PYRAMIDAL TRAVELING FRONTS IN THE BELOUSOV-ZHABOTINSKII REACTION-DIFFUSION SYSTEMS IN $\mathbb{R}^3$

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Abstract. In this article, we consider a diffusion system with the Belousov-Zhabotinskii (BZ for short) chemical reaction. The existence and stability of V-shaped traveling fronts for the BZ system in $\mathbb{R}^2$ had been proved in our previous papers [30, 31]. Here we establish the existence and stability of pyramidal traveling fronts for the BZ system in $\mathbb{R}^3$.

1. Introduction

Consider the reaction-diffusion system

$$
\begin{align*}
    u_t(x,t) &= \Delta u(x,t) + u(x,t)(1 - u(x,t) - rv(x,t)), \\
    v_t(x,t) &= \Delta v(x,t) - bu(x,t)v(x,t),
\end{align*}
$$

(1.1)

where $r$, $b > 0$ are positive parameters and $u$, $v$ correspond to the concentrations of the bromous acid and bromide ion respectively. (1.1) is called the BZ system, which stems from a typical chemical oscillating reaction. The possible existence of such chemical oscillation was predicted by Turing [39] through the method of mathematical calculation, and the chemical phenomenon was observed by Belousov [1]. When the concentrations of the reactants change orderly along with time and space, chemical waves appear [47]. To investigate the mechanism of the BZ reaction, Field and his coworkers formulated a complex model [8] and then simplified it [9]. Later, the simplified model was nondimensionalized by Murray [25, 26] to be (1.1). It is found that the front solution of (1.1) is an appropriate mathematical tool to describe the planar waves [47].

After that, mathematical studies on system (1.1), a lot of progress has been made, mainly including the existence of 1-D traveling wave fronts, admissible traveling speeds and the asymptotic behavior of traveling wave fronts [12, 20, 21, 22, 37, 38, 41]. In fact, studies of traveling wave solutions on reaction-diffusion equations

$$
    u_t(x,t) = \Delta u(x,t) + f(u(x,t)), \quad x \in \mathbb{R}^N, \ t > 0,
$$

which can be originated from the pioneer work of Fisher [6], have attracted a lot of attention [4, 10, 11, 23, 24, 42], which mainly focus on 1-D traveling wave solutions and planar traveling wave solutions in $\mathbb{R}^N (N \geq 2)$. The gradually mature theory of

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1-D traveling wave solutions promotes the research on multidimensional traveling wave solutions, which are proper to describe the traveling wave phenomena in multidimensional space, see [2, 3, 13, 14, 15, 28, 29, 34, 35, 44] for the scalar equation and [3, 16, 17, 18, 19, 27, 36, 43, 45, 46] for the reaction diffusion system.

For the BZ reaction, along with the development of research, nonplanar chemical waves were also observed. In 1995, stable V-shaped chemical waves were observed in the BZ reaction [40], for which we have already made rigorous mathematical proofs [30, 31]. However, mathematical studies of multidimensional nonplanar traveling waves on the BZ system (1.1) are still very few.

In this article, we continue to study the pyramidal traveling fronts of (1.1) and expect to give some theoretical implications to the observation of new nonplanar waves on the BZ system (1.1). We still want to emphasize that the case ‘bistable’ for the BZ system is different from those in Ni and Taniguchi [27] and Wang [45]. Since the acceleration effect of the curvature, it is natural to take $s > 1$.

Now we set $u_1(x, t) = u(x, t), u_2(x, t) = 1 - v(x, t), u = (u_1, u_2)$, then system (1.1) can be rewritten as

$$u_t = \Delta u + F(u), \quad (1.2)$$

where $F(u) = (f_1(u), f_2(u)) = (u_1(1 - r - u_1 + ru_2), bu_1(1 - u_2))$. Under the condition $r > 1$, Theorem 9 tells that system (1.2) admits a unique positive traveling front $U(\xi) = (U_1(\xi), U_2(\xi))$ satisfying

$$U''_1(\xi) - cU'_1(\xi) + f_1(U(\xi)) = 0, \quad U''_2(\xi) - cU'_2(\xi) + f_2(U(\xi)) = 0 \quad (1.3)$$

with

$$U(-\infty) = (0, 0), \quad U(+\infty) = (1, 1), \quad 0 < U_1(\xi) < U_2(\xi) < 1, \quad \forall \xi \in \mathbb{R},$$

where $c \in \left( b/(2\sqrt{r + b})\min(1, b)(r + b) - 0.5b), 2\sqrt{\min(1, b)} \right)$ is the wave speed, see [47] Theorem 6, Theorem 9 and Proposition 10].

Denote $x' = (x_1, x_2), \ x = (x', x_3)$. Without loss of generality, we assume that the traveling solutions travel towards the $-x_3$ direction with speed $s$, then they have the form $u(x, t) = v(x', x_3 + st)$ and satisfy

$$v_t = \Delta v - svx_3 + F(v), \quad x \in \mathbb{R}^3, \ t > 0, \quad (1.4)$$

$$v|_{t=0} = v_0(x), \quad x \in \mathbb{R}^3. \quad (1.5)$$

We aim to find a nontrivial steady state $V(x)$ of the system

$$-\Delta V + svx_3 - F(V) = 0, \quad x \in \mathbb{R}^3. \quad (1.6)$$

Since the acceleration effect of the curvature, it is natural to take $s > c$. Fix $s$ and set

$$m_s = \sqrt{s^2 - c^2}/c.$$

Now we introduce the definition of pyramid. Let $n \geq 3$ be a given integer. Assume $\{(A_j, B_j)\}_{j=1}^n$ is a set of unit vectors, i.e. $A_j^2 + B_j^2 = 1$ for all $j \in \{1, 2, \ldots, n\}$, and satisfies

$$A_j B_{j+1} - A_{j+1} B_j > 0, \quad 1 \leq j \leq n - 1; \quad A_n B_1 - A_1 B_n > 0.$$
We also assume that \((A_j, B_j) \neq (A_i, B_i)\) if \(i \neq j\). Now \((-m_A A_j, -m_B B_j, 1)\) is a normal vector of the plane \(\{x \in \mathbb{R}^3 : -x_3 = m_A (A_j x_1 + B_j x_2)\}\). We put
\[
\begin{align*}
    h_j(x') &= m_A (A_j x_1 + B_j x_2), \\
    h(x') &= \max_{1 \leq j \leq n} h_j(x') = m_A \max_{1 \leq j \leq n} (A_j x_1 + B_j x_2).
\end{align*}
\]
Then \(-x_3 = h(x')\) represents a pyramid in \(\mathbb{R}^3\). Set \(\Omega_j = \{x' \in \mathbb{R}^2 : h(x') = h_j(x')\}\), \(j \in \{1, 2, \ldots, n\}\). Then \(\mathbb{R}^2 = \bigcup_{j=1}^n \Omega_j\). Denote
\[
E := \bigcup_{j=1}^n \partial \Omega_j \subset \mathbb{R}^2.
\]
Now the lateral surfaces of a pyramid are \(S_j = \{x \in \mathbb{R}^3 : -x_3 = h_j(x'), x' \in \Omega_j\}\) for \(j = 1, 2, \ldots, n\). We put
\[
\Gamma_j = \begin{cases} 
    S_j \cap S_{j+1} & \text{if } 1 \leq j \leq n-1, \\
    S_n \cap S_1 & \text{if } j = n.
\end{cases}
\]
Then \(\Gamma_j\) represents an edge of the pyramid and \(\Gamma = \bigcup_{j=1}^n \Gamma_j\) represents the set of all edges. Denote
\[
v^-(x) = U \left( \frac{c}{s} (x_3 + h(x')) \right) = \max_{1 \leq j \leq n} U \left( \frac{c}{s} (x_3 + h_j(x')) \right).
\]
Since \(U(x)\) is a planar traveling wave of (1.2), it is easy to see that \(v^-(x)\) is combined with several such planar traveling waves and thus becomes a nonplanar traveling wave with pyramidal level sets. We also define
\[
D(\gamma) := \{x \in \mathbb{R}^3 : \text{dist}(x, \Gamma) > \gamma\}, \quad \forall \gamma > 0.
\]
We now define the a relation of order in \(\mathbb{R}^3\). We say that \(x < y\) (resp. \(x \leq y\)) An interval \([x_1, x_2] \subset \mathbb{R}^3\) denotes the set of \(x \in \mathbb{R}^3\) with \(x_1 \leq x \leq x_2\). Throughout this paper, we denote \(0 = (0, 0)\) and \(1 = (1, 1)\). The following theorem is the main assertion in this paper.

**Theorem 1.1.** Assume that \(r > 1\) and \(b > 0\). Then for each \(s > c\), there exists a solution \(V(x) = (V_1(x), V_2(x))\) to (1.6) with
\[
    v^-(x) < V(x) < 1 \quad \text{in } \mathbb{R}^3
\]
and
\[
    \lim_{\gamma \to \infty} \sup_{x \in D(\gamma)} \left\{ \frac{|V_i(x) - v^-_i(x)|}{(v^-_i(x))^\beta_i} \right\} = 0, \quad i = 1, 2.
\]
Furthermore, for any \(u_0(x) \in C(\mathbb{R}^3, \mathbb{R}^2)\) with \(u_0(x) \in [0, 1]\) for \(x \in \mathbb{R}^3\) and
\[
    \lim_{\gamma \to \infty} \sup_{x \in D(\gamma)} \left\{ \frac{|V_i(x) - u_0, i(x)|}{(v^-_i(x))^\beta_i} \right\} = 0, \quad i = 1, 2,
\]
the solution \(u(x, t; u_0)\) of (1.2) with initial data \(u_0\) satisfies
\[
    \lim_{t \to \infty} \left\| u_i(\cdot, \cdot; t; u_0) - V_i(\cdot, \cdot; + st) \right\|_{L^\infty(\mathbb{R}^3)} = 0, \quad i = 1, 2.
\]
Here, \(0 < \beta_2 < \beta_1 < \beta^*\) are arbitrary (see (3.4) for \(\beta^*\)).
To use the comparison argument, we consider a modified system
\[ \tilde{\varphi}_t = \Delta \tilde{\varphi} - s \tilde{\varphi}_{x_3} + \tilde{F}(\tilde{\varphi}), \quad x \in \mathbb{R}^3, \ t > 0, \]  
\[ \tilde{\varphi}|_{t=0} = v_0(x, z), \quad x \in \mathbb{R}^3, \]  
where \( \tilde{F}(u) = (\tilde{f}_1(u), \tilde{f}_2(u)) = F(u) + G(u), \ G(u) = (g_1(u), g_2(u)) \) with
\[ g_1(u) = 0, \quad g_2(u) = b(u_1 - 1) \max \{0, u_2 - 1\}. \]
It is clear that both \( F(u) \) and \( \tilde{F}(u) \) are Lipschitz continuous in \( \mathbb{R}^2 \). Obviously,
\[ \partial u_2 f_1 \geq 0, \quad \partial u_1 \tilde{f}_2 \geq 0 \]  
if \((u_1, u_2) \in [0, +\infty) \times [0, +\infty)\).

Then the comparison principle (see [32]) gives
\[ \tilde{\varphi}(x, t; v_0^1) \leq \tilde{\varphi}(x, t; v_0^2), \quad \forall x \in \mathbb{R}^3, \ t \geq 0 \]
if \( 0 \leq v_0^1(x) \leq v_0^2(x) \) in \( \mathbb{R}^3 \), where \( \tilde{\varphi}(x, t; v_0) \) denotes the solution of (1.9) and (1.10). In particular, it holds
\[ \tilde{\varphi}(x, t; v_0) \in [0, 1] \]  
if \( v_0(x) \in [0, 1], \ \forall x \in \mathbb{R}^3 \),
which implies that the interval \( [0, 1] \) is invariant for the solution of (1.9) and (1.10). Thus, for \( v_0(x) \in [0, 1], \ \)the solution \( \tilde{\varphi}(x, t; v_0) \) of (1.9) and (1.10) is also the solution of (1.4) and (1.5), namely, \( \tilde{\varphi}(x, t; v_0) \equiv \varphi(x, t; v_0) \), where \( \varphi(x, t; v_0) \) denotes the solution of (1.4) and (1.5).

For each unit vector \((A_j, B_j)\), (1.6) admits a solution \( U(\xi(x_3 + h_j(x^0))) \), which is called a planar wave. It follows that the function \( v^-(x) \) defined by (1.7) is a subsolution of (1.9), and obviously \( v^-(x) > 0 \). Throughout this paper, we define the operator \( \mathcal{L} \) by
\[ \mathcal{L}[\varphi] := \varphi_t - \Delta \varphi + s \varphi_{x_3} - \tilde{F}(\varphi). \]

The remainder of this paper is organized as follows: in Section 2 we give some notation and known results. In Section 3 we prove the existence result of pyramidal fronts by constructing an appropriate supersolution. And in Section 4 we prove the asymptotic stability of the pyramidal traveling fronts constructed in Section 3.

2. Preliminaries

In this section, we give some notation and known results. By [30] Lemma 1.1 or [37] Lemma 13, we have
\[ \lim_{\xi \to -\infty} \frac{U_2'(\xi)}{U_2(\xi)} = \lambda_2 = c. \]
Thus we can define
\[ N_1 := \sup_{x \in \mathbb{R}} \left| \frac{U_2'(x)}{U_2(x)} \right|, \quad N_2 := \sup_{x \in \mathbb{R}} \left| \frac{U_2''(x)}{U_2(x)} \right|. \]  
[30] Lemma 1.1 also implies that there exist two positive constants \( L_1 < 1 \) and \( L_2 > 1 \) such that
\[ L_1 e^{\max \{\lambda_1, 2\lambda_2\} \xi} < U_1(\xi), U_1'(\xi) < L_2 e^{\min \{\lambda_1, 2\lambda_2\} \xi}, \quad \xi < 0. \]  
\[ L_1 e^{\lambda_1 \xi} < U_2(\xi), U_2'(\xi) < L_2 e^{\lambda_2 \xi}, \quad \xi < 0. \]
Using (1.2), the derivative matrix of $F$ is

$$DF(u) = (f_{ij}(u))_{2 \times 2} = \begin{pmatrix} 1 - r - 2u_1 + ru_2 & ru_1 \\ b(1 - u_2) & -bu_1 \end{pmatrix}$$

where $f_{ij}(u) = \frac{\partial f_i}{\partial u_j}$. Because of $r > 1$, we can find two positive numbers $p_1, p_2$ that satisfy $p_1 > p_2 \geq 1$ and $\frac{p_1}{p_2} > r$. Let $p = (p_1, p_2)^T$, where $T$ means the transpose. We have

$$q = (q_1, q_2)^T := DF(1) \cdot p < 0,$$

Fix an appropriate $\varepsilon_1 \in (0, 1)$ such that

$$DF(u) \cdot p < \frac{1}{2} q, \quad (1 - \varepsilon_1)1 \leq u \leq (1 + \varepsilon_1)1. \quad (2.4)$$

Now we introduce a mollified pyramid, see [34]. Let $\tilde{\rho}(r) \in C^\infty([0, \infty))$ be a function with the following properties:

$$\tilde{\rho}(r) > 0, \quad \tilde{\rho}'(r) \leq 0 \quad \text{for } r \geq 0, \quad \tilde{\rho}(r) \equiv 1 \quad \text{if } 0 \leq r \leq 1, \quad \tilde{\rho}(r) \equiv e^{-r} \quad \text{if } r > 0 \text{ is large enough},$$

$$2\pi \int_0^\infty r\tilde{\rho}(r)dr = 1.$$

Then $\rho(x') := \tilde{\rho}(|x'|)$ belongs to $C^\infty(\mathbb{R}^2)$ and satisfies $\int_{\mathbb{R}^2} \rho(x')dx' = 1$. For a pyramid $-x_3 = h(x')$ we define its corresponding mollified pyramid $-x_3 = \varphi(x')$, where

$$\varphi(x') = \int_{\mathbb{R}^2} \rho(x' - y')dy' = \int_{\mathbb{R}^2} \rho(y')h(x' - y')dy'. \quad (2.5)$$

We set $(a_j, b_j) = m_*(A_j, B_j)$. Then $(a_j, b_j) \in \mathbb{R}^2$ satisfies

$$\frac{s}{\sqrt{1 + a_j^2 + b_j^2}} = c, \quad \text{for } j = 1, 2, \ldots, n.$$

We put

$$S(x') := \frac{s}{\sqrt{1 + \|
abla \varphi(x')\|^2}} - c, \quad (2.6)$$

where $\nabla \varphi = (\varphi_{x_1}, \varphi_{x_2})$. The following two lemmas come from Taniguchi [34].

**Lemma 2.1.** Let $\varphi$ and $S$ be as in (2.5) and (2.6), respectively. Then

$$h(x') < \varphi(x') \leq h(x') + 2m_* \int_0^\infty r^2\tilde{\rho}(r)dr, \quad |\nabla \varphi(x')| < m_*,$$

$$0 < S(x') \leq s - c$$

for all $x' \in \mathbb{R}^2$. In particular,

$$\lim_{\lambda \to \infty} \sup\{S(x')|x' \in \mathbb{R}^2, \text{dist}(x', E) \geq \lambda\} = 0,$$

$$\lim_{\lambda \to \infty} \sup\{\varphi(x') - h(x')|x' \in \mathbb{R}^2, \text{dist}(x', E) \geq \lambda\} = 0$$

and there exists positive constants $\nu_1, \nu_2$ so that

$$0 < \nu_1 \leq \frac{\varphi(x') - h(x')}{S(x')} \leq \nu_2, \quad \forall x' \in \mathbb{R}^2.$$
Lemma 2.2. For all integers \( i_1 \geq 0, i_2 \geq 0 \), one has
\[
C_1 := \sup_{x' \in \mathbb{R}^2} |D_{x_1}^{i_1} D_{x_2}^{i_2} \varphi(x')| < +\infty,
\]
and furthermore, for \( 2 \leq i_1 + i_2 \leq 3 \) one also has
\[
C_2 := \sup_{x' \in \mathbb{R}^2} \frac{|D_{x_1}^{i_1} D_{x_2}^{i_2} \varphi(x')|}{S(x')} < +\infty.
\]

3. Existence of pyramidal traveling fronts

Set \( z' = \alpha x' \), \( z_3 = \alpha x_3 \) and \( z = \alpha x \). Define
\[
\zeta(x) = \frac{x_3 + \varphi(z')/\alpha}{\sqrt{1 + |\nabla \varphi(z')|^2}}, \quad \eta(x) = \frac{c}{s}(x_3 + \varphi(z')/\alpha).
\]
Since \( 1 \leq \sqrt{1 + |\nabla \varphi|^2} < s/c \), we have
\[
\begin{align*}
\frac{s}{c} \eta(x) &< \zeta(x) < \eta(x), \quad \text{if } \zeta(x) < 0, \quad (3.1) \\
\eta(x, z) &< \zeta(x) < \frac{s}{c} \eta(x), \quad \text{if } \zeta(x) > 0. \quad (3.2)
\end{align*}
\]

Now we fix a function \( \omega(x) \in C^\infty(\mathbb{R}) \) with
\[
\omega(x) = 1, \quad \text{if } x \leq -1,
\]
\[
0 < \omega(x) < 1, \quad \omega'(x) < 0, \quad \text{if } -1 < x < 1,
\]
\[
\omega(x) = 0, \quad \text{if } x \geq 1.
\]
In this section, we denote \( \beta := (\beta_1, \beta_2) \) and make it satisfies
\[
0 < \beta_2 < \beta_1 < \beta^* := \frac{\lambda_2}{\lambda_1},
\]
see [30]. Lemmas 1.1 and 1.4] for \( \lambda_1 \) and \( \lambda_2 \). In the proof of the following lemma, we denote
\[
\Pi_i(x) := x^2 - cx + 1 - r, \quad \Pi_3(x) := x^2 - cx.
\]
Obviously, \( \beta^* < 1 \) and \( \Pi_i(\beta_1, \lambda_2) < 0 \) for \( i = 1, 2 \).

Lemma 3.1. There exist a positive constant \( \varepsilon_0^+(\beta) < 1 \) and a positive function \( \alpha_0^+(\varepsilon, \beta) \) such that, for all \( 0 < \varepsilon < \varepsilon_0^+(\beta) \) and \( 0 < \alpha < \alpha_0^+(\varepsilon, \beta) \),
\[
v^+(x; \varepsilon, \beta, \alpha) = U(\varsigma(x)) + \varepsilon S(z')((1 - \omega(\eta(x))) \rho + \omega(\eta(x)) \mathcal{U}^\beta(\eta(x)))
\]
is a supersolution to [19], where \( \mathcal{U}^\beta(\xi) := (U_2^\beta(\xi), U_2^\beta(\xi)) \). Furthermore,
\[
\lim_{\gamma \to +\infty} \sup_{x \in D_{(\gamma)}} \frac{|v^+_i(x; \varepsilon, \beta, \alpha) - v^-_i(x)|}{(v^-_i(x))^\beta_i} \leq 2\varepsilon, \quad i = 1, 2, \quad (3.5)
\]
\[
v^-(x) < v^+(x; \varepsilon, \beta, \alpha), \quad x \in \mathbb{R}^3, \quad (3.6)
\]
\[
\partial_{x_3} v^+(x; \varepsilon, \beta, \alpha) > 0, \quad x \in \mathbb{R}^3. \quad (3.7)
\]
Proof. For the sake of convenience, we denote \( \varsigma(x), \eta(x) \) by \( \varsigma \) and \( \eta \), respectively. We also denote \( v^+(x; \varepsilon, \beta, \alpha) \) by \( v^+(x) \) for simplicity. By a direct computation, we have
\[
\varsigma_{x_1} = -\alpha \frac{\phi z_1 \phi z_2 + \phi z_2 \phi z_1}{1 + |\nabla \phi(z')|^2} \varsigma + \frac{\phi z_1}{\sqrt{1 + |\nabla \phi(z')|^2}}, \quad \eta_{x_1} = \frac{s}{c} \phi z_1,
\]
Throughout the proof, we assume that $\alpha < \epsilon$. From direct computations and (1.3), we have

\[
\tilde{L}[\nu^+(\mathbf{x})] = -\Delta \nu^+(\mathbf{x}) + s\nu^+_{x_3}(\mathbf{x}) - \tilde{F}(\nu^+(\mathbf{x})) \geq 0.
\]

It is easy to check that $\partial_\nu \nu^+(\mathbf{x}) > 0$ holds according to the definition of $\nu^+(\mathbf{x})$. To prove that $\nu^+(\mathbf{x})$ is a supersolution, it suffices to verify that

\[
\tilde{L}[\nu^+(\mathbf{x})_{j}] = \left(1 - \sum_{j=1}^{3} \xi_{x_j}^2 \right)U''(\varsigma) - \left(\sum_{j=1}^{3} \xi_{x_j, x_j} \right)U'(\varsigma)
\]

\[
- \epsilon \alpha^2 \left(\sum_{j=1}^{2} S_{x_j, x_j} \right) \left[(1 - \omega(\eta))p_i + \omega(\eta)U_2^{\beta_i}(\eta)\right]
\]

\[
- 2\epsilon \alpha \left(\sum_{j=1}^{2} S_{x_j, \eta_{x_j}} \right) \left[\omega'(\eta)(-p_i + U_2^{\beta_i}(\eta)) + \omega(\eta)\beta_i U_2^{\beta_i - 1}(\eta)U_2'(\eta)\right]
\]

\[
- \epsilon S(\mathbf{z}') \{ \left[\omega''(\eta) \sum_{j=1}^{3} \eta_{x_j}^2 + \omega'(\eta) \sum_{j=1}^{2} \eta_{x_j, x_j} - c\omega'(\eta) \right] - p_i + U_2^{\beta_i}(\eta) \}
\]

\[
- 2\beta_i \omega'(\eta)U_2'(\eta)U_2^{\beta_i - 1}(\eta) \sum_{j=1}^{3} \eta_{x_j}^2
\]

\[
+ \omega(\eta) \left[ \beta_i(\beta_i - 1)U_2^{\beta_i - 2}(\eta)(U_2'(\eta))^2 \sum_{j=1}^{3} \eta_{x_j}^2 \right]
\]

\[
+ \beta_i U_2^{\beta_i - 1}(\eta)U_2'(\eta) \sum_{j=1}^{3} \eta_{x_j, x_j}
\]

\[
+ \beta_i U_2^{\beta_i - 1}(\eta)U_2''(\eta) \sum_{j=1}^{3} \eta_{x_j}^2 - c\beta_i U_2^{\beta_i - 1}(\eta)U_2'(\eta) \}
\]

\[
+ \left( \frac{s}{\sqrt{1 + |\nabla \varphi(\mathbf{z}')|^2}} - \epsilon \right)U'(\varsigma) - \tilde{f}_i(\nu^+(\mathbf{x})) + f_i(U(\varsigma)).
\]
Let
\[ A_1 := \sup_{\xi \in \mathbb{R}} \left| \sum_{j=1}^{2} S_{x_j}(z') \frac{\xi_j}{S(z')} \right|, \quad A_2 := \sup_{\xi \in \mathbb{R}} \left| \sum_{j=1}^{2} S_{x_j}(z') \eta \xi_j \right|. \]
By Lemmas 2.1, 2.2, we know that \( 0 \leq A_1, A_2 < +\infty \) are well defined. Lemma 2.1 and Lemma 2.2 also imply that there exist positive constants \( A_3, A_4, A_5 \) and \( A_6 \) such that
\[ |1 - \sum_{j=1}^{3} \xi_j^2| \leq \alpha (A_3|\varsigma| + A_4\varsigma^2)S(z') \leq \varepsilon (A_3|\varsigma| + A_4\varsigma^2)S(z'), \]  
(3.8)
\[ \left| \sum_{j=1}^{2} \xi_j \right| \leq \alpha (A_5 + A_6|\varsigma|)S(z') \leq \varepsilon (A_5 + A_6|\varsigma|)S(z'), \]  
(3.9)
\[ \left| \sum_{j=1}^{2} \eta \xi_j \right| \leq \left| \alpha \frac{c}{s} \right| \sum_{j=1}^{2} \phi \xi_j \leq 2\alpha C_1. \]  
(3.10)

Next, we consider three cases.

**Case 1:** \( \varsigma < -X^* \) for some \( X^* > 0 \) large enough. Recalling (3.1), \( \varsigma < \eta \) holds in this case. Assume that \( \varepsilon \leq \frac{1}{2(s-c)} \). And without loss of generality, suppose that \( \eta \leq -X^* < -1 \), where \( X^* > 0 \) is a positive constant such that \( U_2(\eta) \leq \frac{1}{2} \) if \( \eta \leq -X^* \). Under these conditions, we know
\[ v_2^+(x) \leq \frac{1}{2} + \varepsilon S(z') < 1 \quad \text{if} \quad \eta < -X^*, \]
which implies that \( \tilde{f}_2(v^+(x)) = f_2(v^+(x)) \). Then by (3.3) we have
\[ \tilde{L}[v^+(x)], \]
\[ = \left( 1 - \sum_{j=1}^{3} \xi_j^2 \right) U_1''(\varsigma) - \left( \sum_{j=1}^{3} \xi_j \right) U_1'(\varsigma) 
- \varepsilon S(z') U_2^2(\eta) \left\{ \alpha^2 \sum_{j=1}^{2} S_{x_j}(z') + \left( 2\alpha \beta_i \sum_{j=1}^{2} S_{x_j}(z') \eta \xi_j \right) U_2^2(\eta) \right\} 
+ \beta_i (\beta_i - 1) \left( U_2'(\eta) \right)^2 \sum_{j=1}^{3} \eta_j^2 \right) 
+ \left( \frac{s}{\sqrt{1 + |\nabla \phi(z')|^2}} - c \right) U_1'(\varsigma) - f_1(v^+(x)) + f_1(U(\varsigma)). \]

Recall (3.1) and the monotonicity of the wave profile \( U \). Then by (3.8)-(3.10) we have
\[ \tilde{L}[v^+(x)], \]
\[ \geq -\varepsilon S(z') U_2^2(\eta) (A_3|\varsigma| + A_4\varsigma^2) \left| \frac{U_1''(\varsigma)}{U_2^2(\varsigma)} \right| - \varepsilon S(z') U_2^2(\eta) (A_5 + A_6|\varsigma|) \left| \frac{U_1'(\varsigma)}{U_2^2(\varsigma)} \right| 
- \varepsilon S(z') U_2^2(\eta) \left\{ \alpha^2 A_1 + 2\alpha A_2 \right\} \frac{U_2'(\eta)}{U_2(\eta)} + 2\alpha C_1 \frac{U_2'(\eta)}{U_2(\eta)} 
+ \beta_i (\beta_i - 1) \left( \frac{U_2'(\eta)}{U_2(\eta)} \right)^2 \left( 1 + |\nabla \phi(z')|^2 \right) \]
from Lemma 1.1 that there exists $X$ for any $f_{\lambda}$ where $\theta$ as $f_{i}(v^{+}(x)) = f_{i}(U(s)) - f_{i}(v^{+}(x))$. Thus there exists $X$ such that for $v = \lambda U$ with $\lambda \geq 0$ large enough such that $\epsilon \theta_{i} U_{2}^{\beta_{i}}(\eta) \leq \epsilon \theta_{i} U_{2}^{\beta_{i}}(\eta)$, we have

$$ \frac{U_{2}''(x)}{U_{2}(x)} < \frac{3}{2} \lambda_{2}, \quad \left| \frac{U_{2}''(x)}{U_{2}(x)} \right| < -\frac{1}{16} \Pi_{i}(\beta_{1}, \lambda_{2}), $$

$$ \beta_{1}^{2} \left( \frac{U_{2}'(x)}{U_{2}(x)} \right)^{2} - c_{\beta_{1}} \frac{U_{2}'(x)}{U_{2}(x)} + 1 - r \Pi_{1}(\beta_{1}, \lambda_{2}) < 0, $$

$$ \beta_{2}^{2} \left( \frac{U_{2}'(x)}{U_{2}(x)} \right)^{2} - c_{\beta_{2}} \frac{U_{2}'(x)}{U_{2}(x)} \rightarrow \Pi_{2}(\beta_{2}, \lambda_{2}) < 0 $$

as $x \rightarrow -\infty$. Thus there exists $X_{1} > 0$ large enough such that

$$ \frac{U_{2}'(x)}{U_{2}(x)} < \frac{3}{2} \lambda_{2}, \quad \left| \frac{U_{2}''(x)}{U_{2}(x)} \right| < -\frac{1}{16} \Pi_{i}(\beta_{1}, \lambda_{2}), $$

$$ \beta_{1}^{2} \left( \frac{U_{2}'(x)}{U_{2}(x)} \right)^{2} - c_{\beta_{1}} \frac{U_{2}'(x)}{U_{2}(x)} + 1 - r \Pi_{1}(\beta_{1}, \lambda_{2}) < 0, $$

$$ \beta_{2}^{2} \left( \frac{U_{2}'(x)}{U_{2}(x)} \right)^{2} - c_{\beta_{2}} \frac{U_{2}'(x)}{U_{2}(x)} \rightarrow \Pi_{2}(\beta_{2}, \lambda_{2}) < 0 $$

for any $x < -X_{1}$. For the above positive constants $A_{3}, A_{4}, A_{5}$ and $A_{6}$, it follows from Lemma 1.1 that there exists $X_{2} > 0$ large enough such that

$$ (A_{3}|x| + A_{4}x^{2}) \frac{|U_{2}''(x)|}{U_{2}^{\beta_{i}}(x)} < -\frac{1}{16} \Pi_{i}(\beta_{1}, \lambda_{2}), $$

$$ (A_{5} + A_{6}|x|) \frac{|U_{2}'(x)|}{U_{2}^{\beta_{i}}(x)} < -\frac{1}{16} \Pi_{i}(\beta_{1}, \lambda_{2}) $$

for any $x < -X_{2}$. Also there exists $\alpha_{1} \in (0, \beta^{+})$ small enough such that

$$ \alpha^{2} A_{1} + 3 \alpha A_{2} \lambda_{2} + 3 \alpha C_{1} \lambda_{2} < -\frac{1}{16} \Pi_{i}(\beta_{1}, \lambda_{2}), \quad \forall \alpha \in (0, \alpha_{1}), \quad i = 1, 2. $$

For the reaction term $f_{i}$, we have

$$ f_{i}(v^{+}(x)) - f_{i}(U(s)) = \left( \sum_{j=1}^{2} f_{ij} \left( \theta_{i} v^{+}(x) + (1 - \theta_{i}) U(s) \right) U_{2}^{\beta_{i}}(\eta) \right) \epsilon S(z') $$

$$ = \left( \sum_{j=1}^{2} f_{ij} \left( \epsilon \theta_{i} S(z') \frac{U_{2}^{\beta_{i}}(\eta)}{U_{2}(\eta)} \right) \right) \epsilon S(z'), $$

where $\theta_{i} \in (0, 1)$, $i = 1, 2$. If $i = 1$, then $0 < U_{1}(\xi) < U_{2}(\xi) < 1$ yields

$$ f_{12} \left( \epsilon \theta_{1} S(z') \frac{U_{2}^{\beta_{i}}(\eta)}{U_{2}(\eta)} \right) = r \left( U_{1}(\xi) + \epsilon \theta_{1} S(z') U_{2}^{\beta_{i}}(\eta) \right) \leq r(1 + \epsilon s) U_{2}^{\beta_{1}}(\eta). $$

It follows that

$$ f_{1}(v^{+}(x)) - f_{1}(U(s)) $
\[
\begin{align*}
&\leq \left( f_{11} \left( U(s) + \epsilon \theta_1 S(z')U^\beta(\eta) \right) U_2^{\beta_1}(\eta) + r(1 + \epsilon s)U_2^{\beta_2}(\eta) \right) \epsilon S(z') \\
&\leq \left( f_{11} \left( U(s) + \epsilon \theta_1 S(z')U^\beta(\eta) \right) + r(1 + \epsilon s)U_2^{\beta_2}(\eta) \right) \epsilon S(z')U_2^{\beta_1}(\eta),
\end{align*}
\]
and
\[
\begin{align*}
f_{11} \left( U(x) + \epsilon \theta_1 S(z')U^\beta(\eta) \right) + r(1 + \epsilon s)U_2^{\beta_2}(\eta) \to 1 - r \text{ as } x, y \to -\infty. \quad \text{If } i = 2,
\end{align*}
\]
then
\[
\begin{align*}
f_2(v^+(x)) - f_2(U(s)) &= \left( \sum_{j=1}^{2} f_{2j} \left( U(s) + \epsilon \theta_2 S(z')U^\beta(\eta) \right) \right) \epsilon S(z') \\
&= \left( \sum_{j=1}^{2} f_{2j} \left( U(s) + \epsilon \theta_2 S(z')U^\beta(\eta) \right) \right) \epsilon S(z')U_2^{\beta_2}(\eta).
\end{align*}
\]
Note that \((U_2(x))^{\beta_1 - \beta_2} \to 0\) as \(x \to -\infty\), also we have
\[
\sum_{j=1}^{2} f_{2j} \left( U(x) + \epsilon \theta_2 S(z')U^\beta(\eta) \right) (U_2(y))^{\beta_1 - \beta_2} \to 0 \quad \text{as } x, y \to -\infty.
\]
Then we know that there exists \(X_3 > 0\) large enough such that
\[
\begin{align*}
f_{11} \left( U(x) + \epsilon \theta_1 S(z')U^\beta(\eta) \right) + r(1 + \epsilon s)U_2^{\beta_2}(\eta) - (1 - r) < -\frac{1}{16} \Pi_1(\beta_1 \lambda_2), \\
\sum_{j=1}^{2} f_{2j} \left( U(x) + \epsilon \theta_2 S(z')U^\beta(\eta) \right) (U_2(y))^{\beta_1 - \beta_2} < -\frac{1}{16} \Pi_1(\beta_1 \lambda_2)
\end{align*}
\]
for \(x, y < -X_3\).
Let \(X' = \max\{\frac{c}{x}X^*, X_1, X_2, X_3\}\), then for \(c < -X'\) we have
\[
\begin{align*}
\tilde{L}[v^+(x)]_i &\geq -U_2^{\beta_1}(\eta)\epsilon S(z') \left( A_3 |s| + A_4 \right)^2 \frac{|U'_i(s)|}{U_2^{\beta_1}(\eta)} \\
&- U_2^{\beta_1}(\eta)\epsilon S(z') \left( A_5 + A_6 |s| \right) \frac{U'_1(s)}{U_2^{\beta_1}(\eta)} \\
&- \epsilon S(z') U_2^{\beta_1}(\eta) \left\{ \sigma^2 A_1 + 3 \sigma A_2 \lambda_2 + 3 \sigma \lambda_2 \\
&+ \beta_1 \bigg| - \frac{U'_2(\eta)}{U_2(\eta)} \bigg| + \frac{U''_2(\eta)}{U_2(\eta)} \bigg| + \beta_2 \bigg| \frac{U'_2(\eta)}{U_2(\eta)} \bigg| \right\} \\
&- \epsilon S(z') \left( \sum_{j=1}^{2} f_{2j} \left( U(s) + \epsilon \theta_1 S(z')U^\beta(\eta) \right) U_2^{\beta_1}(\eta) \right) \\
&\geq \epsilon S(z') U_2^{\beta_1}(\eta) \left( \frac{1}{16} \Pi_1(\beta_1 \lambda_2) + \frac{1}{16} \Pi_1(\beta_1 \lambda_2) + \frac{1}{16} \Pi_1(\beta_1 \lambda_2) \\
&+ \frac{1}{16} \Pi_1(\beta_1 \lambda_2) - \frac{1}{2} \Pi_1(\beta_1 \lambda_2) + \frac{1}{16} \Pi_1(\beta_1 \lambda_2) \right) > 0.
\end{align*}
\]
Case 2: $\zeta > X''$ for some $X'' > 0$ large enough. Without loss of generality, suppose that \( \eta > 1 \). By [30, Lemma 1.4], we can take $X'_1 > 0$ large enough such that

\[
(A_3|x| + \alpha A_4 x^2)|U''_i(x)| < -\frac{q_i}{8} \quad \text{and} \quad (A_3 + A_0|x|)|U'_i(x)| < -\frac{q_i}{8}
\]

for all $x > X'_1$ and $i = 1, 2$.

Fix a constant $\alpha_2 \in (0, \beta^*)$ such that $\alpha_2 A_1 p_1 < \min_{i=1,2} \{ -\frac{q_i}{8} \}$ for any $\alpha$ in $(0, \alpha_2)$. For the reaction term $f_i$, we have

\[
f_i(v^+(x)) - f_i(u(\zeta)) = \left( \sum_{j=1}^{2} f_{ij}(U(\zeta) + \theta_i \varepsilon S(z')p) p_j \right) \varepsilon S(z'), \quad i = 1, 2.
\]

Since $U_i(x) \to 1$ as $x \to +\infty$ for $i = 1, 2$, there exists $X'_2 > 0$ large enough such that for all $\varepsilon \in (0, \frac{\varepsilon_1}{p_i(s-c)})$ (see [2.4] for $\varepsilon_1$), it holds

\[
1 - \varepsilon_1 < U_i(x) + \varepsilon \theta_i S(z')p_i < 1 + \varepsilon_1, \quad x > X'_2, \quad i = 1, 2,
\]

and then

\[
\sum_{j=1}^{2} f_{ij}(U(x) + \varepsilon \theta_i S(z')p) p_j < \frac{1}{2} q_i, \quad x > X'_2, \quad i = 1, 2.
\]

According to the definition of $\tilde{f}_2$, we know

\[
\tilde{f}_2(v^+(x)) \leq f_2(v^+(x)) + b(\varepsilon S(z'))^2 p_1 p_2.
\]

Take $X'' = \max\{X'_1, X'_2, 1\}$, then for $\zeta > X''$, we have

\[
\tilde{L}[v^+(x)]_i \geq \left( 1 - \sum_{j=1}^{3} \sum_{k=1}^{2} U''_i(\zeta) - \left( \sum_{j=1}^{3} \varepsilon S(z') + \varepsilon \alpha_2 S(z') \sum_{j=1}^{2} S_{z_j} \varepsilon \xi_{1j} \right) - \varepsilon \alpha_2 S(z') \sum_{j=1}^{2} S_{z_j} \varepsilon \xi_{1j} \right)
\]

\[
\left( \frac{q_i}{8} + \frac{q_i}{8} + \frac{q_i}{8} - \frac{q_i}{2} - b \varepsilon p_1 p_2(s-c) \right) > 0
\]

provided that $\varepsilon < \min_{i=1,2} \{ -\frac{\varepsilon_1}{8p_1 p_2(s-c)} \}$.

Case 3: $-X' \leq \zeta \leq X''$. Define $u_* := \min_{-X' \leq x \leq X''} \min_{i=1,2} U'_i(x)$ and

\[
M_{ij} := \sup_{x \in [-\varepsilon_1, (1+\varepsilon_1)]} f_{ij}(u), \quad M_0 := \sup_{1 \leq i, j \leq 2} M_{ij},
\]

\[
M_1 := \sup_{x \in \mathbb{R}, i = 1, 2} |U'_i(x)|, \quad M_2 := \sup_{x \in \mathbb{R}, i = 1, 2} |x||U'_i(x)|,
\]

\[
M_3 := \sup_{x \in \mathbb{R}, i = 1, 2} |x||U''_i(x)|, \quad M_4 := \sup_{x \in \mathbb{R}, i = 1, 2} x^2|U''_i(x)|.
\]

We have

\[
\tilde{L}[v^+(x)]_i \geq -\alpha S(z')(A_3|\zeta| + \alpha A_4|\zeta|^2)|U''_i(\zeta)| - \varepsilon S(z')(A_5 + A_0|\zeta|)|U'_i(\zeta)|
\]
Next we prove that \( (1.9) \) holds. Let

\[
\xi(x) = c \frac{1}{s} (x_3 + h(x')), \quad \nu(x) = \frac{1}{\sqrt{1 + \|\varphi(x')\|^2}} (x_3 + h(x'))
\]

and recall that

\[
\eta(x) = c \frac{1}{s} (x_3 + \varphi(x')/\alpha), \quad \varsigma(x) = \frac{1}{\sqrt{1 + \|\varphi(x')\|^2}} (x_3 + \varphi(x')/\alpha).
\]

If \( \varsigma(x) \geq \xi(x) \), then it is obvious that \( \nu^-(x) < \nu^+(x) \) since \( U_{i}(y)(i = 1, 2) \) are monotone increasing in \( y \). Thus we need only consider the case \( \varsigma(x) < \xi(x) \). It follows from the definitions of \( \varsigma(x) \) and \( \xi(x) \) that

\[
\varsigma(x) - \xi(x) = \left( \frac{1}{\sqrt{1 + \|\varphi(x')\|^2}} - \frac{c}{s} \right) (x_3 + h(x')) + \frac{\varphi(x')/\alpha - h(x')}{\sqrt{1 + \|\varphi(x')\|^2}}
\]

\[
= \frac{1}{s} \frac{S(x') (x_3 + h(x')) + \varphi(x')/\alpha - h(x')}{{\sqrt{1 + \|\varphi(x')\|^2}}} < 0.
\]

Since \( \frac{1}{\sqrt{1 + \|\varphi(x')\|^2}} - \frac{c}{s} > 0 \) and \( \nu_1 \leq \frac{\varphi(x')- h(x')}{s S(x')} \leq \nu_2 \), we have

\[
x_3 + h(x') < -s \frac{\varphi(x')/\alpha - h(x')}{{\sqrt{1 + \|\varphi(x')\|^2}}} S(x') \leq -\frac{c \nu_1}{\alpha} < 0,
\]

which implies that \( \varsigma(x) < \xi(x) \leq \xi(-\frac{c \nu_1}{\alpha}) = -\frac{c^2 \nu_1}{\alpha s} < 0 \) and \( \xi \eta(x) < \varsigma(x) < \eta(x) < 0 \). Then we have

\[
\nu_1^+(x) - \nu_1^-(x) \geq U_i(\nu(x)) + \varepsilon S(x') U_2^\delta(\eta(x)) - U_i(\xi(x))
\]

\[
= \left( \frac{1}{\sqrt{1 + \|\varphi(x')\|^2}} - \frac{c}{s} \right) (x_3 + h(x')) U_1^\epsilon(\theta \nu(x) + (1 - \theta) \xi(x))
\]
We consider three cases. In other words, prove that according to the definition of \( \eta(x) \), we have \( \nu(x) < \varsigma(x) < \xi(x) < \eta(x) < 0 \), and hence

\[
U'_i(\theta_i \nu(x) + (1 - \theta_i) \xi(x)) \leq \begin{cases} 
L_2 e^{\min\{\lambda_1, 2\lambda_2\} \xi(x)}, & i = 1, \\
L_2 e^{\lambda_2 \xi(x)}, & i = 2,
\end{cases}
\]

\[
U^{\beta_i}_i(\eta(x)) \geq L_1 e^{\lambda_1 \beta_i \eta(x)} \geq L_1 e^{\lambda_1 \beta_i \xi(x)}, \quad i = 1, 2.
\]

Then for \( i = 2 \), we have

\[
v^+_2(x) - v^-_2(x) \geq \frac{1}{s} S(\mathbf{z}') \xi(x) U'_2(\theta_2 \nu(x) + (1 - \theta_2) \xi(x)) + \varepsilon S(\mathbf{z}') U^{\beta_2}_2(\eta(x))
\]

\[
\geq S(\mathbf{z}') \left( \frac{L_2}{s} \xi(x) e^{\lambda_2 \xi(x)} + \varepsilon L_1 e^{\lambda_1 \beta_2 \xi(x)} \right)
\]

\[
\geq S(\mathbf{z}') e^{\lambda_1 \beta_2 \xi(x)} \left( \frac{L_2}{s} \xi(x) e^{(\lambda_2 - \lambda_1 \beta_2) \xi(x)} + \varepsilon L_1 \right)
\]

\[
\geq S(\mathbf{z}') e^{\lambda_1 \beta_2 \xi(x)} \left( \frac{L_2}{s(\lambda_2 - \lambda_1 \beta_2)^2} \xi(x) \left( \sup_{\omega > 0} \omega^2 e^{-\omega \xi(x)} + \varepsilon L_1 \right) \right)
\]

\[
\geq S(\mathbf{z}') e^{\lambda_1 \beta_2 \xi(x)} \left( - \frac{4L_2 \alpha}{c^2 e^2 (\lambda_2 - \lambda_1 \beta_2)^2 \nu_1} + \varepsilon L_1 \right) > 0,
\]

provided that

\[
\alpha < \alpha_4 := \min \left\{ \frac{\varepsilon L_1 c^2 e^2 (\lambda_2 - \beta_2 \lambda_1)^2 \nu_1}{4L_2}, \frac{\varepsilon L_1 c^2 e^2 (\min\{\lambda_1, 2\lambda_2\} - \beta_1 \lambda_1)^2 \nu_1}{4L_2} \right\}.
\]

A similar argument will lead to \( v^+_1(x) - v^-_1(x) > 0 \) for the above \( \alpha \) and we omit it. Thus we have that \( \mathbf{v}^+(x) > \mathbf{v}^-(x) \) for all \( x \in \mathbb{R}^3 \).

Next, we prove (3.5). By Lemma 2.1 we know that for each fixed \( \alpha \), there exits a positive constant \( \eta_0 = \frac{1}{\alpha} 2\pi m \int_0^\infty \rho(r) dr \) such that

\[
\xi(x) \leq \eta(x) \leq \xi(x) + \eta_0, \quad \forall x \in \mathbb{R}^3.
\]

Recall \( N_1 := \sup_{x \in \mathbb{R}} \left| \frac{U'_2(x)}{U_2(x)} \right| \). Since \( U_2(x + y) e^{-N_1 y} \) is decreasing in \( y \), we know \( U_2(x + y) e^{-N_1 y} \leq U_2(x) \) for any \( y \geq 0 \). Using this fact and (3.11), we can get

\[
U_2(\xi(x)) \leq U_2(\eta(x)) \leq U_2(\xi(x)) e^{N_1 m_\alpha}, \forall x \in \mathbb{R}^3.
\]

In other words,

\[
1 \leq \frac{U_2(\eta(x))}{U_2(\xi(x))} \leq e^{N_1 m_\alpha}, \quad \forall x \in \mathbb{R}^3.
\]

According to the definition of \( \mathbf{v}^+(x) \) and \( \mathbf{v}^-(x) \), and using (3.12), it is sufficient to prove that

\[
\lim_{\gamma \to +\infty} \sup_{x \in D(\gamma)} \frac{|U_i(\varsigma(x)) - U_i(\xi(x))|}{(U_2(\eta(x)))^{\beta_i}} = 0, \quad i = 1, 2.
\]

We consider three cases.

**Case 1:** \( \xi(x) = x_3 + h(x') \to +\infty. \) Then we have \( 0 < \xi(x) < \varsigma(x) \) and \( \eta(x) < \varsigma(x) < \frac{\xi(x)}{e \eta(x)} \), which implies that

\[
\frac{|U_i(\varsigma(x)) - U_i(\xi(x))|}{(U_2(\eta(x)))^{\beta_i}} \to 0 \quad \text{as} \quad \xi(x) \to +\infty, \quad i = 1, 2.
\]
Case 2: \( \xi(x) = \frac{x}{s}(x_3 + h(x')) \to -\infty \). We have

\[
\zeta(x) = \frac{1}{\sqrt{1 + |\nabla \varphi(\alpha x')|^2}} \frac{s}{c} \xi(x) + \frac{\varphi(\alpha x')/\alpha - h(x')}{\sqrt{1 + |\nabla \varphi(\alpha x')|^2}}.
\]

Since \( 0 < \varphi(\alpha x')/\alpha - h(x') \leq m_\alpha \) and \( \frac{s}{c} < \frac{1}{\sqrt{1 + |\nabla \varphi(\alpha x')|^2}} \leq 1 \), it follows that

\[
\frac{s}{c} \xi(x) \leq \zeta(x) \leq \xi(x) + m_\alpha,
\]

which implies \( \zeta(x) \to -\infty \) as \( \xi(x) \to -\infty \) and vice versa. Thus, we know

\[
\xi(x) \leq \eta(x) \leq \frac{c}{s} \xi(x) < \frac{c}{s} \eta(x) < 0.
\]

Using [30, Lemma 1.1], we have

\[
\frac{U_i(\xi)}{\bar{U}_2(\xi)} = \frac{\lambda_1 e^{(1-\beta_i)} + O(e^{2(1-\sigma)}\xi)}{A\lambda_2 e^{\lambda_2 - \sigma} + O(e^{(1-\sigma)}\xi)} \leq 0 \quad \text{as} \quad \xi \to -\infty.
\]

Thus we know \( \mathcal{C} := \sup_{\xi \leq 0, i=1,2} \frac{U_i(\xi)}{\bar{U}_2(\xi)} < \infty \). Then by (3.11), (3.13) and (2.3), we have

\[
\lim_{\gamma \to +\infty} \sup_{\xi \leq 0, i=1,2} \frac{|U_i(\zeta(x)) - U_i(\eta(x))|}{(U_2(\eta(x)))^{\beta_i}} \leq 0
\]

And notice that \( \eta(x) \to -\infty \) is equivalent to \( \xi(x) \to -\infty \) by (3.11).

Case 3: \( \xi(x) = \frac{x}{s}(x_3 + h(x')) \) is bounded. It is obvious by (3.11) that \( \eta(x) \) is bounded in this case. Suppose \( R_0 > 0 \) is a constant such that \( |\xi(x)| \leq R_0 \). For each \( \gamma > 2R_0 \) and \( x \in D(\gamma) \), there must hold \( \text{dist}(x', E) > \gamma \) and thus

\[
\lim_{\gamma \to +\infty} \sup_{\text{dist}(x', E) > \gamma} (\varphi(x') - h(x')) = 0, \quad \lim_{\gamma \to +\infty} \sup_{\text{dist}(x', E) > \gamma} S(x') = 0,
\]

and hence

\[
\lim_{\gamma \to +\infty} \sup_{x \in D(\gamma)} \frac{|U_i(\zeta(x)) - U_i(\xi(x))|}{(U_2(\eta(x)))^{\beta_i}} = 0, \quad i = 1, 2.
\]

Finally, let \( \alpha_i^+(\varepsilon, \theta) = \min\{\varepsilon, \alpha_1, \alpha_2, \alpha_3, \alpha_4\} \) and

\[
\varepsilon_i^+(\theta) = \min\left\{ \frac{\varepsilon_i}{p_i(s - c)}, \min_{i=1,2} \left\{ \frac{q_i}{8(s - c)b_1p_2} \right\} \right\}.
\]

The proof is complete.
Now we give the existence result for traveling curved fronts.

**Theorem 3.2.** For each \( s > c \), \((1.2)\) admits a pyramidal traveling front \( u(x, t) = V(x', x_3 + st) \). \( V(x) \) satisfies \((1.6)\) with \( \partial_{x_3} V(x) > 0 \) and

\[
\begin{align*}
\dot{v}^- (x) < V(x) < \dot{v}^+ (x; \varepsilon, \beta, \alpha), \quad \forall x \in \mathbb{R}^3,
\end{align*}
\]

where \( 0 < \varepsilon < \varepsilon_0^+ (\beta) \) and \( 0 < \alpha < \alpha_0^+ (\varepsilon, \beta) \). Furthermore, we have

\[
\lim_{\gamma \to +\infty} \sup_{x \in D(\gamma)} |V_i(x) - v_i^- (x)| / (v_i (x))^{\beta_i} = 0, \quad i = 1, 2.
\]

**Proof.** According to the parabolic estimates, we know that there exists a constant \( C > 0 \) such that the solution \( v(x; t) \) of \((1.4)\) and \((1.5)\) with \( v_0(x) \in [0, 1] \) satisfies

\[
\|v(\cdot, t; v_0)\|_{C^1(\mathbb{R}^3)} < C, \quad \forall \ t \geq 1.
\]

Since \( v^- \) is a subsolution, we have \( v(x, t_1; v^-) < v(x, t_2; v^-) \) for all \( x \in \mathbb{R}^3 \) and \( 0 < t_1 < t_2 \), see \(33\) for more details. Thus,

\[
V(x) := \lim_{t \to +\infty} v(x, t; v^-), \quad x \in \mathbb{R}^3,
\]

is well defined and independent of \( \varepsilon, \alpha, \) and \( \beta \). It follows that \( v(\cdot, t; v^-) \) converges monotonically to \( V(\cdot) \) under the norm \( \|\cdot\|_{C^2_{loc}(\mathbb{R}^3)} \), namely,

\[
\lim_{t \to +\infty} \|v(\cdot, t; v^-) - V(\cdot, \cdot)\|_{C^2_{loc}(\mathbb{R}^3)} = 0.
\]

By the comparison principle, we know \( v^- (x) < V(x) < v^+ (x; \varepsilon, \beta, \alpha) \). And the proof of \((3.14)\) is similar to that of \(44\). In view of the monotonicity of \( V(x) \) on the variable \( x_3 \), we come to the conclusion that \( \partial_{x_3} V(x) \geq 0 \) for all \( x \in \mathbb{R}^3 \). Then the strong maximum principle implies the strict inequality. \( \square \)

### 4. Stability of traveling curved fronts

This section discusses the stability of the pyramidal traveling fronts constructed in Section 3 by improving the arguments of Taniguchi \(35\) and Wang \(43, 45\). Consider the Cauchy problem

\[
\begin{align*}
\frac{\partial}{\partial t} \tilde{w}(\xi, \eta, t) - \Delta \tilde{w}(\xi, \eta, t) + \tilde{s} \frac{\partial}{\partial \eta} \tilde{w}(\xi, \eta, t) - F(\tilde{w}) &= 0, \\
\tilde{w}(\xi, \eta, 0) &= \tilde{w}_0(\xi, \eta),
\end{align*}
\]

where \( \tilde{w}(\xi, \eta, t) = (\tilde{w}_1(\xi, \eta, t), \tilde{w}_2(\xi, \eta, t)) \) and \( (\xi, \eta) \in \mathbb{R}^2, t > 0 \). The following theorem is established in \(30, 31\).

**Theorem 4.1.** Assume \( h > 0 \) and \( r > 1 \). Then for each \( \tilde{s} > c \), there exists a steady state \( \Phi(\xi, \eta; \tilde{s}) \) of \((4.1)\) satisfying \( \Phi(\xi, \eta; \tilde{s}) > \tilde{v}^- (\xi, \eta) \) and

\[
\lim_{R \to \infty} \sup_{\xi_2 + \eta^2 > R^2} \left| \Phi_i(\xi, \eta) - \tilde{v}_i^- (\xi, \eta) / (\tilde{v}_2 (\xi, \eta))^{\beta_i} \right| = 0, \quad i = 1, 2,
\]

where

\[
\tilde{v}^- (\xi, \eta) = U \left( \frac{c}{\tilde{s}} (\eta + \sqrt{s^2 - c^2} |\xi|) \right).
\]
and \( \beta_i (i = 1, 2) \) is defined as in (3.4). Furthermore, for any \( \tilde{w}_0(\xi, \eta) \in [0, 1] \) with \( \tilde{w}_0 \in C(\mathbb{R}^2, \mathbb{R}^2) \) and

\[
\lim_{R \to \infty} \sup_{\xi^2 + \eta^2 > R^2} \left| \frac{\tilde{\nu}_i^-(\xi, \eta) - \tilde{\nu}_0^-(\xi, \eta)}{(\tilde{\nu}_2^- (\xi, \eta))^\gamma} \right| = 0, \quad i = 1, 2,
\]

the solution \( \tilde{w}(\xi, \eta, t; \tilde{w}_0) \) of (4.1) with initial data \( \tilde{w}_0 \) satisfies

\[
\lim_{t \to \infty} \left\| \frac{w_i(\cdot, \cdot, t; \tilde{w}_0) - \Phi_i(\cdot, \cdot, st)}{(\tilde{\nu}_2^- (\cdot, \cdot))^\gamma} \right\|_{L^\infty(\mathbb{R}^2)} = 0, \quad i = 1, 2.
\]

For any subset \( D \subseteq \mathbb{R}^3 \), we denote the characteristic function of \( D \) by \( \chi_D \), namely,

\[
\chi_D(x) = \begin{cases} 
1, & x \in D, \\
0, & x \in D^c,
\end{cases}
\]

where \( D^c \) denotes the complementary set of \( D \). Let \( h_{ij}(x, t) \in C(\mathbb{R}^3 \times \mathbb{R}) (i, j = 1, 2) \) be given continuous functions satisfying

\[
0 \leq h_{ij}(x, t) \leq M_{ij}, \quad i \neq j; \quad \sup_{x \in \mathbb{R}^3, t > 0} |h_{ii}(x, t)| \leq M_i,
\]

where \( M_{ij} \in \mathbb{R} \) \((i, j = 1, 2)\) are constants. Now consider the linear system

\[
L_t[w_i(x, t)] - \sum_{j=1}^{2} h_{ij}(x, t)w_j(x, t) = 0, \quad x \in \mathbb{R}^3, \quad t > 0, \quad (4.2)
\]

\[
w_i(x, 0) = w_{i, 0}(x), \quad x \in \mathbb{R}^3, \quad i = 1, 2,
\]

where \( L_t := \frac{\partial}{\partial t} - \sum_{k=1}^{3} \alpha_{ij} \frac{\partial^2}{\partial x_j \partial x_i} + s \frac{\partial}{\partial x_3} \) is a linear operator. We have the following result.

**Lemma 4.2.** Let \( w(x, t) = (w_1(x, t), w_2(x, t)) \) be a solution of (4.2). Then there exist positive constants \( \tilde{A}, \tilde{B} \) and \( \lambda_0 \) such that

\[
\sup_{i=1, 2} \sup_{x \in \mathbb{R}^3} \left| \frac{w_i(x, t)}{(v_2(x))^\beta_i} \right| \leq 6c_{\beta_0} \frac{\tilde{A}}{\tilde{B}} \int_{-2\pi}^{2\pi} e^{-\tilde{B}r^2} dr \sup_{i=1, 2} \sup_{x \in D(\gamma)^c} \left| \frac{w_i(0)(x)}{(v_2(x))^\beta_i} \right|
\]

\[
+ e^{\lambda_0 t} \left( \frac{\pi}{\tilde{B}} \right)^{3/2} \max_{i=1, 2} \sup_{x \in \mathbb{R}^3} \left| \frac{w_i(0)(x)}{(v_2(x))^\beta_i} \right|, \quad \forall t > 0, \quad (4.3)
\]

for any \( \gamma > 0 \), where \( D(\gamma)^c \) is the complementary set of \( D(\gamma) \). Moreover, we have

\[
\sup_{i=1, 2} \sup_{x \in \mathbb{R}^3} \left| \frac{w_i(x, t)}{(v_2(x))^\beta_i} \right| \leq e^{\lambda_0 t} \left( \frac{\pi}{\tilde{B}} \right)^{3/2} \max_{i=1, 2} \sup_{x \in \mathbb{R}^3} \left| \frac{w_i(0)(x)}{(v_2(x))^\beta_i} \right|, \quad \forall t > 0. \quad (4.4)
\]

**Proof.** Let \( \tilde{w}(x, t) = e^{-\lambda_0 t} w(x, t) \), where \( \lambda_0 = \sum_{i=1}^{2} M_{ii} \). Then \( \tilde{w}(x, t) \) satisfies

\[
L_t \tilde{w}_i(x, t) + (\lambda_0 - h_{ii}(x, t))\tilde{w}_i(x, t) - h_{ij}(x, t)\tilde{w}_j(x, t) = 0, \quad x \in \mathbb{R}^3, \quad t > 0, \quad j \neq i, \quad (4.5)
\]

\[
\tilde{w}_{i, 0}(x) = |w_{i, 0}(x)|, \quad x \in \mathbb{R}^3.
\]

A similar discussion as (4.3) implies that

\[
|\tilde{w}_i(x, t)| \leq \tilde{w}_i(x, t), \quad x \in \mathbb{R}^3, \quad t > 0, \quad i = 1, 2.
\]
By [7] Theorems 2 and 3, Chapter 9, we know that there exists a $2 \times 2$ matrix function $\Gamma(x, y, t, s) = \tilde{\Gamma}(x - y, t - s)$ such that
\[
\tilde{w}(x, t) = \int_{\mathbb{R}^3} \Gamma(x - y, t) \cdot \tilde{w}_0(y) dy,
\]
and there exist positive numbers $\tilde{A} \geq 1$ and $\tilde{B} \leq 1$ such that
\[
|\Gamma(x - y, t - s)| = \sum_{i,j=1}^2 |\Gamma_{ij}| \leq \tilde{A}(t - s)^{-3/2} \exp\{-\tilde{B}|x - y|^2/t - s\}
\]
for any $0 \leq s < t \leq 2$. By the uniqueness of solutions, the solution of (4.5) can be written as
\[
\tilde{w}(x, t) = \int_{\mathbb{R}^3} \Gamma(x - y_1, 1) dy_1 \int_{\mathbb{R}^3} \Gamma(y_1 - y_2, 1) dy_2 \cdots \int_{\mathbb{R}^3} \Gamma(y_{k-1} - y_k, 1) dy_k \tilde{w}_0(y) dy
\]
for any $t > 0$, where $k = \max\{[t - 1], 0\}$ and $[t]$ represents the largest integer no more than $t$. Therefore
\[
\frac{\tilde{w}_i(x, t)}{v_2(x)} = \int_{\mathbb{R}^3} \Gamma_i(x - y_1, 1) dy_1 \int_{\mathbb{R}^3} \Gamma(y_1 - y_2, 1) dy_2 \cdots \int_{\mathbb{R}^3} \Gamma(y_{k-1} - y_k, 1) dy_k \tilde{w}_0(y) \frac{dy}{v_2(x)}
\]
for any $t > 0$ and $i = 1, 2$. Then by a same computation as in [13] Lemma 4.2, we have
\[
|I| \leq \tilde{A}^{k+1}(1 + t)^{3/2} \left( \int_{\mathbb{R}^3} e^{-\tilde{B}|x|^2} dx \right)^k \int_{\mathbb{R}^3} t^{-3/2} e^{-\tilde{B}|x|^2} \chi_{D(\gamma)^c}(y) \left| \frac{\tilde{w}_0(y)}{v_2(x)} \right| dy
\]
\[
\leq \tilde{A}^{k+1}e^{2\tilde{B}} \left( \frac{2\pi}{B} \right)^{3k/2} \int_{D(\gamma)^c} t^{-3/2} e^{-\tilde{B}|x|^2} \left( \frac{v_2(x) - y}{v_2(x)} \right)^{3i} \left| \tilde{w}_0(x - y) \right| dy.
\]
Moreover, since $U_2(x + y)e^{-N_1y}y$ is decreasing in $y$, we know $U_2(x + y)e^{-N_1y} \leq U_2(x)$ for any $y \geq 0$. With the help of this fact, we have
\[
\frac{v_2(x - y)}{v_2(x)} = \frac{U_2 \left( \frac{\xi}{z}(x_3 - y_3 + h(x' - y')) \right)}{U_2 \left( \frac{\xi}{z}(x_3 + h(x')) \right)} \leq \frac{U_2 \left( \frac{\xi}{z}(x_3 + h(x' + \max\{1, m_x\} \Sigma_{i=1}^3 y_i)) \right)}{U_2 \left( \frac{\xi}{z}(x_3 + h(x')) \right)} \leq e^{N_1 \frac{\xi}{z} \max\{1, m_x\} \Sigma_{i=1}^3 |y_i|}
\]
\[ |I| \leq C_{k,t} t^{-3/2} \int_{B(x,\gamma)c} e^{-\frac{B|x-y|^2}{4t}} e^{N_i \sum_{i=1}^{3} |x_i-y_i|} dy \max_{i=1,2} \frac{|\tilde{w}_{i,0}(y)|}{(v_2(y))^\beta}, \]
\[ = C_{k,t} t^{-3/2} e^{\frac{3N_i^2}{4t}} \int_{B(x,\gamma)c} e^{-\frac{B|x-y|^2}{4t}} \sum_{i=1}^{3} |y_i-x_i| \frac{e^{-\frac{N_i^2}{2\pi}}}{(v_2(y))^\beta} dy \max_{i=1,2} \frac{|\tilde{w}_{i,0}(y)|}{(v_2(y))^\beta}, \]
\[ \leq C_{k,t} t^{-3/2} e^{\frac{3N_i^2}{4t}} \int_{\mathbb{R}^3} e^{-\frac{B|x-y|^2}{4t}} \chi_{D(\gamma)}(y) \frac{e^{-\frac{N_i^2}{2\pi}}}{(v_2(y))^\beta} dy \max_{i=1,2} \frac{|\tilde{w}_{i,0}(y)|}{(v_2(y))^\beta}, \]
\[ \leq C_{k,t} \left( \frac{\pi}{B} \right)^{3/2} e^{\frac{3N_i^2}{4t}} \max_{i=1,2} \frac{|\tilde{w}_{i,0}(y)|}{(v_2(y))^\beta}. \]

Note that if \( x \in D(2\gamma) \), then \( \gamma \leq |\text{dist}(x, \Gamma) - \text{dist}(y, \Gamma)| \leq |x-y| \) for all \( y \in D(\gamma)c \).

Therefore,
\[ |I| \leq C_{k,t} t^{-3/2} \int_{B(x,\gamma)c} e^{-\frac{B|x-y|^2}{4t}} e^{N_i \sum_{i=1}^{3} |x_i-y_i|} dy \max_{i=1,2} \frac{|\tilde{w}_{i,0}(y)|}{(v_2(y))^\beta}, \]
\[ = C_{k,t} t^{-3/2} e^{\frac{3N_i^2}{4t}} \int_{B(x,\gamma)c} e^{-\frac{B|x-y|^2}{4t}} \sum_{i=1}^{3} |y_i-x_i| \frac{e^{-\frac{N_i^2}{2\pi}}}{(v_2(y))^\beta} dy \max_{i=1,2} \frac{|\tilde{w}_{i,0}(y)|}{(v_2(y))^\beta}, \]
\[ \leq C_{k,t} t^{-3/2} e^{\frac{3N_i^2}{4t}} \int_{\mathbb{R}^3} e^{-\frac{B|x-y|^2}{4t}} \chi_{D(\gamma)}(y) \frac{e^{-\frac{N_i^2}{2\pi}}}{(v_2(y))^\beta} dy \max_{i=1,2} \frac{|\tilde{w}_{i,0}(y)|}{(v_2(y))^\beta}, \]
\[ \leq C_{k,t} \left( \frac{\pi}{B} \right)^{3/2} e^{\frac{3N_i^2}{4t}} \max_{i=1,2} \frac{|\tilde{w}_{i,0}(y)|}{(v_2(y))^\beta}. \]

Note that \( 0 < t - k < 2 \) and let \( \lambda_0 := \lambda_0' + 2 + \ln \left( \frac{2\sqrt{2\pi eA}}{B\sqrt{B}} \right) + \frac{3N_i^2}{4t} \), then (4.3) follows. To prove the inequality (4.4), we just need to replace \( D(\gamma) \) by \( \mathbb{R}^3 \) in the term II. Then the proof is complete. \( \square \)

By a proper coordinate change, we show that the pyramidal traveling front \( \mathbf{V}(\mathbf{x}) \) converges to a two dimensional V-form front on edges of the pyramid at infinity. For each positive integer \( j \in \{ 1, 2, \ldots, n \} \), we consider a plane perpendicular to an edge \( \Gamma_j = S_j \cap S_{j+1} \). Then the cross section of \( -x_3 = \max\{ h_j(\mathbf{x}') \}, h_{j+1}(\mathbf{x}') \} \) in this plane is V-shaped. Let \( \mathbf{V}^j \) be the two dimensional V-form front as in Theorem 4.1 corresponding to the cross section of \( -x_3 = \max\{ h_j(\mathbf{x}') \}, h_{j+1}(\mathbf{x}') \} \). We now make some preparations before giving the formulation of \( \mathbf{V}^j \).

It is easy to see that the expression of \( \Gamma_j \) is
\[ \frac{x_1}{B_j-B_{j+1}} = \frac{x_2}{A_{j+1}-A_j} = \frac{x_3}{m(A_j B_{j+1} - A_{j+1} B_j)}, \quad x_3 < 0. \]

Define
\[ p_j := A_j B_{j+1} - A_{j+1} B_j > 0, \quad 1 \leq j \leq n; \]
\[ q_j := \sqrt{(A_j - A_{j+1})^2 + (B_j - B_{j+1})^2} > 0, \quad 1 \leq j \leq n. \]
Then the direction of $\Gamma_j$ is given by
\[
\frac{1}{\sqrt{m^2_j p_j^2 + q_j^2}} (B_{j+1} - B_j, A_j - A_{j+1}, m_s(A_{j+1} B_j - A_j B_{j+1}))
\]
and the traveling direction of the two dimensional V-form wave $V^j$ is perpendicular to the direction of $\Gamma_j$ and given by
\[
\frac{1}{q_j \sqrt{m^2_j p_j^2 + q_j^2}} (m_s(B_{j+1} - B_j) p_j, m_s(A_j - A_{j+1}) p_j, q_j^2).
\]

Let $s_j$ be the speed of $V^j$ and $2\theta_j \in (0, \pi)$ be the angle between $S_j$ and $S_{j+1}$. It is not difficult to see that
\[
s_j \sin \theta_j = c, \quad \sin \theta_j = \frac{\sqrt{m^2_j p_j^2 + q_j^2}}{q_j \sqrt{1 + m^2_s}}, \quad s_j = \frac{sq_j}{\sqrt{m^2_j p_j^2 + q_j^2}}.
\]
The speed of $V^j$ toward the $x_3$-axis equals
\[
s_j \sqrt{m^2_j p_j^2 + q_j^2/ q_j} = c \sqrt{1 + m^2_s} = s,
\]
which coincides with the speed of $V$. Now we define a matrix
\[
R_j = \begin{pmatrix}
\frac{A_{j+1} - A_j}{q^j} & \frac{m_s(B_{j+1} - B_j) p_j}{q^j \sqrt{m^2_j p_j^2 + q_j^2}} & \frac{B_{j+1} - B_j}{q^j \sqrt{m^2_j p_j^2 + q_j^2}} \\
\frac{B_{j+1} - B_j}{q^j} & \frac{m_s(A_j - A_{j+1}) p_j}{q^j \sqrt{m^2_j p_j^2 + q_j^2}} & \frac{A_j - A_{j+1}}{q^j \sqrt{m^2_j p_j^2 + q_j^2}} \\
q^j & q^j & \sqrt{m^2_j p_j^2 + q_j^2}
\end{pmatrix}
\]
and make the following coordinate transformation:
\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = R_j \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = R_j^T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.
\]
Define $V^j(x) := \Phi(\xi, \eta; s_j)$. Since $R_j$ is an orthogonal matrix, the graph of $V^j(x)$ is the same as that of $\Phi(\xi, \eta; s_j)$ except the position in the space. Direct calculations show that $V^j(x)$ satisfies (1.6) with speed $s_j$ for each $j \in \{1, 2, \ldots, n\}$. Thus we call $V^j$ a planar V-form front corresponding to the edge $\Gamma_j$. Set
\[
Q_j := \{ x \in \mathbb{R}^3 : \text{dist}(x, \Gamma) = \text{dist}(x, \Gamma_j) \}.
\]
Then we have $\mathbb{R}^3 = \bigcup_{j=1}^n Q_j$. Define $\hat{V}(x) = \max_{1 \leq j \leq n} V^j(x)$. From the monotonicity of $V^j$ in $x_3$, $V(x)$ is strictly monotone in $x_3$. In addition, $\hat{V}(x)$ has the following properties.

**Lemma 4.3.** $\hat{V}(x)$ satisfies $v^- (x) < \hat{V}(x) < V(x)$ for $x \in \mathbb{R}^3$ and
\[
\lim_{\gamma \to \infty} \sup_{x \in D(\gamma)} \frac{|\hat{V}(x) - v^- (x)|}{(v^- (x))^\delta_0} = 0.
\]

**Proof.** By Theorem 4.1 we have
\[
\max \{ U\left(\frac{c}{s}(x_3 + h_j(x'))\right), U\left(\frac{c}{s}(x_3 + h_{j+1}(x'))\right) \} < V^j(x), \quad x \in \mathbb{R}^3.
\]
It follows that $v^-(x) = U\left(\frac{c}{s}(x + h(x'))\right) < \hat{V}(x)$ for $x \in \mathbb{R}^3$. Moreover, taking the left and right sides of the inequality

$$\max \left\{ U\left(\frac{c}{s}(x + h_j(x'))\right), U\left(\frac{c}{s}(x + h_{j+1}(x'))\right) \right\} < v^-(x)$$

as initial values of (1.6) respectively, we obtain $V_j(x) \leq V(x)$ for $x \in \mathbb{R}^3$. Then we have $\hat{V}(x) \leq V(x)$ for $x \in \mathbb{R}^3$. Finally, (4.0) follows from (3.14).

Let $v(x, t; v_0)$ be the solution of (1.4) with initial value $v_0 \in [0, 1]$ which satisfies (1.8). By Lemma 4.2 we have

$$\max_{i=1,2} \sup_{x \in D(\gamma)} \frac{|v_i(x, t; v_0) - V_i(x)|}{(v_2(x))^\beta_i} \leq 6e^{\lambda_0 t} \pi \frac{A}{B} \int_0^{\infty} e^{-\delta r^2} dr \max_{i=1,2} \sup_{x \in D(\gamma)} \frac{|V_i(x) - v_i,0(x)|}{(v_2(x))^\beta_i} + e^{\lambda_0 t} \left(\frac{\pi}{B}\right)^{3/2} \max_{i=1,2} \sup_{x \in D(\gamma)} \frac{|v_i(x) - v_i,0(x)|}{(v_2(x))^\beta_i},$$

for any $\gamma > 0$ and $t > 0$. It follows that

$$\lim_{\gamma \to +\infty} \sup_{x \in D(\gamma)} \frac{|v_i(x, t; v_0) - V_i(x)|}{(v_2(x))^\beta_i} = 0,$$

which implies that

$$\lim_{\gamma \to +\infty} \sup_{x \in D(\gamma), x \in Q_j} \frac{|v_i(x, t; v_0) - V_i(x)|}{(v_2(x))^\beta_i} = 0,$$

and

$$\lim_{\gamma \to +\infty} \sup_{x \in D(\gamma)} \frac{|v_i(x, t; v_0) - \hat{V}_i(x)|}{(v_2(x))^\beta_i} = 0,$$

for any fixed $t > 0$. Using Theorem 4.1 and (4.7), we obtain the following result through a similar discussion as Wang et al. [45, Proposition 4.5] or a slight modification of the proof in [33 Proposition 1].

**Proposition 4.4.** Assume that $v_0 \in [0, 1]$ satisfies (1.8). For any given $\epsilon > 0$, one can choose a $T^* > 0$ large enough such that

$$\lim_{R \to \infty} \max_{1 \leq i \leq n |x| \geq R, x \in Q_j} \sup_{1 \leq j \leq n} \frac{|v_i(x, t; v_0) - V_i(x)|}{(v_2(x))^\beta_i} < \epsilon$$

for any fixed $t \geq T^*$.

**Lemma 4.5.** Assume that $v_0 \in [0, 1]$ satisfies (1.8). Let $V$ be defined by (3.15), then for any given $\epsilon > 0$ one can choose a $T^* > 0$ large enough such that

$$\lim_{R \to \infty} \max_{i=1,2} \sup_{|x| \geq R} \frac{|v_i(x, t; v_0) - V_i(x)|}{(v_2(x))^\beta_i} < \epsilon$$

for any fixed $t \geq T^*$. In particular, one has

$$\lim_{R \to \infty} \max_{i=1,2} \sup_{|x| \geq R} \frac{|\hat{V}_i(x) - V_i(x)|}{(v_2(x))^\beta_i} = 0.$$
Lemma 4.7. Let \( \delta > 0 \) be sufficiently large such that, for any \( \delta > 0 \),
\[
\lim_{R \to \infty} \max_{1 \leq j \leq n, |x| \geq R} \sup_{x \in Q_j} \frac{|V_j(x) - V_j^*(x)|}{(v^2_j(x))^{\beta_i}} < \epsilon.
\]
Thus because of the arbitrariness of \( \epsilon \), we have
\[
\lim_{R \to \infty} \max_{1 \leq j \leq n, |x| \geq R} \sup_{x \in Q_j} \frac{|V_j(x) - V_j^*(x)|}{(v^2_j(x))^{\beta_i}} = 0.
\]
Then (4.11) follows from (4.12). And (4.10) follows from (4.12) and (4.9). The proof is complete. \( \Box \)

Equality (4.11) shows that a pyramidal traveling front \( V \) converges to two dimensional V-form fronts \( \Phi \) near edges. By a similar proof to that of Wang [45], we can get the following lemma.

Lemma 4.6. Let \( V \) be defined by (3.10). Then for any \( \delta \in (0, 1) \) we have
\[
\min_{i=1,2} \inf_{V_j(x) \in [\delta, 1-\delta]} \frac{\partial}{\partial x_3} V_j(x) > 0, \quad j = 1, 2, 3, \ldots, n;
\]
\[
\min_{i=1,2} \inf_{V_j(x) \in [\delta, 1-\delta]} \frac{\partial}{\partial x_3} V_j(x) > 0.
\]
Inequality (3.7) and Lemma 4.6 yield that for any \( M > 0 \), there exists a positive \( C_* \) such that
\[
\min_{1 \leq j \leq n} \frac{\partial}{\partial x_3} V_j(x) \geq C_*, \quad \min_{1 \leq j \leq n} \frac{\partial}{\partial x_3} V_j^*(x) \geq C_*.\]
Then we can construct the following two supersolutions.

Lemma 4.7. Let \( V \) be as in (3.15). For any \( 0 < \delta < \min \{ \min_{i=1,2} \left\{ \frac{\prod \beta_j}{24(\lambda_1 \lambda_2)} \right\}, 1 \} \),
there exists a positive constant \( \kappa \) small enough and a positive constant \( \rho = \rho(\kappa) \) sufficiently large such that, for any \( \delta \in (0, \delta_0) \) where
\[
\delta_0 := \min \left\{ \frac{1}{2(s-c+1)}, \frac{\varepsilon_1}{P_1}, \min_{i=1,2} \frac{q_i}{8bP_1} \right\},
\]
the function
\[
W^+(x, t; \delta) = V(x', x_3 + \xi + \rho \delta (1 - e^{-\kappa t})) + \delta e^{-\kappa t} \left( \omega(\xi) \frac{p}{P_1} + (1 - \omega(\xi)) U_{\beta}^3(\theta) \right)
\]
is a supersolution to (1.9), where \( \theta = \varepsilon \frac{1}{s}(x_3 + \xi + \rho \delta (1 - e^{-\kappa t})) + \varphi(\partial x')/\theta \) and \( \xi \in \mathbb{R} \) is a constant. See (2.4) for \( \varepsilon_1 \).

Proof. By a direct computation, we have
\[
\mathcal{L}[W^+], \quad \mathcal{L}[W^+] = \rho \delta e^{-\kappa t} \partial_{x_3} V_i - \delta e^{-\kappa t} \left[ \omega(\xi)p_i + (1 - \omega(\xi)) U_{\beta}^3(\theta) \right]
\]
\[
+ \sum_{j=1}^{3} \frac{\rho \delta^2 e^{-2\kappa t}}{s} \left[ (p_i - U_{\beta}^3(\theta)) \omega'(\xi) + \beta_i (1 - \omega(\xi)) U_{\beta}^{3-1}(\theta) U_{\beta}^2(\theta) \right]
\]
\[
- \delta e^{-\kappa t} \left( \omega''(\xi)(p_i - U_{\beta}^3(\theta)) \sum_{j=1}^{3} \theta_{x_j}^2 + \omega'(\xi)(p_i - U_{\beta}^3(\theta)) \sum_{j=1}^{3} \theta_{x_j} \right)
\]
\[
- \omega'(\xi)(p_i - U_{\beta}^3(\theta)) - 2 \beta_i \omega'(\xi) U_{\beta}^{3-1}(\theta) U_{\beta}^2(\theta) \sum_{j=1}^{3} \theta_{x_j}^2,
\]
\[ + (1 - \omega(\theta)) \left[ \beta_i(\beta_i - 1)U_2^{\beta_i - 2}(\theta)(U_2'(\theta))^2 \sum_{j=1}^{3} \theta_{x_j} + \beta_i U_2^{\beta_i - 1}(\theta)U_2''(\theta) \sum_{j=1}^{3} \theta_{x_j}^2 \right. \\
\left. + \beta_i U_2^{\beta_i - 1}(\theta)U_2'(\theta) \sum_{j=1}^{3} \theta_{x_j x_j} - c \beta_i U_2^{\beta_i - 1}(\theta)U_2'(\theta) \right] \right\} \\
- f_i(W^+) - g_i(W^+) + f_i(V). \]

Then the rest of the proof is almost the same as that of [30 Lemma 4.2], we omit it. □

By a similar argument to that of Lemma 4.7, we obtain the following lemma.

**Lemma 4.8.** There exists a positive \( \kappa \) constant small enough and a positive constant \( p = p(\kappa) \) sufficiently large such that, for any \( \delta \in (0, \delta_0) \) \( (\delta_0 \text{ is given in Lemma 4.7}) \), \( \xi \in \mathbb{R} \) and \( 0 < \alpha < \min \{ \alpha^+ (\varepsilon, \beta), \min_{i=1,2} \left( - \frac{\Pi_{i=1}^2 \lambda_i}{24C_{1,2}} \right) \} \), the function \( w^+(x,t;\delta) = v^+(x',x_3 + \xi_t + \rho \delta(1 - e^{-\kappa t}); \varepsilon, \beta, \alpha) + \delta e^{-\kappa t} \left( \omega(\hat{\theta})p + (1 - \omega(\hat{\theta}))U(\hat{\theta}) \right) \) is a supersolution to (1.9), where \( \hat{\theta} = \frac{x_3 + \xi_t + \rho \delta(1 - e^{-\kappa t}) + \varphi(\alpha x')}{\alpha} \) and \( \xi_t \in \mathbb{R} \) is a constant.

**Lemma 4.9.** Let \( V \) be defined by (3.15). Then it satisfies

\[ \lim_{R \to \infty} \sup_{|x_3 + h(x')| \geq R} \frac{\partial_{x_3} V_i(x)}{(v_2'(x))^\beta_i} = 0, \quad i = 1, 2. \]  

**(4.13)**

**Proof.** We split the proof into two steps.

**Step 1.** We prove that

\[ \lim_{R \to \infty} \sup_{|x_3 + h(x')| \geq R} \frac{\partial_{x_3} V_i(x)}{(v_2'(x))^\beta_i} = 0, \quad i = 1, 2. \]

It is sufficient to prove that \( \lim_{R \to \infty} \sup_{|x_3 + h(x')| \geq R} \partial_{x_3} V_i(x) = 0 \), since

\[ \lim_{R \to \infty} \sup_{|x_3 + h(x')| \geq R} v_i^+(x) = \lim_{R \to \infty} \sup_{|x_3 + h(x')| \geq R} U_2 \left( \frac{C}{8}(x_3 + h(x')) \right) = 1. \]

Obviously, the assumption \( x_3 + h(x') \to \infty \) implies that \( \text{dist}(x,\Gamma) \to \infty \). It follows that

\[ \lim_{R \to \infty} \sup_{|x_3 + h(x')| \geq R} |V(x) - 1| = 0, \]

and thus \( \lim_{R \to \infty} \sup_{|x_3 + h(x')| \geq R} |F(V(x))| = 0 \). Applying the interior Schauder’s estimate to

\[ -\Delta V_i + s \partial_{x_3} V_i = f_i(V) \quad \text{in} \quad B(x_0, 2), \quad \forall x_0 \in \mathbb{R}^3, \quad i = 1, 2, \]

we have

\[ \lim_{R \to \infty} \sup_{i = 1, 2} \left\{ \|V_i\|_{W^{2,p}(B(x_0, 1))} \middle| x_0 \in \mathbb{R}^3, |x_3 + h(x')| \geq R \right\} = 0 \]

for \( p > 3 \). Therefore

\[ \lim_{R \to \infty} \sup_{|x_3 + h(x')| \geq R} \partial_{x_3} V_i(x) = 0, \quad i = 1, 2. \]
**Step 2.** We prove that

\[
\lim_{R \to \infty} \sup_{x_3 + h(x') \leq -R} \frac{\partial x_3 V_i(x)}{(v_2^i(x))^\beta_i} = 0, \quad i = 1, 2.
\]

Note that the set \( \{ x \in \mathbb{R}^3 \mid x_3 + h(x') \leq -R \} \subseteq D(R) = \{ x \in \mathbb{R}^3 \mid \text{dist}(x, \Gamma) \geq R \} \).

It follows from (3.14) that for \( i = 1, 2 \),

\[
\lim_{R \to \infty} \sup_{x_3 + h(x') \leq -R} \frac{|V_i(x) - v_i^-(x)|}{(v_2^i(x))^\beta_i} \leq \lim_{R \to +\infty} \sup_{x \in D(R)} \frac{|V_i(x) - v_i^-(x)|}{(v_2^i(x))^\beta_i} = 0.
\]

Then a similar discussion as that in [30] Lemma 4.6] shows that (4.13) holds for the case \( x_3 + h(x') \to -\infty \). The proof is complete. \( \square \)

To prove the stability result, we also need the following lemma, which can be referred to [49] Lemma 4.8.

**Lemma 4.10.** Assume that \( \delta \in (0, \varepsilon_1) \), where \( \varepsilon_1 \) is in (2.4). For any \( x \in \mathbb{R}^3 \) with

\[
\delta \leq \tilde{V}_i(x) = \max_{1 \leq j \leq n} V_i^j(x) \leq 1 - \delta,
\]

we have

\[
\inf_{0 < \rho < \rho_0} \frac{\tilde{V}_i(x', x_3 + \rho) - \tilde{V}_i(x)}{\rho} \geq \min_{1 \leq j \leq n} \min_{\hat{V}_i^j(x) \leq 1 - \frac{1}{2}} \min_{i = 1, 2} \inf \frac{\partial}{\partial x_3} V_i^j(x) > 0,
\]

where \( \rho_0 \) is a positive constant depending only on \( \delta \).

Let \( v^+(x; \varepsilon, \beta, \alpha) \) be as in Lemma 3.1. Define

\[
V^*(x) := \lim_{t \to \infty} \tilde{v}(x, t; v^+), \quad x \in \mathbb{R}^3.
\]

Since \( v^+ \) is a supersolution, \( V^*(x) \) is well defined and satisfies

\[
-\Delta V^* + s V^*_{x_3} - \tilde{F}(V^*) = 0,
\]

and it may depend on \( \varepsilon, \alpha \) and \( \beta \). By the comparison principle, we have

\[
v^- (x) < \tilde{V}(x) < V(x) \leq V^*(x) < v^+(x; \varepsilon, \beta, \alpha), \quad x \in \mathbb{R}^3.
\]

From (3.5), we have

\[
\lim_{\gamma \to +\infty} \sup_{x \in D(\gamma)} \frac{|V_i^*(x) - v_i^-(x)|}{(v_2^i(x))^\beta_i} = 0, \quad i = 1, 2.
\]

Then applying Proposition 4.1 to \( V^* \) we obtain

\[
\lim_{R \to \infty} \sup_{i = 1, 2} \frac{\tilde{V}_i(x) - V_i^*(x)}{(v_2^i(x))^\beta_i} = 0. \tag{4.14}
\]

It follows immediately that

\[
\lim_{R \to \infty} \sup_{i = 1, 2} \frac{|\tilde{V}_i(x) - V_i^*(x)|}{(v_2^i(x))^\beta_i} = 0. \tag{4.15}
\]

The next lemma says that \( V^*(x) \) is independent of \( \varepsilon, \alpha \) and \( \beta \).

**Lemma 4.11.** One has \( V(x) = V^*(x) \) in \( \mathbb{R}^3 \).
Proof. Assume that $V(x) \neq V^*(x)$. We take $\delta \in (\frac{\delta_0}{2}, \delta_0)$, where $\delta_0$ is given in Lemma 4.7. By the definition of $V^*$ and (4.15), there exists $\lambda > 0$ sufficiently large such that

$$V^*(x) \leq V(x', x_3 + \lambda) + \delta(v_2^* (\theta))^{\beta_i} \leq V(x', x_3 + \lambda) + \delta(\omega(\theta)p + (1 - \omega(\theta))\lambda^3(\theta)),$$

where $\theta = \xi(x_3 + \lambda + \varphi(\partial x') / \partial)$. Then the comparison principle implies that

$$V^*(x) \leq W^+(x', x_3 + \lambda, t; \delta), \quad x \in \mathbb{R}^3, \quad t > 0.$$

Letting $t \to \infty$, we obtain

$$V^*(x) \leq V(x', x_3 + \lambda + \rho \delta), \quad x \in \mathbb{R}^3.$$

Define

$$\Lambda := \inf\{\lambda > 0 : V^*(x) \leq V(x', x_3 + \lambda), \quad x \in \mathbb{R}^3\}.$$

Then $\Lambda \geq 0$, and $V^*(x) \leq V(x', x_3 + \lambda)$ for all $x \in \mathbb{R}^3$. The assumption $V(x) \neq V^*(x)$ implies that $\Lambda \neq 0$. Thus the strong maximum principle yields that either $V^*(x) \equiv V(x', x_3 + \lambda)$ or $V^*(x) < V(x', x_3 + \lambda)$. We assert that the former case is impossible. To see this, we choose a sequence $\{x'_m\}_{m \in \mathbb{N}} \subseteq \mathbb{R}^2$ satisfying $h(x'_m) \to \infty$ and $\text{dist}(x'_m, E) \to \infty$. Then by the fact that $V^- < V \leq V^* < V^+$, we have

$$\lim_{m \to \infty} V^*(x'_m, -h(x'_m)) = U(0), \quad \liminf_{m \to \infty} V^*(x'_m, -h(x'_m) + \Lambda) \geq U\left(\frac{c}{s}\Lambda\right),$$

which contradicts $V^*(x) \equiv V(x', x_3 + \lambda)$. Now we assume that

$$V^*(x) < V(x', x_3 + \lambda), \quad x \in \mathbb{R}^3.$$

By Lemma 4.9 for any fixed $\rho > 0$ defined in Lemma 4.7, we can take a $R_* = R_*(\rho) > 0$ large enough such that

$$\sup_{|x_3 + h(x')| \geq R_* - \frac{3}{2} \frac{(v_2 (x))^{\beta_i}}{\lambda^3}} \frac{\partial x_3 V_i(x)}{v_i(x)} \leq \frac{1}{\lambda}, \quad i = 1, 2.$$

Define

$$\mathcal{D} := \{x \in \mathbb{R}^3 : |x_3 + h(x')| \leq R^*\}.$$

Choose a constant $\sigma$ satisfying $0 < \sigma < \left\{\frac{\delta_0}{2}, \frac{\Lambda}{4}, \frac{\ln \frac{2}{\delta}}{2N_1p}\right\}$, where $N_1 = \sup_{x \in \mathbb{R}^3} U(x) / U_2(x)$. Using Lemma 4.10 for $x \in \mathcal{D}$, we have

$$\hat{V}_i(x', x_3 + \Lambda) - \hat{V}_i(x) \geq \min \{g_0, \Lambda\} \min_{\xi \leq n} \inf_{1 \leq j \leq n} \frac{1}{2} \frac{\partial}{\partial x_3} V_i^j(x) > 0,$$

where

$$\delta_* = \min_{i = 1, 2} \min \left\{\frac{\delta_0}{2}, 1 - \max_{1 \leq j \leq n} \sup_{x \in \mathcal{D}} V_i^j(x', x_3 + \Lambda), \min_{1 \leq j \leq n} \inf_{x \in \mathcal{D}} V_i^j(x)\right\},$$

and $g_0$ is defined in Lemma 4.10 associated with $\delta_*$. Thus, for $x \in \mathcal{D}$, it follows that

$$\lim_{R \to \infty} \inf_{|x| > R, x \in \mathcal{D}} (V_i(x', x_3 + \Lambda) - V^*_i(x))$$

$$\geq \lim_{R \to \infty} \inf_{|x| > R, x \in \mathcal{D}} (\hat{V}_i(x', x_3 + \Lambda) - V^*_i(x))$$

$$\geq \lim_{R \to \infty} \inf_{|x| > R, x \in \mathcal{D}} (\hat{V}_i(x', x_3 + \Lambda) - \hat{V}_i(x)) + \lim_{R \to \infty} \inf_{|x| > R, x \in \mathcal{D}} (\hat{V}_i(x) - V^*_i(x)).$$
> 0,
since \( \lim_{R \to \infty} \sup_{|x| > R, x \in D} (V_i^+(x) - \hat{V}_i(x)) = 0 \) by (4.14). Thus we can choose a small \( \sigma \) such that
\[
V_i(x', x_3 + \Lambda - 2\rho \sigma) > V_i^+(x) \quad \text{in } D.
\]
In the domain \( \mathbb{R}^3 \setminus D \), we have
\[
\frac{V_i(x', x_3 + \Lambda - 2\rho \sigma) - V_i(x', x_3 + \Lambda)}{(v_2(x', x_3 + \Lambda - 2\rho \sigma))^\beta_i} \geq -2\sigma \int_0^1 \partial_{\tau x} V_i(x', x_3 + \Lambda - 2\rho \sigma) d\tau
\]
\[
= -2\sigma \left( \frac{v_2(x', x_3 + \Lambda)}{v_2(x', x_3 + \Lambda - 2\rho \sigma)} \right)^\beta_i \int_0^1 \partial_{\tau x} V_i(x', x_3 + \Lambda - 2\rho \sigma) d\tau
\]
\[
\geq -2\rho \sigma e^{2N_i \rho \sigma \beta_i} \int_0^1 \partial_{\tau x} V_i(x', x_3 + \Lambda - 2\rho \sigma) d\tau
\]
\[
\geq -2\rho \sigma \frac{3}{2} \frac{1}{\beta \rho} = -\sigma.
\]
In other other words,
\[
V_i(x', x_3 + \Lambda) \leq V_i(x', x_3 + \Lambda - 2\rho \sigma) + \sigma U_2^\beta_i \left( \frac{c}{s} (x_3 + \Lambda - 2\rho \sigma + \phi(\alpha x'))/\alpha \right).
\]
Combining the above two cases, we obtain
\[
V_i^+(x) \leq V_i(x', x_3 + \Lambda - 2\rho \sigma) + \sigma U_2^\beta_i \left( \frac{c}{s} (x_3 + \Lambda - 2\rho \sigma + \phi(\alpha x'))/\alpha \right),
\]
for all \( x \in \mathbb{R}^3 \). Then Lemma 4.7 and the comparison principle yield
\[
V_i^+(x) \leq W_i^+(x', x_3 + \Lambda - 2\rho \sigma, t; \sigma), \quad x \in \mathbb{R}^3, \ t > 0.
\]
Letting \( t \to \infty \), we have
\[
V_i^+(x) \leq V_i(x', x_3 + \Lambda - 2\rho \sigma), \quad x \in \mathbb{R}^3.
\]
This contradicts the definition of \( \Lambda \). Thus \( \Lambda \equiv 0 \). The proof is complete. \( \Box \)

**Theorem 4.12.** Assume \( b > 0 \) and \( r > 1 \). Fix a couple of \( \beta_1, \beta_2 \in (0, \beta^*) \) with \( \beta_2 < \beta_1 \). Assume that \( v_0 \in C(\mathbb{R}^3, \mathbb{R}^2) \) with \( v_0 \in [0, 1], v_0(x) \geq v^-(x) \) for \( x \in \mathbb{R}^3 \) and
\[
\lim_{\gamma \to \infty} \sup_{x \in D(\gamma)} \left| \frac{v_{i,0}(x) - V_{i,0}(x)}{(v_2(x))^{\beta_i}} \right| = 0, \quad i = 1, 2.
\]
Then the solution of \( [1.2] \) with initial value \( v_0 \) satisfies
\[
\lim_{t \to \infty} \left\| \frac{v_i(x, t; v_0) - V_i(x)}{(v_2(x))^{\beta_i}} \right\|_{L^\infty(\mathbb{R}^3)} = 0, \quad i = 1, 2. \quad (4.16)
\]

**Proof.** Under the condition \( v_0 \in [0, 1] \), the solution \( \tilde{v}(x, t; v_0) \) of (1.9) and (1.10) is also the solution of (1.4) and (1.5), namely, \( \tilde{v}(x, t; v_0) \equiv v(x, t; v_0) \). For a random
For any fixed $t \geq T^*$, the interior Shauder estimate to $\tilde{v}_i(x,t;v^-) - V_i(x)$ and $\tilde{v}_i(x,t;v^+) - V_i(x)$, and noticing that $v^-$ is bounded on $B_{R_0}(0)$, we have
\[
\lim_{t \to \infty} \sup_{x \in B_{R_0}(0)} \left| \frac{\tilde{v}_i(\cdot,t;v^-) - V_i(\cdot)}{(v_2(x))^{\beta_i}} \right| = 0, \quad \lim_{t \to \infty} \sup_{x \in B_{R_0}(0)} \left| \frac{\tilde{v}_i(\cdot,t;v^+) - V_i(\cdot)}{(v_2(x))^{\beta_i}} \right| = 0,
\]
for $i = 1, 2$. Since $m > 1$ can be taken arbitrarily large, we have
\[
\lim_{t \to \infty} \left\| \tilde{v}_i(\cdot,t;v^-) - V_i(\cdot) \right\|_{L^\infty(\mathbb{R}^3)} = 0, \quad \lim_{t \to \infty} \left\| \tilde{v}_i(\cdot,t;v^+) - V_i(\cdot) \right\|_{L^\infty(\mathbb{R}^3)} = 0,
\]
for $i = 1, 2$. Let $\delta \in (0,\frac{\beta_i}{3})$ and $\epsilon < \min\{\epsilon^+_i(\beta),\frac{\rho}{4\beta_i}\}$. Taking $\alpha \in (0,\alpha^+(\epsilon,\beta))$ small enough and using Lemma 4.5, we have
\[
v_i(x,T^*;v_0) \leq V_i(x) + \delta(v^-(x))^{\beta_i} \leq v_i^+(x) + \delta(v^-(x))^{\beta_i}.
\]
A similar discussion as in [35] shows that
\[
\lim_{t \to \infty} \left\| \frac{v_i(x,t;v^-) - V_i(x)}{(v_2(x))^{\beta_i}} \right\|_{L^\infty(\mathbb{R}^3)} = 0, \quad \lim_{t \to \infty} \left\| \frac{v_i(x,t;v^+) - V_i(x)}{(v_2(x))^{\beta_i}} \right\|_{L^\infty(\mathbb{R}^3)} = 0
\]
for $i = 1, 2$. Take $\hat{t}$ large enough such that
\[
v_i(x,\hat{t};v^-) \leq v_i(x,\hat{t};v^+) < V_i(x) + \delta(v^-(x))^{\beta_i}, \quad t \geq \hat{t}, \; i = 1, 2. \tag{4.17}
\]
Since $w^+(x,t;\delta)$ is a supersolution, there exists a $\hat{t} > 0$ such that
\[
v_i(x,t + T^* + 1;v_0) \leq V_i(x',x_3 + \rho\delta) + \delta e^{-\lambda_0\delta}(v^-(x))^{\beta_i}, \quad t \geq \hat{t}, \; i = 1, 2.
\]
Denote $v^+\delta(x) = v^+(x',x_3 + \rho\delta)$. Then Lemma 4.2 implies that
\[
v_i(x,\hat{t} + T^* + \hat{t} + 1;v_0) - v_i^+\delta(x) \leq \delta(v^-(x))^{\beta_i}, \quad i = 1, 2.
\]
Then by (4.17), we have
\[
v_i(x,\hat{t} + T^* + \hat{t} + 1;v_0) \leq V_i(x',x_3 + \rho\delta) + 2\delta(v^-(x))^{\beta_i}.
\]
Thus it follows from Lemma 4.7 that
\[
v(x,t + \hat{t} + T^* + \hat{t} + 1;v_0) \leq W^+(x',x_3 + \rho\delta,t;2\delta).
\]
Therefore, by letting $t \to \infty$ we have
\[
V_i(x) \leq v_i(x,t;v_0) \leq V_i(x',x_3 + \rho\delta + 2\rho\delta) + 2\delta(v^-(x))^{\beta_i},
\]
\[
0 \leq v_i(x,t;v_0) - V_i(x) \leq V_i(x',x_3 + \rho\delta + 2\rho\delta) - V_i(x) + 2\delta(v^-(x))^{\beta_i}.
\]
\[
\leq (v^-(x))^{\beta_i} \left( \frac{\partial_x v_2(x',x_3 + 3\rho\delta \tau)}{(v_2(x'))^{\beta_i}} + 2\delta \right)
\]
\[
\leq (v^-(x))^{\beta_i} \left( M e^{\frac{\beta_i}{2N_1}} 3\rho\delta + 2\right) \delta,
\]
where $\tau \in (0, 1)$ and
\[ M_* = \max_{x \in \mathbb{R}^3} \frac{\partial_{x_3} V_i(x)}{(v_2(x))^\beta}. \]
Because of the arbitrariness of $\delta$, (4.16) follows. Thus the proof is complete. □

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