NONLINEAR DEGENERATE ELLIPTIC EQUATIONS IN WEIGHTED SOBOLEV SPACES

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Abstract. We study the existence of solutions for the nonlinear degenerated elliptic problem

\[- \text{div} a(x, u, \nabla u) = f \quad \text{in } \Omega,
\]

\[u = 0 \quad \text{on } \partial \Omega,
\]

where \( \Omega \) is a bounded open set in \( \mathbb{R}^N \), \( N \geq 2 \), \( a \) is a Carathéodory function having degenerate coercivity

\[a(x, u, \nabla u) \nabla u \geq \nu(x) b(|u|) |\nabla u|^p, \quad 1 < p < N, \]

\( \nu(\cdot) \) is the weight function, \( b \) is continuous and \( f \in L^r(\Omega) \).

1. Introduction

In this article we prove the existence of solutions for some nonlinear elliptic equations with principal part having degenerate coercivity. The model case is

\[- \text{div} \left( \frac{\nu(\cdot) |\nabla u|^{p-2} \nabla u}{(1 - |u|)^\alpha} \right) = f \quad \text{in } \Omega,
\]

\[u = 0 \quad \text{on } \partial \Omega,
\]

with \( \Omega \) a bounded open subset of \( \mathbb{R}^N \), \( N \geq 2 \), \( p > 1 \), \( \alpha \geq 0 \), \( \nu(\cdot) \) is weight function defined on \( \Omega \) and \( f \) a measurable function on whose summability we will make different assumptions. It is clear from the above example that the differential operator is defined on \( W^{1,p}_0(\Omega, \nu) \), but that it may not be coercive on the same space as \( u \) near to 1. Because of this lack of coercivity, standard existence theorems for solutions of nonlinear elliptic equations cannot be applied. We consider the nonlinear degenerate elliptic problem

\[A(u) = - \text{div}(a(x, u, \nabla u)) = f \quad \text{in } \Omega,
\]

\[u = 0 \quad \text{on } \partial \Omega,
\]

where, \( \Omega \) is a bounded open subset of \( \mathbb{R}^N \), \( N \geq 2 \), \( 1 < p < N \), and \( a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) is a Carathéodory function, such that the following assumption holds

\[a(x, s, \xi) \cdot \xi \geq \nu(x) b(|s|) |\xi|^p,
\]

for almost every \( x \) in \( \Omega \), for every \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^N \), with

\[b(|s|) = 1/(1 - |s|)^\alpha,
\]

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under various assumptions on $f$. As stated before, due to assumption (1.2), the operator $A$ may not be coercive on $W_0^{1,p}(\Omega, \nu)$, when the solutions approach the critical values $\pm1$. To overcome this difficulties, we will reason by approximation, cutting by means of truncatures the nonlinearity $a(x,s,\xi)$ in order to get coercive differential operator on $W_0^{1,p}(\Omega, \nu)$, and give a sense to the equation when the solutions near to $\pm1$ and to manage the set $\{x \in \Omega : |u(x)| = 1\}$. For the case $\nu(\cdot)$ being a constant, the existence of solutions to problem (1.1) is proved in [11], when $f$ a measurable function on whose summability have make different assumptions, the analogous problems was treated by many other authors. See, for example, [3, 4, 9, 10, 8] where problems such as

$$-\text{div} \left( \frac{1}{(1 \pm |u|)^{\alpha}} |\nabla u|^{p-2} \nabla u \right) = f,$$

are considered.

This article is organized as follows: In section 2, we recall some preliminaries on Weighted Sobolev spaces and properties of rearrangement. In section 3, we first prove the propositions that we will use to prove some a priori estimates of the solutions, then we prove the existence of weak and entropy solution with respect to the summability of $f$.

2. Preliminaries

Assumptions. Let $b : [0, l] \to (0, \infty)$, with $l > 0$, be a continuous function such that

$$\lim_{s \to l^-} b(s) = +\infty. \quad (2.1)$$

We define

$$A(s) = \int_0^s b(t)^{\frac{1}{p-1}} dt, \quad \text{for } s \in [0, l),$$

$$A(l^-) = \lim_{s \to l^-} \int_0^s b(t)^{\frac{1}{p-1}} dt = +\infty.$$  

We study Dirichlet problems of the form

$$-\text{div} \, a(x,u,\nabla u) = f \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega, \quad (2.2)$$

where $\Omega$ is a bounded open set in $\mathbb{R}^N$, $N \geq 2$, $1 < p < N$, and $a : \Omega \times (-l,l) \times \mathbb{R}^N \to \mathbb{R}^N$, is a Carathéodory function and $\nu : \Omega \to \mathbb{R}_+$ satisfies the following assumptions:

$$a(x,s,\xi) : \xi \geq b(|s|)\nu(x)|\xi|^p,$$

$$\nu \in L^r(\Omega), \quad r \geq 1, \quad \nu^{-1} \in L^t(\Omega), \quad t \geq N, \quad 1 + \frac{1}{t} < p < N(1 + \frac{1}{l}). \quad (2.3)$$

for a.e. $x \in \Omega$, for all $s \in (-l,l)$ and all $\xi \in \mathbb{R}^N$;

$$|a(x,s,\xi)| \leq \nu(x)[h(x) + b(|s|)|\xi|^{p-1}], \quad (2.4)$$

for a.e. $x \in \Omega$, for all $s \in (-l,l)$, for all $\xi \in \mathbb{R}^N$, and $h \in L^p(\Omega, \nu)$;

$$(a(x,s,\xi) - a(x,s,\xi')) \cdot (\xi - \xi') > 0, \quad (2.5)$$

for a.e. $x \in \Omega$, for all $s \in (-l,l)$ and all $\xi \in \mathbb{R}^N$, $\xi \neq \xi'$. Moreover, $f$ is a measurable function on whose summability we will make several assumptions.

For stating existence results in the next section, we need some classes of solutions.
Definition 2.1. We say that \( u \in W^{1,p}_0(\Omega, \nu) \) is a weak solution to problem (2.2) if
\[
\int_{\Omega} a(x, u, \nabla u) \cdot \nabla \varphi \, dx = \int_{\Omega} f \varphi \, dx, \quad \forall \varphi \in W^{1,p}_0(\Omega, \nu).
\] (2.6)

Definition 2.2. A measurable function \( u \in W^{1,p}_0(\Omega, \nu) \) is an entropy solution to problem (2.2) if
\[
|u| \leq l \quad \text{a.e. in} \ \Omega \quad (2.7)
\]
and for all \( 0 < k < l \),
\[
\int_{\Omega} a(x, u, \nabla u) \cdot \nabla T_k(u - \varphi) \, dx \leq \int_{\Omega} T_k(u - \varphi) \, dx, \quad (2.8)
\]
for any \( \varphi \in W^{1,p}_0(\Omega, \nu) \cap L^\infty(\Omega) \) such that \( \|\varphi\|_{L^\infty(\Omega)} < l - k \).

Weighted Sobolev spaces. Let \( 1 \leq p < N \), and \( \nu : \Omega \to \mathbb{R} \) be a weight function, i.e. a function which is measurable and positive almost everywhere in \( \Omega \). The weighted Lebesgue spaces \( L^p(\Omega, \nu) \) is defined as
\[
L^p(\Omega, \nu) = \left\{ u : \text{measurable, real-valued function}, \int_{\Omega} \nu(x)|u(x)|^p \, dx < \infty \right\}.
\]
which is a Banach space (uniformly convex and hence reflexive if \( p > 1 \)) equipped with the norm
\[
\|u\|_{L^p(\Omega, \nu)} = \left( \int_{\Omega} \nu(x)|u(x)|^p \, dx \right)^{1/p}.
\]
By \( W^{1,p}(\Omega, \nu) \) we denote the completion of the space \( C^1(\overline{\Omega}) \) with respect to the norm
\[
\|u\|_{W^{1,p}(\Omega, \nu)} = \|u\|_{L^p(\Omega, \nu)} + \|\nabla u\|_{L^p(\Omega, \nu)}.
\]
Moreover we denote by \( W^{1,p}_0(\Omega, \nu) \) the closure of \( C^1(\overline{\Omega}) \) in \( W^{1,p}(\Omega, \nu) \) which is normed by
\[
\|u\|_{W^{1,p}_0(\Omega, \nu)} = \|\nabla u\|_{L^p(\Omega, \nu)}.
\]
We denote by \( W^{-1,p'}(\Omega, 1/\nu) \) the dual space of \( W^{1,p}_0(\Omega, \nu) \); for more details see [16].

Rearrangement properties. We recall some definitions about decreasing rearrangement of functions. Let \( \Omega \) be a bounded open set of \( \mathbb{R}^N \) and \( u : \Omega \to \mathbb{R} \) a measurable function.

Definition 2.3. The distribution function of \( u \) is defined as
\[
\mu_u(t) = |\{ x \in \Omega : |u(x)| > t \}|, \quad t \geq 0.
\]
The function \( \mu_u \) is decreasing and right continuous.

Definition 2.4. The decreasing rearrangement of \( u \) is defined as
\[
u_*(s) := \sup\{ t \geq 0 : \mu_u(t) > s \}, \quad s \geq 0.
\]
The function \( \nu_* \) is the generalized inverse of \( \mu_u \). We recall that
\[
\int_{\Omega} |u|^p \, dx = p \int_0^{+\infty} t^{p-1} \mu_u(t) \, dt, \quad \text{for} \ p \geq 1. \quad (2.9)
\]
Then the \( L^p \)-norm, for \( 1 \leq p < +\infty \), is invariant with respect to rearrangement, that is,
\[
\|u\|_{L^p(\Omega)} = \|u_*\|_{L^p[0,|\Omega|]}.
\]
Moreover, if \( u \in L^\infty(\Omega) \), by definition \( u_*(0) = \text{ess sup}_\Omega |u| \). For more details about rearrangements we refer the reader to [6, 13, 18]. We recall that a measurable function \( u \) satisfies
\[
\mu_u(t) = \frac{c}{t^{1/\tau}}, \quad \forall t > 0,
\]
for some constant \( c \). We observe that the above condition is equivalent to
\[
u_u(s) = \frac{c}{s^{1/\tau}}, \quad \forall s > 0,
\]
and we define
\[
\|u\|_{MP(\Omega)} = \sup_{s > 0} u_*(s)^{1/\tau}.
\]
We observe that the above condition is equivalent to
\[
L^p(\Omega) \subset MP(\Omega) \subset L^q(\Omega),
\]
for \( 1 \leq q < p \). Now, we give a sense to the gradient of a function \( u \in L^1(\Omega) \) such that the truncates of \( u \) are Sobolev functions.

**Lemma 2.5 ([7]).** For each measurable function \( u : \Omega \to \mathbb{R} \) such that for every \( k > 0 \) the truncated function \( T_k(u) \) belong to \( W^{1,1}_{loc}(\Omega) \), there exists a unique measurable function \( v : \Omega \to \mathbb{R}^N \) such that
\[
\nabla T_k(u) = v\chi_{|u| < k} \quad \text{a.e. in } \Omega.
\]
Furthermore, \( u \in W^{1,1}_{loc}(\Omega) \) if and only if \( v \in L^1_{loc}(\Omega) \), and then \( v = \nabla u \) in the usual weak sense.

Now we recall some Sobolev-type inequalities which will be used later.

**Lemma 2.6 ([16]).** Let \( \nu \) be a nonnegative function on \( \Omega \) such that \( \nu \in L^r(\Omega) \), \( r \geq 1, \nu^{-1} \in L^t(\Omega), t \geq N \). And let \( p, p^* \) be two real number that satisfy \( t \geq N/p \), \( 1 + \frac{1}{t} < p < N(1 + \frac{1}{t}) \), \( 1/p^* = 1/p(1 + \frac{1}{t}) - \frac{1}{N} \). Then
\[
\|u\|_{p^*} \leq c_0 \|\nabla u\|_{L^p(\nu)}, \quad \forall u \in W^{1,p}_{0}(\Omega, \nu).
\]

**Lemma 2.7.** Suppose that \( \lambda > 0 \) and \( 1 \leq \gamma < +\infty \). Let \( \psi \) a non-negative measurable function on \((0, +\infty)\). Then the
\[
\int_0^{+\infty} \left( t^{-\lambda} \int_0^t \psi(s)ds \right)^\gamma \frac{dt}{t} \leq c \int_0^{+\infty} (t^{1-\lambda} \psi(t))^{\gamma} \frac{dt}{t}, \quad (2.11)
\]
\[
\int_0^{+\infty} \left( t^{\lambda} \int_t^{+\infty} \psi(s)ds \right)^\gamma \frac{dt}{t} \leq c \int_0^{+\infty} (t^{1+\lambda} \psi(t))^{\gamma} \frac{dt}{t}. \quad (2.12)
\]

Also we shall need the following proposition of weak approximation (see [5]). Let \( u \in W^{1,p}_0(\Omega) \), and for \( s \in [0, |\Omega|] \), let \( G(s) \) be a measurable subset of \( \Omega \) such that \( |G(s)| = s \)
\[
s_1 < s_2 \Rightarrow G(s_1) \subset G(s_2)
\]
\[
G(s) = \{ x \in \Omega : |u(x)| > t \} \quad \text{if } s = \mu(t).
\]
For a given a function \( \varphi \in L^1(\Omega) \), we set
\[
\phi(s) = \frac{d}{ds} \int_{G(s)} \varphi(x) \, dx.
\]
Lemma 2.8 ([7]). If $\varphi \in L^p(\Omega)$ with $p > 1$, then there exists a sequence $(\varphi_n)$ such that $\varphi^*_n(s) = \varphi^*(s)$ and $\varphi_n \rightharpoonup \varphi$ weakly in $L^p(0, |\Omega|)$.

3. Main result

The following Proposition gives a sufficient condition for the gradient of a function to belong to some Marcinkiewicz space. These are the generalized results of [7] in the Weighted Sobolev spaces $W_0^{1,p}(\Omega, \nu)$.

Proposition 3.1. Let $1 < p < N$, and $u \in \mathcal{T}_0^{1,p}(\Omega, \nu)$ be such that

$$\int_{|u| < k} |\nabla u|^p \nu(x) \, dx \leq Mk^\lambda$$

for every $k > 0$. Then $u \in \mathcal{M}^{p_1}(\Omega)$ where $p_1 = p^*(1 - \lambda/p)$. More precisely, there exists a $c$ such that $\text{meas}\{|u| > k\} = \text{meas}\{x \in \Omega : |u(x)| > k\} \leq ck^{-p_1}$.

Proof. For $k > 0$, from [23], we have

$$\|T_k(u)\|_{p_1} \leq c_1 \|\nabla T_k(u)\|_{L^p(\nu)} \leq c_1 k^\lambda/p.$$

For $0 < \varepsilon \leq k$, we have $\{x \in \Omega : |u| > \varepsilon\} = \{x \in \Omega : |T_k(u)| > \varepsilon\}$. Hence

$$\text{meas}\{|u| > \varepsilon\} \leq \left( \frac{\|T_k(u)\|_{p_1}}{\varepsilon} \right)^{p_1} \leq c_1 k^\lambda p_1 p/(\varepsilon^{p-1}).$$

Setting $\varepsilon = k$, we obtain $\text{meas}\{|u| > \varepsilon\} \leq c_1 k^{-p_1}$, where $p_1 = p^*(1 - \lambda/p)$. \hfill $\square$

Proposition 3.2. Let $1 < p < N$, and $u \in \mathcal{T}_0^{1,p}(\Omega, \nu)$ be such that

$$\int_{|u| < k} |\nabla u|^p \nu(x) \, dx \leq Mk^\lambda$$

for every $k > 0$. Then $\nu^{1/p} \nabla u \in \mathcal{M}^{p_2}(\Omega)$ where $p_2 = pp_1/(\lambda + p_1)$. More precisely, there exists a $c$ such that $\text{meas}\{\nu^{1/p} |\nabla u| > h\} \leq ch^{-p_2}$.

Proof. For $k, h > 0$. Set $\phi(k,\alpha) = \text{meas}\{\nu(x)|\nabla u|^p > \alpha, |u| > k\}$. From Proposition 3.1 we have

$$\phi(k,0) \leq c_1 k^{-p_1}.$$

Using that the function $\alpha \mapsto \phi(k,\alpha)$ is non-increasing, for $k, \lambda > 0$ we obtain

$$\phi(0,\alpha) \leq \frac{1}{\alpha} \int_0^\alpha \phi(0,s) \, ds$$

$$= \frac{1}{\alpha} \int_0^\alpha \phi(0,s) + \phi(k,0) - \phi(k,0) \, ds$$

$$\leq \phi(k,0) + \frac{1}{\alpha} \int_0^\alpha \phi(0,s) - \phi(k,0) \, ds$$

$$\leq \phi(k,0) + \frac{1}{\alpha} \int_0^\alpha \phi(0,s) - \phi(k,s) \, ds. \tag{3.1}$$

Since $\phi(0,s) - \phi(k,s) = \text{meas}\{\nu(x)|\nabla u|^p > s, |u| < k\}$ we have

$$\frac{1}{\alpha} \int_0^\alpha \phi(0,s) - \phi(k,s) \, ds = \frac{1}{\alpha} \int_{|u| < k} \nu(x)|\nabla u|^p \, dx \leq c k^\lambda \frac{\alpha}{\alpha},$$

which by (3.1) gives

$$\phi(0,\alpha) \leq c_1 k^{-p_1} + c_2 \frac{k^\lambda}{\alpha} \leq c_1 k^{-p_1} + c_2 \frac{k^\lambda}{\alpha}. \tag{3.2}$$
By minimizing (3.2) in \( k \) and setting \( \alpha = h^p \) we obtain
\[
\meas\{v^{1/p}|\nabla v| > k\} \leq ch^{-pp/(\lambda+p)}.
\]
\[\square\]

3.1. A priori estimate. Let \( \varepsilon \) be positive and sufficiently small. We consider the problem
\[
- \text{div} \ a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) = f_\varepsilon \quad \text{in } \Omega,
\]
\[
u u_\varepsilon = 0 \quad \text{on } \partial \Omega,
\]
where \( a_\varepsilon(x, s, \xi) = a(x, \mathcal{T}_\varepsilon(s), \xi) \), with \( x \in \Omega, s \in \mathbb{R} \) and \( \xi \in \mathbb{R}^N \) and \( f_\varepsilon \in L^\infty(\Omega) \). We use some classical results (see, for example [1, 2]) to assure that problem (3.3) has at least one solution \( u_\varepsilon \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \). Then, we define \( b_\varepsilon(t) = b(T_{l_\varepsilon(t)}) \) for all \( t \in [0, +\infty) \), and
\[
A_\varepsilon(s) = \int_0^s b_\varepsilon(r)^{1/(p-1)} dr.
\]
First, we prove an integral inequality for weak solutions of problem (3.3).

**Proposition 3.3.** Let \( u_\varepsilon \) be a weak solution of (3.3). Then
\[
A_\varepsilon(u_\varepsilon^*(s)) \leq C_N \int_{s}^{\|\Omega\|} r^{-\nu'/\nu'}[D(r)]^{\nu'/p} \left( \int_0^r f_\varepsilon^*(\sigma) d\sigma \right)^{p'/p} dr, \quad s \in [0, \|\Omega\|],
\]
where \( D : [0, \|\Omega\|] \to \mathbb{R} \) is a measurable function such that
\[
\int_{|u_\varepsilon| > y} \nu^{-t}(x) dx = \int_0^{\mu(y)} (D(r))^{t} dr.
\]
**Proof.** Let \( \phi = T_h(u_\varepsilon - T_\theta(u_\varepsilon)) \) be a test function in (3.3). Then we have
\[
\frac{1}{h} \int_{|u_\varepsilon| \leq \theta + h} b(|u_\varepsilon|)\nu(x)|\nabla u_\varepsilon|^p dx \leq \int_{|u_\varepsilon| > \theta} |f| dx
\]
Applying Hardy-Littlewood inequality and passing to the limit on \( h \) to 0, we obtain
\[
b(\theta) \left( -\frac{d}{d\theta} \int_{|u_\varepsilon| > \theta} \nu(x)|\nabla u_\varepsilon|^p dx \right) \leq \int_0^{\mu_{u_\varepsilon}(\theta)} f_\varepsilon^*(s) ds.
\]
On the other hand by Hölder inequality, we obtain
\[
-\frac{d}{d\theta} \int_{|u_\varepsilon| > \theta} |\nabla u_\varepsilon| dx \leq \left( -\frac{d}{d\theta} \int_{|u_\varepsilon| > \theta} \nu(x)|\nabla u_\varepsilon|^p dx \right)^{1/p}
\]
\[
\times \left( -\frac{d}{d\theta} \int_{|u_\varepsilon| > \theta} \nu^{-p'/p}(x) dx \right)^{1/p'}
\]
\[
\leq \left( -\frac{d}{d\theta} \int_{|u_\varepsilon| > \theta} \nu(x)|\nabla u_\varepsilon|^p dx \right)^{1/p}
\]
\[
\times \left( -\frac{d}{d\theta} \int_{|u_\varepsilon| > \theta} \nu^{-t}(x) dx \right)^{1/r_1 p'} (-\mu_{u_\varepsilon}(\theta))^{1/r_2 p'}.
\]
where \( 1/r_1 + 1/r_2 = 1 \) and \( p'/r_1/p = t \). By Lemma 2.8 since \( \nu^{-1} \in L^t(\Omega), t > 1 \) there exists \( D \in L^t([0, \|\Omega\|]) \) such that
\[
-\frac{d}{d\theta} \int_{|u_\varepsilon| > \theta} \nu^{-t}(x) dx = -\mu_{u_\varepsilon}(\theta)[D(\mu_{u_\varepsilon}(\theta))]^t.
\]
Then inequality (3.6), becomes
\[
-\frac{d}{d\theta} \int_{|u_\theta|>\theta} |\nabla u_\theta| \, dx \leq \left( -\frac{d}{d\theta} \int_{|u_\theta|>\theta} \nu(x)|\nabla u_\theta|^p \, dx \right)^{1/p} \\
\times \left( (-\mu'_{u_\theta}(\theta))^{1/p'} [D(\mu_{u_\theta}(\theta))]^{1/r_1} \right).
\] (3.7)

From isoperimetric inequality and Fleming-Rishel formula (see [15]), it follows that
\[
C_N b(\theta)^{1/p}(\mu_{u_\theta}(\theta))^{1/N'} \leq \left( -\frac{d}{d\theta} \int_{|u_\theta|>\theta} \nu(x)|\nabla u_\theta|^p \, dx \right)^{1/p} \\
\times \left( (-\mu'_{u_\theta}(\theta))^{1/p'} [D(\mu_{u_\theta}(\theta))]^{1/r_1} \right),
\] (3.8)
which by (3.5) gives
\[
b(\theta)^{1/(p-1)} \leq C_N(\mu_{u_\theta}(\theta))^{-p'/N'}(-\mu'_{u_\theta}(\theta))[D(\mu_{u_\theta}(\theta))]^{t/r_1} \left( \int_0^{\mu_{u_\theta}(\theta)} f_\epsilon^p(s) \, ds \right)^{1/p'}
\]
integrating between 0 and \(u_\epsilon(s)\) we obtain
\[
A(u_\epsilon(s)) \leq C_N \int_0^{u_\epsilon(s)} \left[ (\mu_{u_\theta}(\theta))^{-p'/N'}(-\mu'_{u_\theta}(\theta))[D(\mu_{u_\theta}(\theta))]^{t/r_1} \right] \left( \int_0^{\mu_{u_\theta}(\theta)} f_\epsilon^p(s) \, ds \right)^{1/p'} \, d\theta,
\] (3.9)
which gives the results. \(\square\)

**Remark 3.4.** Since \(1 + \frac{1}{r} < p < N(1 + \frac{1}{r})\), and \(t \geq N/p\), we have \(q'p / p \geq 1\) and \(q/r_1^t \geq 1\), where \(r_1 = t(p-1)\), which allows us to apply the Proposition 2.11 and Proposition 2.12 to prove estimation (3.10) and (3.11), below.

**Proposition 3.5.** Let \(u_\epsilon\) be a solution of (3.3).

(a) If \(1 < r < t N/(tp - N)\), then
\[
\|(A_\epsilon(|u_\epsilon|))q\|_{L^1(\Omega)} \leq c\|f\|_{L^q(\Omega)}^{q'p / p},
\] (3.10)
where \(q = rt N(p-1)/(t(N-rp) + r N)\).

(b) If \(r = 1\), then
\[
\|A_\epsilon(|u_\epsilon|)\|_{L^{N(r-1)/(N+t(N-p))}(\Omega)} \leq c\|f\|_{L^q(\Omega)}^{p' / p},
\] (3.11)

**Proof.** Case 1 < \(r < t N/(tp - N)\). Let us observe that \(A_\epsilon\) being monotone, by Proposition 3.3 properties of rearrangements, (2.12) and (2.11), we obtain
\[
\|(A_\epsilon(|u_\epsilon|))q\|_{L^1(\Omega)} \leq C_N \int_0^{+\infty} \left[ \int_s^{\int_{\Omega}} r^{-p'/N'} [D(r)]^{p'/p} \left( \int_0^r f_\epsilon(\sigma) \, d\sigma \right)^{p'/p'} \, dr \right]^{q / s} \, ds \leq C_N \int_0^{+\infty} \left[ \int_s^{\int_{\Omega}} r^{-\frac{p' q}{N'}} \left( \int_0^r f_\epsilon(\sigma) \, d\sigma \right)^{p'/p'} \, dr \right]^{q / s} \, ds
\]
\[
\leq C_N \int_0^{+\infty} \left[ \int_s^{\int_{\Omega}} r^{-\frac{p' q}{N'}} \left( \int_0^r f_\epsilon(\sigma) \, d\sigma \right)^{p'/p'} \, dr \right]^{q / s} \, ds
\]
\[
\leq C_N \int_0^{+\infty} \left[ \int_s^{\int_{\Omega}} r^{-\frac{p' q}{N'}} \left( \int_0^r f_\epsilon(\sigma) \, d\sigma \right)^{p'/p'} \, dr \right]^{q / s} \, ds
\]
\[
\leq C_N \int_0^{+\infty} \left[ \int_s^{\int_{\Omega}} r^{-\frac{p' q}{N'}} \left( \int_0^r f_\epsilon(\sigma) \, d\sigma \right)^{p'/p'} \, dr \right]^{q / s} \, ds
\]
\[ \leq C_N \int_{0}^{+\infty} \left[ s^{\frac{\alpha + t}{q'} - \frac{\nu + t}{p'}} s^{\frac{r}{p'}} f_*(s) \right] \frac{q'^{p}}{s} ds \]
\[ \leq C_N \int_{0}^{+\infty} \left[ s^{\frac{\alpha + t}{q'} - \frac{\nu + t}{p'}} s^{\frac{r}{p'}} f_*(s) \right] \frac{q'^{p}}{s} ds, \]

where \( \frac{q'}{p} \geq 1, \frac{\nu + t}{p'} = t, \) and \( C_N \) a constant that vary from line to line. Since \( f_\epsilon \in M^t(\Omega) \) we conclude that

\[ \|(A_\epsilon(|u_\epsilon|))''\|_{L^1(\Omega)} \leq C_N \int_{0}^{+\infty} (f_*(s))^{\frac{t}{q} - \frac{\nu}{p} + \frac{\nu + t}{p'}} ds \]
\[ \leq C_N \|f_*\|_{L^t([0,\Omega])}. \]

where

\[ r = -\frac{t r N (p - 1)}{t (N - r p) + r N}. \]

Case \( r = 1 \). By Proposition 3.3 and Hölder inequality, we have

\[ A_\epsilon(u_\epsilon(s)) \leq C_N \int_{0}^{[\Omega]} \left[ s^{r - \frac{\nu}{p'}} |D(r)|^{p'} \right] f_*(\sigma) ds \]
\[ \leq C_N \|D\|_{L^t([0,\Omega])} \left( \int_{0}^{[\Omega]} \left[ s^{r - \frac{\nu}{p'}} \right]^{\frac{p}{Nt(p - 1) - N + tp}} ds \right) \]
\[ \leq C_N \|D\|_{L^t([0,\Omega])} s^{1 - \frac{\nu}{Nt(p - 1) - N + tp}}, \]

which implies the result. \( \square \)

**Remark 3.6.** Since \( p/N < 1 + \frac{1}{t} \), (see (2.3)), we have

\[ \frac{N t p}{N t (p - 1) - N + tp} > 1. \]

**Proposition 3.7.** Let \( u_\epsilon \) be a solution of (3.3).

(a) If \( \frac{N t p}{N t (p - 1) - N + tp} < r < \frac{t N p}{N t (p - 1) + pt - N} \), then

\[ \|\nabla A_\epsilon(|u_\epsilon|)\|_{L^t(\Omega, r)} \leq c_1. \]

(b) If

\[ \max \left( 1, \frac{t N p}{N t (p - 1) p + pt - N} \right) < r < \frac{t N p}{N t (p - 1) + pt - N}, \]

then

\[ \|\nabla A_\epsilon(|u_\epsilon|)\|_{L^t(\Omega, r, \beta)} \leq c_2, \]

where \( \beta = \frac{r N t (p - 1) p}{r N + N t p - p t r} \).

(c) If

\[ 1 \leq r \leq \max \left( 1, \frac{t N p}{N t (p - 1) p + pt - N} \right), \]

then

\[ \|u^{1/p} \nabla A_\epsilon(|u_\epsilon|)\|_{M^\beta(\Omega)} \leq c_3, \]

where \( \beta = \frac{r N t (p - 1) p}{r N + N t p - p t r} \).
Proof. Let $u_\varepsilon$ be a solution of (3.3), by the definition of $A_\varepsilon$ we can use as test function $v = [T_b(A_\varepsilon(|u_\varepsilon|)) - T_\theta(A_\varepsilon(|u_\varepsilon|))] \text{sign}(u_\varepsilon)$ and obtain

$$
\int_{\theta < A_\varepsilon(|u_\varepsilon|) \leq \theta + \varepsilon} \nu(x)|\nabla A_\varepsilon(|u_\varepsilon|)|^p \, dx \leq \int_{A_\varepsilon(|u_\varepsilon|) > \theta} |f_\varepsilon| \, dx, \quad (3.16)
$$

Case 1: \(\frac{Nt}{Nt(p-1)-N+p} < r < \frac{tN}{tp-N}\). Passing to the limit in (3.16), we obtain

$$
\frac{d}{d\theta} \int_{A_\varepsilon(|u_\varepsilon|) \leq \theta} \nu(x)|\nabla A_\varepsilon(|u_\varepsilon|)|^p \, dx \leq \int_0^{\mu_\varepsilon(\theta)} f_\varepsilon^*(s) \, ds, \quad (3.17)
$$

where we have denoted with $\mu_\varepsilon(\theta)$ the distribution functions of $A_\varepsilon(|u_\varepsilon|)$. Integrating (3.17) between 0 and $+\infty$ and using a Hölder inequality, we have

$$
\int_{\Omega} \nu(x)|\nabla A_\varepsilon(|u_\varepsilon|)|^p \, dx \leq \int_0^{+\infty} \frac{d}{d\theta} \int_0^{\mu_\varepsilon(\theta)} f_\varepsilon^*(s) \, ds
$$

$$
= \int_0^{\int_{\Omega} A_\varepsilon(u_\varepsilon^*(s)) f_\varepsilon^*(s) \, ds
$$

$$
\leq ||f||_{L^r(\Omega)} ||A_\varepsilon(|u_\varepsilon|)||_{L^r(\Omega)}. \quad (3.18)
$$

We observe that if $r$ is such that \(\frac{Nt}{Nt(p-1)-N+p} \leq r < \frac{tN}{tp-N}\), by (3.10) the right-hand side of the above inequality is controlled by a constant depending on the norm of $f_\varepsilon$ in $L^r(\Omega)$; so by (3.18) inequality (3.13) follows.

Case 2: max \(1, \frac{Nt(p-1)+pt-N}{tp-N}\) \(< r < \frac{Nt}{Nt(p-1)+pt-N}\). Applying the Hölder inequality in (3.16) and reasoning as before, we obtain

$$
\int_{\Omega} |\nabla A_\varepsilon(|u_\varepsilon|)|^2 \nu^{\beta/p}(x) \, dx
$$

$$
\leq \int_0^{+\infty} \left( \int_0^{\mu_\varepsilon(\theta)} f_\varepsilon^*(s) \, ds \right)^{\beta/p} (-\mu_\varepsilon(\theta))^{1-\frac{\beta}{p}} \, d\theta
$$

$$
\leq \left( \int_0^{+\infty} (1 + \theta)^\eta (-\mu_\varepsilon(\theta)) \, d\theta \right)^{1-\frac{\beta}{p}} \times \left( \int_0^{+\infty} (1 + \theta)^{q(1-\frac{\beta}{p})} \left( \int_0^{\mu_\varepsilon(\theta)} f_\varepsilon^*(s) \, ds \right) \, d\theta \right)^{\beta/p}. \quad (3.19)
$$

By the properties of rearrangements, we can write the first integral on the right-hand side of (3.19) as

$$
\int_0^{+\infty} (1 + \theta)^\eta (-\mu_\varepsilon(\theta)) \, d\theta = \int_0^{[\Omega]} (1 + A_\varepsilon(u_\varepsilon^*))^\eta \, ds, \quad (3.20)
$$

and by (3.10) this quantity is bounded by a constant depending on the norm of $f_\varepsilon$ in $L^r(\Omega)$. On the other hand, integrating by parts the second integral on the right-hand side of (3.19) we have

$$
\int_0^{+\infty} (1 + \theta)^{q(1-\frac{\beta}{p})} \left( \int_0^{\mu_\varepsilon(\theta)} f_\varepsilon^*(s) \, ds \right) \, d\theta
$$

$$
\leq c \int_0^{[\Omega]} f_\varepsilon^*(s)((1 + A_\varepsilon(u_\varepsilon^*))^{q(1-\frac{\beta}{p})+1}) \, ds
$$

$$
\leq c ||f_\varepsilon||_{L^r(\Omega)} \left( \int_0^{[\Omega]} ((1 + A_\varepsilon(u_\varepsilon^*))^q) \, ds \right)^{1-\frac{\beta}{p}}. \quad (3.21)
$$
Applying again (3.10), by (3.19) it follows the estimate (3.14).

Case 3: \(1 \leq r \leq \max \left(1, \frac{tN_p}{Nt(p-1)p+pt-N} \right)\). Integrating inequality (3.17) between 0 and \(k\), we obtain

\[
\int_{A_{\epsilon}(u_{\epsilon}) \leq k} \nu(x)|\nabla A_{\epsilon}(|u_{\epsilon}|)|^p \, dx \leq \int_0^k \nu_x(t) f_x^*(s) \, ds. \tag{3.22}
\]

If \(r = 1\), from (3.22) we obtain

\[
\int_{A_{\epsilon}(u_{\epsilon}) \leq k} \nu(x)|\nabla A_{\epsilon}(|u_{\epsilon}|)|^p \, dx \leq k \|f_x\|_{L^1(\Omega)}.
\]

by (3.11) and (2.3) we obtain the assertion.

If \(1 \leq r \leq \max(1, tN_pNt(p-1)p+pt-N)\), then by (3.10) it follows that \(A_{\epsilon}(|u_{\epsilon}|) \in M^q(\Omega)\), with \(q = \frac{rNt}{Nt + N} + \frac{rNt}{Nt - pt} - pt\); so we obtain

\[
\int_{A_{\epsilon}(u_{\epsilon}) \leq k} \nu(x)|\nabla A_{\epsilon}(|u_{\epsilon}|)|^p \, dx \leq ck^{1 - \frac{q}{r}}
\]

by Proposition 3.2, we conclude the result. \(\Box\)

Replacing \(\nabla A_{\epsilon}(|u_{\epsilon}|)\) by \(\nabla u_{\epsilon}\) the above estimates also hold; furthermore it follows that

\[
\int_{\Omega} \nu(x)|\nabla u_{\epsilon}|^\gamma \, dx \leq c,
\]

with \(\gamma < \frac{Nt(p-1)}{Nt + N + Nt - pt}\), where \(c\) is a constant depending on the \(L^1(\Omega)\) norm of \(f_x\). Using (3.5), the \(T_k(u_{\epsilon})\) are uniformly bounded in \(W^{1,p}_0(\Omega, \nu)\) for any \(k > 0\). Hence, there exists a function \(u \in W^{1,\gamma}_0(\Omega, \nu)\) such that

\[
u(x)|\nabla u_{\epsilon}|^\gamma \, dx \leq c,
\]

\[
u(x)|\nabla u_{\epsilon}|^\gamma \, dx \leq c\tag{3.23}
\]

and, for any \(k > 0\),

\[
T_k(u_{\epsilon}) \rightarrow T_k(u) \quad \text{weakly in } W^{1,p}_0(\Omega, \nu). \tag{3.24}
\]

**Remark 3.8.** Choosing \(k > l\), we have

\[
u(x)|\nabla u_{\epsilon}|^\gamma \, dx \leq c\tag{3.25}
\]

Indeed, let us suppose \(f \in L^1(\Omega)\). Using \(T_{2l}(|u_{\epsilon}|) - T_l(|u_{\epsilon}|)\) as test function in (3.3), by (2.3) we obtain

\[
b(l - \varepsilon) \int_{\Omega} (T_{2l}(|u_{\epsilon}|) - T_l(|u_{\epsilon}|)) |u_{\epsilon}|^p \, dx \leq l\|f_x\|_{L^1(\Omega)}.
\]

Letting \(\varepsilon \rightarrow 0\), from condition (2.1), we conclude that, for almost all \(x \in \Omega, |u| \leq l\), which give the result by (3.24).

Next we prove a lemma needed for proving the existence result.

**Lemma 3.9.** Let \(u_{\epsilon}\) be a weak solution to problem (3.3). Suppose \(f \in L^1(\Omega)\), and let \(f_x \in L^\infty(\Omega)\) be such that \(f_x \rightarrow f \) in \(L^1(\Omega)\). Then

\[
\nabla u_{\epsilon} \rightarrow \nabla u \quad \text{a.e. in } \{|u| < l\}.
\]
Proof. We adapt the proof presented in [11]. By Remark 3.8, we have \( u_\varepsilon \to u \) in measure. We will prove that \( u_\varepsilon \to u \) in measure on \( \{|u| < m\} \). Let \( \lambda > 0 \) and \( \eta > 0 \) for \( 0 < k < l \), and \( M > 0 \), we set

\[
E_1 = \{|u| < l\} \cap \{|\nabla u_\varepsilon| > M\} \cup \{|\nabla u| > M\} \cup \{|u_\varepsilon| > k\} \cup \{|u| > k\},
\]

\[
E_2 = \{|u| < l\} \cap \{|u_\varepsilon - u| > \eta\},
\]

\[
E_3 = \{|u_\varepsilon - u| \leq \eta, |\nabla u_\varepsilon| \leq M, |\nabla u| \leq M, |u_\varepsilon| \leq k, |u| \leq k, |\nabla (u_\varepsilon - u)| \geq \lambda\}
\]

\[ \cap \{|u| < l\}. \]

Observe that \( \{|u| < l\} \cap \{|\nabla u_\varepsilon| \geq \lambda\} \subset E_1 \cup E_2 \cup E_3. \)

Since \( u_\varepsilon \) and \( \nabla u_\varepsilon \) are bounded in \( L^1(\Omega) \), for any \( \sigma > 0 \) we can fix \( M \) and \( k < l \) such that \( |E_1| < \sigma/3 \) independently of \( \varepsilon \). By the monotonicity Assumption (2.5), there exists a real valued function \( \gamma \) such that

\[
\text{meas}\{x \in \Omega : \gamma(x) = 0\} = 0,
\]

\[
(a(x, s, \xi) - a(x, s, \xi'))(\xi - \xi') \geq \gamma(x),
\]

for any \( s \in (-l, l), \xi, \xi' \in \mathbb{R}^N, |s| \leq k, |\xi|, |\xi'| \leq M \), and \( |\xi - \xi'| \geq \lambda \). Denoting by \( \chi_\eta \) the characteristic function of \( [0, \eta] \), we obtain

\[
\int_{E_3} \gamma(x) \, dx \leq \int_{E_3} \left[ a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) - a(x, u_\varepsilon, \nabla u_\varepsilon)\right][\nabla u_\varepsilon - u](\nabla u_\varepsilon - u) \, dx
\]

\[ \leq \int_{\{|u_\varepsilon| \leq k, |u| \leq k\}} \left[ (a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) - a(x, u_\varepsilon, \nabla T_k(u))) \right] \, dx
\]

\[ \times \left( \nabla u_\varepsilon - T_k(u) \right) \chi_\eta(|u_\varepsilon - T_k(u)|) \, dx
\]

\[ \leq \int_{\Omega} \left[ (a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) - a(x, u_\varepsilon, \nabla T_k(u))) \right] \, dx
\]

\[ \times \left( \nabla u_\varepsilon - T_k(u) \right) \chi_\eta(|u_\varepsilon - T_k(u)|) \, dx
\]

\[ - \int_{\Omega} a_\varepsilon(x, u_\varepsilon, \nabla T_k(u)) \cdot (\nabla u_\varepsilon - T_k(u)) \chi_\eta(|u_\varepsilon - T_k(u)|) \, dx
\]

\[ := J_1 - J_2. \]

For the term \( J_1 \), using \( T_\eta(u_\varepsilon - T_k(u)) \), we have

\[
|J_1| = \left| \int_{\Omega} f_{x_\varepsilon} T_\eta(|u_\varepsilon - T_k(u)|) \, dx \right| \leq \eta \| f \|_{L^1(\Omega)}.
\]

Choosing \( \eta > 0 \) such that \( k + \eta < l \), there exists \( \varepsilon_0 > 0 \) such that for all \( \varepsilon < \varepsilon_0 \),

\[
a_\varepsilon(x, u_\varepsilon, \nabla T_k(u)) = a(x, u_\varepsilon, \nabla T_k(u)) \quad \text{in} \quad \{x \in \Omega : |u_\varepsilon - T_k(u)| \leq \eta\};
\]

and since \( \{x \in \Omega : |u_\varepsilon - T_k(u)| \leq \eta\} \subset \{x \in \Omega : |u_\varepsilon| \leq k + \eta\} \) we obtain

\[
J_2 = \int_{\Omega} a(x, u_\varepsilon, \nabla T_k(u)) \cdot \nabla T_\eta(u_\varepsilon - T_k(u)) \, dx
\]

\[ = \int_{\Omega} a(x, T_{k+\eta}(u_\varepsilon), \nabla T_k(u)) \cdot (\nabla T_{k+\eta}(u_\varepsilon - T_k(u))) \chi_\eta(|u_\varepsilon - T_k(u)|) \, dx.\]
By (3.24), it follows that
\[ T_{k+\eta}(u_\epsilon) \to T_{k+\eta}(u) \quad \text{weakly in } W^{1,p}_0(\Omega, \nu), \]
on the other hand
\[ |a(x, T_{k+\eta}(u_\epsilon), \nabla T_k(u))| \leq b(|T_{k+\eta}(u_\epsilon)|) \nu(x)|\nabla T_{k+\eta}(u)|^{p-1} \]
using Vitali’s theorem we have
\[ a(x, T_{k+\eta}(u_\epsilon), \nabla T_k(u)) \to a(x, T_{k+\eta}(u), \nabla T_k(u)) \quad \text{strongly in } L^p(\Omega, \nu^{-1/(p-1)}). \]

Letting \( \epsilon \) and \( \eta \) tend to 0 respectively in \( J_2 \), we obtain
\[
\lim_{\epsilon \to 0} \int_\Omega a(x, u_\epsilon, \nabla T_k(u)) \cdot \nabla \chi_\eta(|u_\epsilon - T_k(u)|) \, dx = 0.
\]
and
\[
\lim_{\eta \to 0} \int_\Omega a(x, T_{k+\eta}(u), \nabla T_k(u)) \cdot (\nabla T_{k+\eta}(u - T_k(u))) \chi_\eta(|u_\epsilon - T_k(u)|) \, dx = 0.
\]
For \( \eta \) small enough \( \eta \|f\|_{L^1(\Omega)} < \delta/2 \), by Kolmogorov theorem, we have \( |E_3| < \sigma \) independently of \( \epsilon \). Fix \( \eta \), by the fact that \( u_\epsilon \to u \) in measure, we choose \( \epsilon_1 \) such that \( |E_2| \leq \eta \) for \( \epsilon \leq \epsilon_1 \). This implies that \( \nabla u_\epsilon \to \nabla u \) in measure in \( \{|u| < l\} \), consequently
\[ \nabla u_\epsilon \to \nabla u \quad \text{a.e. in } \{|u| < l\}. \]

We observe that since \( u_\epsilon \to u \) a.e. in \( \Omega \) (see (3.23)), we have
\[
\{x \in \Omega : |u(x)| = l\} = \left\{ x \in \Omega : \lim_{\epsilon \to 0} \int_0^{[u_\epsilon(x)]} b_\epsilon(t) \, dt \geq \int_0^l b(t) \, dt \right\}. \tag{3.26}
\]

**Theorem 3.10.** Let \( f \) be a function in \( L^s(\Omega) \), with \( r > tN/(tp - N) \). Assume that (2.1), (2.3) hold. Then there exists a weak solution \( u \in W^{1,p}_0(\Omega, \nu) \) of problem (2.2) such that \( \|u\|_{L^\infty(\Omega)} < l \).

**Proof.** For \( f_\epsilon = f \) with \( \epsilon > 0 \). By classical results see for example [2, 11], there exists a solution \( u_\epsilon \in W^{1,p}_0(\Omega, \nu) \) of the approximated problem (2.2). Estimate (3.4) implies
\[
A_\epsilon(\|u_\epsilon\|_{L^\infty}) \leq C(f) = C_N \int_0^{[2]} r^{-p'/N'} [D(r)]^{p'/p} \left( \int_0^{r} f_\epsilon^*(\sigma) \, d\sigma \right)^{p'/p} \, dr. \tag{3.27}
\]
Since \( A \) is bijective in \([0, l] \), we can take \( B = A^{-1}(C(f)) \) and then we choose \( \epsilon_0 > 0 \) such that \( b(s) \leq b(l - \epsilon) \) for any \( s \in [0, B] \). By definition of \( b_\epsilon \) and \( A_\epsilon \) we have, for any \( \epsilon < \epsilon_0 \),
\[ A_\epsilon(s) = A(s), \quad s \in [0, B]. \]
Moreover, being \( A_\epsilon \) increasing, it follows that, for any \( \epsilon < \epsilon_0 \),
\[ A_\epsilon(s) \leq C(f) \iff s \in [0, B], \]
so by (3.27) we obtain
\[ \|u_\epsilon\|_{L^\infty} \leq B < l. \]
By (2.2) and Lemma 3.9, we have
\[ a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \rightarrow a(x, u, \nabla u) \quad \text{strongly in } L^{p'}(\Omega, \nu^{-1/(p-1)}), \]
\[ f_\varepsilon \rightarrow f \quad \text{strongly in } L^\infty(\Omega). \]

Passing to the limit in the weak formulation of problem (3.3), we conclude that there exists a weak solution \( u \) of (2.2), which satisfies \( \|u\|_{L^\infty(\Omega)} < l \).

\[ \text{Theorem 3.11.} \quad \text{Let } u \in W^{1,p}_0(\Omega, \nu) \text{ be a weak solution to the approximate problem (3.3). By passing to the limit we can show that } \lim_{\varepsilon \to 0} u_\varepsilon \to u \text{ a.e. in } \Omega. \]

\[ \text{Proof.} \quad \text{Let } u_\varepsilon \in W^{1,p}_0(\Omega, \nu) \text{ be a weak solution to the approximate problem (3.3). By Remark (3.8), we have } u_\varepsilon \to u \text{ a.e. in } \Omega, \text{ since } A(l^-) = +\infty. \]

(2.2) implies that
\[ A_\varepsilon(|u_\varepsilon|) \to A(|u|) \quad \text{a.e. in } \Omega. \]

By (2.3) and (3.13), we obtain
\[ A_\varepsilon(|u_\varepsilon|) \to A(|u|) \quad \text{weakly in } W^{1,p}_0(\Omega, \nu), \]
\[ \text{Since } A(|u|) \text{ is bounded in } L^1(\Omega) \text{ and } \text{meas}\{x \in \Omega : |u(x)| = l\} = 0, \text{ by (2.3) we have} \]
\[ a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \to a(x, u, \nabla u) \quad \text{a.e. } \Omega. \]

On the other hand by (2.3) and (3.13),
\[ |a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)| \text{ is bounded in } L^{p'}(\Omega, \nu^{-1/(p-1)}); \]

passing to the limit in the weak formulation (3.3), we obtain
\[ \int_{\Omega} a(x, u, \nabla u) \cdot \nabla \varphi \, dx = \int_{\Omega} f \varphi \, dx, \quad \text{for all } \varphi \in W^{1,p}_0(\Omega, \nu). \]

\[ \text{Theorem 3.12.} \quad \text{Let } f \in L^r(\Omega), \text{ with } 1 \leq r < \frac{Ntp}{Nt(p-1)-Ntp}. \text{ Under hypothesis (2.1) -- (2.5), there exists a solution } u \in W^{1,p}_0(\Omega, \nu) \text{ of problem (2.2), in the sense of Definition (2.2) such that } \text{meas}\{x \in \Omega : |u(x)| = l\} = 0. \]

\[ \text{Proof.} \quad \text{Let } u_\varepsilon \text{ be a weak solution of the approximate problem (3.3), by passing to the limit we can show that } |u| < l \text{ a.e. in } \Omega. \text{ Take } T_k(u_\varepsilon - \varphi), \text{ with } \varphi \in W^{1,p}_0(\Omega, \nu) \cap L^\infty(\Omega) \text{ as test function in (3.3) we obtain} \]
\[ \int_{|u_\varepsilon - \varphi| \leq k} a(x, T_{l^-}(u_\varepsilon), \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \, dx \]
\[ - \int_{|u_\varepsilon - \varphi| \leq k} a(x, T_{l^-}(u_\varepsilon), \nabla u_\varepsilon) \cdot \nabla \varphi \, dx \]
\[ = \int_{\Omega} f_\varepsilon T_k(u_\varepsilon - \varphi) \, dx. \]

Since \( \{u_\varepsilon - \varphi\} \subseteq \{|u_\varepsilon| \leq k + \|\varphi\|_{L^\infty(\Omega)} = M\}, \text{ for } 1 < k < l \text{ and } \|\varphi\|_{L^\infty(\Omega)} < l - k, \text{ we obtain } M < l \text{ and consequently } |a(x, T_M(u_\varepsilon), \nabla T_M(u_\varepsilon))| \text{ is bounded in } L^{p'}(\Omega, \nu^{-1/(p-1)}), \]
\[ \lim_{\varepsilon \to 0} \int_{|u_\varepsilon - \varphi| \leq k} a(x, T_{l^-}(u_\varepsilon), \nabla u_\varepsilon) \cdot \nabla \varphi \, dx = \int_{|u-\varphi| \leq k} a(x, u, \nabla u) \cdot \nabla \varphi \, dx. \]
Moreover since $f_\varepsilon$ strongly convergent to $f$ in $L^1(\Omega)$, and $T_k(u_\varepsilon - \varphi)$ weakly* convergent to $T_k(u - \varphi)$ in $L^\infty(\Omega)$, we have
\begin{equation}
\lim_{\varepsilon \to 0} \int_\Omega f_\varepsilon T_k(u_\varepsilon - \varphi) \, dx = \int_\Omega f T_k(u - \varphi) \, dx. \tag{3.32}
\end{equation}

On the other hand $a(x, T_{1-\varepsilon}(u_\varepsilon), \nabla u_\varepsilon) \cdot \nabla u_\varepsilon$ being non-negative, and almost everywhere convergent to $a(x, u, \nabla u) \cdot \nabla u$, by Fatou’s lemma we conclude that
\begin{equation}
\liminf_{\varepsilon \to 0} \int_{|u_\varepsilon - \varphi| \leq k} a(x, T_{1-\varepsilon}(u_\varepsilon), \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \, dx \leq \int_{|u - \varphi| \leq k} a(x, u, \nabla u) \cdot \nabla u \, dx. \tag{3.33}
\end{equation}

Combining (3.31), (3.32) and (3.33) we obtain
\[ \int_\Omega a(x, u, \nabla u) \cdot \nabla T_k(u - \varphi) \, dx \leq \int_\Omega f T_k(u - \varphi) \, dx, \quad \text{for all } \varphi \in W_0^{1,p}(\Omega, \nu). \]

\[ \square \]

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