Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 114, pp. 1–17.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

ASYMPTOTIC BEHAVIOR OF POSITIVE RADIAL SOLUTIONS TO ELLIPTIC EQUATIONS APPROACHING CRITICAL GROWTH

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Abstract. We study the asymptotic behavior of radially symmetric solutions to the subcritical semilinear elliptic problem

$$
\begin{gathered}
-\Delta u = u^{\frac{N+2}{N-2}}/[\log(e+u)]^{\alpha} \quad \text{in } \Omega = B_R(0) \subset \mathbb{R}^N, \\
u > 0, \quad \text{in } \Omega, \\
u = 0, \quad \text{on } \partial\Omega,
\end{gathered}
$$

as $\alpha \to 0^+$. Using asymptotic estimates, we prove that there exists an explicitly defined constant $L(N,R) > 0$, only depending on N and R, such that

$$
\begin{aligned}
&\limsup_{\alpha\to 0^+} \frac{\alpha u_\alpha(0)^2}{[\log(e+u_\alpha(0))]^{1+\frac{\alpha(N+2)}{2}}} \\
&\le L(N,R) \\
&\le 2^* \liminf_{\alpha\to 0^+} \frac{\alpha u_\alpha(0)^2}{[\log(e+u_\alpha(0))]^{\frac{\alpha(N-4)}{2}}}.
\end{aligned}
$$

1. Introduction and main results

We consider the classical Dirichlet boundary value problem

$$
\begin{gathered}
-\Delta u = f(u) \quad \text{in } \Omega \\
u > 0 \quad \text{in } \Omega \\
u = 0 \quad \text{in } \partial\Omega
\end{gathered}
\tag{1.1}
$$

for $u \in C^2(\overline{\Omega})$, in which Ω is an open bounded regular domain in $\mathbb{R}^N$, $N > 2$, and f is locally-Lipschitz in $[0,\infty)$ and superlinear at infinity (i.e. $\liminf f(u)/u > \lambda_1$ as $u \to \infty$ where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions). We denote by $2^* := 2N/(N-2)$ the critical Sobolev exponent. Namely, $H^1(\Omega)$ is compactly embedded in $L^p(\Omega)$ if and only if $p < 2^*$. The extended real number $f^\star := \lim_{u\to\infty} f(u)/u^{2^*-1}$ discriminates the problem (1.1) into three types: *critical* if $f^\star \in (0,\infty)$, *supercritical* if $f^* = \infty$, and *subcritical* if $f^\star = 0$.

Pohozaev [15] discover that for the power nonlinearity $f(u) = u^p$ with $p \ge 2^*-1$, there are no positive solutions to (1.1) in star-shaped domains. Bahri, Coron and

2010 *Mathematics Subject Classification.* 35B33, 35B45, 35B09, 35J60.
Key words and phrases. A priori bounds; positive solutions; semilinear elliptic equations; Dirichlet boundary conditions; growth estimates; subcritical nonlinearites.

Submitted November 11, 2019. Published November 18, 2020.

Ding show that (1.1) has a solution for some classes of non star-shaped domains, see [3, 9]. The equivalence between uniform $L^{2^*}(\Omega)$ a-priori bounds and uniform $L^\infty(\Omega)$ a-priori bounds in the subcritical case is proved in [4].

Assume that the nonlinearity is a pure subcritical power $f(u) = u^{2^*-1-\varepsilon}$, $\varepsilon > 0$, and $\Omega = B_R$ (the open ball of radius R). Atkinson and Peletier [2] studied the asymptotic behavior as $\varepsilon \to 0^+$ of solutions to (1.1), and proved that

$$\lim_{\varepsilon\to 0^+} \varepsilon u_\varepsilon(0)^2 = L(N,R),$$

and for all $r \neq 0$,

$$\lim_{\varepsilon\to 0^+} \frac{u_\varepsilon(r)}{\sqrt{\varepsilon}} = \widetilde{L}(N,R)\Big(\frac{1}{r^{N-2}} - \frac{1}{R^{N-2}}\Big).$$

Here $L(N,R)$ and $\widetilde{L}(N,R)$ are constants only dependent on N, and R, defined by

$$L(N,R) := \frac{4}{N-2}[N(N-2)]^{\frac{N-2}{2}} \frac{\Gamma(N)}{\Gamma(N/2)^2} \frac{1}{R^{N-2}}, \tag{1.2}$$

$$\widetilde{L}(N,R) := \frac{(N-2)^{\frac12}}{2}[N(N-2)]^{\frac{N-2}{4}} \frac{\Gamma(N/2)}{\Gamma(N)^{1/2}} R^{\frac{N-2}{2}} = \frac{[N(N-2)]^{\frac{N-2}{2}}}{L(N,R)^{1/2}}, \tag{1.3}$$

where Γ denotes the Gamma function. See also [11] with similar results for least energy solutions on general domains.

We focus our attention on problem (1.1) with nonlinearity

$$f(u) = f_\alpha(u) := \frac{|u|^{2^*-2}u}{[\log(e+|u|)]^\alpha}. \tag{1.4}$$

When $\alpha > \frac{2}{N-2}$, there are a-priori L^∞ bounds for classical positive solutions in bounded, C^2 domains, see [5, 6, 13, 14].

In [12], the existence of a-priori L^∞ bounds for positive solutions is extended for Hamiltonian elliptic systems $-\Delta u = f(v), -\Delta v = g(u)$ with Dirichlet homogeneous boundary conditions with

$$f(v) = \frac{v^p}{[\ln(e+v)]^\alpha}, \quad g(u) = \frac{u^q}{[\ln(e+u)]^\beta}, \quad \frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N},$$

and $\alpha, \beta > \frac{2}{N-2}$.

Also for the p-Laplacian there are a-priori bounds for $C^{1,\mu}(\overline{\Omega})$ positive solutions of elliptic equations $-\Delta_p u = f(u)$ with Dirichlet homogeneous boundary conditions when

$$f(u) = \frac{u^{p^\star-1}}{[\ln(e+u)]^\alpha}, \quad p^* = \frac{Np}{N-p}, \quad \alpha > \frac{p}{(N-p)};$$

see [7]. This leads to a natural question: Is this lower bound on α a technical or an intrinsic condition?

In this article we analyze the asymptotic behavior of solutions to

$$\begin{gathered} -\Delta u = u^{\frac{N+2}{N-2}}/[\log(e+u)]^\alpha \quad \text{in } \Omega = B_R(0) \subset \mathbb{R}^N, \\ u > 0, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.5}$$

as $\alpha \to 0^+$. Firstly, we prove that for each $\alpha \in \big(0, \frac{2}{N-2}\big]$ fixed, the set of positive solutions to (1.5) is a priori bounded. Henceforth, the bound from below on α in [5, 6, 7, 12] are technical rather than intrinsic, at least when Ω is the open ball of radius R. Secondly, we provide estimates for the growth of $u_\alpha(0)$ and $u_\alpha(r)$ as $\alpha \to 0^+$. We adapt the techniques introduced by Atkinson and Peletier for the case of subcritical powers in [1, 2].

Our first main result is on the existence of solutions to (1.5), and of L^∞ a priori bounds for each $\alpha > 0$ fixed. The existence of solutions is already known due to a result of Figueiredo, Lions and Nussbaum [8, Thm. 2.8] employing different techniques involving elliptic regularity theory and topological variational methods.

Theorem 1.1. *Fix $\alpha \in \big(0, \frac{2}{N-2}\big]$, let $f = f_\alpha$ be as in (1.4) and assume $\Omega = B_R$. Then the following results hold:*

(i) *There exists a radially symmetric solution to (1.5), $u = u_\alpha(r) > 0$.*
(ii) *There are constants $A = A_\alpha(N,R)$, $B = B_\alpha(N,R) > 0$ depending only on α, N and R, such that for every $u = u_\alpha > 0$, radially symmetric solution to (1.5),*

$$A_\alpha(N,R) \le \|u_\alpha\|_{L^\infty(\Omega)} \le B_\alpha(N,R), \quad \text{for each } \alpha \in \big(0, \frac{2}{N-2}\big].$$

Our second main result is an estimate of the asymptotic behavior of $u_\alpha(0) = \|u_\alpha\|_{L^\infty(\Omega)}$ as $\alpha \to 0^+$.

Theorem 1.2. *Let $f = f_\alpha$ be as in (1.4) with $\alpha \in \Big(0, \frac{2}{N-2}\Big]$, and $\Omega = B_R$. Then, there exists a constant $L(N,R) > 0$ only depending on N and R (defined by (1.2)), such that for any $u_\alpha = u_\alpha(r)$, radially symmetric positive solution to (1.5), we have*

$$\limsup_{\alpha\to 0^+} \frac{\alpha u_\alpha(0)^2}{[\log(e+u_\alpha(0))]^{1+\frac{\alpha(N+2)}{2}}} \le L(N,R), \tag{1.6}$$

$$\liminf_{\alpha\to 0^+} \frac{\alpha u_\alpha(0)^2}{[\log(e+u_\alpha(0))]^{\frac{\alpha(N-4)}{2}}} \ge \frac{1}{2^*} L(N,R). \tag{1.7}$$

Our third main result is an estimate of the asymptotic behavior of $u_\alpha(r)$ as $\alpha \to 0^+$, when $r \neq 0$.

Theorem 1.3. *Let $f_\alpha(u)$ be as in (1.4) with $\alpha \in \big(0, \frac{2}{N-2}\big]$, and $\Omega = B_R$. Then, there exists a constant $\widetilde{L}(N,R) > 0$ only depending on N and R, such that for all $u_\alpha = u_\alpha(r)$, radially symmetric solution to (1.5) and for every $r \neq 0$, we have*

$$\begin{aligned} &\liminf_{\alpha\to 0^+} \Big[[\log(e+u_\alpha(0))]^{\frac{1}{2}[1-\alpha(\frac{N-6}{2})]} \frac{u_\alpha(r)}{\sqrt{\alpha}}\Big] \\ &\ge \widetilde{L}(N,R)\Big(\frac{1}{r^{N-2}} - \frac{1}{R^{N-2}}\Big), \end{aligned} \tag{1.8}$$

$$\begin{aligned} &\limsup_{\alpha\to 0^+} \Big[[\log(e+u_\alpha(0))]^{-\alpha\frac{N+4}{4}} \frac{u_\alpha(r)}{\sqrt{\alpha}}\Big] \\ &\le \sqrt{2^*}\,\widetilde{L}(N,R)\Big(\frac{1}{r^{N-2}} - \frac{1}{R^{N-2}}\Big), \end{aligned} \tag{1.9}$$

where $\widetilde{L}(N,R)$ is defined by (1.3).

In Section 2, keeping $\alpha \in (0, \frac{2}{N-2}]$ and $u_\alpha(0) = d > 0$ fixed, we obtain lower and upper estimate for radial solutions $u = u_\alpha(r)$ of (1.5). In Section 3 we prove Theorem 1.1 keeping $\alpha \in (0, \frac{2}{N-2}]$ fixed, and allowing d to vary. In Section 4 we prove Theorem 1.2 letting $\alpha \to 0^+$. Finally, in Section 5 we prove Theorem 1.3.

2. Basic lemmas

In this Section we estimate $u_\alpha(r)$ through several estimates of an auxiliary function, keeping $\alpha \in \big(0, \frac{2}{N-2}\big]$ and $d > 0$ fixed.

From Gidas, Ni and Nirenberg [10], it is well known that any positive solution u_α of (1.5) is radially symmetric and $\frac{\partial u_\alpha}{\partial r} < 0$ for $0 < r < R$. The search for radial solutions of (1.5) leads to the ODE problem

$$
\begin{gathered}
u'' + \frac{N-1}{r} u' + f(u) = 0 \quad \text{for } r \in [0, R), \\
u(r) > 0 \quad \text{for } r \in [0, R), \\
u(R) = 0, \quad u'(0) = 0.
\end{gathered} \tag{2.1}
$$

where, from now on $f(u) = f_\alpha(u)$ is defined by (1.4). Let us consider the associated initial-value problem

$$
\begin{gathered}
u'' + \frac{N-1}{r} u' + f(u) = 0, \quad \text{for } r > 0, \\
u(r) > 0, \\
u(0) = d, \quad u'(0) = 0.
\end{gathered} \tag{2.2}
$$

The Contraction Mapping Principle with parameters is applicable to (2.2) and for each $\alpha \in \big(0, \frac{2}{N-2}\big]$ and $d > 0$ the initial-value problem (2.2) has a unique solution $u(r) = u_\alpha(r, d)$ depending continuously on α and d.

Since (2.2) is equivalent to

$$
\begin{gathered}
\big(r^{N-1} u'\big)' + r^{N-1} f\big(u(r)\big) = 0, \quad 0 < r < R, \\
u(r) > 0, \\
u(0) = d, \quad u'(0) = 0,
\end{gathered}
$$

integrating on $[0, r]$ we have

$$
r^{N-1} u'(r) = -\int_0^r s^{N-1} f\big(u(s)\big)\, ds < 0,
$$

and the solutions are decreasing. It is clear that there exist solution to (2.1) if there exists some d such that $u_\alpha(R, d) = 0$. Set

$$
t := \Big(\frac{N-2}{r}\Big)^{N-2}, \quad y(t) := u(r), \quad \big(y(t) = y_\alpha(t, d) = u_\alpha(r, d)\big), \tag{2.3}
$$

problem (2.2) becomes the backward problem

$$
\begin{gathered}
y'' + t^{-\frac{2(N-1)}{N-2}} f(y(t)) = 0 \quad \text{for } t < \infty, \\
y(t) > 0, \\
\lim_{t \to +\infty} y(t) = d, \quad \lim_{t \to +\infty} y'(t) = 0.
\end{gathered} \tag{2.4}
$$

When the nonlinearity is $f(s) = As^p$, for some $A > 0$, equation (2.4) is known as the Emden-Fowler equation.

Integrating y'' on $(t, +\infty)$, see (2.4),

$$
y'(t) = \int_t^\infty s^{-\frac{2(N-1)}{N-2}} f\big(y(s)\big)\, ds \tag{2.5}
$$

Integrating now y' on $(t, +\infty)$, and from Fubini's Theorem

$$y(t) = d - \int_t^\infty (s-t)s^{-\frac{2(N-1)}{N-2}} f(y(s))\, ds. \tag{2.6}$$

Throughout this section we keep $\alpha \in \big(0, \frac{2}{N-2}\big]$ and $d > 0$ fixed. Define

$$T(d) = T_\alpha(d) := \inf\{t > 0 : y(t) > 0\}. \tag{2.7}$$

By definition $T(d) \geq 0$, and since continuous dependence on the parameters, $T(d)$ is continuous. We will prove in Lemma 2.4 that $T(d) > 0$, therefore we can define $R(d) := (N-2)/T(d)^{\frac{1}{N-2}}$. Obviously, $u = u_\alpha(r, d)$ is a solution to (2.1) on $(0, R)$ if and only if for each $\alpha \in \big(0, \frac{2}{N-2}\big]$, there exists some $d > 0$ (depending on α), such that $R(d) = R$, or in other words,

$$T(d) := \Big(\frac{N-2}{R}\Big)^{N-2}. \tag{2.8}$$

Let

$$\mathcal{D}_\alpha := \{d = d_\alpha > 0 : T_\alpha(d) = [(N-2)/R]^{N-2}\}. \tag{2.9}$$

By [8, Thm 2.8], problem (2.1) has a solution. In other words, $\mathcal{D}_\alpha \neq \emptyset$. Our first aim is to prove that, for α fixed, the set $\mathcal{D}_\alpha$ is bounded. We denote

$$z(t) = z_\alpha(t, d) := dt\Big[t^{\frac{2}{N-2}} + \frac{(N-2)f(d)}{Nd}\Big]^{-\frac{N-2}{2}}. \tag{2.10}$$

By direct computations we can show that z satisfies the Emden-Fowler equation

$$\begin{gathered} z''(t) + t^{-\frac{2(N-1)}{N-2}} \frac{1}{[\log(e+d)]^\alpha} z(t)^{2^*-1} = 0, \quad \text{for } t > 0 \\ z(t) > 0 \\ z(0) = 0, \quad \lim_{t\to+\infty} z(t) = d, \quad \lim_{t\to+\infty} z'(t) = 0. \end{gathered} \tag{2.11}$$

Obviously $z'' < 0$, and integrating z'' on $(t, +\infty)$, then $z' > 0$. Moreover, in its integral form, (2.11) is equivalent to

$$z(t) = d - \frac{1}{[\log(e+d)]^\alpha} \int_t^\infty (s-t)s^{-\frac{2(N-1)}{N-2}} z(s)^{2^*-1}\, ds. \tag{2.12}$$

The function z will be useful in estimating y. For instance we have the following result proved in [1, Lemma 1.(iii) and Remark 1].

Lemma 2.1. *Fix $\alpha \in \big(0, \frac{2}{N-2}\big]$ and $d > 0$. Let $y = y(t, d)$ solve (2.4), and $z = z(t, d)$ solve (2.11). Then*

$$y(t, d) < z(t, d) \quad \text{for every } t > T(d).$$

Using (2.11) it is easy to see that for $t \geq 0$, the function z is increasing and concave. Then for every $t > 0$, $z(t) < \min\{z'(0)t, d\}$. A direct computation using (2.10) shows that $z'(0) = N_1 M(d)$ where

$$N_1 := \Big(\frac{N}{N-2}\Big)^{\frac{N-2}{2}}, \quad \text{and} \quad M = M(d) := \frac{\log(e+d)^{\frac{\alpha(N-2)}{2}}}{d}. \tag{2.13}$$

Hence, we have the following consequence of Lemma 2.1.

Lemma 2.2. *Fix $\alpha \in \big(0, \frac{2}{N-2}\big]$ and $d > 0$. Let $y = y(t,d)$ solve (2.4). Then*

$$y(t) < \min\{N_1\, M(d)t,\, d\} \quad \text{for every } t > T(d), \tag{2.14}$$

where N_1, and $M(d)$ are defined by (2.13)

For further estimates we introduce the Pohozaev functional

$$H(t) := \frac{1}{2}t\big(y'(t)\big)^2 - \frac{1}{2}y(t)y'(t) + \big(\frac{1}{t}\big)^{\frac{N}{N-2}} F\big(y(t)\big), \quad \text{for } t \geq T(d), \tag{2.15}$$

where $F(s) = \int_0^s f(t)\,dt$. The following lemma states some properties of H.

Lemma 2.3. *Fix $\alpha \in \big(0, \frac{2}{N-2}\big]$ and $d > 0$. Let $y = y(t,d)$ solve (2.4). Then the Pohozaev functional (2.15) satisfies $H'(t) < 0$ for $t > T(d)$ and $H(t) \searrow 0$ as $t \to \infty$. In particular $H(t) > 0$ for $t \geq T(d)$.*

Proof. Integrating $F(t)$ by parts,

$$F(t) = \frac{1}{2^*}\Big[tf(t) + \alpha \int_0^t \frac{s^{2^*}}{[\log(e+s)]^{\alpha+1}(e+s)}\,ds\Big]. \tag{2.16}$$

Differentiating (2.15) and using (2.4), we have

$$H'(t) = -\frac{\alpha}{2}\big(\frac{1}{t}\big)^{\frac{2(N-1)}{N-2}} \int_0^{y(t)} \frac{s^{2^*}}{[\log(e+s)]^{\alpha+1}(e+s)}\,ds < 0, \tag{2.17}$$

which proves the first claim of the lemma.

Substituting (2.16) in (2.15), we obtain

$$H(t) = \frac{1}{2}t(y')^2 - \frac{1}{2}yy' + \frac{1}{2^*}\big(\frac{1}{t}\big)^{\frac{N}{N-2}}\Big[yf(y) \tag{2.18}$$

$$+ \alpha \int_0^{y(t)} \frac{s^{2^*}}{[\log(e+s)]^{\alpha+1}(e+s)}\,ds\Big]. \tag{2.19}$$

By L'Hopital's Rule and (2.4),

$$\lim_{t\to\infty} ty'(t) = \lim_{t\to\infty} \big(\frac{1}{t}\big)^{\frac{2}{N-2}} f\big(y(t)\big) = 0, \tag{2.20}$$

hence $t(y')^2 \to 0$ as $t \to \infty$. Therefore, the first term in the right hand side of (2.18) tends to 0 as $t \to \infty$. Since the asymptotic behavior of y, and y' as $t \to \infty$. The second, third and fourth terms in the right hand side of (2.18) also tend to 0 as $t \to \infty$. Then $H(t) \to 0$ as $t \to \infty$.

Since $H' < 0$, $H(t) \searrow 0$ as $t \to \infty$, consequently $H(t) > 0$ for $t \geq T(d)$. This completes the proof. □

The above lemmas are useful for proving the positiveness of $T(d)$.

Lemma 2.4. *Fix $\alpha \in \big(0, \frac{2}{N-2}\big]$. Let $T = T(d)$ be defined by (2.7). Then*

$$T(d) > 0, \quad \text{for every } d > 0.$$

Proof. Assume by contradiction that $T(d) = 0$. From Lemma 2.3, $H(0) > 0$. Moreover, from $F(s) = \int_0^s f(t)\,dt \leq \frac{s^{2^*}}{2^*}$, and Lemmas 2.2 and 2.3, we have

$$t^{-(\frac{N}{N-2})}F(y(t)) \leq \frac{1}{2^*}t^{-(\frac{N}{N-2})}y(t)^{2^*} \leq \frac{1}{2^*}\big(N_1 M(d)\big)^{2^*} t^{\frac{N}{N-2}} \to 0 \quad \text{as } t \to 0^+.$$

This and (2.15) imply that $H(0) = -\frac{1}{2}y(0)y'(0) = 0$, contradicting Lemma 2.3. □

We now look for a lower estimate for y. Let

$$\widetilde{T}(d) = \widetilde{T}_\alpha(d) := \frac{d^2}{\log(e+d)^{\frac{\alpha(N-2)}{2}}}, \tag{2.21}$$

then, for every $\varepsilon > 0$,

$$z\big(\varepsilon\widetilde{T}(d)\big) = c_\varepsilon d, \quad \text{with} \quad c_\varepsilon := \frac{\varepsilon}{\left[\frac{N-2}{N} + \varepsilon^{\frac{2}{N-2}}\right]^{\frac{N-2}{2}}}. \tag{2.22}$$

Observe that

$$\frac{c_\varepsilon}{\varepsilon} \to N_1 \quad \text{as } \varepsilon \to 0, \quad \text{and} \quad \widetilde{T}(d) = \frac{d}{M(d)}, \tag{2.23}$$

see (2.13). Next, we state a lower bound of y.

Lemma 2.5. *Let $y = y(t,d)$ solve (2.4), and $z = z(t,d)$ solve (2.11). For every $\varepsilon > 0$, there exists $d_0 = d_0(\varepsilon)$ and some $c'_{\varepsilon,d} > 0$ for $d \ge d_0$, such that*

$$y(t) > \big[1 - \alpha\big(\frac{3}{2}\big)^\alpha c'_{\varepsilon,d}\big] z(t) \quad \textit{for every } t > \varepsilon\widetilde{T}(d).$$

Proof. Fix any $\varepsilon > 0$, and any $d > 0$. Take $t > \varepsilon\widetilde{T}(d)$. Since ($z > y$ and $f \nearrow$), from (2.12), using the Mean Value Theorem with $\theta \in (z,d)$, with $\theta > z > c_\varepsilon d$, using (2.12), and $d < z/c_\varepsilon$, we deduce that

$$\begin{aligned}
y(t) &> d - \int_t^\infty (s-t)s^{-\frac{2(N-1)}{N-2}} f(z)\,ds \\
&= z - \int_t^\infty (s-t)s^{-\frac{2(N-1)}{N-2}} z^{2^*-1}\Big[\frac{1}{[\log(e+z)]^\alpha} - \frac{1}{[\log(e+d)]^\alpha}\Big]\,ds \\
&= z - \alpha\int_t^\infty (s-t)s^{-\frac{2(N-1)}{N-2}} z^{2^*-1}\frac{d-z}{[\log(e+\theta)]^{\alpha+1}(\theta+e)}\,ds \\
&\ge z - \frac{\alpha d}{[\log(e+c_\varepsilon d)]^{\alpha+1}(c_\varepsilon d+e)}\int_t^\infty (s-t)s^{-\frac{2(N-1)}{N-2}} z^{2^*-1}\,ds \\
&\ge z - \frac{\alpha}{c_\varepsilon[\log(e+c_\varepsilon d)]^{\alpha+1}}\int_t^\infty (s-t)s^{-\frac{2(N-1)}{N-2}} z^{2^*-1}\,ds \\
&= z - \frac{\alpha[\log(e+d)]^\alpha}{c_\varepsilon[\log(e+c_\varepsilon d)]^{\alpha+1}}(d-z) \\
&\ge z\Big[1 - \alpha\frac{(1-c_\varepsilon)}{c_\varepsilon^2}\frac{[\log(e+d)]^\alpha}{[\log(e+c_\varepsilon d)]^{\alpha+1}}\Big].
\end{aligned}$$

Consequently, for all $\varepsilon > 0$, and $d > 0$ fixed,

$$y(t) \ge \Big[1 - \alpha\frac{(1-c_\varepsilon)}{c_\varepsilon^2}\frac{[\log(e+d)]^\alpha}{[\log(e+c_\varepsilon d)]^{\alpha+1}}\Big] z(t), \quad \text{for any } t > \varepsilon\widetilde{T}(d). \tag{2.24}$$

Let us keep $\varepsilon > 0$ fixed and allow d to be large. Since $\frac{\log(d+e)}{\log(e+c_\varepsilon d)} \to 1$ as $d \to \infty$, there exists $d_0 = d_0(\varepsilon)$ such that $\frac{\log(d+e)}{\log(e+c_\varepsilon d)} < 3/2$, for all $d \ge d_0$, in fact we can define

$$d_0 = d_0(\varepsilon) := \frac{1}{c_\varepsilon^3},$$

where c_ε is defined by (2.22). Now, taking

$$c'_{\varepsilon,d} := \frac{1-c_\varepsilon}{c_\varepsilon^2}\frac{1}{\log(e+c_\varepsilon d)}, \tag{2.25}$$

the proof is complete. $\square$

Lemma 2.6. *Let $y = y_\alpha(t,d)$ solve (2.4), and $z = z_\alpha(t,d)$ solve (2.11). For every $\varepsilon > 0$, there exists $d_1 = d_1(\varepsilon)$, such that for all $d \geq d_1$*

$$y(t) > \gamma(\alpha) z(t) \quad \text{for every } t > \varepsilon \widetilde{T}(d). \tag{2.26}$$

where

$$\gamma(\alpha) := 1 - \frac{N-2}{4}\alpha \geq \frac{1}{2}, \quad \text{for all } \alpha \in \big(0, \frac{2}{N-2}\big]. \tag{2.27}$$

In particular, for every $\varepsilon \in \big(0, \big(\frac{2}{N}\big)^{\frac{N-2}{2}}\big)$,

$$y\big(\varepsilon \widetilde{T}(d)\big) \geq \frac{1}{2}\varepsilon d, \quad \text{for all } d \geq d_1.$$

Proof. For $\varepsilon > 0$ fixed, let us define

$$d_1 = d_1(\varepsilon) := \frac{1}{c_\varepsilon} \exp\big[\frac{4}{N-2}\big(\frac{3}{2}\big)^\alpha \frac{1-c_\varepsilon}{c_\varepsilon^2}\big], \tag{2.28}$$

where c_ε is given by (2.22). Hence,

$$1 - \alpha\big(\frac{3}{2}\big)^\alpha c'_{\varepsilon,d} \geq 1 - \frac{N-2}{4}\alpha \geq \frac{1}{2}, \quad \text{for all } d \geq d_1, \text{ and } \alpha \in \big(0, \frac{2}{N-2}\big],$$

which, combined with Lemma 2.5, proves (2.26).

In particular, for $\varepsilon \in (0, \varepsilon_0)$, and $d \geq d_1(\varepsilon)$,

$$y\big(\varepsilon \widetilde{T}(d)\big) \geq \frac{1}{2}\, z\big(\varepsilon \widetilde{T}(d)\big) = \frac{1}{2} \frac{\varepsilon d}{\big[\frac{N-2}{N} + \varepsilon^{\frac{2}{N-2}}\big]^{\frac{N-2}{2}}} \geq \frac{1}{2} \frac{\varepsilon d}{\big[\frac{N-2}{N} + \varepsilon_0^{\frac{2}{N-2}}\big]^{\frac{N-2}{2}}},$$

choosing $\varepsilon_0 := \big(\frac{2}{N}\big)^{\frac{N-2}{2}}$ we obtain

$$y\big(\varepsilon \widetilde{T}(d)\big) \geq \frac{1}{2}\, \varepsilon d > 0,$$

which compltes the proof. $\square$

3. Further estimates and proof of Theorem 1.1

In this Section we estimate $u = u_\alpha(r,d)$ through several estimates of the auxiliary function $y = y_\alpha(t,d)$ and in particular of $T = T_\alpha(d)$, keeping $\alpha \in (0, \frac{2}{N-2}]$ fixed and allowing d to vary. As an immediate consequence of Lemmas 2.5-2.6 we have the following lemma.

Lemma 3.1. *Let $\widetilde{T}(d)$ be defined by (2.21). Then*

$$T(d) = o(\widetilde{T}(d)) \quad \text{as } d \to \infty. \tag{3.1}$$

Proof. Lemma 2.6 state in particular that for any $\varepsilon > 0$ small enough, there exists $d_1 = d_1(\varepsilon)$, such that for all $d \geq d_1$,

$$y\big(\varepsilon \widetilde{T}(d)\big) \geq \frac{1}{2}\, \varepsilon d > 0.$$

Therefore, from definition of $T(d)$, for any $\varepsilon > 0$, and $d \geq d_1(\varepsilon)$, $T(d) < \varepsilon \widetilde{T}(d)$. $\square$

Now, we introduce the *Hardy asymptotic notation.* For $f, g : \mathbb{R} \to \mathbb{R}_+$, we say that

$$f(d) \lesssim g(d) \text{ as } d \to d_0, \text{ with } 0 \le d_0 \le \infty, \quad \text{if } \limsup_{d\to d_0} \frac{|f(d)|}{|g(d)|} < +\infty.$$

In a similar way we use the notation $f(d) \gtrsim g(d)$ as $d \to d_0$, if $\limsup_{d\to d_0} \frac{|g(d)|}{|f(d)|} < +\infty$. Finally we will use the notation

$$f(d) = \Theta(g(d)) \quad \text{as } d \to d_0, \text{ with } 0 \le d_0 \le \infty,$$

to denote $f \lesssim g$ and $g \lesssim f$ as $d \to d_0$. The following lemma relate to estimations of $y(t)$ and $y'(t)$ for specific values of t when d is large.

Lemma 3.2. *Let $y = y(t, d)$ solve (2.4). Let $T = T(d)$, $\widetilde{T} = \widetilde{T}(d)$ and $M = M(d)$ be defined by (2.7), (2.21) and (2.13) respectively. Then, the following holds:*

(i) *$y(2T) = o(d)$, as $d \to \infty$.*

(ii) *There exists a constant $C_{N,\alpha}$ depending only on N and α, explicitly defined by (3.2), such that*

$$y(\widetilde{T}(d)) \ge C_{N,\alpha}\, d, \quad \text{as } d \to \infty.$$

(iii) *$y'(2T) = \Theta\big(M(d)\big)$, as $d \to \infty$.*

(iv) *$y(t, d) = \Theta\Big(M(d)\big(t - T(d)\big)\Big)$, as $d \to \infty$, uniformly for every $t \in [2T, \widetilde{T}]$.*

Proof. (i) Using (2.14) with $t = 2T(d)$, (2.21)-(2.23) and (3.1), we obtain

$$\frac{y(2T)}{d} \le 2N_1 \frac{T(d)}{\widetilde{T}(d)} \to 0 \quad \text{as } d \to +\infty.$$

(ii) Taking $\varepsilon = 1$ in Lemma 2.6, and from (2.22), we can write

$$y(\widetilde{T}(d)) \ge \Big(1 - \frac{N-2}{4}\alpha\Big) z(\widetilde{T}(d)) \ge C_{N,\alpha}\, d,$$

where

$$C_{N,\alpha} := \Big(1 - \frac{N-2}{4}\alpha\Big)\Big(\frac{N}{2(N-1)}\Big)^{\frac{N-2}{2}}. \tag{3.2}$$

(iii) Using that $y'' < 0$, Lemma 2.1, (2.10), and Lemma 3.1, we deduce

$$\begin{aligned} y'(2T) &< \frac{y(2T) - y(T)}{T} \le \frac{z(2T)}{T} \\ &= \frac{2d}{[(2T)^{\frac{2}{N-2}} + \frac{f_\alpha(d)}{(\frac{N}{N-2})d}]^{\frac{N-2}{2}}} \\ &\le 2d\Big(\frac{N}{N-2}\Big)^{\frac{N-2}{2}} \frac{[\log(e+d)]^{\alpha\frac{N-2}{2}}}{d^2} \\ &\le 2N_1 M(d). \end{aligned}$$

On the other hand, using again $y'' < 0$, (i), (ii), and Lemma 3.1 we obtain

$$y'(2T) > \frac{y(\widetilde{T}(d)) - y(2T)}{\widetilde{T}(d) - 2T} \ge \frac{C_{N,\alpha} d - y(2T)}{\widetilde{T}(d) - 2T} \ge \frac{C_{N,\alpha} - \varepsilon}{1+\varepsilon} M(d) \ge \frac{1}{2} C_{N,\alpha} M(d).$$

(iv) Since $y'' < 0$, $y(T) = 0$, and Lemma 2.2, it follows that

$$\frac{y(t,d)}{t - T(d)} \le \frac{y(2T)}{T(d)} \lesssim M(d)$$

uniformly with respect to $t \in [2T, \widetilde{T}]$. On the other hand, using $y'' < 0$, (ii), Lemma 3.1, and (2.23)

$$\frac{y(t,d)}{t-T(d)} \geq \frac{y(\widetilde{T})}{\widetilde{T}-T} \gtrsim \frac{d}{\widetilde{T}(d)} = M(d),$$

uniformly with respect to $t \in [2T, \widetilde{T}]$. This completes the proof. □

To prove the lower and upper bounds in Theorem 1.1 we need the following two lemmas.

Lemma 3.3. *Let $T = T(d)$ be defined by (2.7). Then*

$$0 < T(d) \leq \Big(\frac{N-2}{2^*}\Big)^{\frac{N-2}{2}} \frac{d^2}{[\log(e+d)]^{\frac{\alpha(N-2)}{2}}}\,, \tag{3.3}$$

and in particular

$$T(d) \lesssim d^2 \quad \text{as } d \to 0^+.$$

Proof. Since (2.6), Lemma 2.2, and f is increasing, it follows that

$$0 = y(T) \geq d - f(d)\int_T^\infty (s-T)s^{-\frac{2(N-1)}{N-2}}\,ds = d - \frac{f(d)}{\frac{2^*}{N-2}}\big(\frac{1}{T}\big)^{\frac{2}{N-2}},$$

then (3.3) holds. We complete the proof by letting $d \to 0$. □

Lemma 3.4. *Let $T = T(d)$ be defined by (2.7) and keep $\alpha \in \Big(0, \frac{2}{N-2}]$ fixed. Then*

$$T(d) \gtrsim \frac{d^2}{[\log(e+d)]^{\frac{\alpha(N-2)}{2}+1}}, \quad \text{as } d \to \infty.$$

Proof. From Lemmas 2.3 and 3.1, it is clear that

$$H(2T) > H(2T) - H(\widetilde{T}) = \int_{2T}^{\widetilde{T}} (-H'(s))\,ds > \int_{\widetilde{T}/2}^{\widetilde{T}} (-H'(s))\,ds.$$

By L'Hopital's Rule, it is easy to prove that for $m > 1$ and $\beta > 0$,

$$\lim_{t\to\infty} \frac{\int_0^t \frac{s^{m-1}}{[\log(e+s)]^\beta}\,ds}{\frac{t^m}{\log(t+e)^\beta}} = \frac{1}{m}, \quad \lim_{t\to\infty} \frac{\int_0^t \frac{s^m}{[\log(e+s)]^\beta(s+e)}\,ds}{\frac{t^m}{\log(t+e)^\beta}} = \frac{1}{m}. \tag{3.4}$$

Therefore,

$$F(t) = \Theta(tf(t)), \quad \text{as } t \to \infty. \tag{3.5}$$

We notice that

$$M(d)(s - T(d)) = \Theta(d) \quad \text{uniformly for } s \in [\widetilde{T}/2, \widetilde{T}], \tag{3.6}$$

see (2.23) and Lemma 3.1. Now using (2.17) and part (iv) of Lemma 3.2 we deduce the following:

$$\begin{aligned}
H(2T) &\gtrsim \int_{\widetilde{T}/2}^{\widetilde{T}} s^{-\frac{2(N-1)}{N-2}} \frac{y(s)^{2^*}}{[\log(e+y(s))]^{\alpha+1}}\,ds \quad \text{(by (3.4) and (3.6))}\\
&\gtrsim \int_{\widetilde{T}/2}^{\widetilde{T}} s^{-\frac{2(N-1)}{N-2}} \frac{\big(M(d)(s-T)\big)^{2^*}}{[\log(e+M(d)(s-T))]^{\alpha+1}}\,ds \quad \text{(by Lemma 3.2 (iv))}\\
&\gtrsim \frac{[\log(e+d)]^{\alpha N-\alpha-1}}{d^{2^*}} \int_{\widetilde{T}/2}^{\widetilde{T}} s^{-\frac{2(N-1)}{N-2}} (s-T)^{2^*}\,ds \quad \text{(using (3.6))}
\end{aligned}$$

$$\begin{aligned}&\gtrsim \frac{[\log(e+d)]^{\alpha N-\alpha-1}}{d^{2^*}}\big(\widetilde{T}\big)^{\frac{N}{N-2}} \quad \text{(by Lemma 3.1)}\\ &= [\log(e+d)]^{\frac{\alpha(N-2)}{2}-1}.\end{aligned}$$

Note that $\frac{\alpha(N-2)}{2}-1\le 0$. Using Lemma 3.2 (iii) and (iv), we have

$$\begin{aligned}H(2T) &< Ty'(2T)^2+(2T)^{-(\frac{N}{N-2})}F\big(y(2T)\big) \quad \text{(by (2.15))}\\ &\lesssim T\big(M(d)\big)^2+\frac{1}{T^{\frac{N}{N-2}}}\frac{y(2T)^{2^*}}{[\log(e+y(2T))]^{\alpha}} \quad \text{(by Lemma 3.2 (iii) and (3.5))}\\ &\lesssim T\big(M(d)\big)^2+\frac{M(d)^{2^*}T^{\frac{N}{N-2}}}{[\log(e+M(d)T)]^{\alpha}} \quad \text{(by Lemma 3.2 (iv))}\end{aligned} \tag{3.7}$$

Denoting $S(d):=T(d)\big(M(d)\big)^2$, we can write

$$5H(2T)\lesssim S(d)+S(d)^{\frac{N}{N-2}}[\log(e+S(d)/M(d))]^{-\alpha}.$$

From Lemma 3.1 we know that $S(d)=o\Big([\log(e+d)]^{\frac{\alpha(N-2)}{2}}\Big)$, and from Lemma 3.4 that

$$S(d)\gtrsim [\log(e+d)]^{\frac{\alpha(N-2)}{2}-1}, \quad \text{as } d\to\infty.$$

Hence $\frac{S(d)}{M(d)}\gtrsim\frac{d}{\log(e+d)}$. Moreover, since $\log\Big(e+\frac{d}{\log(e+d)}\Big)=\Theta\big(\log(e+d)\big)$ as $d\to\infty$, we have

$$\Big[\log\Big(e+\frac{S(d)}{M(d)}\Big)\Big]^{-\alpha}\lesssim[\log(e+d)]^{-\alpha}, \quad \text{and} \quad \frac{S(d)^{\frac{2}{N-2}}}{[\log(e+S(d)/M(d))]^{\alpha}}=o(1).$$

Consequently $H(2T)\lesssim S(d)$ and

$$T(d)\gtrsim\frac{d^2}{[\log(e+d)]^{\frac{\alpha(N-2)}{2}+1}}.$$

□

Proof of Theorem 1.1. (i) Fix $\alpha\in(0,\frac{2}{N-2}]$. From Lemmas 3.3 and3.4, and the continuity of $T(d)$, there exists a $d=d_\alpha\in(0,\infty)$ such that $T(d_\alpha)=[(N-2)/R]^{N-2}$. The corresponding solutions of the IVP (2.2) is a radial solution of the BVP (2.1).

(ii) Fix $\alpha\in\Big(0,\frac{2}{N-2}\Big]$. Assume on the contrary that there exists a sequence of solutions to (2.1), denoted by u_n, such that $d_n:=u_n(0)=\|u_n\|_\infty\to0$ as $n\to\infty$. By Lemma 3.3, $T_n:=T(d_n)\to0$ as $d_n\to0^+$. But $u_n=u_{\alpha,n}$ is a solution to (2.1), and therefore $y_n:=y_{\alpha,n}$ is a solution to (2.4) with $T(d_n)=[(N-2)/R]^{N-2}$ constant, contradicting that $T(d_n)\to0$ as $d_n\to0^+$. Therefore, there is a constant $A>0$ such that $A<\|u\|_\infty$.

On the other hand, assume on the contrary that there exists a sequence of solutions to (2.1), denoted by u_n, such that $d_n:=u_n(0)=\|u_n\|_\infty\to\infty$ as $n\to\infty$. By Lemma 3.4, $T(d_n)\to\infty$ as $d_n\to\infty$. But reasoning as before, $T(d_n)=[(N-2)/R]^{N-2}$, a constant value, contradicting that $T(d_n)\to\infty$ as $d_n\to\infty$. Therefore, there exists a constant $B>0$ such that $\|u\|_\infty<B$. This completes the proof. □

4. Proof of Theorem 1.2

In this Section, we consider only values of $d = d_\alpha \in \mathcal{D}_\alpha$, where $\mathcal{D}_\alpha$ is defined by (2.9), and allow α to vary. As a consequence $T = T_\alpha(d)$ is fixed and defined by

$$T = T_\alpha(d) = \Big(\frac{N-2}{R}\Big)^{N-2}, \quad \forall d = d_\alpha \in \mathcal{D}_\alpha,\ \forall \alpha \in \Big(0, \frac{2}{N-2}\Big],$$

and $u_\alpha(r,d)$ is a solution of (2.1) for $d \in \mathcal{D}_\alpha$.

Lemma 4.1. *Let $\mathcal{D}_\alpha$ be defined by* (2.9)*. Then*

$$\lim_{\alpha\to 0^+} \inf \mathcal{D}_\alpha = +\infty.$$

Proof. Assume by contradiction that there is a sequence $\alpha_n \searrow 0$ and some $M_0 > 0$ such that $\inf D_{\alpha_n} < M_0$. Then, there is a subsequence $d_n \in D_{\alpha_n}$ such that $d_n < M_0$ for every n. Hence, there is an $\varepsilon_0 > 0$ depending only on M_0, such that

$$\varepsilon_0 \widetilde{T}(d_n) = \varepsilon_0 \frac{d_n^2}{[\log(e+d_n)]^{\frac{\alpha(N-2)}{2}}} \le \big(\frac{N-2}{R}\big)^{N-2} = T, \quad \text{for every } n.$$

Then, firstly from (2.24), and secondly from $d_n < M_0$, there is an $\alpha_0 > 0$ such that for every $\alpha_n \in (0, \alpha_0)$,

$$0 = y(T, d_n) > \Big[1 - \alpha_n \frac{1 - c_{\varepsilon_0}}{c_{\varepsilon_0}^2} \frac{[\log(e+d_n)]^{\alpha_n}}{[\log(e + c_{\varepsilon_0} d_n]^{\alpha_n+1}}\Big] z(T, d_n) > 0,$$

which is a contradiction. □

To obtain new estimates, we will use the incomplete beta function defined as

$$B(x,a,b) = \int_x^\infty t^{a-1}(1+t)^{-a-b}\, dt, \quad a, b > 0.$$

In [2, Lemma A2] a slightly variant of the following relation is proved

$$\begin{aligned} &\int_t^\infty s^{-\frac{2(N-1)}{N-2}} z^r(s,d)\, ds \\ &= N_1 \frac{N}{2} d^{r-2^*} [\log(e+d_\alpha)]^{\alpha\frac{N}{2}} B\Big(\Big(\frac{N_1 t}{\widetilde{T}}\Big)^{\frac{2}{N-2}}, \frac{r - \frac{N}{N-2}}{\frac{2}{N-2}}, \frac{N}{2}\Big), \end{aligned} \tag{4.1}$$

with $r > \frac{N}{N-2}$. We denote

$$I(\alpha) := \frac{T\big(y_\alpha'(T)\big)^2}{\alpha} = \int_T^\infty t^{-\frac{2(N-1)}{N-2}} \Big(\int_0^{y_\alpha(t)} \frac{s^{2^*}}{[\log(e+s)]^{\alpha+1}(e+s)}\, ds\Big) dt. \tag{4.2}$$

This equality is a consequence of (2.17) and (2.15).

Lemma 4.2. *Let $y = y_\alpha(t,d)$ solve* (2.4)*, and let $\mathcal{D}_\alpha$ and I_α be defined by* (2.9) *and* (4.2) *respectively. Then*

(i)
$$\limsup_{\alpha\to 0^+} \sup_{d_\alpha \in \mathcal{D}_\alpha} \Big[\frac{d_\alpha^2}{[\log(e+d_\alpha)]^{\alpha N}} T\big(y_\alpha'(T)\big)^2\Big] \le N_1^2 T.$$

(ii)
$$\liminf_{\alpha\to 0^+} \inf_{d_\alpha \in \mathcal{D}_\alpha} \Big[\frac{d_\alpha^2}{[\log(e+d_\alpha)]^{\alpha(N-2)}} T\big(y_\alpha'(T)\big)^2\Big] \ge N_1^2 T. \tag{4.3}$$

(iii)

$$\limsup_{\alpha\to 0^+}\sup_{d_\alpha\in\mathcal{D}_\alpha}\Big[\frac{1}{[\log(e+d_\alpha)]^{\alpha\frac{N}{2}}}I(\alpha)\Big]\le N_1\frac{N}{2}\frac{\Gamma(\frac{N}{2})^2}{\Gamma(N)}. \tag{4.4}$$

(iv)

$$\liminf_{\alpha\to 0^+}\inf_{d_\alpha\in\mathcal{D}_\alpha}\Big[[\log(e+d_\alpha)]^{1-\frac{\alpha(N-2)}{2}}I(\alpha)\Big]\ge N_1\frac{N-2}{4}\frac{\Gamma(\frac{N}{2})^2}{\Gamma(N)}. \tag{4.5}$$

Proof. (i) From (2.5), Lemma 2.1, and (4.1) with $t=T$ and $r=2^*-1$, we have

$$\begin{aligned} y'_\alpha(T)&=\int_T^\infty t^{-\frac{2(N-1)}{N-2}}f\big(y_\alpha(t)\big)\,dt\\ &\le\int_T^\infty t^{-\frac{2(N-1)}{N-2}}f\big(z_\alpha(t)\big)\,dt\\ &\le\int_T^\infty t^{-\frac{2(N-1)}{N-2}}z_\alpha(t)^{2^*-1}\,dt\\ &=N_1\frac{N}{2}\frac{[\log(e+d_\alpha)]^{\alpha\frac{N}{2}}}{d_\alpha}B\Big(\Big(\frac{TN_1}{\widetilde{T}}\Big)^{\frac{2}{N-2}},1,\frac{N}{2}\Big)\\ &\le N_1\frac{N}{2}\frac{[\log(e+d_\alpha)]^{\alpha\frac{N}{2}}}{d_\alpha}B\big(0,1,\frac{N}{2}\big)\\ &=N_1\frac{[\log(e+d_\alpha)]^{\alpha\frac{N}{2}}}{d_\alpha}. \end{aligned} \tag{4.6}$$

Hence

$$\limsup_{\alpha\to 0^+}\sup_{d_\alpha\in\mathcal{D}_\alpha}\Big[\frac{d_\alpha}{[\log(e+d_\alpha)]^{\alpha\frac{N}{2}}}\,y'_\alpha(T)\Big]\le N_1, \tag{4.7}$$

which proves part (i).

(ii) Fix an arbitrary $\varepsilon>0$. From (2.5), Lemma 3.1, Lemma 2.6 and (4.1), there exists a d_1 only depending on ε (see (2.28)), such that for every $d_\alpha\ge d_1$

$$\begin{aligned} y'_\alpha(T)&>\int_{\varepsilon\widetilde{T}}^\infty s^{-\frac{2(N-1)}{N-2}}\frac{y_\alpha(s)^{2^*-1}}{[\log(e+y_\alpha(s))]^\alpha}\,ds\\ &>\frac{\gamma(\alpha)^{2^*-1}}{[\log(e+d_\alpha)]^\alpha}\int_{\varepsilon\widetilde{T}}^\infty s^{-\frac{2(N-1)}{N-2}}z_\alpha(s)^{2^*-1}\,ds\\ &=N_1\frac{N}{2}\frac{\gamma(\alpha)^{2^*-1}}{[\log(e+d_\alpha)]^\alpha}\frac{[\log(e+d_\alpha)]^{\alpha\frac{N}{2}}}{d_\alpha}B\Big((\varepsilon N_1)^{\frac{2}{N-2}},1,\frac{N}{2}\Big). \end{aligned} \tag{4.8}$$

The inequality $d_\alpha\ge d_1$ for α small enough, holds thanks to Lemma 4.1. Hence

$$\inf_{d_\alpha\in\mathcal{D}_\alpha}\Big[\frac{d_\alpha}{[\log(e+d_\alpha)]^{\frac{\alpha(N-2)}{2}}}\,y'_\alpha(T)\Big]\ge N_1\frac{N}{2}\gamma(\alpha)^{2^*-1}B\Big((\varepsilon N_1)^{\frac{2}{N-2}},1,\frac{N}{2}\Big), \tag{4.9}$$

for an arbitrary $\varepsilon>0$ fixed. Because $\gamma(\alpha)\to 1$ as $\alpha\to 0^+$, see (2.27), and by continuity of the incomplete beta function with respect to its first argument, $B\big((\varepsilon N_1)^{\frac{2}{N-2}},1,\frac{N}{2}\big)\to B\big(0,1,\frac{N}{2}\big)$ as $\varepsilon\to 0$. Therefore,

$$\liminf_{\alpha\to 0^+}\inf_{d_\alpha\in\mathcal{D}_\alpha}\Big[\frac{d_\alpha}{[\log(e+d_\alpha)]^{\frac{\alpha(N-2)}{2}}}\,y'_\alpha(T)\Big]\ge N_1. \tag{4.10}$$

part (ii) has been proved.

(iii) Since the integrand in (4.2) is increasing, by Lemma 2.1 and (4.1), we have

$$
\begin{aligned}
I(\alpha) &= \int_T^\infty t^{-\frac{2(N-1)}{N-2}} \Big(\int_0^{y_\alpha(t)} \frac{s^{2^*}}{[\log(e+s)]^{\alpha+1}(e+s)}\, ds\Big)\, dt \\
&\le \int_T^\infty t^{-\frac{2(N-1)}{N-2}} \frac{y_\alpha(t)^{2^*+1}}{[\log(e+y_\alpha(t)]^{\alpha+1}\big(e+y_\alpha(t)\big)}\, dt \\
&\le \int_T^\infty t^{-\frac{2(N-1)}{N-2}} y_\alpha(t)^{2^*}\, dt \\
&< \int_T^\infty t^{-\frac{2(N-1)}{N-2}} z_\alpha(t)^{2^*}\, dt \\
&= N_1 \frac{N}{2} [\log(e+d_\alpha)]^{\alpha\frac{N}{2}} B\Big(\Big(\frac{TN_1}{\widetilde{T}}\Big)^{\frac{2}{N-2}}, \frac{N}{2}, \frac{N}{2}\Big).
\end{aligned}
\tag{4.11}
$$

Hence

$$
\limsup_{\alpha\to 0^+} \sup_{d_\alpha\in\mathcal{D}_\alpha} \frac{I(\alpha)}{[\log(e+d_\alpha)]^{\alpha\frac{N}{2}}} \le N_1\frac{N}{2} B\Big(0, \frac{N}{2}, \frac{N}{2}\Big) = N_1 \frac{N}{2}\, \frac{\Gamma(\frac{N}{2})^2}{\Gamma(N)}. \tag{4.12}
$$

This proves part (iii).

(iv) Fix an arbitrary $\varepsilon > 0$ and $\delta \in (0,1)$. From (4.2), Lemma 3.1, (3.4), Lemma 2.6 and (4.1), there exists a d_1 only depending on ε (see (2.28)), such that for every $d_\alpha \ge d_1$,

$$
\begin{aligned}
I(\alpha) &= \int_T^\infty t^{-\frac{2(N-1)}{N-2}} \Big(\int_0^{y_\alpha(t)} \frac{s^{2^*}}{[\log(e+s)]^{\alpha+1}(e+s)}\, ds\Big)\, dt \\
&\ge \int_{\varepsilon\widetilde{T}}^\infty t^{-\frac{2(N-1)}{N-2}} \Big(\int_0^{y_\alpha(t)} \frac{s^{2^*}}{[\log(e+s)]^{\alpha+1}(e+s)}\, ds\Big)\, dt \\
&\ge \frac{1-\delta}{2^*} \int_{\varepsilon\widetilde{T}}^\infty \frac{t^{-\frac{2(N-1)}{N-2}} y_\alpha(t)^{2^*}}{[\log(e+y_\alpha(t)]^{\alpha+1}}\, dt \\
&\ge \frac{(1-\delta)\gamma(\alpha)^{2^*}}{2^*[\log(e+d)]^{\alpha+1}} \int_{\varepsilon\widetilde{T}}^\infty t^{-\frac{2(N-1)}{N-2}} z_\alpha(t)^{2^*}\, dt \\
&= N_1\frac{N}{2} \frac{(1-\delta)\gamma(\alpha)^{2^*}}{2^*} [\log(e+d_\alpha]^{\alpha(\frac{N-2}{2})-1} B\Big((\varepsilon N_1)^{\frac{2}{N-2}}, \frac{N}{2}, \frac{N}{2}\Big).
\end{aligned}
\tag{4.13}
$$

Since $\gamma(\alpha) \to 1$ as $\alpha \to 0^+$, see (2.27), it follows that

$$
\inf_{d_\alpha\in\mathcal{D}_\alpha} \Big[[\log(e+d_\alpha)]^{1-\frac{\alpha(N-2)}{2}} I(\alpha)\Big] \ge \frac{N-2}{4} N_1(1-\delta) B\Big((\varepsilon N_1)^{\frac{2}{N-2}}, \frac{N}{2}, \frac{N}{2}\Big),
$$

for an arbitrary $\varepsilon > 0$ fixed. Again, by the continuity of the incomplete beta function with respect to its first argument, $B\big((\varepsilon N_1)^{\frac{2}{N-2}}, 1, \frac{N}{2}\big) \to B\big(0, 1, \frac{N}{2}\big)$ as $\varepsilon \to 0$, and

$$
\begin{aligned}
\liminf_{\alpha\to 0^+} \inf_{d_\alpha\in\mathcal{D}_\alpha} \Big[[\log(e+d_\alpha)]^{1-\frac{\alpha(N-2)}{2}} I(\alpha)\Big] &\ge \frac{N-2}{4} N_1(1-\delta) B\big(0, \frac{N}{2}, \frac{N}{2}\big) \\
&= \frac{N-2}{4} N_1(1-\delta) \frac{\Gamma(\frac{N}{2})^2}{\Gamma(N)}.
\end{aligned}
$$

for $\delta \in (0,1)$ arbitrary, this completes the proof of (iv) and of the Lemma. □

Proof of Theorem 1.2. Recall that $u_\alpha(0) = d_\alpha$. Using (4.2), Lemma 4.2 (i) and (4.5), and from definition of T, see (2.8), we have

$$
\begin{aligned}
&\limsup_{\alpha\to 0^+}\Big(\frac{\alpha u_\alpha(0)^2}{[\log(e+u_\alpha(0))]^{1+\frac{\alpha(N+2)}{2}}}\Big)\\
&= \limsup_{\alpha\to 0^+}\sup_{d_\alpha\in\mathcal{D}_\alpha}\Big(\frac{\alpha d_\alpha^2}{[\log(e+d_\alpha)]^{1+\frac{\alpha(N+2)}{2}}}\Big)\\
&\le \limsup_{\alpha\to 0^+}\sup_{d_\alpha\in\mathcal{D}_\alpha}\Big(\frac{d_\alpha^2 T y_\alpha'(T)^2}{[\log(e+d_\alpha)]^{\alpha N}}\Big)\limsup_{\alpha\to 0^+}\sup_{d_\alpha\in\mathcal{D}_\alpha}\Big(\frac{[\log(e+d_\alpha)]^{-1+\frac{\alpha(N-2)}{2}}}{I(\alpha)}\Big)\\
&\le \frac{2}{N}N_1\, 2^*\,\frac{\Gamma(N)}{\Gamma(N/2)^2}T\\
&= \frac{4}{N-2}[N(N-2)]^{(N-2)/2}\frac{\Gamma(N)}{\Gamma(N/2)^2}\frac{1}{R^{N-2}} = L(N,R),
\end{aligned}
$$

and (1.6) has been proved.

Now we prove (1.7). Using (4.2), Lemma 4.2, (4.3) and (4.4) we have

$$
\begin{aligned}
&\liminf_{\alpha\to 0^+}\Big(\frac{\alpha u_\alpha(0)^2}{[\log(e+u_\alpha(0))]^{\alpha(N-4)/2}}\Big)\\
&= \liminf_{\alpha\to 0^+}\inf_{d_\alpha\in\mathcal{D}_\alpha}\Big(\frac{\alpha d_\alpha^2}{[\log(e+d_\alpha)]^{\alpha(N-4)/2}}\Big)\\
&\ge \liminf_{\alpha\to 0^+}\inf_{d_\alpha\in\mathcal{D}_\alpha}\Big(\frac{d_\alpha^2 T y_\alpha'(T)^2}{[\log(e+d_\alpha)]^{\alpha(N-2)}}\Big)\liminf_{\alpha\to 0^+}\inf_{d_\alpha\in\mathcal{D}_\alpha}\Big(\frac{[\log(e+d_\alpha)]^{\alpha\frac{N}{2}}}{I(\alpha)}\Big)\\
&\ge \frac{2}{N}N_1\frac{\Gamma(N)}{\Gamma(\frac{N}{2})^2}T\\
&= \frac{2}{N}[N(N-2)]^{\frac{N-2}{2}}\frac{\Gamma(N)}{\Gamma(N/2)^2}\frac{1}{R^{N-2}} = \frac{1}{2^*}L(N,R).
\end{aligned}
\tag{4.14}
$$

Assertion (1.7) has been proved. This completes the proof of Theorem 1.2. □

5. Proof of Theorem 1.3

Theorem 1.3 will be a consequence of Lemma 4.2 and the following lemma.

Lemma 5.1. *Let $y = y_\alpha(t,d)$ solve* (2.4), *and let $\mathcal{D}_\alpha$ be defined by* (2.9). *Then, the following estimates hold*

(i) *For every $t \ge T$,*

$$
\limsup_{\alpha\to 0^+}\sup_{d_\alpha\in\mathcal{D}_\alpha}\Big[\frac{d_\alpha}{[\log(e+d_\alpha)]^{\alpha\frac{N}{2}}}y_\alpha(t)\Big] \le N_1(t-T). \tag{5.1}
$$

(ii)

$$
\liminf_{\alpha\to 0^+}\inf_{d_\alpha\in\mathcal{D}_\alpha}\Big[\frac{d_\alpha}{[\log(e+d_\alpha)]^{\frac{\alpha(N-2)}{2}}}y_\alpha(t)\Big] \ge N_1(t-T). \tag{5.2}
$$

Proof. (i) Using the concavity of y, we deduce $y_\alpha'(t) \le y_\alpha'(T)$ for every $t \ge T$. Now, integrating (4.6) we obtain (5.1).

(ii) Fix an arbitrary $\varepsilon > 0$. Let us take $t \in (T, \varepsilon\widetilde{T})$. Since concavity of y, from (2.5), Lemma 2.6 and (4.1), there exists a d_1 only depending on ε, see (2.28), such that for every $d_\alpha \ge d_1$,

$$\begin{aligned} y_\alpha'(t) &\ge y_\alpha'\big(\varepsilon\widetilde{T}(d)\big) \\ &= \int_{\varepsilon\widetilde{T}}^{\infty} s^{-\frac{2(N-1)}{N-2}} \frac{y_\alpha(s)^{2^*-1}}{[\log(e+y_\alpha(s)]^\alpha}\,ds \\ &> \frac{\gamma(\alpha)^{2^*-1}}{[\log(e+d_\alpha)]^\alpha}\int_{\varepsilon\widetilde{T}}^{\infty} s^{-\frac{2(N-1)}{N-2}} z_\alpha(s)^{2^*-1}\,ds \\ &= N_1\frac{N}{2}\frac{\gamma(\alpha)^{2^*-1}}{[\log(e+d_\alpha)]^\alpha}\frac{[\log(e+d_\alpha)]^{\alpha\frac{N}{2}}}{d_\alpha}B\Big((N_1\varepsilon)^{\frac{2}{N-2}},1,\frac{N}{2}\Big). \end{aligned}$$

Then

$$\inf_{d_\alpha\in\mathcal{D}_\alpha}\Big[\frac{d_\alpha}{[\log(e+d_\alpha)]^{\frac{\alpha(N-2)}{2}}}y_\alpha'(t)\Big] \ge N_1\frac{N}{2}\gamma(\alpha)^{2^*-1}B\Big((N_1\varepsilon)^{\frac{2}{N-2}},1,\frac{N}{2}\Big).$$

Since $\gamma(\alpha)\to 1$ as $\alpha\to 0^+$, for every $t>T$,

$$\liminf_{\alpha\to 0^+}\inf_{d_\alpha\in\mathcal{D}_\alpha}\Big[\frac{d_\alpha}{[\log(e+d_\alpha)]^{\frac{\alpha(N-2)}{2}}}y_\alpha'(t)\Big] \ge N_1\frac{N}{2}B\Big((N_1\varepsilon)^{\frac{2}{N-2}},1,\frac{N}{2}\Big),$$

for an arbitrary $\varepsilon>0$ fixed. By continuity of the incomplete beta function with respect to its first argument, $B\Big((\varepsilon N_1)^{\frac{2}{N-2}},1,\frac{N}{2}\Big)\to B\big(0,1,\frac{N}{2}\big)$ as $\varepsilon\to 0$, and

$$\liminf_{\alpha\to 0^+}\inf_{d_\alpha\in\mathcal{D}_\alpha}\Big[\frac{d_\alpha}{[\log(e+d_\alpha)]^{\frac{\alpha(N-2)}{2}}}y_\alpha'(t)\Big] \ge N_1\frac{N}{2}B\big(0,1,\frac{N}{2}\big)=N_1.$$

This completes the proof. □

Proof of Theorem 1.3. (i) First we prove (1.8). From (5.2), (2.3), (2.8) and (2.13) we can write

$$\liminf_{\alpha\to 0^+}\inf_{d_\alpha\in\mathcal{D}_\alpha}\Big[\frac{d_\alpha}{[\log(e+d_\alpha)]^{\alpha\frac{N-2}{2}}}u_\alpha(r)\Big] \ge [N(N-2)]^{\frac{N-2}{2}}\Big(\frac{1}{r^{N-2}}-\frac{1}{R^{N-2}}\Big).$$

From (1.6) we deduce that

$$\liminf_{\alpha\to 0^+}\inf_{d_\alpha\in\mathcal{D}_\alpha}\frac{[\log(e+d_\alpha)]^{\frac12+\alpha\frac{N+2}{4}}}{\sqrt{\alpha}\,d_\alpha} \ge \sqrt{\frac{1}{L(N,R)}}.$$

Multiplying both inequalities, we deduce that

$$\liminf_{\alpha\to 0^+}\inf_{d_\alpha\in\mathcal{D}_\alpha}\Big[[\log(e+d_\alpha)]^{\frac12-\alpha\frac{N-6}{4}}\frac{u_\alpha(r)}{\sqrt{\alpha}}\Big] \ge \widetilde{L}(N,R)\Big(\frac{1}{r^{N-2}}-\frac{1}{R^{N-2}}\Big).$$

(ii) Next we prove (1.9). From (5.1), (2.3), (2.8) and (2.13), we can write

$$\limsup_{\alpha\to 0^+}\sup_{d_\alpha\in\mathcal{D}_\alpha}\Big[\frac{d_\alpha}{[\log(e+d_\alpha)]^{\alpha\frac{N}{2}}}u_\alpha(r)\Big] \le [N(N-2)]^{\frac{N-2}{2}}\Big(\frac{1}{r^{N-2}}-\frac{1}{R^{N-2}}\Big).$$

From (1.7) we deduce

$$\limsup_{\alpha\to 0^+}\sup_{d_\alpha\in\mathcal{D}_\alpha}\frac{[\log(e+d_\alpha)]^{\alpha\frac{N-4}{4}}}{d_\alpha}\frac{1}{\sqrt{\alpha}} \le \sqrt{\frac{2^*}{L(N,R)}}.$$

Multiplying both inequalities we deduce

$$\limsup_{\alpha\to 0^+}\sup_{d_\alpha\in\mathcal{D}_\alpha}\Big[\frac{1}{[\log(e+d_\alpha)]^{\alpha\frac{N+4}{4}}}\frac{u_\alpha(r)}{\sqrt{\alpha}}\Big]$$

$$\leq \sqrt{2^\star}\,\widetilde{L}(N,R)\Big(\frac{1}{r^{N-2}}-\frac{1}{R^{N-2}}\Big).$$

This completes the proof. □

Acknowledgments. Rosa Pardo is supported bythe Spanish Ministerio de Ciencia e Innovación (MICINN) under Projects MTM2016-75465 and PID2019-103860GB-l00, and by the Grupo de Investigación CADEDIF 920894, UCM.

References

[1] F. V. Atkinson, L. A. Peletier; Emden-Fowler equations involving critical exponents. *Nonlinear Anal., Theory, Methods & Applications*, **10** (1986), no. 8,755–776.

[2] F. V. Atkinson, L. A. Peletier; Elliptic equations with nearly critical growth. *J. Differential Equations*, **70** (1987), no. 3, 349–365.

[3] A. Bahri, J. M. Coron; On a nonlinear elliptic equation involving the critical sobolev exponent: The effect of the topology of the domain. *Comm. Pure Appl. Math.*, **41** (1988), no. 3, 253-294.

[4] A. Castro, N. Mavinga, R. Pardo; Equivalence between uniform $L^{2^\star}(\Omega)$ a-priori bounds and uniform $L^\infty(\Omega)$ a-priori bounds for subcritical elliptic equations. *Topol. Methods Nonlinear Anal.*, **53** (2019), no. 1, 43–56.

[5] A. Castro, R. Pardo; A priori bounds for positive solutions of subcritical elliptic equations. *Rev. Mat. Complut.* **28** (2015), 715–731.

[6] A. Castro, R. Pardo; A priori estimates for positive solutions to subcritical elliptic problems in a class of non-convex regions. *Discrete Contin. Dyn. Syst. Ser. B*, **22** (2017), no. 3, 783–790.

[7] L. Damascelli, R. Pardo; A priori estimates for some elliptic equations involving the p-Laplacian. *Nonlinear Anal.*, **41** (2018), 475 - 496.

[8] D. G. de Figueiredo, P. L. Lions and R. D. Nussbaum; A priori estimates and existence of positive solutions of semilinear elliptic equations. *J. Math. Pures Appl. (9)*, **61** (1982), no. 1, 41–63.

[9] W.-Y. Ding; Positive solutions of $\Delta u + u^{(n+2)/(n-2)} = 0$ on contractible domains. *J. Partial Differential Equations*, **2** (1989), no. 4, 83 - 88.

[10] B. Gidas, Wei Ming Ni, L. Nirenberg; Symmetry and related properties via the maximum principle. *Comm. Math. Phys.* **68** (1979), no. 3, 209–243.

[11] Z.-C. Han; Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **8** (1991), no. 2, 159–174.

[12] N. Mavinga, R. Pardo; A priori bounds and existence of positive solutions for subcritical semilinear elliptic systems. *J. Math. Anal. Appl.*, **449** (2017), no. 2, 1172–1188.

[13] R. Pardo; On the existence of a priori bounds for positive solutions of elliptic problems, I. *Revista Integración. Temas de Matemáticas.* **37** (2019), no. 1, 77–111.

[14] R. Pardo; On the existence of a priori bounds for positive solutions of elliptic problems, II. *Revista Integración. Temas de Matemáticas.* **37** (2019), no. 1, 113–148.

[15] S. I. Pohozaev; On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. *Dokl. Akad. Nauk SSSR*, **165** (1965), 36–39.

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