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GLOBAL SOLUTIONS FOR FRACTIONAL VISCOELASTIC EQUATIONS WITH LOGARITHMIC NONLINEARITIES

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In memory of my mother Celia Lapa C.

ABSTRACT. In this article we study a fractional viscoelastic equation of Kirchhoff type with logarithmic nonlinearities. Under suitable conditions we prove the existence of global solutions and the exponential decay of the energy.

1. INTRODUCTION

We consider the problem of finding $u = u(x,t)$ weak solutions to the nonlinear heat equation of Kirchhoff type with variable exponent of nonlinearity, viscoelastic term and logarithmic source terms, involving the fractional Laplacian,

$$\begin{gathered}
\big(1 + a|u|^{r(x)-2}\big)u_t + M(\|u\|_{w_0}^2)(-\Delta)^s u - \int_0^t g(t-\tau)(-\Delta)^s u(\tau)d\tau \\
= \big(|u|^{\rho-2}u + |u|^{\sigma-2}u\big)\log|u| =: f(u) \quad \text{in } \Omega\times]0,\infty[, \\
u = 0 \quad \text{in } (\mathbb{R}^N\backslash\Omega)\times[0,\infty[, \\
u(x,0) = u^0(x) \quad \text{in } \Omega,
\end{gathered} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain, $M(t) = t^{\alpha-1} + 1$, $0 \leq t0$, $s \in]0,1[$, $2 < N/s$, $\alpha > 1$, $g : [0,\infty[\to]0,\infty[$ belongs to $C^1([0,\infty[)$, $g(0) > 0$, $l = 1 - \int_0^\infty g(\tau)d\tau > 0$, $g'(t) \leq 0$, $\rho,\sigma > 2$, and r is a continuous function.

This type of problems without viscoelastic term (that is $g = 0$), with $r(x)$ constant , $M(t) = 1$ and f a polynomial, have been considered by many authors with the standard Laplace operator $(-\triangle)^s$, $s = 1$, and can be seen as special case of doubly nonlinear parabolic type equations

$$(\varphi(u))_t - \Delta u = f(u),$$

which appear in the mathematical modeling of various physical processes such as flows of incompressible turbulent fluids or gases in pipes, processes of filtration in porous media, glaciology, see [2, 3, 13, 35] and the further references therein.

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The questions of solvability and the long time behavior of solutions to the doubly nonlinear nonlocal parabolic equation

$$(\varphi(u))_t - \operatorname{div} \sigma(\nabla u) = \int_0^t g(t-\tau) div\sigma(\nabla u(\tau)) d\tau + f(x,t,u),$$

were studied in [4, 19, 30, 31, 32, 34]. This equation arises from the study of heat conduction in materials with memory. On the other hand, many fractional and non-local operators are actively studied in recent years. This type of operators arises in a quite natural way in many interesting applications, such as, finance, physics, game theory, Lévy stable diffusion processes, crystal dislocation; see [5, 22, 36] and their references. The first result concerning fractional Kirchhoff problems was obtained by Fiscella and Valdinoci [18]. Pan et al [26] investigated for the first time the existence of global weak solutions for degenerate Kirchhoff-type diffusion problems involving fractional p-Laplacian, by combining the Galerkin method with potential well theory, for the special function $M(t) = t^{\lambda-1}(t \geq 0)$. Mingqi et al [24] proved the existence and blow-up of solutions for a similar equation with more general conditions on M which cover the degenerate case. Recently, logarithmic nonlinearity appears frequently in partial differential equations which describes important physical phenomena, see [12, 14, 20, 23, 37] and the references therein. Ding and Zhou [14] studied the semilinear parabolic problem of Kirchhoff type with logarithmic nonlinearity,

$$u_t - M([u]_s^2)\mathcal{L}_K u = |u|^{p-2} u \log|u|.$$

They obtained results of global solutions and of finite time blow-up of solutions, when the initial energy is subcritical and critical, by using the potential well method. In the works mentioned above, there are only a few about global existence and exponential decay rate for doubly nonlinear parabolic equations involving variable exponent, viscoelastic term in the fractional setting, and logarithmic nonlinear terms. Motivated by this, we study global solutions for (1.1) by using Galerkin's method and similar arguments as those in Tartar [33]. Also, we give the exponential decay rate of the energy via the energy perturbation method. It is worth mentioning that we do not use the logarithmic Sobolev inequality to obtain our results.

The article is organized as follows. In Section 2, we give the preliminaries for our research. In Section 3, by using the Galerkin approximation method we obtain a global solution, and finally, we obtain the exponential decay under certain class of initial data.

2. Preliminaries

In this section, we present some material and assumptions needed in the rest of this paper. We denote $Q = \mathbb{R}^{2N} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$, $\mathcal{C}\Omega := \mathbb{R}^N \setminus \Omega$, and

$$W = \big\{u : \mathbb{R}^N \to \mathbb{R} : u|_\Omega \in L^2(\Omega),\ \iint_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy < \infty\big\},$$

where $u|_\Omega$ represents the restriction to Ω of function $u(x)$. Also, we define the linear subspace of W,

$$W_0 = \big\{u \in W : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\big\}.$$

The linear space W is endowed with the norm

$$\|u\|_W := \|u\|_{L^2(\Omega)} + \Big(\iint_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{1/2}.$$

It is easily seen that $\|\cdot\|_W$ is a norm on W and $C_0^\infty(\Omega) \subseteq W_0$. The functional

$$\|u\|_{W_0} = \Big(\iint_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{1/2},$$

is a equivalent norm on $W_0 = \{u \in W : u(x) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$ which is a closed linear subspace of W. Furthermore $(W_0, \|\cdot\|_{W_0})$ is a Hilbert space with inner product

$$\langle u, v\rangle_{W_0} = \iint_Q \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dxdy.$$

Now we review the main embedding results for the space W_0.

Lemma 2.1 ([27, 28, 29]). *The embedding $W_0 \hookrightarrow L^r(\Omega)$ is continuous for any $r \in [1, 2_s^*]$, and compact for any $r \in [1, 2_s^*[$.*

Lemma 2.2 ([25, Lemma 2.1]). *Let $N \geq 1$, $0 < s < 1$, $p > 1$, $q \geq 1$, $\tau > 0$ and $0 < \theta < 1$ be such that $\frac{1}{\tau} = \theta(\frac{1}{p} - \frac{s}{N}) + \frac{1-\theta}{q}$. Then*

$$\|u\|_{L^\tau(\mathbb{R}^N)} \leq \|u\|^\theta_{W^{s,p}(\mathbb{R}^N)} \|u\|^{1-\theta}_{L^q(\mathbb{R}^N)}, \quad \forall u \in C_0^1(\mathbb{R}^N).$$

Now, we recall some background concerning the generalized Lebesgue-Sobolev spaces. We refer the reader to [15, 16, 17] for details. Set

$$C_+(\overline{\Omega}) = \{p(x) : p(x) \in C(\overline{\Omega}),\ p(x) > 1,\ \text{for all } x \in \overline{\Omega}\}.$$

For $p \in C_+(\overline{\Omega})$ we define

$$p^+ = \max\{p(x) : x \in \overline{\Omega}\}, \quad p^- = \min\{p(x); x \in \overline{\Omega}\},$$

and the space

$$L^{p(x)}(\Omega) = \big\{u : u \text{ is a measurable real-valued function}, \int_\Omega |u(x)|^{p(x)} dx < \infty\big\},$$

with

$$\|u\|_{p(x)} \equiv \|u\|_{L^{p(x)}(\Omega)} = \inf\big\{\lambda > 0 : \int_\Omega \big|\frac{u(x)}{\lambda}\big|^{p(x)} dx \leq 1\big\}$$

which is a Banach space [21]. We also define the space

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u(x)\|_{p(x)} + \|\nabla u(x)\|_{p(x)}.$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Of course the norm $\|u\| = \|\nabla u\|_{p(x)}$ is an equivalent norm in $W_0^{1,p(x)}(\Omega)$.

Proposition 2.3 ([16]).
(i) *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\int_\Omega |uv| dx \leq \big(\frac{1}{p^-} + \frac{1}{p'^-}\big)\|u\|_{p(x)}\|v\|_{p'(x)} \leq 2\|u\|_{p(x)}\|v\|_{p'(x)}.$$

(ii) *If $p_1(x), p_2(x) \in C_+(\overline{\Omega})$ and $p_1(x) \leq p_2(x)$ for all $x \in \overline{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.*

Proposition 2.4 ([16]). *Set $\rho(u) = \int_\Omega |\nabla u(x)|^{p(x)}\,dx$, then for $u \in W_0^{1,p(x)}(\Omega)$ and $(u_k) \subset W_0^{1,p(x)}(\Omega)$, we have*

(1) $\|u\| < 1$ *(resp.* $= 1; > 1$*) if and only if* $\rho(u) < 1$ *(resp.* $= 1; > 1$*);*
(2) *for* $u \neq 0$, $\|u\| = \lambda$ *if and only if* $\rho(u/\lambda) = 1$*;*
(3) *if* $\|u\| > 1$*, then* $\|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$*;*
(4) *if* $\|u\| < 1$*, then* $\|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$*;*
(5) $\|u_k\| \to 0$ *(resp.* $\to \infty$*) if and only if* $\rho(u_k) \to 0$ *(resp.* $\to \infty$*).*

For $x \in \Omega$, let us define

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

Proposition 2.5 ([17])**.** *If* $q \in C_+(\overline{\Omega})$ *and* $q(x) \leq p^*(x)$ $(q(x) < p^*(x))$ *for* $x \in \overline{\Omega}$*, then there is a continuous (compact) embedding* $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$*.*

Lemma 2.6. *Let* $2 < r < \rho < 2_s^*$*.For each* $\epsilon > 0$*, there exists a positive constant* C_ϵ *such that*

$$\|v\|_\rho^\rho \leq \epsilon \|v\|_{W_0}^2 + C_\epsilon \|v\|_r^{kr},$$

for all $v \in W_0 \cap L^r(\Omega)$ *where*

$$k = \frac{2\rho(1-\theta)}{r(2-\rho\theta)}, \quad \theta = \Big(\frac{1}{r} - \frac{1}{\rho}\Big)\Big(\frac{s}{N} - \frac{1}{2} + \frac{1}{r}\Big)^{-1}.$$

The above lemma immediately follows from Lemma 2.2 and Young's inequality.

Lemma 2.7 ([21, Theorem 1, pag 23])**.** *Suppose that* $r \in L_+^\infty(\Omega)$*,* $r^- \geq 2$*,* $w \in L^{r(x)}(\Omega \times]0,T[)$ *and*

$$\frac{\partial}{\partial t}(|w|^{r(x)-2}w) \in L^{r'(x)}(\Omega \times]0,T[).$$

Then, for any $s, \tau \in [0,T]$ *with* $s < \tau$ *we have formula of integration by parts,*

$$\int_s^\tau \int_\Omega w\Big(\frac{1}{r(x)-1}|w|^{r(x)-2}w\Big)\,dx\,dt = \int_\Omega \frac{1}{r(x)}|w(\tau)|^{r(x)}\,dx - \int_\Omega \frac{1}{r(x)}|w(s)|^{r(x)}\,dx.$$

3. Existence of global solutions and exponential decay

In this section, we focus our attention on global solutions and exponential decay for problem (1.1).

Definition 3.1. Let $T > 0$. A weak solution of (1.1) is a function $u \in L^\infty(0,T;W_0)$, with $u_t \in L^2(0,T;L^2(\Omega))$ and $(|u|^{r(x)/2})_t \in L^2(\Omega \times]0,T[)$ such that

$$\begin{aligned} &\int_0^T \int_\Omega \big(1 + a|u|^{r(x)-2}\big)u_t w\,dx\,dt + M(\|u\|_{w_0}^2)\int_0^T \langle u, w\rangle_{W_0}\,dt \\ &- \int_0^T \int_0^t g(t-\tau)\langle u(\tau), w\rangle_{W_0} d\tau\,dt \\ &= \int_0^T \int_\Omega f(u)w\,dx\,dt, \end{aligned}$$

for all $w \in L^2(0,T;W_0)$ and $u(x,0) = u^0(x) \in W_0$, where for $s \in \mathbb{R}$, $f(s) = \big(|s|^{\rho-2}s + |s|^{\sigma-2}s\big)\log|s|$.

Theorem 3.2 (Local solution)**.** *Assume* $u^0 \in W_0 \setminus \{0\}$*,* $2 < r^- < 2_s^*$*,* r^+*,* ρ*,* $\sigma \in]2, 2_s^*[$ *, then problem (1.1) has a unique weak solution* u *for* T *small enough.*

Proof. We prove the existence of weak solutions by using the Faedo-Galerkin method with ideas from [7]. We choose a sequence $\{w_\nu\}_{\nu\in\mathbb{N}} \subseteq C_0^\infty(\Omega)$ such that

$$C_0^\infty(\Omega) \subseteq \overline{\cup_{\nu=1}^\infty V_m}^{C^1(\overline{\Omega})}$$

and $\{w_\nu\}$ is a standard orthonormal basis with respect to the Hilbert space $L^2(\Omega)$ and an orthogonal basis in W_0, where $V_m = spam\{w_1, w_2, ...w_m\}$. Now, we construct approximate solutions u_m $(m = 1, 2, \dots)$, of the problem (1.1), in the form

$$u_m(x,t) = \sum_{i=1}^m g_{jm}(t) w_j(x),$$

where the coefficient functions g_{jm} satisfy the system of ordinary differential equations

$$\begin{aligned} &\int_\Omega \big(1 + a|u_m(t)|^{r(x)-2}\big) u_{mt}(t) w_j \, dx + M(\|u_m(t)\|_{w_0}^2)\langle u_m(t), w_j\rangle_{W_0} \\ &- \int_0^t g(t-\tau)\langle u_m(\tau), w_j\rangle_{W_0} d\tau dt \\ &= \int_\Omega f(u_m) w_j \, dx, \quad j = 1, 2, \dots m; \end{aligned} \tag{3.1}$$

$$u_m(x,0) = u_m^0(x) \to u^0(x) \quad \text{in } W_0.$$

Let us show that the system (3.1) is locally solvable. It is clear that (3.1) can be rewritten in the form

$$\begin{aligned} \frac{d}{dt}\Phi(g_m(t)) &= -M\Big(\|\sum_{i=1}^m g_{jm}(t) w_j(x)\|_{W_0}^2\Big) B g_m(t) \\ &\quad + \int_0^t g(t-\tau) B g_m(\tau) d\tau + F(g_m(t)), \end{aligned} \tag{3.2}$$

where

$$g_m(t) = (g_{m1}(t), g_{m2}(t), \dots, g_{mm}(t))^T, \quad B = [\langle w_i, w_j\rangle]_{1\le i,j\le m},$$
$$\Phi(\eta) = (\Phi_1(\eta), \Phi_2(\eta), \dots, \Phi_m(\eta))^T \quad \text{with } \eta = (\eta_1, \eta_2, \dots, \eta_m) \in \mathbb{R}^m,$$
$$\Phi_i(\eta) = \int_\Omega \Big\{ \sum_{j=1}^m \eta_j w_j + \frac{a}{r(x)-1}\Big|\sum_{k=1}^m \eta_k w_k\Big|^{r(x)-2} \sum_{k=1}^m \eta_k w_k \Big\} w_i \, dx$$
$$i = 1, 2, \dots, m;$$
$$F(\eta) = \Big(\int_\Omega f\Big(\sum_{k=1}^m \eta_j w_j\Big) w_1 \, dx, \dots, \int_\Omega f\Big(\sum_{k=1}^m \eta_j w_j\Big) w_m \, dx\Big)^T.$$

This system is equivalent to

$$\begin{aligned} \Phi(g_m(t)) = \Phi(g_m(0)) + \int_0^t \Big[-M\Big(\|\sum_{i=1}^m g_{jm}(t) w_j(x)\|_{W_0}^2\Big) B g_m(t) \\ + \int_0^\xi g(\xi-\tau) B g_m(\tau) d\tau + F(g_m(\xi))\Big] d\xi. \end{aligned}$$

and the fact that the map $s \mapsto f(s)$ is increasing for large s, we obtain

$$(\Phi(\zeta) - \Phi(\eta), \zeta - \eta)_{\mathbb{R}^m} \ge C_m |\zeta - \eta|_{\mathbb{R}^m}^2 \tag{3.3}$$

for $\zeta, \eta \in \mathbb{R}^m$, where C_m is a constant such that, for any g_m in $\mathbb{R}^m$,

$$\int_\Omega |u_m|^2\,dx \ge C_m |g_m|^2_{\mathbb{R}^m}\,.$$

So, by the elementary inequality $s\log s \ge s-1, \forall s>0$, we deduce that Φ is monotone coercive. Also it is obviously continuous. So, by the Brouwer theorem Φ is onto. In view of (3.3), Φ^{-1} is locally Lipchitz continuous.

Consider the map $L: C(0,T,\mathbb{R}^m) \to C(0,T,\mathbb{R}^m)$, defined by

$$\begin{aligned} L(g_m)(t) = \Phi^{-1}\Big(\Phi(g_m(0)) + \int_0^t \Big[-M\Big(\|\sum_{i=1}^m g_{jm}(t)w_j(x)\|^2_{W_0}\Big) B g_m(t) \\ + \int_0^\xi g(\xi-\tau)Bg_m(\tau)d\tau + F(g_m(\xi))\Big]\,d\xi\Big), \quad t\in[0,T]. \end{aligned}$$

It is not hard to prove that L is completely continuous and that there exist (sufficient small) $T_m>0$ and (sufficient large) $R>0$ such that $L(\overline{B_R}) \subseteq \overline{B_R}$, where $\overline{B_R}$ is the ball in $C(0,T_m,\mathbb{R}^m)$ with center the origin and radius R. Consequently, by Schauder's theorem, the operator L has a fixed point in $C(0,T_m,\mathbb{R}^m)$. This fixed point is a solution of (3.2). So, we can obtain an approximate solution $u_m(t)$ of (3.1) in V_m over $[0,T_m[$ and it can be extended to the whole interval $[0,T]$, for all $T>0$, as a consequence of the a priori estimates that shall be proven in the next step.

First estimate. Multiplying (3.1) by $g_{jm}(t)$ and adding in $j=1,\dots,m$, we have

$$\begin{aligned} &\int_\Omega \Big(1+a|u_m(t)|^{r(x)-2}\Big)\,u_{mt}(t)u_m(t)\,dx + M(\|u_m(t)\|^2_{w_0})\langle u_m(t),u_m(t)\rangle_{W_0} \\ &-\int_0^t g(t-\tau)\langle u_m(\tau),u_m(t)\rangle_{W_0}d\tau dt \\ &=\int_\Omega (|u_m(t)|^\rho + |u_m(t)|^\sigma)\log|u_m(t)|\,dx \end{aligned} \tag{3.4}$$

which implies, integrating with respect to the time variable from 0 to t on both sides, using Lemma 2.7 that

$$\begin{aligned} S_m(t) = S_m(0) + \int_0^t d\lambda \int_0^\lambda g(\lambda-\tau)\langle u_m(\tau),u_m(\lambda)\rangle_{W_0}\,d\tau \\ + \int_0^t\int_\Omega (|u_m(\tau)|^\rho + |u_m(\tau)|^\sigma)\log|u_m(\tau)|\,dx\,d\tau, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} S_m(t) = \int_\Omega |u_m(t)|^2\,dx + a\int_\Omega \frac{1}{r(x)}|u_m(t)|^{r(x)}\,dx \\ + \int_0^t \big(\|u_m(\tau)\|^{2\alpha}_{W_0} + \|u_m(\tau)\|^2_{W_0}\big)\,d\tau. \end{aligned}$$

Let us introduce the function $\Theta(\lambda) = \int_0^\lambda g(\lambda-\tau)\|u_m(\tau)\|_{W_0}$. Estimating the second term on right-hand side of (3.5) we have

$$\begin{aligned} &\int_0^t d\lambda \int_0^\lambda g(\lambda-\tau)\langle u_m(\tau), u_m(\lambda)\rangle_{W_0}\,d\tau \\ &\le \frac{1}{2}\int_0^t \left(\|u_m(\tau)\|_{W_0}^{2\alpha} + \|u_m(\tau)\|_{W_0}^2\right)\,d\tau + \frac{1}{2}\int_0^t \Theta^2(\lambda)\,d\lambda. \end{aligned} \tag{3.6}$$

But, using Young Inequality and noting that $\int_0^\infty g(\tau)d\tau < 1$, we obtain

$$\int_0^t \Theta^2(\lambda)d\lambda \le \int_0^\infty g(\tau)d\tau \int_0^t \left(\|u_m(\tau)\|_{W_0}^{2\alpha} + \|u_m(\tau)\|_{W_0}^2\right)\,d\tau. \tag{3.7}$$

Let $\varrho_\rho := 2_s^* - \rho$, $\varrho_\sigma := 2_s^* - \sigma$. Since $\log(|u|^\varrho) \le |u|^\varrho$ it follows that

$$\begin{aligned} \int_\Omega |u_m(t)|^\rho \log|u_m(t)|\,dx &= \frac{1}{\varrho_\rho}\int_\Omega |u_m(t)|^\rho \log(|u_m(t)|^{\varrho_\rho})\,dx \\ &\le \frac{1}{\varrho_\rho}\int_\Omega |u_m(t)|^{\rho+\varrho_\rho}\,dx \end{aligned} \tag{3.8}$$

Plugging (3.6)-(3.8) into (3.5), it follows that

$$\begin{aligned} S_m(t) \le S_m(0) &+ \frac{1}{2}\Big(1 + \int_0^\infty g(\tau)d\tau\Big)\int_0^t \left(\|u_m(\tau)\|_{W_0}^{2\alpha} + \|u_m(\tau)\|_{W_0}^2\right)\,d\tau \\ &+ \int_0^t \Big(\frac{1}{\varrho_\rho}\|u_m(\tau)\|_{\varrho_\rho+\rho}^{\varrho_\rho+\rho} + \frac{1}{\varrho_\sigma}\|u_m(\tau)\|_{\varrho_\sigma+\sigma}^{\varrho_\sigma+\sigma}\Big)\,d\tau. \end{aligned} \tag{3.9}$$

To estimate the last term in (3.9) we use Lemma 2.6,

$$\begin{aligned} &\int_0^t \Big(\frac{1}{\varrho_\rho}\|u_m(\tau)\|_{\varrho_\rho+\rho}^{\varrho_\rho+\rho} + \frac{1}{\varrho_\sigma}\|u_m(\tau)\|_{\varrho_\sigma+\sigma}^{\varrho_\sigma+\sigma}\Big)\,d\tau \\ &\le \epsilon\int_0^t \left(\|u_m(\tau)\|_{W_0}^{2\alpha} + \|u_m(\tau)\|_{W_0}^2\right)\,d\tau + C_0\int_0^t \left(S_m^{k_1}(\lambda) + S_m^{k_2}(\lambda)\right)\,d\lambda, \end{aligned} \tag{3.10}$$

where

$$k_1 = \frac{2(\varrho_\rho+\rho)(1-\theta)}{r^-[2-(\rho+\varrho_\rho)\theta]} > 1, \quad k_2 = \frac{2(\varrho_\sigma+\sigma)(1-\theta)}{r^-[2-(\varrho_\sigma+\sigma)\theta]} > 1.$$

Taking ϵ suitably small in (3.10), it follows from (3.5)-(3.10) that

$$S_m(t) \le \hat{C}_0 + \hat{C}_1\int_0^t \left(S_m^{k_1}(\lambda) + S_m^{k_2}(\lambda)\right)\,d\lambda. \tag{3.11}$$

Hence, by employing [10, Theorem 1.2], there exists a constant T_0 such that

$$S_m(t) \le C_{T_0}, \quad \forall t \in [0, T_0]. \tag{3.12}$$

Second estimate. Multiplying (3.1) by $g'_{jm}(t)$ and adding in $j = 1, \dots, m$, it follows that

$$\begin{aligned} &\frac{d}{dt}\Big\{\frac{1}{2\alpha}\|u_m(t)\|_{W_0}^{2\alpha} + \frac{1}{2}\Big(1 - \int_0^t g(\tau)d\tau\Big)\|u_m(t)\|_{W_0}^2 + \frac{1}{2}(g \diamond u)(t) \\ &- \frac{1}{\rho}\int_\Omega |u_m(t)|^\rho \log|u_m(t)|\,dx + \frac{1}{\rho^2}\|u_m(t)\|_\rho^\rho \\ &- \frac{1}{\sigma}\int_\Omega |u_m(t)|^\sigma \log|u_m(t)|\,dx + \frac{1}{\sigma^2}\|u_m(t)\|_\sigma^\sigma\Big\} \\ &+ \|u_{mt}(t)\|_2^2 + a\int_\Omega |u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\,dx \\ &= \frac{1}{2}(g' \diamond u)(t) - \frac{1}{2}g(t)\|u_m(t)\|_{W_0}^2. \end{aligned} \tag{3.13}$$

where

$$(g \diamond u)(t) = \int_0^t g(t-\tau)\|u(t) - u(\tau)\|_{W_0}^2\,d\tau.$$

Integrating (3.13) on $[0,t]$, $t \le T_0$ we obtain

$$\begin{aligned} &\int_0^t \|u_{mt}(t)\|_2^2 + a\int_0^t\int_\Omega |u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\,dx + \frac{1}{2\alpha}\|u_m(t)\|_{W_0}^{2\alpha} + \frac{l}{2}\|u_m(t)\|_{W_0}^2 \\ &\le \frac{1}{2\alpha}\|u_m(0)\|_{W_0}^{2\alpha} + \frac{1}{2}\|u_m(0)\|\|_{W_0}^2 + \frac{1}{\rho}\int_\Omega |u_m(t)|^\rho \log|u_m(t)|\,dx \\ &\quad + \frac{1}{\rho^2}\|u_m(0)\|_\rho^\rho + \frac{1}{\sigma}\int_\Omega |u_m(t)|^\sigma \log|u_m(t)|\,dx + \frac{1}{\sigma^2}\|u_m(0)\|_\sigma^\sigma \\ &\quad - \frac{1}{\rho}\int_\Omega |u_m(0)|^\rho \log|u_m(0)|\,dx - \frac{1}{\sigma}\int_\Omega |u_m(0)|^\sigma \log|u_m(0)|\,dx. \end{aligned}$$

From the assumptions on u^0, (3.8), Lemma 2.6 and the estimate (3.12), it follows that

$$\begin{aligned} &\int_0^t \|u_{mt}(t)\|_2^2 + a\int_0^t\int_\Omega |u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\,dx \\ &+ \frac{1}{2\alpha}\|u_m(t)\|_{W_0}^{2\alpha} + \frac{l}{2}\|u_m(t)\|_{W_0}^2 \le M_1, \end{aligned} \tag{3.14}$$

for some constant $M_1 > 0$. By the above estimates (3.12) and (3.14), $\{u_m\}$ have subsequences still denoted by $\{u_m\}$ such that

$$\begin{gathered} u_m \to u \quad \text{weakly* in } L^\infty(0,T_0;W_0), \\ u_{mt} \to u_t \quad \text{weakly in } L^2(0,T_0;L^2(\Omega)), \\ \Big(|u_m|^{r(x)/2}\Big)_t \to \chi \quad \text{weakly in } L^2(0,T_0;L^2(\Omega)). \end{gathered} \tag{3.15}$$

Also, reasoning as in [12], taking into account the compact embedding of W_0 into $L^\beta(\Omega), \beta = \rho, \sigma$, we have

$$\begin{aligned} &|u_m|^{\rho-2}u_m \log|u_m| \to |u|^{\rho-2}u\log|u| \quad \text{weakly* in } \ L^\infty(0,T_0;L^{\frac{\rho}{\rho-1}}(\Omega)), \\ &|u_m|^{\sigma-2}u_m \log|u_m| \to |u|^{\sigma-2}u\log|u| \quad \text{weakly* in } \ L^\infty(0,T_0;L^{\frac{\sigma}{\sigma-1}}(\Omega)) \end{aligned} \tag{3.16}$$

Employing the same arguments as in [9] we can prove that

$$\chi = \big(|u|^{r(x)/2}\big)_t, \quad |u_m|^{r(x)/2}u_{mt} \to |u|^{r(x)/2}u_t \quad \text{weakly in } L^2(\Omega\times]0,T_0[) \tag{3.17}$$

Therefore, passing to the limit in (3.1) as $m \to +\infty$, by (3.15)– (3.17), we can show that u satisfies the initial condition $u(0) = u^0$ and

$$\begin{aligned}
&\int_0^T \int_\Omega \big(1 + a|u|^{r(x)-2}\big) u_t w \, dx \, dt + M(\|u\|_{w_0}^2) \int_0^T \langle u, w\rangle_{W_0} \, dt \\
&- \int_0^T \int_0^t g(t-\tau)\langle u(\tau), w\rangle_{W_0} d\tau dt \\
&= \int_0^T \int_\Omega f(u) w \, dx \, dt,
\end{aligned}$$

for all $w \in L^2(0, T_0; W_0)$.

The uniqueness property of solutions can be derived as in [13, Theorem 3, p. 1095], observing that $\big(u + \frac{a}{r(x)-1}|u|^{r(x)-2}u\big) \in L^2(\Omega \times]0, T_0[)$, $F(s) = f(s)\log(|s|)$ is locally Lipschitz continuous and $Au = M(\|u\|_{w_0}^2)(-\Delta)^s u$ is a monotone operator. We omit the details. □

Next, we consider the existence of global solutions and their energy decay for problem (1.1). For this purpose we define the energy associated with problem (1.1) by

$$\begin{aligned}
E(t) = &\frac{1}{2\alpha}\|u(t)\|_{W_0}^{2\alpha} + \frac{1}{2}\Big(1 - \int_0^t g(\tau) d\tau\Big)\|u(t)\|_{W_0}^2 + \frac{1}{2}(g \diamond u)(t) \\
&- \frac{1}{\rho}\int_\Omega |u(t)|^\rho \log|u(t)| \, dx + \frac{1}{\rho^2}\|u(t)\|_\rho^\rho \\
&- \frac{1}{\sigma}\int_\Omega |u(t)|^\sigma \log|u(t)| \, dx + \frac{1}{\sigma^2}\|u(t)\|_\sigma^\sigma.
\end{aligned} \tag{3.18}$$

Then, we easily can check that

$$\begin{aligned}
\frac{d}{dt}E(t) = &\frac{1}{2}(g' \diamond u)(t) - \frac{1}{2}g(t)\|u(t)\|_{W_0}^2 - \|u_t(t)\|_2^2 \\
&- a\int_\Omega |u(t)|^{r(x)-2} u_t^2(t) \, dx \le 0
\end{aligned} \tag{3.19}$$

for any regular solution. This remains valid for weak solutions by simple density argument. This shows that $E(t)$ is a nonincreasing function.

Before going on, we introduce the following notation

$$B_1 = \sup_{u \in W_0, \, u \neq 0} \frac{\|u\|_{\rho + \varrho_\rho}}{\sqrt{l\|u\|_{W_0}^2}}, \quad B_2 = \sup_{u \in W_0, \, u \neq 0} \frac{\|u\|_{\sigma + \varrho_\sigma}}{\sqrt{l\|u\|_{W_0}^2}},$$

$$\gamma_1 = \frac{1}{\rho\varrho_\rho}B_1^{\rho + \varrho_\rho}, \quad \gamma_2 = \frac{1}{\sigma\varrho_\sigma}B_2^{\sigma + \varrho_\sigma}.$$

Define the function

$$h(\lambda) = \frac{1}{4}\lambda^2 - \gamma_1\lambda^{\rho + \varrho_\rho} - \frac{3}{2}\gamma_2\lambda^{\sigma + \varrho_\sigma}. \tag{3.20}$$

Then

$$h'(\lambda) = \frac{1}{2}\lambda - (\rho + \varrho_\rho)\gamma_1\lambda^{\rho + \varrho_\rho - 1} - \frac{3}{2}(\sigma + \varrho_\sigma)\gamma_2\lambda^{\sigma + \varrho_\sigma - 1}.$$

So, choosing $\lambda \in \mathbb{R}$ such that

$$0 \le \lambda^{\rho + \varrho_\rho - 1} \le \frac{1}{4(\rho + \varrho_\rho)\gamma_1} \quad \text{and} \quad 0 \le \lambda^{\sigma + \varrho_\sigma - 1} \le \frac{1}{6(\sigma + \varrho_\sigma)\gamma_2}$$

we obtain that these λs satisfy the inequality

$$\frac{1}{2}\lambda \geq (\rho+\varrho_\rho)\gamma_1\lambda^{\rho+\varrho_\rho-1} + \frac{3}{2}(\sigma+\varrho_\sigma)\gamma_2\lambda^{\sigma+\varrho_\sigma-1}$$

and $h'(\lambda) \geq 0$ for $0 \leq \lambda \leq \lambda_1$ where

$$\lambda_1 = \min\Big\{\big[\frac{1}{4(\rho+\varrho_\rho)\gamma_1}\big]^{1/(\rho+\varrho_\rho-1)}, \big[\frac{1}{6(\sigma+\varrho_\sigma)\gamma_1}\big]^{1/(\sigma+\varrho_\sigma-1)}\Big\}.$$

Thus $h(0)=0$ and $h(\lambda)\geq 0$ for all $\lambda\in[0,\lambda_1[$ Therefore, from (3.20), we have

$$\frac{1}{4}\lambda^2 - \gamma_1\lambda^{\rho+\varrho_\rho} - \frac{3}{2}\gamma_2\lambda^{\sigma+\varrho_\sigma} \geq 0, \quad \forall \lambda\in[0,\lambda_1[. \tag{3.21}$$

Now, if one considers

$$l\|u(t)\|^2_{W_0} + (g\diamond u)(t) < \lambda_1^2, \tag{3.22}$$

from (3.21), we obtain

$$\begin{aligned}
&\frac{1}{4}\Big(\sqrt{l\|u(t)\|^2_{W_0} + (g\diamond u)(t)}\Big)^2 + \frac{1}{2}\gamma_2\Big(\sqrt{l\|u(t)\|^2_{W_0} + (g\diamond u)(t)}\Big)^{\sigma+\varrho_\sigma}\\
&\leq \frac{1}{2}\Big(\sqrt{l\|u(t)\|^2_{W_0} + (g\diamond u)(t)}\Big)^2 - \gamma_1\Big(\sqrt{l\|u(t)\|^2_{W_0} + (g\diamond u)(t)}\Big)^{\rho+\varrho_\rho}\\
&\quad - \gamma_2\Big(\sqrt{l\|u(t)\|^2_{W_0} + (g\diamond u)(t)}\Big)^{\sigma+\varrho_\sigma}
\end{aligned}$$

which implies

$$\begin{aligned}
E(t) &\geq \frac{l}{2}\|u(t)\|^2_{W_0} + \frac{1}{2}(g\diamond u)(t) - \gamma_1\Big(\sqrt{l\|u(t)\|^2_{W_0} + (g\diamond u)(t)}\Big)^{\rho+\varrho_\rho}\\
&\quad - \gamma_2\Big(\sqrt{l\|u(t)\|^2_{W_0} + (g\diamond u)(t)}\Big)^{\sigma+\varrho_\sigma}\\
&\geq \frac{1}{4}\Big(\sqrt{l\|u(t)\|^2_{W_0} + (g\diamond u)(t)}\Big)^2\\
&\quad + \frac{1}{2}\gamma_2\Big(\sqrt{l\|u(t)\|^2_{W_0} + (g\diamond u)(t)}\Big)^{\sigma+\varrho_\sigma}, \quad t\geq 0.
\end{aligned} \tag{3.23}$$

Now, we are ready to state our main result.

Theorem 3.3. *Assume that hypotheses of Theorem 3.2 are satisfied. Consider $u_0\in W_0$, satisfying*

$$0 < l^{1/2}\|u_0\|_{W_0} < \lambda_1, \tag{3.24}$$

$$\big(\frac{4}{l}E_1\big)^{1/2} < \lambda_1, \tag{3.25}$$

where

$$\begin{aligned}
E_1 &= \frac{1}{2\alpha}\|u^0\|^{2\alpha}_{W_0} + \frac{1}{2}\|u^0\|^2_{W_0} - \frac{1}{\rho}\int_\Omega |u^0|^\rho \log|u^0|\,dx + \frac{1}{\rho^2}\|u^0\|^\rho_\rho\\
&\quad - \frac{1}{\sigma}\int_\Omega |u^0|^\sigma \log|u^0|\,dx + \frac{1}{\sigma^2}\|u^0\|^\sigma_\sigma.
\end{aligned} \tag{3.26}$$

Then the problem admits a global weak solution in time. In addition, if there exists a constant $\xi_0>0$ such that $g'(t)\leq -\xi_0 g(t)$, then this solution satisfies

$$E(t) \leq L_0 e^{-\gamma t}, \quad \forall t\geq 0, \tag{3.27}$$

where L_0 and γ are positive constants.

Proof. We will get global estimates for $u_m(t)$ solution of the approximate system (3.1) under the conditions (3.24)–(3.25) for u^0. For this, it suffices to show that

$$E_m(t)+\int_0^t\|u_{mt}(t)\|_2^2+a\int_0^t\int_\Omega|u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\,dx,$$

where $E_m(t)$ is defined in (3.18) with $u(t)$ replaced by $u_m(t)$, is bounded and independently of t. From (3.13) and the definition of energy, we have

$$E_m(t)+\int_0^t\|u_{mt}(t)\|_2^2+a\int_0^t\int_\Omega|u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\,dx\le E_m(0). \tag{3.28}$$

From the convergence $u_{0m}\to u^0$ in W_0 we see that $E_m(0)<\frac{l}{4}\lambda_1^2$ for sufficiently large m. We claim that there exists an integer ν_0 such that

$$\sqrt{l\|u_m(t)\|_{W_0}^2+(g\diamond u_m)(t)}<\lambda_1\quad\forall t\in[0,T_m[,m\ge\nu_0. \tag{3.29}$$

Supposing that the claim is proved, $h\left(\sqrt{l\|u_m(t)\|_{W_0}^2+(g\diamond u_m)(t)}\right)\ge 0$ and from (3.23), (3.28)–(3.29) we obtain

$$\|u_m(t)\|_{W_0}^{2\alpha}+\|u_m(t)\|_{W_0}^2+\int_0^t\|u_{mt}(t)\|_2^2+a\int_0^t\int_\Omega|u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\,dx\le C.$$

where C is a constant independent of m. Thus, we obtain the global solution.
Proof of Claim: Suppose (3.29) is not true. Thus, for each $m>\nu_0$, there exists $t_1\in[0,T_m[$ such that

$$\sqrt{l\|u_m(t_1)\|_{W_0}^2+(g\diamond u_m)(t_1)}\ge\lambda_1. \tag{3.30}$$

Here, we observe that, from (3.24) and the convergence $u_{0m}\to u^0$ in W_0 there exists ν_1 such that

$$l^{1/2}\|u_m(0)\|_{W_0}<\lambda_1\qquad\forall m>\nu_1.$$

Hence, by continuity there exists

$$t^*=\inf\{t\in[0,T_m[:\sqrt{l\|u_m(t)\|_{W_0}^2+(g\diamond u_m)(t)}\ge\lambda_1\},$$

such that

$$\sqrt{l\|u_m(t^*)\|_{W_0}^2+(g\diamond u_m)(t^*)}=\lambda_1. \tag{3.31}$$

By (3.23), we see that

$$\begin{aligned}E_m(t^*)&\ge h\left(\sqrt{l\|u_m(t^*)\|_{W_0}^2+(g\diamond u_m)(t^*)}\right)\\&\ge\frac14(l\|u(t^*)\|_{W_0}^2+(g\diamond u)(t^*))=\frac14\lambda_1^2\end{aligned} \tag{3.32}$$

which contradicts $E_m(t)\le E_m(0)<\frac{l}{4}\lambda_1^2$. Therefore our claim is true. The above estimates permit us to pass to the limit in the approximate equation.

To show the uniform decay of the solution we introduce the perturbed energy functional

$$F(t)=E(t)+\epsilon\Phi(t), \tag{3.33}$$

where ϵ is a positive constant which shall be determined later, and

$$\Phi(t)=\int_\Omega(|u|^2+\frac{a}{r(x)}|u|^{r(x)})\,dx. \tag{3.34}$$

It is straightforward to see that $F(t)$ and $E(t)$ are equivalent in the sense that there exist two positive constants β_1 and β_2 depending on ϵ such that for $t \geq 0$,

$$\beta_1 E(t) \leq F(t) \leq \beta_2 E(t). \tag{3.35}$$

By taking the time derivative of the function F defined in (3.33), using (3.19), and performing several integration by parts, we obtain

$$\begin{aligned} \frac{d}{dt}F(t) = &\frac{1}{2}(g' \diamond u)(t) - \frac{1}{2}g(t)\|u(t)\|_{W_0}^2 - \|u_t(t)\|_2^2 - a\int_\Omega |u(t)|^{r(x)-2}u_t^2(t)\,dx \\ &- \epsilon\|u(t)\|_{W_0}^{2\alpha} - \epsilon\|u(t)\|_{W_0}^2 + \epsilon\int_\Omega uf(u)\log|u(t)|\,dx \\ &+ \epsilon\int_0^t g(t-\tau)\langle u(\tau), u(t)\rangle_{W_0}\,d\tau. \end{aligned} \tag{3.36}$$

On the other hand, from (3.23) and the monotonicity of $E(t)$ we have $\|u(t)\|_{W_0} \leq 4E(0)/\sqrt{l}$, which implies

$$\begin{aligned} &\Big|\int_\Omega uf(u)\log|u|\Big| \\ &\leq \frac{1}{\varrho_\rho}\|u(t)\|_{\varrho_\rho+\rho}^{\varrho_\rho+\rho} + \frac{1}{\varrho_\sigma}\|u(t)\|_{\varrho_\sigma+\sigma}^{\varrho_\sigma+\sigma} \\ &\leq \frac{C_{*\rho}^{\varrho_\rho+\rho}}{\varrho_\rho}\|u(t)\|_{W_0}^{\varrho_\rho+\rho} + \frac{C_{*\sigma}^{\varrho_\sigma+\sigma}}{\varrho_\sigma}\|u(t)\|_{W_0}^{\varrho_\sigma+\sigma} \\ &\leq \Big[\frac{C_{*\rho}^{\varrho_\rho+\rho}}{\varrho_\rho l}\Big(\frac{4}{\sqrt{l}}E(0)\Big)^{\varrho_\rho+\rho-2} + \frac{C_{*\sigma}^{\varrho_\sigma+\sigma}}{\varrho_\sigma l}\Big(\frac{4}{\sqrt{l}}E(0)\Big)^{\varrho_\sigma+\sigma-2}\Big] l\|u(t)\|_{W_0}^2 \\ &\equiv \theta l\|u(t)\|_{W_0}^2. \end{aligned} \tag{3.37}$$

where

$$\frac{1}{c_{*\delta}} = \inf_{u\in W_0\setminus 0}\frac{\|u\|_{W_0}}{\|u\|_\delta},$$

with $\delta = \rho, \sigma$. From Young's inequality and the fact that $\int_0^t g(\tau)\,d\tau \leq \int_0^\infty g(\tau)\,d\tau = 1-l$, it follows that

$$\begin{aligned} &\int_0^t g(t-\tau)\langle u(\tau), u(t)\rangle_{W_0}\,d\tau \\ &\leq \frac{1}{2}\|u(t)\|_{W_0}^2 + \frac{1}{2}\Big\{\int_0^t g(t-\tau)\left(\|u(\tau)-u(t)\|_{W_0} + \|u(t)\|_{W_0}\right)\,d\tau\Big\}^2 \\ &\leq \frac{1}{2}\|u(t)\|_{W_0}^2 + \frac{1}{2}(1+\eta)\Big(\int_0^t g(t-\tau)\|u(t)\|_{W_0}\,d\tau\Big)^2 \\ &\quad + \frac{1}{2}(1+\frac{1}{\eta})\Big(\int_0^t g(t-\tau)\|u(\tau)-u(t)\|_{W_0}\,d\tau\Big)^2 \\ &\leq \frac{1}{2}\|u(t)\|_{W_0}^2 + \frac{1}{2}(1+\eta)(1-l)^2\|u(t)\|_{W_0}^2 + \frac{1}{2}(1+\frac{1}{\eta})(1-l)(g\diamond u)(t). \end{aligned} \tag{3.38}$$

for any $\eta > 0$. Now, letting $\eta = \frac{l}{1-l} > 0$, (3.38) yields

$$\int_0^t g(t-\tau)\langle u(\tau), u(t)\rangle_{W_0}\,d\tau \leq \frac{2-l}{2}\|u(t)\|_{W_0}^2 + \frac{1-l}{2l}(g\diamond u)(t). \tag{3.39}$$

Substituting (3.39) into (3.36), we obtain

$$\begin{aligned}\frac{d}{dt}F(t) \le -\frac{1}{2}\Big(\xi_0 - \epsilon\frac{1-l}{l}\Big)(g\diamond u)(t) - \epsilon\|u(t)\|_{W_0}^{2\alpha} - \frac{\epsilon l}{2}\|u(t)\|_{W_0}^2 \\ + \epsilon\int_\Omega uf(u)\log|u(t)|\,dx.\end{aligned} \tag{3.40}$$

Using the definition of $E(t)$ and (3.37), for any positive constant M, we have

$$\begin{aligned}\frac{d}{dt}F(t) &\le -M\epsilon E(t) + \epsilon\big(\frac{M}{2\alpha} - 1\big)\|u(t)\|_{W_0}^{2\alpha} \\ &\quad + \epsilon\Big\{\big[\frac{1}{2} + \frac{c_{*\rho}^{\rho}}{\rho^2}(\frac{4E(0)}{\sqrt{l}})^{\rho-2} + \frac{c_{*\sigma}^{\sigma}}{\sigma^2}(\frac{4E(0)}{\sqrt{l}})^{\sigma-2}\big]M \\ &\quad + (M+2)\theta l - l\Big\}\|u(t)\|_{W_0}^2 + \frac{1}{2}\big[\epsilon(\frac{1-l}{l} + \frac{M}{2}) - \xi_0\big](g\diamond u)(t).\end{aligned} \tag{3.41}$$

At this point, we choose $1 > M > 0$ and $E(0)$ small sufficiently such that $\frac{M}{2\alpha} - 1 < 0$ and

$$\Big[\frac{1}{2} + \frac{c_{*\rho}^{\rho}}{\rho^2}(\frac{4E(0)}{\sqrt{l}})^{\rho-2} + \frac{c_{*\sigma}^{\sigma}}{\sigma^2}(\frac{4E(0)}{\sqrt{l}})^{\sigma-2}\Big]M + (M+2)\theta l - l < 0.$$

After M is fixed, we choose ϵ small enough such that

$$\epsilon\big(\frac{1-l}{l} + \frac{M}{2}\big) - \xi_0 < 0.$$

Inequality (3.41) becomes $\frac{d}{dt}F(t) \le -M\epsilon E(t)$. Then by (3.35), we have

$$\frac{d}{dt}F(t) \le -M\beta_2\epsilon F(t).$$

So $F(t) \le Ce^{-Kt}$ where $K = M\beta_2\epsilon > 0$. Consequently, by using (3.35) once again, we conclude the result. Hence, the proof of Theorem 3.3 is complete. □

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