\
&\quad - \frac{1}{2_s^*}\int_{B_\rho(0)} g|tu_{\epsilon,\eta}|^{2_s^*} dx - C\int_{B_\rho(0)} |u_1||tu_{\epsilon,\eta}|^{2_s^*-1} dx.
\end{aligned} \tag{3.22}$$

We now consider

$$\begin{aligned}
J_\lambda(tu_{\epsilon,\eta}) &= \frac{a}{2}\|tu_{\epsilon,\eta}\|^2 + bC_m\|u_1\|^{2m} + bD_m\|tu_{\epsilon,\eta}\|^{2m} + C|f_-|_\infty |tu_{\epsilon,\eta}|_q^q \\
&\quad - \frac{1}{2_s^*}\int_{B_\rho(0)} g|tu_{\epsilon,\eta}|^{2_s^*} dx - C\int_{B_\rho(0)} |u_1||tu_{\epsilon,\eta}|^{2_s^*-1} dx.
\end{aligned}$$

Claim 1. There exist t_ϵ and $t_2 > 0$ independent of ϵ and λ such that

$$t_2 \leq t_\epsilon \leq t_1, \quad J_\lambda(t_\epsilon u_{\epsilon,\eta}) = \sup_{t\geq 0} J_\lambda(tu_{\epsilon,\eta}), \quad \frac{d}{dt}J_\lambda(tu_{\epsilon,\eta})|_{t=t_\epsilon} = 0, \tag{3.23}$$

where t_1 is given in Lemma 3.7. Since $\lambda \in (0,\lambda_3)$, from Lemma 3.1 we have

$$0 < \alpha < \alpha - c_\lambda^+ \leq c_\lambda^- - c_\lambda^+ \leq \sup_{t\geq 0} I_\lambda(u_1 + tu_{\epsilon,\eta}) - c_\lambda^+ \leq J_\lambda(t_\epsilon u_{\epsilon,\eta}),$$

which implies $t_2 \leq t_\epsilon$ for some $t_2 > 0$.

To find the estimate of $\sup_{t\geq 0} J_\lambda(tu_{\epsilon,\eta})$, we define

$$h(t) = \frac{at^2}{2}\|u_{\epsilon,\eta}\|^2 - \frac{t^{2_s^*}}{2_s^*}\int_{B_\rho(0)} g|u_{\epsilon,\eta}|^{2_s^*} dx.$$

By (3.19) and Lemma 3.6, we obtain

$$\sup_{t\geq 0} h(t) \leq \frac{s}{N}(aS)^{\frac{N}{2s}} + O\big(\epsilon^{N-2s}\big). \tag{3.24}$$

From (3.23), (3.24) and Lemmas 3.6 and 2.7, we deduce that

$$
\begin{aligned}
J_\lambda(tu_{\epsilon,z}) &\le \frac{s}{N}\left(aS\right)^{\frac{N}{2s}} + bC_m\|u_1\|^{2m} + bD_m t_1^{2m}\|u_{\epsilon,\eta}\|^{2m} + O\left(\epsilon^{N-2s}\right) \\
&\quad + Ct_1^q|f_-|_\infty |u_{\epsilon,\eta}|_q^q - Ct_2^{2_s^*-1}\int_{B_\rho(0)} |u_1||u_{\epsilon,\eta}|^{2_s^*-1}dx \\
&\le \frac{s}{N}(aS)^{\frac{N}{2s}} + O(\epsilon^{N-2s}) + Cb\lambda^{\frac{2m}{2-q}} + Cb + C\epsilon^{\frac{q(N-2s)}{2}} - C\epsilon^{\frac{N-2s}{2}} \\
&\le \frac{s}{N}(aS)^{\frac{N}{2s}} + Cb\lambda^{\frac{2m}{2-q}} + Cb - C\epsilon_0^{\frac{N-2s}{2}}
\end{aligned}
\tag{3.25}
$$

for some $\epsilon_0 > 0$. Thus, there exist two positive numbers $\Lambda_* \in (0, \Lambda_0]$ and $\bar{b} > 0$ such that for any $\lambda \in (0, \Lambda_*)$ and $b \in (0, \bar{b})$ it holds

$$
Cb\lambda^{\frac{2m}{2-q}} + Cb + D\lambda^{\frac{2}{2-q}} < C\epsilon_0^{\frac{N-2s}{2}}.
$$

Combining this and (3.22)-(3.25) and by Lemma 3.8, we arrive at the desired result. □

Proof of Theorem 1.1. Clearly, (i) follows from Proposition 3.5. (ii) Let $\lambda \in (0, \Lambda_*)$. From Proposition 3.5 and Lemma 3.7, we obtain $u_1 \in A_1$ and $u_1 + t_1 u_{\epsilon,\eta} \in A_2$. We define a path $\gamma(s) = u_1 + st_1u_{\epsilon,\eta}$ for $s \in [0,1]$. Since $\gamma(0) \in A_1$ and $\gamma(1) \in A_2$, there exists $s \in (0,1)$ such that $u_1 + st_1u_{\epsilon,\eta} \in N_\lambda^-$, which implies $c_\lambda^- \le \sup_{t\ge 0} I_\lambda(u_1 + tu_{\epsilon,\eta})$. According to Lemma 3.8, we obtain $c_\lambda^- < c_\lambda^*$ for any $\lambda \in (0, \Lambda_*)$ and $b \in (0, \bar{b})$. In view of Corollary 2.8, N_λ^- is a closed set. By Proposition 3.5, there exists $u_2 \in N_\lambda^-$ such that $I'_\lambda(u_2) = 0$ and $I_\lambda(u_2) = c_\lambda^-$. This indicates that u_2 is also a positive solution to problem (1.1). □

Proof of Theorem 1.2. (i) From Lemma 2.5, we obtain $N_\lambda^+ = N_\lambda$ and define $c_\lambda = \inf_{u\in N_\lambda} I_\lambda(u)$. It is clear that $c_\lambda < 0$. By Proposition 3.5, there exists a $(PS)_{c_\lambda}$ sequence $\{u_n\} \subset N_\lambda^+$ for I_λ. It follows from Lemma 2.2 that $\{u_n\}$ is bounded in E_0. Hence, up to a subsequence, there is $u \in E_0$ satisfying (3.9). Denote $v_n = u_n - u$, for $b \ge 1/S^m$. Then

$$
\int_\Omega g(x)|u|^{2_s^*}dx - b\|u\|^{2m} < 0.
$$

Combining this (3.11) and (3.12) yields

$$
a\|v_n\|^2 \le a\|v_n\|^2 + b\|u\|^{2(m-1)}\|v_n\|^2 + b\|v_n\|^{2m} - \int_\Omega g(x)|u|^{2_s^*}dx = o_n(1),
$$

which implies $u_n \to u$ in E_0. According to Proposition 3.5, we see that u is a positive solution to problem (1.1).

The proof of (ii) is similar to that of Theorem 1.1, so we omit it. □

4. Proof of Theorem 1.3

In this section, we assume that $m > \frac{N}{N-2s}$, $f_- \equiv 0$ and condition (A1) holds. Let us start with a compactness result.

Lemma 4.1. *I_λ satisfies the (PS) condition if*

$$
c < c_{\lambda,b}^* := \frac{s}{N}(aS)^{\frac{N}{2s}} - D_1 b - D_2\lambda^{\frac{2}{2-q}}.
$$

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence satisfying (3.2). We claim that $\{u_n\}$ is bounded in E_0. By way of contradiction, we assume that there is a subsequence of the original sequence such that $\|u_n\| \to \infty$, as $n \to \infty$. We define $w_n = u_n/\|u_n\|$. By the Sobolev and Hölder's inequalities, we obtain

$$\int_\Omega g(x)|w_n|^{2_s^*}dx \le S^{-2_s^*/2} \quad \text{and} \quad \int_\Omega \lambda f(x)|w_n|^q dx \le \lambda S^{-q/2}|f|_{q^*}.$$

Therefore,

$$\begin{aligned}\frac{c+o_n(1)}{\|u_n\|^{2_s^*}} &= \frac{a}{2}\frac{\|u_n\|^2}{\|u_n\|^{2_s^*}} + \frac{b}{2m}\frac{\|u_n\|^{2m}}{\|u_n\|^{2_s^*}} - \frac{1}{q}\frac{\int_\Omega \lambda f(x)|u|^q dx}{\|u_n\|^{2_s^*}} - \frac{1}{2_s^*}\int_\Omega g(x)|w_n|^{2_s^*}dx \\ &\ge \frac{b}{2m}\|u_n\|^{2m-2_s^*} - \frac{S^{-2_s^*/2}}{2_s^*} + o_n(1) \to \infty.\end{aligned}$$

In view of $m > N/(N-2s)$, this yields a contradiction.

Hence, up to a subsequence, there exists $u \in E_0$ satisfying (3.9). Similar to the proof of Lemma 3.4, setting $v_n = u_n - u$, we can suppose that $\lim_{n\to\infty}\|v_n\| = d_1 > 0$. Using Hölder's inequality, (3.15) and $m > \frac{N}{N-2s}$, we deduce that

$$\begin{aligned}c &= \lim_{n\to\infty}\Big(I_\lambda(u_n) - \frac{1}{2_s^*}\langle I_\lambda'(u_n), u_n\rangle\Big) \\ &= \lim_{n\to\infty}\{\frac{as}{N}\|u_n\|^2 - \Big(\frac{1}{2_s^*} - \frac{1}{2m}\Big)b\|u_n\|^{2m} - \Big(\frac{1}{q} - \frac{1}{2_s^*}\Big)\lambda\int_\Omega f|u_n|^q dx\} \\ &\ge \frac{as}{N}d_1^2 + \frac{as}{N}\|u\|^2 - D_1 b - \Big(\frac{1}{q} - \frac{1}{2_s^*}\Big)\lambda|f|_{q^*}\|u\|^q \\ &\ge \frac{s}{N}(aS)^{\frac{N}{2s}} - D_1 b - D_2\lambda^{\frac{2}{2-q}},\end{aligned}$$

where $D_1 = D_1(N,m,s)$ and $D_2 = D_2(N,q,S,a,|f|_{q^*})$. This contradicts the hypothesis of $c < c^*_{\lambda,b}$.. ☐

Lemma 4.2. *There exist $\lambda_5 > 0$ and $r > 0$ such that for any $\lambda \in (0,\lambda_5)$ it holds*

$$\inf_{u\in E_0,\|u\|=r} I_\lambda(u) = \tilde{\alpha} > 0.$$

In particular, when $\lambda = 0$, there exists an $r_0 > r$ such that $I_0(u) > 0$ for all $u \in B_{r_0}\backslash\{0\}$.

Proof. For $u \in E_0$, we have

$$\begin{aligned}I_\lambda(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{2m}\|u\|^{2m} - \frac{1}{q}\int_\Omega \lambda f(x)|u|^q dx - \frac{1}{2_s^*}\int_\Omega g(x)|u|^{2_s^*}dx \\ &\ge \frac{a}{2}\|u\|^2 - \frac{\lambda|f|_{q^*}}{qS^{q/2}}\|u\|^q - \frac{1}{2_s^* S^{2_s^*/2}}\|u\|^{2_s^*} \\ &\ge \Big(\frac{a}{2}\|u\|^{2-q} - \frac{\lambda|f|_{q^*}}{qS^{q/2}} - \frac{1}{2_s^* S^{2_s^*/2}}\|u\|^{2_s^*-q}\Big)\|u\|^q.\end{aligned} \tag{4.1}$$

We define

$$l(t) = \frac{a}{2}t^{2-q} - \frac{1}{2_s^*}S^{-2_s^*/2}t^{2_s^*-q}$$

for $t \ge 0$. In view of $2 < 2_s^*$, for each $u \in E_0$ with

$$\|u\| = r := \big[\frac{a2_s^* S^{2_s^*/2}(2-q)}{2(2_s^*-q)}\big]^{1/(2_s^*-2)},$$

we obtain $\max_{t\geq 0} l(t) = l(r) > 0$. Thus, by taking

$$\lambda < \lambda_5 = \frac{l(r)q}{S^{-q/2}|f|_{q^*}},$$

we obtain

$$I_\lambda(u) \geq \Big(l(r) - \lambda\frac{|f|_{q^*}}{qS^{2/q}}\Big)r^q =: \tilde{\alpha} > 0.$$

When $\lambda = 0$, from (4.1), there exists an $r_0 = [(2_s^* - q)/(2-q)]^{1/(2_s^*-2)}r > r$ such that $I_0(u) > 0$ for $u \in B_{r_0}\backslash\{0\}$. $\square$

Lemma 4.3. *Let λ_5 be given in Lemma 4.2. Then there exist two positive constants b_1 and $0 < \lambda_6 \leq \lambda_5$ such that for each $b \in (0, b_1)$ and $\lambda \in (0, \lambda_6)$, problem* (1.1) *admits a positive solution u_b with $I_\lambda(u_b) < 0$.*

Proof. It follows from Lemma 4.2 that there is an $r > 0$ such that $I_\lambda(u) \geq 0$ for $\|u\| = r$. For $u \in E_0\backslash\{0\}$ and $t > 0$ sufficiently small, we have

$$I_\lambda(tu) = \frac{at^2}{2}\|u\|^2 + \frac{bt^{2m}}{2m}\|u\|^{2m} - \frac{t^q}{q}\int_\Omega \lambda f(x)|u|^q dx - \frac{t^{2_s^*}}{2_s^*}\int_\Omega g(x)|u|^{2_s^*}dx < 0.$$

Thus, we obtain

$$m_\lambda := \inf\{I_\lambda(u) : u \in \bar{B}_r\} < 0. \tag{4.2}$$

By Ekeland's variational principle [9], there exists a minimizing sequence $\{u_n\} \subset \bar{B}_r$ such as

$$I_\lambda(u_n) \to m_\lambda, \quad \|I_\lambda'(u_n)\|_{E_0^{-1}} \to 0,$$

as $n \to \infty$. On the other hand, it is easy to see that there exist $b_1 > 0$ and $\lambda_6 \leq \lambda_5$ such that

$$c_{\lambda,b}^* > 0, \;\; b \in (0, b_1), \;\lambda \in (0, \lambda_6),$$

where $c_{\lambda,b}^*$ is given in Lemma 4.1. It follows from (4.2) and Lemma 4.1 that there exists $u_b \in E_0$ such that $u_n \to u_b$, i.e. u_b is a nontrivial solution of problem (1.1). By Proposition 3.5, we see that u_b is a positive solution of (1.1). $\square$

Lemma 4.4. *Let r and b_1 be given in Lemmas 4.2 and 4.3, respectively. Then there exist $0 < b_2 \leq b_1$ and $e_1 \in E_0$ with $\|e_1\| > r$ such that $I_\lambda(e_1) < 0$ for $b \in (0, b_2)$.*

Proof. If $b = 0$, we consider the functional I_λ denoted by

$$I_{\lambda,0}(tu_{\epsilon,\eta}) = \frac{at^2}{2}\|u_{\epsilon,\eta}\|^2 - \frac{t^q}{q}\int_{B_\rho(0)} \lambda f(x)|u_{\epsilon,\eta}|^q dx - \frac{t^{2_s^*}}{2_s^*}\int_{B_\rho(0)} g(x)|u_{\epsilon,\eta}|^{2_s^*}dx.$$

Recalling Fatou's Lemma and Lemma 3.6 (iii), we can see that for small $\epsilon > 0$ it holds

$$\lim_{t\to\infty}\frac{I_{\lambda,0}(tu_{\epsilon,\eta})}{t^{2_s^*}} \leq -\frac{1}{2_s^*}S^{\frac{N}{2s}} + C\epsilon^N < 0.$$

Namely, there exists a large $T > 0$ satisfying $\|Tu_{\epsilon,\eta}\| > r$ and $I_{\lambda,0}(Tu_{\epsilon,\eta}) < 0$. Since $I_\lambda(Tu_{\epsilon,\eta}) \to I_{\lambda,0}(Tu_{\epsilon,\eta})$ as $b \to 0^+$, we deduce that there exists $0 < b_2 \leq b_1$ such that $I_\lambda(Tu_{\epsilon,\eta}) < 0$ for any $b \in (0, b_2)$. $\square$

Lemma 4.5. *Let λ_6 and b_2 be given in Lemmas 4.3 and 4.4, respectively. Then there exists $\alpha_b < 0$ such that for any $b \in (0, b_2)$ it holds*

$$\alpha_b \leq \bar{m}_\lambda := \inf\{I_\lambda(u) : u \in E_0\} < 0.$$

Furthermore, for any $b \in (0, b_2)$ and $\lambda \in (0, \lambda_6)$, problem (1.1) *admits a positive solution u_λ with $I_\lambda(u_\lambda) = \bar{m}_\lambda$.*

Proof. Note that

$$\begin{aligned} I_\lambda(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{2m}\|u\|^{2m} - \frac{1}{q}\int_\Omega \lambda f(x)|u|^q dx - \frac{1}{2_s^*}\int_\Omega g(x)|u|^{2_s^*} dx \\ &\geq \frac{b}{2m}\|u\|^{2m} - \frac{1}{q}\lambda |f|_{q^*} S^{-q/2}\|u\|^q - \frac{1}{2_s^*} S^{-\frac{2_s^*}{2}}\|u\|^{2_s^*}. \end{aligned}$$

Let

$$\bar{A} = \frac{2m\lambda|f|_{q^*}}{bqS^{q/2}}, \quad \bar{B} = \frac{2m}{2_s^* b S^{2_s^*/2}}, \quad \Phi_{\bar{A},\bar{B}}(t) = t^{2m} - \bar{A}t^q - \bar{B}t^{2_s^*}.$$

From [22, Lemm 2.3], for any $b > 0$ there exist $t_3, t_4 > 0$ such that

$$\alpha_b = \min_{t \geq 0} \Phi_{\bar{A},\bar{B}}(t) = \Phi_{\bar{A},\bar{B}}(t_3) < 0$$

and $\Phi_{\bar{A},\bar{B}}(t) \geq 0$ for $t \geq t_4$. While, by Lemma 4.4, it is easy to see that for any $b \in (0, b_2)$ it holds

$$\bar{m}_\lambda := \inf\{I_\lambda(u) : u \in E_0\} < 0. \tag{4.3}$$

Then using a similar strategy of Lemma 4.3, we obtain that u_λ is a positive solution of problem (1.1) . □

To obtain two distinct solutions to (1.1), we need to show that the infimum $\bar{m}_\lambda < m_\lambda$.

Lemma 4.6. *There exists $0 < \lambda_7 \leq \lambda_6$ such that $\bar{m}_\lambda < m_\lambda$ for $b \in (0, b_2)$ and $\lambda \in (0, \lambda_7)$.*

Proof. For $\lambda = 0$, let m_0 be given as in (4.3). So, for any $b \in (0, b_2)$, problem (1.1) admits a positive solution $u_{0,b}$ satisfying $I_0(u_{0,b}) = m_0 := \inf\{I_0(u) : u \in E_0\} < 0$. Taking into account $f_- = 0$, we deduce

$$\bar{m}_\lambda \leq I_\lambda(u_{0,b}) = I_0(u_{0,b}) - \lambda\int_\Omega f|u_{0,b}|^q dx = m_0 - \lambda\int_\Omega f|u_{0,b}|^q dx \leq m_0. \tag{4.4}$$

In view of Lemma 4.2 and (4.2), we have $m_\lambda \to 0$ as $\lambda \to 0$. Then, there exists $0 < \lambda_7 \leq \lambda_6$ such that $m_0 < m_\lambda$ for any $b \in (0, b_2)$ and $\lambda \in (0, \lambda_7)$. Combining this and (4.4), we arrive at the desired result. □

It follows from Lemmas 4.2 and 4.4 that I_λ has the mountain pass geometry. Using the mountain pass theorem [24], there exists a $(PS)_{c_{\lambda,b}}$ sequence $\{u_n\} \subset E_0$, that is

$$I_\lambda(u_n) \to c_{\lambda,b}, \quad \|I'_\lambda(u_n)\|_{E_0^{-1}} \to 0.$$

We note that $c_{\lambda,b}$ has the characteristic property

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

where

$$\Gamma := \{\gamma \in \mathcal{C}([0,1], E_0) : \gamma(0) = u_b, \gamma(1) = u_b + Tu_{\epsilon,\eta}\}.$$

Similar to Lemma 3.8, we can obtain the following lemma.

Lemma 4.7. *There exist $0 < \Lambda^* \leq \lambda_7$ and $0 < b^* \leq b_2$ such that for any $\lambda \in (0, \Lambda^*)$ and $b \in (0, b^*)$ it holds*

$$c_{\lambda,b} \leq \sup_{t \geq 0} I_\lambda(u_b + tu_{\epsilon,\eta}) < c^*_{\lambda,b}.$$

Proof of Theorem 1.3. By Lemmas 4.2 and 4.4, it is easy to see that I_λ has the mountain pass geometry. i.e. there exists a bounded $(PS)_{c_{\lambda,b}}$ sequence $\{u_n\}$. It follows from Lemmas 4.1 and 4.7 that, up to a subsequence, there exists $u_{\lambda,b} \in E_0$ such that $I_\lambda(u_{\lambda,b}) = c_{\lambda,b} > 0$. By Lemmas 4.3 and 4.5 and 4.6, problem (1.1) admits three positive solutions u_b, u_λ and $u_{\lambda,b}$. □

Acknowledgments. H. B. Chen was supported by the National Natural Science Foundation of China (No. 11671403). J. Yang was Supported by the Scientific Research Fund of Hunan Provincial Education Department (No. 19B450), and by the Natural Science Foundation of Hunan Province (No. 2019JJ50473).

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