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INGHAM TYPE APPROACH FOR UNIFORM OBSERVABILITY INEQUALITY OF THE SEMI-DISCRETE COUPLED WAVE EQUATIONS

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Abstract. This article concerns an observability inequality for a system of coupled wave equations for the continuous models as well as for the space semi-discrete finite difference approximations. For finite difference and standard finite elements methods on uniform numerical meshes it is known that a numerical pathology produces a blow-up of the constant on the observability inequality as the mesh-size h tends to zero. We identify this numerical anomaly for coupled wave equations and we prove that there exists a uniform observability inequality in a subspace of solutions generated by low frequencies. We use the Ingham type approach for getting a uniform boundary observability.

1. Introduction

This article concerns an observability inequality problem for $1-d$ coupled wave equations for the continuous model as well as for their numerical counterpart for the space semi-discrete finite difference approximation. The goal of the present study is to analyze the semi-discrete counterpart in finite difference of an observability inequality. It is well recognized that observability inequalities are relevant for control and stabilization theories as well as for inverse problems. In that direction, the understanding of these questions in finite dimensional is very important to theoretical and numerical analysis.

In the literature, we found several contributions dealing with numerical questions on uniform observability and its consequence to the control and stabilization of wave equations. However, there exist only a few works dealing with analogue questions for coupled systems of wave equations and numerical counterpart. Almeida Júnior et al [1] first contributed in that direction for uniform boundary observability. See also Akri and Maniar [2, 3] and Xu [17]. To fix our ideas we address the problem of the well known observability inequality in one-dimensional setting. It is well-known that for the wave equation

$$u_{tt} - u_{xx} = 0, \quad \text{in } (0,L)\times(0,T), \tag{1.1}$$

$$u(0,t) = u(L,t) = 0, \quad \forall t \geq 0, \tag{1.2}$$

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$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \forall x \in (0,L), \tag{1.3}$$

the total energy of the solution is estimated uniformly by the energy concentrated near the endpoint $x = L$. More precisely, for any $T > 2L$ there exists a positive constant $C(T)$ satisfying

$$E(0) \leq C(T) \int_0^T \big|u_x(L,t)\big|^2 dt, \tag{1.4}$$

for every finite energy solution of (1.1)–(1.3) where the energy is

$$E(t) := \frac{1}{2}\int_0^L u_t^2\,dx + \frac{1}{2}\int_0^L u_x^2\,dx. \tag{1.5}$$

Estimate (1.4) is known as boundary observability (observability inequality/inverse inequality) and the best constant C is the so-called observability constant. We refer the readers to Lions [12] and Komornik [11] for an analysis of the equivalence between controllability and observability through the Hilbert Uniqueness Method (HUM). On the other hand, semi-discrete schemes generate high frequency numerical spurious oscillations because standard discrete approximations of the wave equations are, in general, non-uniformly observable. These spurious oscillations weakly converge to zero as $h \to 0$ and this fact is perfectly compatible with the convergence property. But, a uniform constant $C(T)$ for estimate (1.4) is needed and this is not the case of elementary numerical schemes such as finite difference and standard finite elements. A numerical evidence to the lack of numerical observability to boundary observability problem was first observed by Glowinski et al [6, 7, 8], in connection with the exact boundary controllability of the wave equation and the numerical implementation of the so-called HUM method. The non-uniform observability inequality to semi-discrete versions of (1.4) was first solved by Infante and Zuazua [9]. They noticed that the problem to estimate the total numerical energy in terms of the numerical energy concentrated at the boundary is not uniform as $h \to 0$ for the discrete dynamic equation

$$u_j''(t) - \frac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{h^2} = 0, \quad j = 1,2,\dots,J,\ 0 < t < T, \tag{1.6}$$

$$u_0(t) = u_{J+1}(t) = 0, \quad 0 < t < T, \tag{1.7}$$

$$u_j(0) = u_j^0, \quad u_j'(0) = u_j^1, \quad j = 0,1,2,\dots,J+1. \tag{1.8}$$

The total energy of (1.6)–(1.8) is conserved, i.e.,

$$E_h(t) := \frac{h}{2}\sum_{j=0}^{J}\Big[|u_j'(t)|^2 + \big|\frac{u_{j+1}(t) - u_j(t)}{h}\big|^2\Big] = E_h(0), \quad \forall 0 < t < T. \tag{1.9}$$

In [9] there is an analysis of a discrete version of (1.4), i.e.,

$$E_h(0) \leq C(T) \int_0^T |u_J(t)/h|^2 dt, \tag{1.10}$$

with $u_{J+1}(t) = 0$ for $0 < t < T$. The problem is to know if the positive constant $C(T)$ blows-up as $h \to 0$. According to the result given in [9, Theorem 1.1],

$$\sup_{u_j \text{ solution of (1.6)-(1.8)}} \frac{E_h(0)}{\int_0^T |\frac{u_J(t)}{h}|^2 dt} \to \infty \quad \text{as } h \to 0. \tag{1.11}$$

Otherwise, there exists a positive counterpart to (1.11) in a subspace of solutions generated by low frequencies. See [9, Theorem 1.2] for similar results for the standard finite element methods. In both cases, the authors used standard numerical schemes on uniform meshes. Since the pioneering work by Infante and Zuazua [9], a large number of papers and numerical techniques have been introduced to deal with the uniform observability and questions related to control and stabilization theories. We refer the readers to the surveys [5, 18] for the developments in numerical analysis dealing with uniform observability on uniform meshes. See also Ervedoza et al [4] who opened new perspectives in the uniform observability problem by taking non-uniform numerical meshes for finite difference and standard finite element approximations of the 1D wave equation.

This article concerns the theoretical numerical analysis of an observability inequality concerning a coupled system of wave equations. We use a particular scheme in finite difference semi-discretization on a uniform mesh. Let us consider the 1D coupled wave equations

$$u_{tt} - u_{xx} + \alpha v = 0, \quad (0, L) \times (0, T), \tag{1.12}$$

$$v_{tt} - v_{xx} + \alpha u = 0, \quad (0, L) \times (0, T), \tag{1.13}$$

$$u(x,t) = v(x,t) = 0, \quad x = 0, L,\ 0 < t < T, \tag{1.14}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \forall x \in (0, L), \tag{1.15}$$

$$v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \quad \forall x \in (0, L), \tag{1.16}$$

where the positive coupling term α is proportional to the displacements $u(x,t)$ and $v(x,t)$. Here, t and x are the time and space variables, respectively. The functions $u(x,t)$ and $v(x,t)$ are the displacements of two vibrating strings, measured from their equilibrium positions and the coupling is given by coupling terms αv and αu. This system is motivated by analogous problem in ordinary differential equations for coupled harmonic oscillators (see [13]) and it is well-posed in the energy space $\mathcal{H} := \big(H_0^1(0,L) \times L^2(0,L)\big)^2$. That is to say, denoting $U = (u, u_t, v, v_t)$ we have that for any $U_0 \in \big(H_0^1(0,L) \times L^2(0,L)\big)^2$ there exists a unique solution U in $C^0\big([0,T]; D(\mathcal{A})\big) \cap C^1\big([0,T]; \mathcal{H}\big)$, where $D(\mathcal{A}) := \big(H_0^1(0,L) \cap H^2(0,L) \times H_0^1(0,L)\big)^2$.

The energy of solutions to system (1.12)–(1.16) is

$$\mathcal{E}(t) := \frac{1}{2}\int_0^L u_t^2\,dx + \frac{1}{2}\int_0^L u_x^2\,dx + \frac{1}{2}\int_0^L v_t^2\,dx + \frac{1}{2}\int_0^L v_x^2\,dx + \alpha\int_0^L uv\,dx. \tag{1.17}$$

In this work, we obtain an observability inequality of system (1.12)–(1.16) and we study such observability in the light of a semi-discrete scheme in finite difference.

The rest of this article is organized as follows. In section 2, we show a necessary condition to obtain the positivity of the energy and we build an observability inequality. In section 3, we introduce a semi-discrete numerical scheme in finite difference and we show a uniform observability inequality no filtering. In section 4, we prove results on uniform observability for filtered solutions using Ingham inequalities and we improve the results obtained in section 3.

2. Energy properties at continuous level

In this section, we establish some properties concerning the system (1.12)–(1.16). We focus on the observability inequality built from multipliers. First, we develop the Fourier series of the system and after that we show two important properties concerning the energy in (1.17): its positivity and its conservation law.

2.1. **Fourier expansion of the solutions.** Here, we show that the solutions of (1.12)–(1.16) admit a Fourier expansion on the specific basis of eigenvectors. To do this, we decoupled the system into two systems, namely: for $\psi := u - v$ we obtain from system (1.12)–(1.16) the system given by

$$\psi_{tt} - \psi_{xx} - \alpha\psi = 0, \quad \text{in } (0,L)\times(0,T), \tag{2.1}$$

$$\psi(0,t) = \psi(L,t) = 0, \quad 0 < t < T, \tag{2.2}$$

$$\psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \quad \forall x \in (0,L), \tag{2.3}$$

as well as for the system

$$\phi_{tt} - \phi_{xx} + \alpha\phi = 0, \quad \text{in } (0,L)\times(0,T), \tag{2.4}$$

$$\phi(0,t) = \phi(L,t) = 0, \quad 0 < t < T, \tag{2.5}$$

$$\phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \phi_1(x), \quad \forall x \in (0,L), \tag{2.6}$$

where $\phi := u + v$. Both systems are conservative and their energies are

$$G(t) := \frac{1}{2}\int_0^L \psi_t^2\,dx + \frac{1}{2}\int_0^L \psi_x^2\,dx - \frac{\alpha}{2}\int_0^L \psi^2\,dx, \tag{2.7}$$

$$F(t) := \frac{1}{2}\int_0^L \phi_t^2\,dx + \frac{1}{2}\int_0^L \phi_x^2\,dx + \frac{\alpha}{2}\int_0^L \phi^2\,dx. \tag{2.8}$$

The results obtained for the energy of the decoupled systems are extended to the coupled system by taking $\mathcal{E}(t) := (F(t) + G(t))/2$ for $t \geq 0$.

Proposition 2.1. *Assume that $\alpha \leq \pi^2/L^2$. Then, the solutions of the system* (2.1)–(2.3) *admit the Fourier expansion, on the basis of eigenvectors,*

$$\psi = \sum_{k\geq 1}\Big[a_k \sin\Big(\sqrt{\nu_k^-}\,t\Big) + b_k \cos\Big(\sqrt{\nu_k^-}\,\ t\Big)\Big]\sin\Big(\frac{k\pi x}{L}\Big), \tag{2.9}$$

where a_k, b_k are the Fourier coefficients, and $\nu_k^- = \left(\frac{k\pi}{L}\right)^2 - \alpha$ are the eigenvalues for $k \geq 1$.

Proof. The solutions of (2.1)–(2.3) can be written as $\psi(x,t) = f(x)T(t)$. By substituting this decomposition into (2.1) we obtain

$$\frac{T''(t) - \alpha T(t)}{T(t)} = \frac{f''(x)}{f(x)} = -\nu^2, \ \forall t \geq 0. \tag{2.10}$$

Thus, we obtain the eigenvalue problem

$$f'' + \nu^2 f = 0, \tag{2.11}$$

$$f(0) = f(L) = 0. \tag{2.12}$$

To obtain non-trivial solutions, we must take $\nu > 0$. Under this assumption, it is immediate that the solution of (2.11)–(2.12) is

$$f(x) = c_1 \sin(\nu x), \tag{2.13}$$

where c_1 is constant. By taking the solution at $x = L$ it results that $\sin(\nu L) = 0$, from where we obtain $\nu = \nu_k = k\pi/L$, $k \geq 1$. On the other hand, from the equation $T''(t) + (\nu^2 - \alpha)T(t) = 0$ we obtain

$$T(t) = ae^{i\sqrt{\nu^2-\alpha}\,t} + be^{-i\sqrt{\nu^2-\alpha}\,t}. \tag{2.14}$$

The solutions are valid for $\nu^2 > \alpha$. Hence, for $\alpha \leq \pi^2/L^2 \leq k^2\pi^2/L^2$ for all $k \geq 1$ and using the linear combination of solutions, we obtain

$$T_k(t) = a_k \sin\Big(\sqrt{\nu_k^-}\,t\Big) + b_k \cos\Big(\sqrt{\nu_k^-}\,t\Big), \quad \forall t \geq 0, \tag{2.15}$$

where $\nu_k^- := \big(\frac{k\pi}{L}\big)^2 - \alpha$, and a_k, b_k are the Fourier coefficients for $k \geq 1$. The proof is complete. $\square$

One important question is the positivity of the energy of solutions for both systems (2.1)–(2.3) and (1.12)–(1.16). By taking the initial data

$$\psi^0(x) = \sum_{k\geq 1} b_k \sin\Big(\frac{k\pi x}{L}\Big), \quad \psi^1(x) = \sum_{k\geq 1} a_k\sqrt{\nu_k^-}\sin\Big(\frac{k\pi x}{L}\Big),$$

we obtain that

$$G(0) = \frac{L}{4}\sum_{k\geq 1} b_k^2\Big[\Big(\frac{k\pi}{L}\Big)^2 - \alpha\Big] + \frac{L}{4}\sum_{k\geq 1} a_k^2\nu_k^- \geq 0, \tag{2.16}$$

if $\alpha \leq \pi^2/L^2$. Analogously, we have the following two results.

Proposition 2.2. *The solutions of* (2.4)–(2.6) *admit the Fourier expansion, on the basis of eigenvectors,*

$$\phi = \sum_{k\geq 1}\Big[c_k \sin\Big(\sqrt{\nu_k^+}\,t\Big) + d_k\cos\Big(\sqrt{\nu_k^+}\,t\Big)\Big]\sin\Big(\frac{k\pi x}{L}\Big), \tag{2.17}$$

where c_k, d_k are the Fourier coefficients and $\nu_k^+ = \big(\frac{k\pi}{L}\big)^2 + \alpha$ are the eigenvalues for $k \geq 1$.

Proposition 2.3. *Assume that $\alpha \leq \pi^2/L^2$. Then the solutions of* (1.12)–(1.16) *admit the Fourier expansion, on the basis of eigenvectors,*

$$\begin{aligned} u = \frac{1}{2}\sum_{k\geq 1}\Big[&a_k \sin\Big(\sqrt{\nu_k^-}\,t\Big) + b_k\cos\Big(\sqrt{\nu_k^-}\,t\Big) + c_k\sin\Big(\sqrt{\nu_k^+}\,t\Big) \\ &+ d_k\cos\Big(\sqrt{\nu_k^+}\,t\Big)\Big]\sin\Big(\frac{k\pi x}{L}\Big), \end{aligned} \tag{2.18}$$

$$\begin{aligned} v = \frac{1}{2}\sum_{k\geq 1}\Big[&c_k \sin\Big(\sqrt{\nu_k^+}\,t\Big) + d_k\cos\Big(\sqrt{\nu_k^+}\,t\Big) - a_k\sin\Big(\sqrt{\nu_k^-}\,t\Big) \\ &- b_k\cos\Big(\sqrt{\nu_k^-}\,t\Big)\Big]\sin\Big(\frac{k\pi x}{L}\Big), \end{aligned} \tag{2.19}$$

where a_k, b_k, c_k, d_k are the Fourier coefficients and $\nu_k^{\pm} = \big(\frac{k\pi}{L}\big)^2 \pm \alpha$ are the eigenvalues.

The proof of the two above results follows from Propositions 2.1 and 2.2. By taking the initial data

$$u^0(x) = \frac{1}{2}\sum_{k\geq 1}(b_k + d_k)\sin\Big(\frac{k\pi x}{L}\Big),$$

$$
\begin{aligned}
u^1(x) &= \frac{1}{2}\sum_{k\geq 1}\Big(a_k\sqrt{\nu_k^-}+c_k\sqrt{\nu_k^+}\Big)\sin\Big(\frac{k\pi x}{L}\Big),\\
v^0(x) &= \frac{1}{2}\sum_{k\geq 1}(d_k-b_k)\sin\Big(\frac{k\pi x}{L}\Big),\\
v^1(x) &= \frac{1}{2}\sum_{k\geq 1}\Big(c_k\sqrt{\nu_k^+}-a_k\sqrt{\nu_k^-}\Big)\sin\Big(\frac{k\pi x}{L}\Big),
\end{aligned}
$$

and after some calculations we obtain the positivity of $\mathcal{E}(t)$, i.e.,

$$
\begin{aligned}
\mathcal{E}(0) = \frac{L}{4}\sum_{k\geq 1} d_k^2\Big[\Big(\frac{k\pi}{L}\Big)^2+\alpha\Big] + \frac{L}{4}\sum_{k\geq 1} b_k^2\Big[\Big(\frac{k\pi}{L}\Big)^2-\alpha\Big]\\
+\frac{L}{4}\sum_{k\geq 1}\Big(c_k^2\nu_k^+ + a_k^2\nu_k^-\Big)\geq 0,
\end{aligned}
$$

if $\alpha\leq \pi^2/L^2$.

2.2. **Observability inequality for the coupled system.** One of the main physical properties concerning the hyperbolic systems of wave propagations is the positivity of the energy of solutions. Before getting the observability inequality for system (1.12)–(1.16), we show that it's energy is positive and it obeys the energy conservation law.

Theorem 2.4. *Let $U=(u,u_t,v,v_t)$ be the solution of* (1.12)–(1.16). *Then, for all $\alpha\leq\pi^2/L^2$, holds*

$$
\mathcal{E}(t)\geq \frac{\pi^2-\alpha L^2}{2\pi^2}\Big[\int_0^L u_x^2\,dx+\int_0^L v_x^2\,dx\Big]\geq 0,\quad \forall t\in[0,T], \tag{2.20}
$$

where $\mathcal{E}(t)$ is the energy defined (1.17).

Proof. We multiply formally the Eqs. (1.12) and (1.13) by u_t and v_t respectively, and then we add the two resulting equations considering the boundary conditions (1.14) to obtain

$$
\frac{d}{dt}\Big[\frac{1}{2}\int_0^L u_t^2\,dx+\frac{1}{2}\int_0^L u_x^2\,dx+\frac{1}{2}\int_0^L v_t^2\,dx+\frac{1}{2}\int_0^L v_x^2\,dx+\alpha\int_0^L uv\,dx\Big]=0, \tag{2.21}
$$

for all $t\in[0,T]$. Then, the energy conservation property is assured, i.e.,

$$
\mathcal{E}(t)=\mathcal{E}(0),\quad \forall t\in[0,T], \tag{2.22}
$$

where $\mathcal{E}(t)$ is given by (1.17). On the other hand, taking into account that $2uv \geq -u^2 - v^2$ and using Poincaré's inequality [16] we have

$$
\begin{aligned}
\mathcal{E}(t) &\geq \frac{1}{2}\Big[\int_0^L u_t^2\,dx + \int_0^L u_x^2\,dx + \int_0^L v_t^2\,dx + \int_0^L v_x^2\,dx \\
&\quad - \alpha\int_0^L u^2\,dx - \alpha\int_0^L v^2\,dx\Big] \\
&\geq \frac{1}{2}\Big[\int_0^L u_t^2\,dx + \int_0^L u_x^2\,dx + \int_0^L v_t^2\,dx + \int_0^L v_x^2\,dx \\
&\quad - \alpha\frac{L^2}{\pi^2}\int_0^L u_x^2\,dx - \alpha\frac{L^2}{\pi^2}\int_0^L v_x^2\,dx\Big] \\
&= \frac{1}{2}\Big[\int_0^L u_t^2\,dx + \Big(1-\alpha\frac{L^2}{\pi^2}\Big)\int_0^L u_x^2\,dx + \int_0^L v_t^2\,dx \\
&\quad + \Big(1-\alpha\frac{L^2}{\pi^2}\Big)\int_0^L v_x^2\,dx\Big],
\end{aligned}
\tag{2.23}
$$

and since $\alpha \leq \pi^2/L^2$, it follows that

$$
\mathcal{E}(t) \geq \frac{\pi^2 - \alpha L^2}{2\pi^2}\Big[\int_0^L u_x^2\,dx + \int_0^L v_x^2\,dx\Big], \tag{2.24}
$$

assuring the positivity of $\mathcal{E}(t)$. $\square$

The next theorem gives our result on the observability inequality.

Theorem 2.5. *Let $U = (u, u_t, v, v_t)$ be the solution of the system* (1.12)–(1.16). *Then for all $\alpha \leq \pi^2/L^2$ there exists $T_0 > 0$ such that for all $T > T_0$ there exists $C(T,\alpha) > 0$ for which*

$$
\begin{aligned}
\mathcal{E}(0) &\leq C(T,\alpha)\Big[\alpha\int_0^T\int_0^L (u^2+v^2)\,dx\,dt + \frac{L}{2}\int_0^T u_x^2(L,t)\,dt \\
&\quad + \frac{L}{2}\int_0^T v_x^2(L,t)\,dt\Big],
\end{aligned}
\tag{2.25}
$$

where $\mathcal{E}(t)$ is the energy given by (1.17).

Proof. Multiplying the Eq. (1.12) by xu_x we have

$$
\int_0^T\int_0^L \big(u_{tt} - u_{xx} + \alpha v\big)xu_x\,dx\,dt = 0. \tag{2.26}
$$

From the boundary conditions in (1.14) we have

$$
\int_0^T\int_0^L u_{tt}xu_x\,dx\,dt = \Big[\int_0^L u_tu_xx\,dx\Big]_0^T + \frac{1}{2}\int_0^T\int_0^L u_t^2\,dx\,dt\,, \tag{2.27}
$$

$$
-\int_0^T\int_0^L u_{xx}u_xx\,dx\,dt = \frac{1}{2}\int_0^T\int_0^L u_x^2\,dx\,dt - \frac{L}{2}\int_0^T u_x^2(L,t)\,dt. \tag{2.28}
$$

Combining (2.26), (2.27) and (2.28), one has

$$
\begin{aligned}
&X_u(t)\big|_0^T + \frac{1}{2}\int_0^T\int_0^L u_t^2\,dx\,dt + \frac{1}{2}\int_0^T\int_0^L u_x^2\,dx\,dt + \alpha\int_0^T\int_0^L vxu_x\,dx\,dt\\
&= \frac{L}{2}\int_0^T u_x^2(L,t)\,dt,
\end{aligned}
\tag{2.29}
$$

where $X_u(t) = \int_0^L xu_xu_t\,dx$. Analogously, multiplying the Eq. (1.13) by xv_x we obtain

$$
\begin{aligned}
&X_v(t)\big|_0^T + \frac{1}{2}\int_0^T\int_0^L v_t^2\,dx\,dt + \frac{1}{2}\int_0^T\int_0^L v_x^2\,dx\,dt + \alpha\int_0^T\int_0^L uxv_x\,dx\,dt\\
&= \frac{L}{2}\int_0^T v_x^2(L,t)\,dt,
\end{aligned}
\tag{2.30}
$$

where $X_v(t) = \int_0^L xv_xv_t\,dx$. Adding (2.29) and (2.30) we obtain

$$
\begin{aligned}
&\big[X_u(t) + X_v(t)\big]\big|_0^T + \int_0^T\Big[\frac{1}{2}\int_0^L u_t^2\,dx + \frac{1}{2}\int_0^L v_t^2\,dx + \frac{1}{2}\int_0^L v_x^2\,dx\\
&+ \frac{1}{2}\int_0^L v_x^2\,dx\Big]dt + \alpha\int_0^T\int_0^L (vu_x + uv_x)x\,dx\,dt\\
&= \frac{L}{2}\int_0^T u_x^2(L,t)\,dt + \frac{L}{2}\int_0^T v_x^2(L,t)\,dt.
\end{aligned}
\tag{2.31}
$$

Moreover,

$$
\alpha\int_0^T\int_0^L (vu_x + uv_x)x\,dx\,dt = \alpha\int_0^T\int_0^L x\frac{d}{dx}(uv)\,dx\,dt = -\alpha\int_0^T\int_0^L uv\,dx\,dt,
$$

from where we arrive at

$$
\begin{aligned}
&[X_u(t) + X_v(t)]\big|_0^T + \int_0^T \mathcal{E}(t)dt\\
&= 2\alpha\int_0^T\int_0^L uv\,dx\,dt + \frac{L}{2}\int_0^T u_x^2(L,t)\,dt + \frac{L}{2}\int_0^T v_x^2(L,t)\,dt.
\end{aligned}
\tag{2.32}
$$

Using Young's inequality we have

$$
|X_u(t) + X_v(t)| \le \frac{L}{2}\int_0^L u_t^2\,dx + \frac{L}{2}\int_0^L u_x^2\,dx + \frac{L}{2}\int_0^L v_t^2\,dx + \frac{L}{2}\int_0^L v_x^2\,dx\,, \tag{2.33}
$$

and taking (2.23) into account we have

$$
\begin{aligned}
&\frac{L}{2}\Big[\int_0^L u_t^2\,dx + \int_0^L u_x^2\,dx + \int_0^L v_t^2\,dx + \int_0^L v_x^2\,dx\Big]\\
&\le \alpha L\int_0^L u^2\,dx + \alpha L\int_0^L v^2\,dx + L\mathcal{E}(t)\\
&\le \alpha\frac{L^3}{\pi^2}\int_0^L u_x^2\,dx + \alpha\frac{L^3}{\pi^2}\int_0^L v_x^2\,dx + L\mathcal{E}(t)\\
&= \alpha\frac{L^3}{\pi^2}\Big[\int_0^L u_x^2\,dx + \int_0^L v_x^2\,dx\Big] + L\mathcal{E}(t)\,.
\end{aligned}
\tag{2.34}
$$

From Theorem (2.4) we conclude that

$$\begin{aligned}
&\frac{L}{2}\Big[\int_0^L u_t^2\,dx+\int_0^L u_x^2\,dx+\int_0^L v_t^2\,dx+\int_0^L v_x^2\,dx\Big]\\
&\le \alpha\frac{L^3}{\pi^2}\Big[\int_0^L u_x^2\,dx+\int_0^L v_x^2\,dx\Big]+L\mathcal{E}(t)\\
&\le \alpha\frac{L^3}{2\pi^2}\frac{2\pi^2}{\pi^2-\alpha L^2}\mathcal{E}(t)+L\mathcal{E}(t)\\
&=\frac{L\pi^2}{\pi^2-\alpha L^2}\mathcal{E}(t),
\end{aligned}\tag{2.35}$$

from where we obtain

$$|X_u(t)+X_v(t)|\le L\frac{\pi^2}{\pi^2-\alpha L^2}\mathcal{E}(t).\tag{2.36}$$

On the other hand, from energy conservation law (2.22), the inequality (2.32) lead us to an observability inequality. Indeed, for $T_0=2L\frac{\pi^2}{\pi^2-\alpha L^2}>0$ we have

$$\mathcal{E}(0)\le C\Big[2\alpha\int_0^T\int_0^L uv\,dx\,dt+\frac{L}{2}\int_0^T u_x^2(L,t)\,dt+\frac{L}{2}\int_0^T v_x^2(L,t)\,dt\Big],\tag{2.37}$$

where $C=C(T,\alpha):=\frac{\pi^2-\alpha L^2}{T(\pi^2-\alpha L^2)-2L\pi^2}$. Finally, using the Young's inequality we obtain the required result. □

Note that (2.25) uses observations on the boundary and observations distributed on the whole space domain. The observability term with α which is integrated on the whole domain is normally removable, but the literature is too brief. We refer the reader to Tebou [15] for a study of several observability estimates for a system of two coupled non-conservative wave equations, and for possible extensions to other similar models with fewer controls. In any case, one can use the compactness-uniqueness argument such performed by Zuazua [19] to prove that the observation distributed on the whole domain can be absorbed. In that direction, it is possible to obtain the observability inequality given by

$$\mathcal{E}(0)\le \hat{C}(T)\frac{L}{2}\int_0^T\big[u_x^2+v_x^2\big](L,t)\,dt,\tag{2.38}$$

where $\hat{C}(T)$ depends also on α.

2.3. **Observability inequalities for the uncoupled systems.** In this section, we obtain observability inequalities for the coupled systems and after that we recover the observability inequality (2.25).

Theorem 2.6. *Let (ϕ,ϕ_t) be the solution of system (2.4)–(2.6). Then, for all $T>2L$ there exists $C(T)>0$ such that*

$$F(0)\le C(T)\Big[\alpha\int_0^T\int_0^L\phi^2\,dx\,dt+\frac{L}{2}\int_0^T\phi_x^2(L,t)\,dt\Big],\tag{2.39}$$

where $F(t)$ is the energy given by

$$F(t):=\frac{1}{2}\int_0^L\phi_x^2\,dx+\frac{1}{2}\int_0^L\phi_x^2\,dx+\frac{\alpha}{2}\int_0^L\phi^2\,dx.\tag{2.40}$$

Proof. Proceeding as in Theorem (2.5), by using multipliers of the type $x\phi_x$, we obtain

$$
\begin{aligned}
&X_\phi(t)\Big|_0^T + \frac{1}{2}\int_0^T\int_0^L \phi_t^2\,dx\,dt + \frac{1}{2}\int_0^T\int_0^L \phi_x^2\,dx\,dt - \frac{\alpha}{2}\int_0^T\int_0^L \phi^2\,dx\,dt\\
&= \frac{L}{2}\int_0^T \phi_x^2(L,t)\,dt,
\end{aligned}
\tag{2.41}
$$

where $X_\phi(t) = \int_0^L x\phi_x\phi_t\,dx$ and from where results that

$$
X_\phi(t)\Big|_0^T + \int_0^T F(t)dt = \alpha\int_0^T\int_0^L \phi^2\,dx\,dt + \frac{L}{2}\int_0^T \phi_x^2(L,t)\,dt. \tag{2.42}
$$

It is immediate that $X_\phi(t)\big|_0^T \geq -2LF(t)$. Moreover, having in mind the energy conservation law for $F(t)$, the inequality (2.42) lead us to the observability inequality for $T > 2L$, i.e.,

$$
F(0) \leq C(T)\Big[\alpha\int_0^T\int_0^L \phi^2\,dx\,dt + \frac{L}{2}\int_0^T \phi_x^2(L,t)\,dt\Big], \tag{2.43}
$$

where $C(T) = 1/(T-2L)$. The proof is complete. □

Theorem 2.7. *Let (ψ,ψ_t) be the solution of the system (2.1)–(2.3). Then, for all $\alpha \leq \pi^2/L^2$ there exists $T_0 > 0$ such that for all $T > T_0$ there exists $C(T,\alpha) > 0$ such that*

$$
G(0) \leq C(T,\alpha)\frac{L}{2}\int_0^T \psi_x^2(L,t)\,dt, \tag{2.44}
$$

where

$$
C(T,\alpha) = \frac{\pi^2 - \alpha L^2}{T(\pi^2-\alpha L^2) - 2L\pi^2} > 0,
$$

and $G(t)$ is the energy given by

$$
G(t) := \frac{1}{2}\int_0^L \psi_x^2\,dx + \frac{1}{2}\int_0^L \psi_x^2\,dx - \frac{\alpha}{2}\int_0^L \psi^2\,dx. \tag{2.45}
$$

Proof. It is immediate that

$$
\begin{aligned}
&X_\psi(t)\Big|_0^T + \frac{1}{2}\int_0^T\int_0^L \psi_t^2\,dx\,dt + \frac{1}{2}\int_0^T\int_0^L \psi_x^2\,dx\,dt + \frac{\alpha}{2}\int_0^T\int_0^L \psi^2\,dx\,dt\\
&= \frac{L}{2}\int_0^T \psi_x^2(L,t)\,dt,
\end{aligned}
\tag{2.46}
$$

where $X_\psi(t) = \int_0^L x\psi_x\psi_t\,dx$ and taking into account the energy defined by (2.45), we obtain

$$
X_\psi(t)\Big|_0^T + \int_0^T G(t)dt = -\alpha\int_0^T\int_0^L \psi^2\,dx\,dt + \frac{L}{2}\int_0^T \psi_x^2(L,t)\,dt. \tag{2.47}
$$

Moreover, using Poincaré's inequality [16], we have

$$
\begin{aligned}
G(t) &\geq \frac{1}{2}\Big[\int_0^L \psi_t^2\,dx + \int_0^L \psi_x^2\,dx - \alpha\frac{L^2}{\pi^2}\int_0^L \psi_x^2\,dx\Big]\\
&= \frac{1}{2}\int_0^L \Big[\psi_t^2 + \Big(1-\alpha\frac{L^2}{\pi^2}\Big)\psi_x^2\Big]\,dx\,.
\end{aligned}
\tag{2.48}
$$

It follows for $\alpha \le \pi^2/L^2$ that

$$G(t) \ge \frac{\pi^2 - \alpha L^2}{2\pi^2} \int_0^L \psi_x^2\, dx, \tag{2.49}$$

implying the positivity of $G(t)$. Now we estimate $X_\psi(t)$ as follows,

$$\begin{aligned}
|X_\psi(t)| &\le \frac{L}{2}\int_0^L \psi_t^2\, dx + \frac{L}{2}\int_0^L \psi_x^2\, dx \\
&\le LG(t) + \alpha \frac{L}{2}\int_0^L \psi^2\, dx \\
&\le LG(t) + \alpha \frac{L^3}{2\pi^2}\int_0^L \psi_x^2\, dx\,.
\end{aligned}$$

Using (2.49), we obtain

$$|X_\psi(t)| \le LG(t) + \alpha \frac{L^3}{2\pi^2}\int_0^L \psi_x^2\, dx \le L\frac{\pi^2}{\pi^2 - \alpha L^2} G(t). \tag{2.50}$$

Finally, having in mind the energy conservation law of $G(t)$, inequality (2.47) lead us to the observability inequality for $T > T_0$ where $T_0 = 2L\frac{\pi^2}{\pi^2 - \alpha L^2}$, i.e.,

$$\begin{aligned}
G(0) &\le \frac{\pi^2 - \alpha L^2}{T(\pi^2 - \alpha L^2) - 2L\pi^2}\Big[-\alpha \int_0^T\int_0^L \psi^2\, dx\, dt + \frac{L}{2}\int_0^T \psi_x^2(L,t)\, dt\Big] \\
&\le \frac{\pi^2 - \alpha L^2}{T(\pi^2 - \alpha L^2) - 2L\pi^2}\frac{L}{2}\int_0^T \psi_x^2(L,t)\, dt,
\end{aligned}$$

which completes the proof. ◻

From Theorems (2.6) and (2.7) we have

$$\begin{aligned}
F(0) + G(0) &\le \frac{1}{T - 2L}\Big[\alpha \int_0^T\int_0^L \phi^2\, dx\, dt + \frac{L}{2}\int_0^T \phi_x^2(L,t)\, dt\Big] \\
&\quad + \frac{\pi^2 - \alpha L^2}{T(\pi^2 - \alpha L^2) - 2L\pi^2}\frac{L}{2}\int_0^T \psi_x^2(L,t)\, dt\,.
\end{aligned}$$

Noting that

$$\begin{aligned}
T(\pi^2 - \alpha L^2) - 2L\pi^2 &= T\pi^2 - T\alpha L^2 - 2L\pi^2 \\
&\le (T\pi^2 + 2\alpha L^3) - T\alpha L^2 - 2L\pi^2 = (T - 2L)(\pi^2 - \alpha L^2),
\end{aligned}$$

we obtain

$$\frac{1}{T - 2L} \le \frac{\pi^2 - \alpha L^2}{T(\pi^2 - \alpha L^2) - 2L\pi^2}.$$

Therefore, using $\phi = u + v$ and $\psi = u - v$, we obtain

$$\begin{aligned}
&F(0) + G(0) \\
&\le \frac{\pi^2 - \alpha L^2}{T(\pi^2 - \alpha L^2) - 2L\pi^2}\Big[\alpha \int_0^T\int_0^L (u+v)^2\, dx\, dt \\
&\quad + \frac{L}{2}\int_0^T \big(u_x + v_x\big)^2(L,t)dt + \frac{L}{2}\int_0^T \big(u_x - v_x\big)^2(L,t)dt\Big] \\
&= \frac{\pi^2 - \alpha L^2}{T(\pi^2 - \alpha L^2) - 2L\pi^2}\Big[4\alpha \int_0^T\int_0^L uv\, dx\, dt + L\int_0^T \big[u_x^2 + v_x^2\big](L,t)dt\Big],
\end{aligned}$$

and using Young's inequality it follows that

$$\mathcal{E}(0) \leq \frac{\pi^2 - \alpha L^2}{T(\pi^2 - \alpha L^2) - 2L\pi^2}\Big[\alpha \int_0^T \int_0^L (u^2 + v^2)\,dx\,dt + \frac{L}{2}\int_0^T \big[u_x^2 + v_x^2\big](L,t)dt\ \Big],$$

for $2\mathcal{E}(0) = F(0) + G(0)$.

3. Finite difference semi-discretization

For our purposes we consider J a nonnegative integer, $h = L/(J+1)$ and the partition of $(0, L)$ given by

$$0 = x_0 < x_1 < \cdots < x_J < x_{J+1} = L, \quad \text{where } x_j = jh,\ \forall j = 0, \ldots, J+1. \tag{3.1}$$

Here, we consider the following semi-discrete system for the 1D coupled wave equations (1.12)–(1.16):

$$u_j''(t) - \Delta_h u_j(t) + \alpha v_j(t) = 0, \quad j = 1, 2, \ldots, J,\ 0 < t < T, \tag{3.2}$$

$$v_j''(t) - \Delta_h v_j(t) + \alpha u_j(t) = 0, \quad j = 1, 2, \ldots, J,\ 0 < t < T, \tag{3.3}$$

$$u_0(t) = u_{J+1}(t) = 0, \quad v_0(t) = v_{J+1}(t) = 0, \quad 0 < t < T, \tag{3.4}$$

$$\begin{gathered} u_j(0) = u_j^0, \quad u_j'(0) = u_j^1, \quad v_j(0) = v_j^0, \quad v_j'(0) = v_j^1, \\ j = 0, 1, 2, \ldots, J+1, \end{gathered} \tag{3.5}$$

where primes $'$ denotes the derivative of u with respect to time t, and the functions $u_j(t)$ and $v_j(t)$ are approximations to $u(x_j, t)$ and $v(x_j, t)$ respectively, being u and v solutions of (1.12)–(1.16). We use Δ_h to denote

$$\Delta_h u_j(t) := \frac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{h^2}, \quad \Delta_h v_j(t) := \frac{v_{j+1}(t) - 2v_j(t) + v_{j-1}(t)}{h^2}$$

The energy of (3.2)–(3.5) is

$$\begin{aligned} \mathcal{E}_h(t) := \frac{h}{2}\sum_{j=0}^{J}\Big[&|u_j'(t)|^2 + \big|\frac{u_{j+1}(t) - u_j(t)}{h}\big|^2 + |v_j'(t)|^2 \\ &+ \big|\frac{v_{j+1}(t) - v_j(t)}{h}\big|^2 + 2\alpha u_j(t)v_j(t)\Big], \end{aligned} \tag{3.6}$$

and it is a conserved quantity of the time t for the system (3.2)–(3.5). Our aim here relies on analysis of a semi-discrete counterpart to the boundary observability (2.38) with respect to mesh size h. To do this we use the Ingham's type approach [10]. First of all, we show that there exists a uniform observability inequality associated with the observability inequality (2.25) (see [14, 18]). We consider the two systems associated to the discrete system (3.2)–(3.5). The first one is obtained by taking $\phi_j(t) := u_j(t) + v_j(t)$ from where we have

$$\phi_j''(t) - \Delta_h \phi_j(t) + \alpha\phi_j(t) = 0, \quad j = 1, 2, \ldots, J,\ 0 < t < T, \tag{3.7}$$

$$\phi_0(t) = \phi_{J+1}(t) = 0, \quad 0 < t < T, \tag{3.8}$$

$$\phi_j(0) = \phi_j^0, \quad \phi_j'(0) = \phi_j^1, \quad j = 0, 1, 2, \ldots, J+1, \tag{3.9}$$

and the second one is obtained for $\psi_j(t) := u_j(t) - v_j(t)$ from where we have

$$\psi_j''(t) - \Delta_h \psi_j(t) - \alpha\psi_j(t) = 0, \quad j = 1, 2, \ldots, J,\ 0 < t < T, \tag{3.10}$$

$$\psi_0(t) = \psi_{J+1}(t) = 0, \quad 0 < t < T, \tag{3.11}$$

$$\psi_j(0) = \psi_j^0, \quad \psi_j'(0) = \psi_j^1, \quad j = 0, 1, 2, \ldots, J+1. \tag{3.12}$$

Note that for $u_j(t) := (\phi_j(t)+\psi_j(t))/2$ and $v_j(t) := (\phi_j(t)-\psi_j(t))/2$ we recover the Eqs. (3.2)-(3.3). Their energies are given, respectively, by

$$F_h(t) := \frac{h}{2}\sum_{j=0}^{J}\Big[|\phi_j'(t)|^2 + \big|\frac{\phi_{j+1}(t)-\phi_j(t)}{h}\big|^2 + \alpha|\phi_j(t)|^2\Big], \tag{3.13}$$

$$G_h(t) := \frac{h}{2}\sum_{j=0}^{J}\Big[|\psi_j'(t)|^2 + \big|\frac{\psi_{j+1}(t)-\psi_j(t)}{h}\big|^2 - \alpha\psi_j^2(t)\Big]. \tag{3.14}$$

It is not difficult to see these energies are conservative for all time t. That is to say, $F_h(t) = F_h(0)$ and $G_h(t) = G_h(0)$ for all $t \in [0,T]$. Naturally, $\mathcal{E}_h(t) = (G_h(t)+F_h(t))/2$ for all $t \in [0,T]$.

Proposition 3.1 (Energy conserving)**.** *For any $h > 0$ and (u,v) solution of* (3.2)–(3.5) *we have*

$$\mathcal{E}_h(t) = \mathcal{E}_h(0), \quad \forall t \in [0,T], \tag{3.15}$$

where $\mathcal{E}_h(t)$ is given in (3.6)*.*

Proof. Multiplying the Eqs. (3.2) and (3.3) by $hu_j'(t)$ and $hv_j'(t)$ respectively, and adding the results for indices $j = 1,2,\dots,J$, we obtain

$$h\sum_{j=1}^{J} u_j''u_j' - h\sum_{j=1}^{J}(\Delta_h u_j)u_j' + h\alpha\sum_{j=1}^{J} v_ju_j' = 0, \tag{3.16}$$

$$h\sum_{j=1}^{J} v_j''v_j' - h\sum_{j=1}^{J}(\Delta_h v_j)v_j' + h\alpha\sum_{j=1}^{J} u_jv_j' = 0. \tag{3.17}$$

Keeping in mind the boundary conditions (3.4) and after some calculations we arrive at

$$-h\sum_{j=1}^{J}(\Delta_h u_j)u_j' = \frac{h}{2}\frac{d}{dt}\sum_{j=0}^{J}\big|\frac{u_{j+1}-u_j}{h}\big|^2, \tag{3.18}$$

$$-h\sum_{j=1}^{J}(\Delta_h v_j)v_j' = \frac{h}{2}\frac{d}{dt}\sum_{j=0}^{J}\big|\frac{v_{j+1}-v_j}{h}\big|^2. \tag{3.19}$$

On the other hand,

$$\alpha h\sum_{j=1}^{J} v_ju_j' + \alpha h\sum_{j=1}^{J} u_jv_j' = \alpha h\frac{d}{dt}\sum_{j=0}^{J} u_jv_j. \tag{3.20}$$

Substituting (3.18), (3.19) and (3.20) into (3.16) and (3.17) we obtain

$$\frac{d}{dt}\sum_{j=0}^{J}\Big[\frac{h}{2}|u_j'|^2 + \frac{h}{2}\sum_{j=0}^{J}\big|\frac{u_{j+1}-u_j}{h}\big|^2 + \frac{h}{2}|v_j'|^2 + \frac{h}{2}\big|\frac{v_{j+1}-v_j}{h}\big|^2 + \alpha h u_jv_j\Big] = 0, \tag{3.21}$$

or

$$\frac{d}{dt}\mathcal{E}_h(t) = 0 \;\Rightarrow\; \mathcal{E}_h(t) = \mathcal{E}_h(0), \quad \forall t \in [0,T]. \tag{3.22}$$

The proof is complete. □

Next, we show the positivity of $\mathcal{E}_h(t)$.

Proposition 3.2 (Energy positivity)**.** *For any $h>0$ and (u,v) solution of* (3.2)–(3.5) *we have*

$$\mathcal{E}_h(t) \geq \frac{\pi^2-\alpha L^2}{\pi^2}\frac{h}{2}\sum_{j=0}^{J}\Big[\big|\frac{u_{j+1}(t)-u_j(t)}{h}\big|^2+\big|\frac{v_{j+1}(t)-v_j(t)}{h}\big|^2\Big], \tag{3.23}$$

since $\alpha \leq \pi^2/L^2$, where $\mathcal{E}(t)$ is given by (3.6)*.*

Proof. Proceeding as in Theorem (2.4), we have

$$\begin{aligned}
2\mathcal{E}_h(t) &= h\sum_{j=0}^{J}\Big[|u_j'(t)|^2+\big|\frac{u_{j+1}(t)-u_j(t)}{h}\big|^2+|v_j'(t)|^2\\
&\quad+\big|\frac{v_{j+1}(t)-v_j(t)}{h}\big|^2+2\alpha u_j(t)v_j(t)\Big]\\
&\geq h\sum_{j=0}^{J}\Big[|u_j'(t)|^2+\big|\frac{u_{j+1}(t)-u_j(t)}{h}\big|^2+|v_j'(t)|^2+\big|\frac{v_{j+1}(t)-v_j(t)}{h}\big|^2\\
&\quad-\alpha|v_j(t)|^2-\alpha|u_j(t)|^2\Big].
\end{aligned}$$

Now, using the embedding Theorem (see [9])

$$h\sum_{j=0}^{J}|u_j(t)|^2 \leq \frac{L^2}{\pi^2}h\sum_{j=0}^{J}\big|\frac{u_{j+1}(t)-u_j(t)}{h}\big|^2, \tag{3.24}$$

and for h sufficiently small, we obtain

$$\begin{aligned}
2\mathcal{E}_h(t) &\geq h\sum_{j=0}^{J}\Big[|u_j'(t)|^2+\Big(1-\alpha\frac{L^2}{\pi^2}\Big)\big|\frac{u_{j+1}(t)-u_j(t)}{h}\big|^2+|v_j'(t)|^2\\
&\quad+\Big(1-\alpha\frac{L^2}{\pi^2}\Big)\big|\frac{v_{j+1}(t)-v_j(t)}{h}\big|^2\Big]\\
&= h\sum_{j=0}^{J}\Big[|u_j'(t)|^2+\frac{\pi^2-\alpha L^2}{\pi^2}\big|\frac{u_{j+1}(t)-u_j(t)}{h}\big|^2+|v_j'(t)|^2\\
&\quad+\frac{\pi^2-\alpha L^2}{\pi^2}\big|\frac{v_{j+1}(t)-v_j(t)}{h}\big|^2\Big],
\end{aligned}$$

from where we obtain the positivity of $\mathcal{E}_h(t)$ for $\alpha \leq \pi^2/L^2$. □

3.1. **Uniform observability inequality: no filtering.** In this section, we prove that there exists a uniform observability inequality which is the discrete counterpart of the observability inequality (2.25). The proof requires a set of results obtained using discrete multipliers. The first result concerns to the system (3.10)–(3.12).

Lemma 3.3. *For any $h>0$ and ψ solution of* (3.10)–(3.12) *we have*

$$TG_h(0)+\chi_h(t)\big|_0^T = \frac{h^3}{4}\sum_{j=0}^{J}\int_0^T\Big(\frac{\psi_{j+1}'-\psi_j'}{h}\Big)^2dt+\frac{L}{2}\int_0^T\big|\frac{\psi_J}{h}\big|^2dt, \tag{3.25}$$

where

$$\chi_h(t) = h\sum_{j=1}^{J}j\psi_j'\Big(\frac{\psi_{j+1}-\psi_{j-1}}{2}\Big). \tag{3.26}$$

Proof. Multiplying the Eq. (3.10) by $j(\psi_{j+1}-\psi_{j-1})/2$ we obtain

$$
\begin{aligned}
&h\sum_{j=1}^{J}\int_0^T j\psi_j''\Big(\frac{\psi_{j+1}-\psi_{j-1}}{2}\Big)dt\\
&-h\sum_{j=1}^{J}\int_0^T j\frac{\psi_{j+1}-2\psi_j+\psi_{j-1}}{h^2}\Big(\frac{\psi_{j+1}-\psi_{j-1}}{2}\Big)dt\\
&-\alpha h\sum_{j=1}^{J}\int_0^T j\psi_j\Big(\frac{\psi_{j+1}-\psi_{j-1}}{2}\Big)dt=0.
\end{aligned}
\tag{3.27}
$$

It is immediate that

$$
\begin{aligned}
&h\sum_{j=1}^{J}\int_0^T j\psi_j''\Big(\frac{\psi_{j+1}-\psi_{j-1}}{2}\Big)dt\\
&=h\sum_{j=1}^{J}j\psi_j'\Big(\frac{\psi_{j+1}-\psi_{j-1}}{2}\Big)\Big|_0^T-h\sum_{j=1}^{J}\int_0^T j\psi_j'\Big(\frac{\psi_{j+1}'-\psi_{j-1}'}{2}\Big)dt\\
&=\chi_h(t)\big|_0^T+\frac{h}{2}\sum_{j=0}^{J}\int_0^T\psi_j'\psi_{j+1}'dt\\
&=\chi_h(t)\big|_0^T-\frac{h^3}{4}\sum_{j=0}^{J}\int_0^T\Big(\frac{\psi_{j+1}'-\psi_j'}{h}\Big)^2dt+\frac{h}{2}\sum_{j=0}^{J}\int_0^T\big|\psi_j'\big|^2dt.
\end{aligned}
\tag{3.28}
$$

On the other hand, it is not difficult to see that

$$
\begin{aligned}
&h\sum_{j=1}^{J}\int_0^T j\frac{\psi_{j+1}-2\psi_j+\psi_{j-1}}{h^2}\,\frac{\psi_{j+1}-\psi_{j-1}}{2}dt\\
&=\frac{h}{2h^2}\sum_{j=0}^{J}\int_0^T j|\psi_{j+1}|^2dt-\frac{h}{2h^2}\sum_{j=0}^{J}\int_0^T j|\psi_j|^2dt-\frac{h}{2h^2}\sum_{j=0}^{J}\int_0^T|\psi_j|^2dt\\
&\quad+\frac{(J+1)h}{2h^2}\int_0^T|\psi_J|^2dt+\frac{h}{h^2}\sum_{j=0}^{J}\int_0^T\psi_j\psi_{j+1}dt\,.
\end{aligned}
$$

Keeping in mind the Dirichlet boundary conditions, we obtain

$$
\begin{aligned}
&-h\sum_{j=1}^{J}\int_0^T j\frac{\psi_{j+1}-2\psi_j+\psi_{j-1}}{h^2}\,\frac{\psi_{j+1}-\psi_{j-1}}{2}dt\\
&=\frac{h}{2}\sum_{j=0}^{J}\int_0^T\Big|\frac{\psi_{j+1}-\psi_j}{h}\Big|^2dt-\frac{L}{2}\int_0^T\Big|\frac{\psi_J}{h}\Big|^2dt.
\end{aligned}
\tag{3.29}
$$

Moreover,

$$
\alpha h\sum_{j=1}^{J}\int_0^T j\psi_j\Big(\frac{\psi_{j+1}-\psi_{j-1}}{2}\Big)dt=-\frac{\alpha}{2}h\sum_{j=0}^{J}\int_0^T\psi_{j+1}\psi_j dt.
\tag{3.30}
$$

Substituting (3.28), (3.29) and (3.30) into (3.27), we arrive at

$$\begin{aligned}
&\frac{h}{2}\sum_{j=0}^{J}\int_0^T \Big[|\psi_j'|^2+\big|\frac{\psi_{j+1}-\psi_j}{h}\big|^2-\alpha|\psi_j|^2\Big]dt\\
&+\frac{\alpha}{2}h\sum_{j=0}^{J}\int_0^T [|\psi_j|^2+\psi_j\psi_{j+1}]dt+\chi_h(t)\big|_0^T\\
&=\frac{h^3}{4}\sum_{j=0}^{J}\int_0^T \Big(\frac{\psi_{j+1}'-\psi_j'}{h}\Big)^2dt+\frac{L}{2}\int_0^T \big|\frac{\psi_J}{h}\big|^2dt.
\end{aligned}$$

To finish the proof, we observe that

$$\frac{\alpha}{2}h\sum_{j=0}^{J}\int_0^T [|\psi_j|^2+\psi_j\psi_{j+1}]dt=\frac{\alpha}{4}h\sum_{j=0}^{J}\int_0^T \big(\psi_{j+1}+\psi_j\big)^2dt,$$

from where we obtain the required result

$$TG_h(0)+\chi_h(t)\big|_0^T=\frac{h^3}{4}\sum_{j=0}^{J}\int_0^T \Big(\frac{\psi_{j+1}'-\psi_j'}{h}\Big)^2dt+\frac{L}{2}\int_0^T \big|\frac{\psi_J}{h}\big|^2dt.$$

□

Lemma 3.4. *The energy $G_h(t)$ of the system* (3.10)–(3.12) *preserves the positivity since that $\alpha \le \pi^2/L^2$, i.e.,*

$$G_h(0)\ge \frac{\pi^2-\alpha L^2}{2\pi^2}h\sum_{j=0}^{J}\big|\frac{\psi_{j+1}-\psi_j}{h}\big|^2\ge 0. \tag{3.31}$$

Proof. The proof is immediate. Indeed, from the energy defined in (3.14) and using the discrete Poincaré's inequality (3.24), we have

$$\begin{aligned}
G_h(t)&\ge \frac{h}{2}\sum_{j=0}^{J}\Big[|\psi_j'|^2+\big|\frac{\psi_{j+1}-\psi_j}{h}\big|^2-\alpha\frac{L^2}{\pi^2}\big|\frac{\psi_{j+1}-\psi_j}{h}\big|^2\Big]\\
&\ge \frac{h}{2}\sum_{j=0}^{J}\Big[|\psi_j'|^2+\Big(1-\alpha\frac{L^2}{\pi^2}\Big)h\sum_{j=0}^{J}\big|\frac{\psi_{j+1}-\psi_j}{h}\big|^2\Big],
\end{aligned}$$

from where the result follows for $\alpha \le \pi^2/L^2$. □

Lemma 3.5. *For any $h>0$, $0\le t\le T$, $\alpha<\pi^2/L^2$ and ψ solution of* (3.10)–(3.12) *we have*

$$|\chi_h(t)|\le L\frac{\pi^2}{\pi^2-\alpha L^2}G_h(0), \tag{3.32}$$

where $\chi_h(t)$ is given in (3.26).

Proof. Based on [9, Lemma 2.5], we have

$$|\chi_h(t)|\le L\Big[\sum_{j=0}^{J}|\psi_j'|^2\Big]^{1/2}\Big[h\sum_{j=0}^{J}\big|\frac{\psi_{j+1}-\psi_j}{h}\big|^2\Big]^{1/2}.$$

Now we use Young's inequality and the energy defined in (3.14) to obtain

$$|\chi_h(t)| \le L\Big[\frac{h}{2}\sum_{j=0}^{J}|\psi_j'|^2 + \frac{h}{2}\sum_{j=0}^{J}\Big|\frac{\psi_{j+1}-\psi_j}{h}\Big|^2\Big] \le L\Big[G_h(t) + \alpha\frac{h}{2}\sum_{j=0}^{J}|\psi_j|^2\Big],$$

and using inequality (3.24) we obtain

$$|\chi_h(t)| \le L\Big[G_h(t) + \alpha\frac{h}{2}\Big(\frac{L}{\pi}\Big)^2\sum_{j=0}^{J}\Big|\frac{\psi_{j+1}-\psi_j}{h}\Big|^2\Big].$$

To complete the proof, we use the inequality (3.31) from Lemma (3.4) to obtain

$$|\chi_h(t)| \le L\Big[G_h(t) + \frac{\alpha}{2}\Big(\frac{L}{\pi}\Big)^2\frac{2\pi^2}{\pi^2-\alpha L^2}G_h(0)\Big],$$

from where we conclude the proof by taking the energy conservation law for energy $G_h(t)$. ☐

Now we prove the uniform observability inequality concerning to system (3.10)–(3.12).

Theorem 3.6. *Let ψ be the solution of the system* (3.10)–(3.12)*. Then for all $\alpha \le \pi^2/L^2$ there exists $T_0 > 0$ such that for each $T > T_0$ there exists $C(T,\alpha) > 0$ which is independent of h such that, for all $h > 0$,*

$$G_h(0) \le C(T,\alpha)\Big[\frac{h^3}{4}\sum_{j=0}^{J}\int_0^T\Big(\frac{\psi_{j+1}'-\psi_j'}{h}\Big)^2dt + \frac{L}{2}\int_0^T\Big|\frac{\psi_J}{h}\Big|^2dt\Big], \tag{3.33}$$

where $G_h(\cdot)$ is the energy in (3.14) *and $C(T,\alpha) = \frac{\pi^2-\alpha L^2}{T(\pi^2-\alpha L^2)-2L\pi^2}$.*

Proof. From Lemmas 3.3 and 3.5, we have

$$TG_h(0) - 2L\frac{\pi^2}{\pi^2-\alpha L^2}G_h(0) \le \frac{h^3}{4}\sum_{j=0}^{J}\int_0^T\Big(\frac{\psi_{j+1}'-\psi_j'}{h}\Big)^2dt + \frac{L}{2}\int_0^T\Big|\frac{\psi_J}{h}\Big|^2dt, \tag{3.34}$$

and for $T > T_0$ where the time T_0 is the same as in Theorem 2.7, we obtain

$$G_h(0) \le C(T,\alpha)\Big[\frac{h^3}{4}\sum_{j=0}^{J}\int_0^T\Big(\frac{\psi_{j+1}'-\psi_j'}{h}\Big)^2dt + \frac{L}{2}\int_0^T\Big|\frac{\psi_J}{h}\Big|^2dt\Big], \tag{3.35}$$

where $C(T,\alpha) = \frac{\pi^2-\alpha L^2}{T(\pi^2-\alpha L^2)-2L\pi^2}$. ☐

In a similar way, any solution ϕ of the system (3.7)–(3.9) obeys a uniform observability inequality. Indeed, we have the following Theorem.

Theorem 3.7. *Let ϕ be the solution of the system* (3.7)–(3.9)*. Then for all $T > 2L$ there exists $C(T) > 0$ which is independent of h such that, for all $h > 0$,*

$$F_h(0) \le C(T)\Big[\frac{h^3}{4}\sum_{j=0}^{J}\int_0^T\Big(\frac{\phi_{j+1}'-\phi_j'}{h}\Big)^2dt + \alpha h\sum_{j=0}^{J}\int_0^T|\phi_j|^2dt + \frac{L}{2}\int_0^T\Big|\frac{\phi_J}{h}\Big|^2dt\Big],$$

where $F_h(\cdot)$ is the energy in (3.13) *and $C(T) = 1/(T-2L)$.*

The insertion, at semi-discrete level, of the extra terms

$$\frac{h^3}{4}\sum_{j=0}^{J}\int_0^T\Big(\frac{\psi'_{j+1}-\psi'_j}{h}\Big)^2dt \quad\text{and}\quad \frac{h^3}{4}\sum_{j=0}^{J}\int_0^T\Big(\frac{\phi'_{j+1}-\phi'_j}{h}\Big)^2dt,$$

is a suitable one to reestablish the uniform observability inequality. For example, in light of the arguments discussed in [18], it can be shown that this extra term has order of $h^2\nu_j^- G_h(0)$ (see ν_j^- from Proposition 3.9) and it is important to overcome the high-frequency when $\nu_j^- = O(h^{-2})$. Moreover, the terms involving observation on the boundary and observation distributed on the whole space domain are necessary for estimate (3.33). See [18] for details of several remedies to overcome the lack of uniform observability.

Now we are in a position to prove the uniform observability inequality to the coupled system (3.2)–(3.5).

Theorem 3.8. *Let (u,v) be the solution of system (3.2)–(3.5). Then for all $\alpha \le \pi^2/L^2$ there exists $T_0>0$ such that for all $T>T_0$ there exists $C(T,\alpha)>0$ such that, for all $h>0$,*

$$\begin{aligned}\mathcal{E}_h(0) \le C(T,\alpha)\Big[&\frac{h^3}{4}\sum_{j=0}^{J}\int_0^T\Big(\frac{u'_{j+1}-u'_j}{h}\Big)^2dt+\frac{h^3}{4}\sum_{j=0}^{J}\int_0^T\Big(\frac{v'_{j+1}-v'_j}{h}\Big)^2dt\\ &+\alpha h\sum_{j=0}^{J}\int_0^T(u_j^2+v_j^2)dt+\frac{L}{2}\int_0^T\Big(\big|\frac{u_J}{h}\big|^2+\big|\frac{v_J}{h}\big|^2\Big)dt\Big].\end{aligned}\tag{3.36}$$

Proof. From uniform observability inequalities built in Theorems 3.6 and 3.7 and taking into account the decompositions of the solutions ϕ_j and ψ_j, we have

$$\begin{aligned}&F_h(0)+G_h(0)\\ &\le C(T)\Big[\frac{h^3}{4}\sum_{j=0}^{J}\int_0^T\Big(\frac{\phi'_{j+1}-\phi'_j}{h}\Big)^2dt+\alpha h\sum_{j=0}^{J}\int_0^T|\phi_j|^2dt+\frac{L}{2}\int_0^T\big|\frac{\phi_J}{h}\big|^2dt\Big]\\ &\quad + C(T,\alpha)\Big[\frac{h^3}{4}\sum_{j=0}^{J}\int_0^T\Big(\frac{\psi'_{j+1}-\psi'_j}{h}\Big)^2dt+\frac{L}{2}\int_0^T\big|\frac{\psi_J}{h}\big|^2dt\Big]\\ &\le C(T,\alpha)\Big[\frac{h^3}{2}\sum_{j=0}^{J}\int_0^T\Big(\frac{u'_{j+1}-u'_j}{h}\Big)^2dt+\frac{h^3}{2}\sum_{j=0}^{J}\int_0^T\Big(\frac{v'_{j+1}-v'_j}{h}\Big)^2dt\\ &\quad +2\alpha h\sum_{j=0}^{J}\int_0^T(u_j^2+v_j^2)dt+L\int_0^T\big|\frac{u_J}{h}\big|^2dt+L\int_0^T\big|\frac{v_J}{h}\big|^2dt\Big],\end{aligned}$$

and therefore we conclude the proof by taking $\mathcal{E}(0)=(F_h(0)+G_h(0))/2$. □

3.2. **Spectral analysis.** In this section, we study a uniform semi-discrete version to the boundary observability (2.38). We solved the semi-discrete versions to the Propositions 2.1, 2.2 and 2.3.

Proposition 3.9. *Assume that $\alpha\le\pi^2/L^2$. Then, the Fourier series expansion of system (3.10)–(3.12) is*

$$\Psi_h(t)=\sum_{k=1}^{J}\Big[a_k\sin\Big(\sqrt{\nu_k^-(h)}\ t\Big)+b_k\cos\Big(\sqrt{\nu_k^-(h)}\ t\Big)\Big]\varphi^k,$$

where a_k, b_k are the Fourier coefficients, $\nu_k^-(h) = \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2L}\right) - \alpha$ are the eigenvalues, and $\varphi^k = (\varphi_{k,1}, \dots, \varphi_{k,J})$ are the eigenvectors $\varphi_{k,j} = \sin\left(\frac{k\pi x_j}{L}\right)$, $j, k = 1, \dots, J$.

Proof. Making $\psi_j(t) := \varphi_j T(t)$ and substituting into (3.10) we obtain

$$\frac{T''(t)}{T(t)} = \Big[\frac{\varphi_{j+1} - 2\varphi_j + \varphi_{j-1}}{h^2} + \alpha\varphi_j\Big]\frac{1}{\varphi_j} = \nu, \tag{3.37}$$

for all $t \geq 0$ and $j = 0, 1, 2, \dots, J, J+1$. Then, we have the eigenvalue problem

$$\frac{\varphi_{j+1} - 2\varphi_j + \varphi_{j-1}}{h^2} - (\nu - \alpha)\varphi_j = 0, \quad j = 1, \dots, J \tag{3.38}$$

$$\varphi_0 = \varphi_{J+1} = 0. \tag{3.39}$$

Proceeding as in Proposition 2.1, we arrive at

$$\nu_k^-(h) = \frac{4}{h^2}\sin^2\Big(\frac{k\pi h}{2L}\Big) - \alpha, \quad \forall k = 1, \dots, J, \tag{3.40}$$

for $\alpha \leq \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2L}\right)$, i.e., $\alpha \leq \pi^2/L^2$ for h sufficiently small. Now, from identity (3.37) we obtain the equation $T''(t) - \nu T(t) = 0$ for all $t > 0$ and solving this equation we obtain

$$T_k(t) = a_k \sin\Big(\sqrt{\nu_k^-(h)}\, t\Big) + b_k \cos\Big(\sqrt{\nu_k^-(h)}\, t\Big), \quad k = 1, \dots, J,\ \forall t > 0, \tag{3.41}$$

where a_k, b_k are the Fourier coefficients. The proof is complete. $\square$

In a similar way, the following holds.

Proposition 3.10. *The Fourier series expansion of system* (3.7)–(3.9) *is*

$$\Phi_h(t) = \sum_{k=1}^{J}\Big[c_k \sin\Big(\sqrt{\nu_k^+(h)}\, t\Big) + d_k \cos\Big(\sqrt{\nu_k^+(h)}\, t\Big)\Big]\varphi^k$$

where c_k, d_k are the Fourier coefficients, $\nu_k^+(h) = \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2L}\right) + \alpha$ are the eigenvalues, and $\varphi^k = (\varphi_{k,1}, \dots, \varphi_{k,J})$ are the eigenvectors, where $\varphi_{k,j} = \sin\left(\frac{k\pi x_j}{L}\right)$ for $j, k = 1, \dots, J$.

The next result follows from Propositions (3.9) and (3.10).

Proposition 3.11. *Assume that $\alpha \leq \pi^2/L^2$. Then the Fourier series expansions of the system* (3.2)–(3.5) *are*

$$\begin{aligned} U_h(t) = \frac{1}{2}\sum_{k=1}^{J}\Big[&a_k \sin(\sqrt{\nu_k^-(h)}\, t) + b_k \cos(\sqrt{\nu_k^-(h)}\, t) + c_k \sin(\sqrt{\nu_k^+(h)}\, t) \\ &+ d_k \cos(\sqrt{\nu_k^+(h)}\, t)\Big]\varphi^k, \end{aligned} \tag{3.42}$$

$$\begin{aligned} V_h(t) = \frac{1}{2}\sum_{k=1}^{J}\Big[&c_k \sin(\sqrt{\nu_k^+(h)}\, t) + d_k \cos(\sqrt{\nu_k^+(h)}\, t) - a_k \sin(\sqrt{\nu_k^-(h)}\, t) \\ &- b_k \cos(\sqrt{\nu_k^-(h)}\, t)\Big]\varphi^k, \end{aligned} \tag{3.43}$$

where a_k, b_k, c_k, d_k are the Fourier coefficients, $\nu_k^{\pm}(h) = \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2L}\right) \pm \alpha$ are the eigenvalues, and $\varphi^k = (\varphi_{k,1}, \dots, \varphi_{k,J})$ are the eigenvectors associated where each component $\varphi_{k,j}$ is $\varphi_{k,j} = \sin\left(\frac{k\pi x_j}{L}\right)$, $j, k = 1, \dots, J$.

The next two lemmas play an important role for getting the uniform boundary observability of system (3.2)–(3.5). These lemmas concern the boundary observability of the spectral problem associated to the uncoupled systems (3.10)-(3.12) and (3.7)–(3.9). The proofs are trivial and we omit them.

Lemma 3.12. *For any eigenvector $\varphi = (\varphi_1, \dots, \varphi_J)$ of (3.10)-(3.12) we have*

$$h\sum_{j=0}^{J}\Big|\frac{\varphi_{j+1}-\varphi_j}{h}\Big|^2 = \frac{2L}{4-(\nu_k^-(h)+\alpha)h^2}\Big|\frac{\varphi_J}{h}\Big|^2, \tag{3.44}$$

where $\nu_k^-(h) = \frac{4}{h^2}\sin^2\Big(\frac{k\pi h}{2L}\Big) - \alpha$ for $k = 1, \dots, J$.

Lemma 3.13. *For each eigenvector $\varphi = (\varphi_1, \dots, \varphi_J)$ of (3.7)–(3.9) we have*

$$h\sum_{j=0}^{J}\Big|\frac{\varphi_{j+1}-\varphi_j}{h}\Big|^2 = \frac{2L}{4-(\nu_k^+(h)-\alpha)h^2}\Big|\frac{\varphi_J}{h}\Big|^2, \tag{3.45}$$

where $\nu_k^+(h) = \frac{4}{h^2}\sin^2\big(\frac{k\pi h}{2L}\big) + \alpha$ for $k = 1, \dots, J$.

Equalities (3.44) and (3.45) provide an explicit relation between the total energy of the eigenvectors and the energy concentrated on $x = L$ according to the quantity measured by $|\varphi_J/h|^2$. In both cases, the observability inequalities are written as

$$h\sum_{j=0}^{J}\Big|\frac{\varphi_{j+1}-\varphi_j}{h}\Big|^2 = \frac{2L}{4-4\sin^2\Big(\frac{k\pi h}{2L}\Big)}\Big|\frac{\varphi_J}{h}\Big|^2, \tag{3.46}$$

and

$$4\sin^2\Big(\frac{J\pi h}{2L}\Big) = 4\cos^2\Big(\frac{\pi h}{2L}\Big) \to 4, \quad \text{as } h \to 0,$$

from where the blow-up happens on the right hand side of (3.44) and (3.45). Therefore, the lack of numerical boundary observability follows in these cases.

4. Uniform observability: non-harmonic Fourier series

In this section, we use the Ingham's theorem [10] to get a uniform boundary observability (2.38).

Theorem 4.1 (Ingham's Inequality)**.** *Let $\{\mu_k\}_{k\in\mathcal{I}}$ be a sequence of real numbers such that*

$$\mu_{k+1} - \mu_k \geq \gamma > 0, \quad \forall k \in \mathcal{I}. \tag{4.1}$$

Then, for any $T > \frac{2\pi}{\gamma}$ there exist positive constants $C_i(T,\gamma) > 0$, $i = 1, 2$ such that

$$C_1(T,\gamma)\sum_{k\in\mathcal{I}}|a_k|^2 \leq \int_0^T\Big|\sum_{k\in\mathcal{I}}a_k e^{i\mu_k t}\Big|^2 dt \leq C_2(T,\gamma)\sum_{k\in\mathcal{I}}|a_k|^2, \tag{4.2}$$

for all sequences of complex numbers $\{a_k\} \in l^2$.

To apply Ingham's inequality, we need an estimate between the roots of consecutive eigenvalues entering in the Fourier expansions to the solutions of the decoupled systems (3.7)–(3.9) and (3.10)-(3.12) into subspaces of filtered solutions. We have in mind that, these numerical solutions, are the filtered solutions generate by spectral problem such that $\lambda h^2 \leq \gamma$. In that direction and taking into account the

Fourier series expansions, given any $0 < \gamma < 4$ we introduce the following class of filtered solutions to the system (3.10)-(3.12),

$$\mathcal{P}_h(\gamma) := \Big\{\psi = \sum_{\lambda_k(h)\leq\gamma h^{-2}} \Big[a_k \sin\Big(\sqrt{\nu_k^-(h)}\, t\Big) + b_k \cos\Big(\sqrt{\nu_k^-(h)}\, t\Big)\Big]\varphi^k\Big\}, \quad (4.3)$$

where $\lambda_k(h) := \frac{4}{h^2}\sin^2\big(\frac{k\pi h}{2L}\big)$ and $a_k, b_k \in \mathbb{R}$. Moreover, we introduce the class of filtered solutions to (3.7)–(3.9) given by

$$\mathcal{Q}_h(\gamma) := \Big\{\phi = \sum_{\lambda_k(h)\leq\gamma h^{-2}} \Big[c_k \sin\Big(\sqrt{\nu_k^+(h)}\ t\Big) + d_k \cos\Big(\sqrt{\nu_k^+(h)}\ t\Big)\Big]\varphi^k\Big\}, \quad (4.4)$$

where $c_k, d_k \in \mathbb{R}$. In that direction, we consider the gap between the consecutive eigenvalues $\nu_j^+(h)$ and $\nu_j^-(h)$.

Lemma 4.2. *Assume that*

$$i\gamma = 4\sin^2\Big(\frac{\pi\varepsilon}{2}\Big) \quad \textit{and} \quad \alpha < \frac{\gamma(\varepsilon)}{h^2}, \quad (4.5)$$

for some $0 \leq \varepsilon < 1$. *Then,*

$$\sqrt{\nu_{j+1}^+(h)} - \sqrt{\nu_j^+(h)} \geq \frac{\pi}{L}\cos\Big(\frac{\pi\varepsilon}{2}\Big) > 0, \quad (4.6)$$

for all eigenvalues $\nu_j^+(h) = \lambda_j(h) + \alpha$ *such* $\lambda h^2 \leq \gamma$ *where* $\lambda_j(h) = \frac{4}{h^2}\sin^2\big(\frac{j\pi h}{2L}\big)$, $j = 1, \dots, J$.

Proof. Note that

$$\sqrt{\lambda_j(h) + \alpha} = \sqrt{\lambda_j(h)}\sqrt{1 + \frac{\alpha}{\lambda_j(h)}} \approx \sqrt{\lambda_j(h)}\Big(1 + \frac{\alpha}{2\lambda_j(h)}\Big), \quad (4.7)$$

for $j = 1, \dots, J$; from where we obtain

$$\sqrt{\lambda_{j+1}(h) + \alpha} - \sqrt{\lambda_j(h) + \alpha} \approx \Big(\sqrt{\lambda_{j+1}(h)} - \sqrt{\lambda_j(h)}\Big)\Big(1 - \frac{\alpha}{2\sqrt{\lambda_{j+1}(h)\lambda_j(h)}}\Big),$$

for $j = 1, \dots, J$. On the other hand, using $\lambda_{j+1}(h) > \lambda_j(h)$, $j = 1, \dots, J$, we obtain

$$\begin{aligned}
\sqrt{\lambda_{j+1}(h) + \alpha} - \sqrt{\lambda_j(h) + \alpha} &\geq \Big(\sqrt{\lambda_{j+1}(h)} - \sqrt{\lambda_j(h)}\Big)\Big(1 - \frac{\alpha}{2\sqrt{\lambda_j(h)\lambda_j(h)}}\Big) \\
&\geq \Big(\sqrt{\lambda_{j+1}(h)} - \sqrt{\lambda_j(h)}\Big)\Big(1 - \frac{\alpha}{2}\frac{h^2}{4\sin^2(j\pi h/2L)}\Big) \\
&\geq \Big(\sqrt{\lambda_{j+1}(h)} - \sqrt{\lambda_j(h)}\Big)\Big(1 - \alpha\frac{h^2}{4\sin^2(\pi\varepsilon/2)}\Big) \\
&\geq \Big(\sqrt{\lambda_{j+1}(h)} - \sqrt{\lambda_j(h)}\Big)\Big(1 - \frac{\alpha h^2}{\gamma(\varepsilon)}\Big),
\end{aligned}$$

for $\gamma = 4\sin^2\big(\frac{\pi\varepsilon}{2}\big)$. Now, by taking $\alpha = \gamma(\varepsilon)/2h^2 \leq \gamma(\varepsilon)/h^2$ we arrive at

$$\sqrt{\lambda_{j+1}(h) + \alpha} - \sqrt{\lambda_j(h) + \alpha} \geq \frac{1}{2}\Big(\sqrt{\lambda_{j+1}(h)} - \sqrt{\lambda_j(h)}\Big). \quad (4.8)$$

The estimate for the right hand side of (4.8) can be found in [1, 9] which completes the proof of this lemma. $\square$

Lemma 4.3. *Assume that $\alpha \le \pi^2/L^2$ and $\gamma = 4\sin^2\left(\frac{\pi\varepsilon}{2}\right)$ for some $0 \le \varepsilon < 1$. Then,*

$$\sqrt{\nu_{j+1}^-(h)} - \sqrt{\nu_j^-(h)} \ge \frac{\pi}{L}\cos\left(\frac{\pi\varepsilon}{2}\right) > 0, \tag{4.9}$$

for all eigenvalues $\nu_j^-(h) = \lambda_j(h) - \alpha$ such $\lambda h^2 \le \gamma$ where $\lambda_j(h) = \frac{4}{h^2}\sin^2\left(\frac{j\pi h}{2L}\right)$, $j = 1, \dots, J$.

Proof. The proof is immediate. Indeed, for $\alpha \le \pi^2/L^2$, one has

$$\sqrt{\lambda_j(h) - \alpha} = \sqrt{\lambda_j(h)}\sqrt{1 - \frac{\alpha}{\lambda_j(h)}} \approx \sqrt{\lambda_j(h)}\Big(1 - \frac{\alpha}{2\lambda_j(h)}\Big), \; j = 1, \dots, J, \tag{4.10}$$

and then

$$\begin{aligned}\sqrt{\lambda_{j+1}(h) - \alpha} - \sqrt{\lambda_j(h) - \alpha} &\approx \Big(\sqrt{\lambda_{j+1}(h)} - \sqrt{\lambda_j(h)}\Big)\Big(1 + \frac{\alpha}{2\sqrt{\lambda_{j+1}(h)\lambda_j(h)}}\Big), \\ &\ge \Big(\sqrt{\lambda_{j+1}(h)} - \sqrt{\lambda_j(h)}\Big), \quad \forall j = 1, \dots, J;\end{aligned}$$

The remainder of the proof can be found in [1, 9]. $\square$

4.1. **Proof of the uniform boundary observability to uncoupled systems.** In this section, we obtain the proof of two theorems concerning the uniform boundary observability to uncoupled systems (3.7)–(3.9) and (3.10)-(3.12).

Theorem 4.4. *For any $0 < \gamma < 4$ there exists $T(\gamma) > 0$ such that for any $T > T(\gamma)$ there exists a positive constant $C(T, \alpha, \gamma(\varepsilon))$ such that*

$$F_h(0) \le C(T, \alpha, \gamma(\varepsilon))\frac{L}{2}\int_0^T \big|\frac{\phi_J}{h}\big|^2 dt, \tag{4.11}$$

for any solution of system (3.7)–(3.9) in the class $\mathcal{Q}_h(\gamma(\varepsilon))$ in (4.4).

Proof. According to Ingham's inequality and in view of Lemma 4.2, for any $0 \le \varepsilon < 1$ and $T > \frac{2L}{\cos(\pi\varepsilon/2)}$ there exist positive constants $C_i(T, \varepsilon) > 0$, $i = 1, 2$ satisfying

$$\begin{aligned}C_1(T, \varepsilon)\sum_{|\mu_k|h \le \sqrt{\gamma(\varepsilon)}} |a_k|^2 &\le \int_0^T \Big|\sum_{|\mu_k|h \le \sqrt{\gamma(\varepsilon)}} a_k e^{i\mu_k t}\Big|^2 dt \\ &\le C_2(T, \varepsilon)\sum_{|\mu_k|h \le \sqrt{\gamma(\varepsilon)}} |a_k|^2,\end{aligned} \tag{4.12}$$

where $\gamma(\varepsilon) = 4\sin^2\left(\frac{\pi\varepsilon}{2}\right)$. By Lemma 3.13, we have

$$\begin{aligned}h\sum_{j=0}^{J}\big|\frac{\varphi_{j+1} - \varphi_j}{h}\big|^2 &= \frac{2L}{4 - (\nu^+(h) - \alpha)h^2}\big|\frac{\varphi_J}{h}\big|^2 \\ &= \frac{2L}{4 - \lambda(h)h^2}\big|\frac{\varphi_J}{h}\big|^2 \\ &\le \frac{L}{2\cos^2\big(\pi\varepsilon/2\big)}\big|\frac{\varphi_J}{h}\big|^2,\end{aligned} \tag{4.13}$$

for any eigenvector φ associated with an eigenvalue λ satisfying $\lambda h^2 \le \gamma(\varepsilon)$.

Now, let us consider the solution ϕ of system (3.7)–(3.9) into the class $\mathcal{Q}_h(\gamma(\varepsilon))$, which is written as

$$\phi = \sum_{|\mu_k(h)|h\le\sqrt{\gamma(\varepsilon)}} a_k e^{i\mu_k(h)t}\varphi^k, \tag{4.14}$$

where $\mu_k(h) = \sqrt{\nu_j^+(h)}$. Then, we can deduce that for $T > \frac{2L}{\cos(\pi\varepsilon/2)}$ it holds

$$\begin{aligned} \frac{L}{2}\int_0^T \big|\frac{\phi_J}{h}\big|^2 dt &= \frac{L}{2h^2}\int_0^T \Big|\sum_{|\mu_k|h\le\sqrt{\gamma(\varepsilon)}} a_k e^{i\mu_k(h)t}\varphi_{k,J}\Big|^2 dt \\ &\ge C_1(T,\varepsilon)\cos^2\big(\pi\varepsilon/2\big) \sum_{|\mu_k|h\le\sqrt{\gamma(\varepsilon)}} |a_k|^2 h \sum_{j=0}^{J} \big|\frac{\varphi_{k,j+1}-\varphi_{k,j}}{h}\big|^2. \end{aligned} \tag{4.15}$$

Moreover, it is not difficult to see that

$$\sum_{|\mu_k|h\le\sqrt{\gamma(\varepsilon)}} \Big[|a_k|^2 h \sum_{j=0}^{J} \big|\frac{\varphi_{k,j+1}-\varphi_{k,j}}{h}\big|^2\Big] = \frac{\nu_J^+(h)-\alpha}{\nu_J^+(h)} F_h(0) = \frac{\lambda_J(h)}{\lambda_J(h)+\alpha} F_h(0).$$

Therefore, substituting the above expression into (4.15), for $T > \frac{2L}{\cos(\pi\varepsilon/2)}$ and $\phi \in \mathcal{Q}_h(\gamma(\varepsilon))$ we have

$$F_h(0) \le \frac{\lambda_J(h)+\alpha}{\lambda_J(h)} \frac{1}{\cos^2\big(\pi\varepsilon/2\big)C_1(T,\varepsilon)} \frac{L}{2}\int_0^T \big|\frac{\phi_J}{h}\big|^2 dt, \tag{4.16}$$

and we conclude the proof by taking

$$C(T,\alpha,\gamma(\varepsilon)) = \frac{\lambda_J(h)+\alpha}{\lambda_J(h)} \frac{1}{C_1(T,\varepsilon)\cos^2\big(\pi\varepsilon/2\big)}.$$

□

We observe that for $T > \frac{2L}{\cos(\pi\varepsilon/2)}$, since $\gamma = \gamma(\varepsilon)$, Theorem 4.4 holds with

$$T(\gamma) = \frac{2L}{\sqrt{1-\gamma/4}}, \quad C(T,\alpha,\gamma(\varepsilon)) = \frac{\lambda_J(h)+\alpha}{\lambda_J(h)} \frac{1}{C_1(T,\varepsilon)(1-\gamma/4)}. \tag{4.17}$$

The time observability is such that: $T(\gamma) \nearrow \infty$ as $\gamma \nearrow 4$ and $T(\gamma) \searrow 2L$ as $\gamma \searrow 0$. Clearly, the Ingham's type approach improves the estimates on observability inequalities (see Theorem 3.7). Indeed, the uniform observability inequality from Theorem 3.7 holds for $T > 2L$ and it is given by

$$F_h(0) \le \frac{1}{T-2L}\Big[\frac{h^3}{4}\sum_{j=0}^{J}\int_0^T \Big(\frac{\phi'_{j+1}-\phi'_j}{h}\Big)^2 dt + \alpha h \sum_{j=0}^{J}\int_0^T |\phi_j|^2 dt + \frac{L}{2}\int_0^T \big|\frac{\phi_J}{h}\big|^2 dt\Big],$$

which represents the discrete counterpart (uniformly) of

$$F(0) \le \frac{1}{T-2L}\Big[\alpha\int_0^T\int_0^L \phi^2\,dx\,dt + \frac{L}{2}\int_0^T \phi_x^2(L,t)\,dt\Big]. \tag{4.18}$$

Here, in contrast to the above result, we obtain a uniform observability inequality given by

$$F_h(0) \le C(T,\alpha,\gamma(\varepsilon))\frac{L}{2}\int_0^T \big|\frac{\phi_J}{h}\big|^2 dt, \tag{4.19}$$

which represents the discrete counterpart (uniformly) of the boundary observability inequality given by

$$F(0) \leq C(T,\alpha)\frac{L}{2}\int_0^T \phi_x^2(L,t)\,dt. \tag{4.20}$$

In that direction, the above constant $C(T,\alpha)$ is the limit of $C(T,\alpha,\gamma(\varepsilon))$ when $h \to 0$ and $\gamma \to 0$. Indeed, when h and γ approaches to zero, one has

$$\begin{aligned} C(T,\alpha,\gamma(\varepsilon)) &= \Big(1+\frac{\alpha}{\lambda_J(h)}\Big)\frac{1}{C_1(T,\varepsilon)(1-\gamma/4)} \\ &\to C(T,\alpha) = \Big[1+\alpha\Big(\frac{L}{J\pi}\Big)^2\Big]\frac{1}{C_1(T,\varepsilon)} > 0, \end{aligned}$$

where $\lambda_J(h) = \frac{4}{h^2}\sin^2\Big(\frac{J\pi h}{2L}\Big) \to \Big(\frac{J\pi}{L}\Big)^2$ when h approaches to zero. Analogously to the previous case, we have the following theorem:

Theorem 4.5. *Assume that $0 < \gamma < 4$. Then, for each $\alpha < \lambda_J(h)$ there exists $T(\gamma) > 0$ such that for any $T > T(\gamma)$ there exists a positive constant $\overline{C}(T,\alpha,\gamma(\varepsilon)) > 0$ such that*

$$G_h(0) \leq \overline{C}(T,\alpha,\gamma(\varepsilon))\frac{L}{2}\int_0^T \big|\frac{\psi_J}{h}\big|^2 dt, \tag{4.21}$$

for any solution of system (3.10)-(3.12) *in the class $\mathcal{P}_h(\gamma(\varepsilon))$ in* (4.3).

Proof. According to Ingham's inequality and in view of Lemma 4.3, for any $0 \leq \varepsilon < 1$ and $T > 2L/\cos(\pi\varepsilon/2)$ there exist positive constants $D_i(T,\varepsilon) > 0$, $i = 1,2$, satisfying

$$\begin{aligned} D_1(T,\varepsilon)\sum_{|\mu_k|h\leq\sqrt{\gamma(\varepsilon)}}|a_k|^2 &\leq \int_0^T\Big|\sum_{|\mu_k|h\leq\sqrt{\gamma(\varepsilon)}} a_k e^{i\mu_k t}\Big|^2 dt \\ &\leq D_2(T,\varepsilon)\sum_{|\mu_k|h\leq\sqrt{\gamma(\varepsilon)}}|a_k|^2, \end{aligned} \tag{4.22}$$

where $\gamma(\varepsilon) = 4\sin^2\Big(\frac{\pi\varepsilon}{2}\Big)$. Now, from Lemma 3.12, we have

$$\begin{aligned} h\sum_{j=0}^{J}\big|\frac{\varphi_{j+1}-\varphi_j}{h}\big|^2 &= \frac{2L}{4-(\nu^-(h)+\alpha)h^2}\big|\frac{\varphi_J}{h}\big|^2 = \frac{2L}{4-\lambda(h)h^2}\big|\frac{\varphi_J}{h}\big|^2 \\ &\leq \frac{L}{2\cos^2\big(\pi\varepsilon/2\big)}\big|\frac{\varphi_J}{h}\big|^2, \end{aligned} \tag{4.23}$$

for any eigenvector φ associated with an eigenvalue λ satisfying $\lambda h^2 \leq \gamma(\varepsilon)$. Let us consider the solution ψ of the system (3.10)-(3.12) in the class $\mathcal{P}_h(\gamma(\varepsilon))$, which is written as

$$\psi = \sum_{|\mu_k(h)|h\leq\sqrt{\gamma(\varepsilon)}} a_k e^{i\mu_k(h)t}\varphi^k, \tag{4.24}$$

where we are denoting $\mu_k(h) = \sqrt{\nu_j^-(h)}$. Then, combining (4.23) and (4.24) and taking $T > \frac{2L}{\cos(\pi\varepsilon/2)}$, we obtain

$$\frac{L}{2}\int_0^T\big|\frac{\psi_J}{h}\big|^2 dt = \frac{L}{2h^2}\int_0^T\Big|\sum_{|\mu_k|h\leq\sqrt{\gamma(\varepsilon)}} a_k e^{i\mu_k(h)t}\varphi_{k,J}\Big|^2 dt$$

$$
\begin{aligned}
&\geq D_1(T,\varepsilon)\cos^2\big(\pi\varepsilon/2\big)\sum_{|\mu_k|h\leq\sqrt{\gamma(\varepsilon)}}|a_k|^2 h\sum_{j=0}^{J}\Big|\frac{\varphi_{k,j+1}-\varphi_{k,j}}{h}\Big|^2\\
&= D_1(T,\varepsilon)\cos^2\big(\pi\varepsilon/2\big)\frac{\lambda_J(h)}{\lambda_J(h)-\alpha}G_h(0),
\end{aligned}
$$

and for any $\psi\in\mathcal{P}_h(\gamma(\varepsilon))$, we have

$$
G_h(0)\leq \overline{C}(T,\gamma)\frac{L}{2}\int_0^T\Big|\frac{\psi_J}{h}\Big|^2dt, \tag{4.25}
$$

where

$$
\overline{C}(T,\alpha,\gamma(\varepsilon))=\frac{\lambda_J(h)-\alpha}{\lambda_J(h)}\frac{1}{D_1(T,\varepsilon)\cos^2(\pi\varepsilon/2)}.
$$

This observability constant is positive for $\alpha<\lambda_J(h)=\frac{4}{h^2}\sin^2\big(\frac{J\pi h}{2L}\big)$. Therefore, the proof is complete. $\square$

Observe that for $T>\frac{2L}{\cos(\pi\varepsilon/2)}$ and since $\gamma=\gamma(\varepsilon)$ the previous theorem holds since

$$
T(\gamma)=\frac{2L}{\sqrt{1-\gamma/4}},\quad \overline{C}(T,\alpha,\gamma(\varepsilon))=\frac{\lambda_J(h)-\alpha}{\lambda_J(h)}\frac{1}{D_1(T,\varepsilon)\cos^2\big(\pi\varepsilon/2\big)}. \tag{4.26}
$$

It is clear that $T(\gamma)\nearrow\infty$ as $\gamma\nearrow 4$ and $T(\gamma)\searrow 2L$ as $\gamma\searrow 0$. The improvement in comparison with Theorem 3.6 is basically on uniform observability inequality and also on time observability. In Theorem 3.6, the uniform observability inequality

$$
G_h(0)\leq C(T,\alpha)\Big[\frac{h^3}{4}\sum_{j=0}^{J}\int_0^T\Big(\frac{\psi'_{j+1}-\psi'_j}{h}\Big)^2dt+\frac{L}{2}\int_0^T\Big|\frac{\psi_J}{h}\Big|^2dt\Big],
$$

holds for $T>T_0=2L\frac{\pi^2}{\pi^2-\alpha L^2}$ and for

$$
C(T,\alpha)=\frac{\pi^2-\alpha L^2}{T(\pi^2-\alpha L^2)-2L\pi^2}. \tag{4.27}
$$

Moreover, the uniform boundary observability

$$
G_h(0)\leq\overline{C}(T,\alpha,\gamma(\varepsilon))\frac{L}{2}\int_0^T\Big|\frac{\psi_J}{h}\Big|^2dt, \tag{4.28}
$$

represents the discrete counterpart (uniformly) of

$$
G(0)\leq\overline{C}(T,\alpha)\frac{L}{2}\int_0^T\psi_x^2(L,t)\,dt, \tag{4.29}
$$

and the constant $\overline{C}(T,\alpha)$ is the limit of $\overline{C}(T,\alpha,\gamma(\varepsilon))$ when $h\to 0$ and $\gamma\to 0$. Indeed, when h and γ approach to zero, one has

$$
\begin{aligned}
\overline{C}(T,\alpha,\gamma(\varepsilon))&=\frac{\lambda_J(h)-\alpha}{\lambda_J(h)}\frac{1}{D_1(T,\varepsilon)\cos^2\big(\pi\varepsilon/2\big)}\\
&\to C(T)=\Big[1-\alpha\Big(\frac{L}{J\pi}\Big)^2\Big]\frac{1}{D_1(T,\varepsilon)}>0,
\end{aligned}
$$

where $\lambda_J(h)=\frac{4}{h^2}\sin^2\big(\frac{J\pi h}{2L}\big)\to\big(\frac{J\pi}{L}\big)^2$ as $h\to 0$.

4.2. **Proof of the uniform boundary observability for coupled systems.** In this section, we use the Theorems 4.4 and 4.5 to show a uniform boundary observability to the system (3.2)-(3.5). In order to do this we consider the class of filtered solutions $\mathcal{R}_h(\gamma)$ which the solutions are given by

$$\begin{aligned} U_h &= \frac{1}{2}\sum_{\lambda_k(h)\le\gamma h^{-2}}\Big[a_k\sin(\sqrt{\nu_k^-(h)}\,t)+b_k\cos(\sqrt{\nu_k^-(h)}\,t)\\ &\quad + c_k\sin(\sqrt{\nu_k^+(h)}\,t)+d_k\cos(\sqrt{\nu_k^+(h)}\,t)\Big]\varphi^k,\\ V_h &= \frac{1}{2}\sum_{\lambda_k(h)\le\gamma h^{-2}}\Big[c_k\sin(\sqrt{\nu_k^+(h)}\,t)+d_k\cos(\sqrt{\nu_k^+(h)}\,t)\\ &\quad - a_k\sin(\sqrt{\nu_k^-(h)}\,t)-b_k\cos(\sqrt{\nu_k^-(h)}\,t)\Big]\varphi^k, \end{aligned}$$

where $a_k, b_k, c_k, d_k \in \mathbb{R}$.

Theorem 4.6. *Assume that $0 < \gamma < 4$. Then, for each $\alpha < \lambda_J(h)$ there exists $T(\gamma) > 0$ such that for any $T > T(\gamma)$ there exists a positive constant $\widetilde{C}(T, \alpha, \gamma(\varepsilon)) > 0$, such that*

$$\mathcal{E}_h(0) \le \widetilde{C}(T,\alpha,\gamma(\varepsilon))\frac{L}{2}\Big[\int_0^T \big|\frac{u_J}{h}\big|^2 dt + \int_0^T \big|\frac{v_J}{h}\big|^2 dt\Big], \tag{4.30}$$

for any solution of system (3.2)-(3.5) *in the class $\mathcal{R}_h(\gamma(\varepsilon))$.*

Proof. The proof follows by a combination of the results obtained in Theorems 4.5 and 4.4 and from decomposition of the energy $\mathcal{E}_h(t)$. Indeed, we have

$$\begin{aligned} 2\mathcal{E}_h(0) &= F_h(0) + G_h(0)\\ &\le \max\big\{C(T,\alpha,\gamma(\varepsilon)),\overline{C}(T,\alpha,\gamma(\varepsilon))\big\}\frac{L}{2}\int_0^T\Big[\big|\frac{\phi_J}{h}\big|^2+\big|\frac{\psi_J}{h}\big|^2\Big]dt, \end{aligned}$$

and taking into account the decompositions $\phi_j = u_j + v_j$ and $\psi_j = u_j - v_j$, we arrive at

$$\mathcal{E}_h(0) \le \widetilde{C}(T,\alpha,\gamma(\varepsilon))\frac{L}{2}\Big[\int_0^T \big|\frac{u_J}{h}\big|^2 dt + \int_0^T \big|\frac{v_J}{h}\big|^2 dt\Big], \tag{4.31}$$

where

$$\widetilde{C}(T,\alpha,\gamma(\varepsilon)) = 2\max\Big[\frac{1}{\cos^2\big(\pi\varepsilon/2\big)D(T,\varepsilon)},\frac{\lambda_J(h)+2\alpha}{\lambda_J(h)}\frac{1}{\cos^2\big(\pi\varepsilon/2\big)C_1(T,\varepsilon)}\Big], \tag{4.32}$$

and the conclusions are similar to the previous cases. □

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References

[1] D. S. Almeida Júnior, A.J. A. Ramos, M. L. Santos; Observability inequality for the finite-difference semi-discretization of the 1-d coupled wave equations, *Adv. Comput. Math.*, **41** (2015), 105–130.

[2] A. El Akri, L. Maniar; Indirect boundary observability of semi-discrete coupled wave equations, *Electron. J. of Differential Eqns.*, **18** (2018), 133, 1–27.

[3] A. El Akri, L. Maniar; Uniform indirect boundary controllability of semi-discrete 1-d coupled wave equations, *Mathematical Control & Related Fields*, **10(4)** (2019), 1–30.

[4] S. Ervedoza, A. Marica, E. Zuazua; Numerical meshes ensuring uniform observability of 1d waves: construction and analysis, *IMA J. Numerical Anal.*, **36 (2)** (2015), 503–542.

[5] S. Ervedoza, E. Zuazua; The wave equation: Control and numerics, *Lecture Notes in Mathematics , CIME Subseries. Springer Verlag*, (2011).

[6] R. Glowinski; Ensuring well-posedness by analogy: Stokes problem and boundary control for the wave equation, *Journal Comput. Physics*, **103** (1992), 189–221.

[7] R. Glowinski, J. L. Lions; Exact and Approximate Controllability for Distributed Parameter System, *Acta Numerica*, (1996), 159–333.

[8] R. Glowinski, C. H. Li, J. L. Lions; A numerical approach to the exact boundary controllability of the wave equation. (I) Dirichlet Controls: Description of the numerical methods, *Jap. J. Appl. Math*, (1990), 1–76.

[9] J. A. Infante, E. Zuazua; Boundary observability for the space semi-discretizations of the 1-D wave equation, *Math. Model. Num. An.*, **33** (1999), 407–438.

[10] A. E. Ingham; Some trigonometrical inequalities with applications to the theory of series, *Mathematische Zeitschrift*, **41** (1936), 367–379.

[11] V. Komornik; Exact Controllability and Stabilization. The Multiplier Method, *John Wiley & Sons-Masson*, 1994.

[12] J. L. Lions; Contrôlabilité exacte, stabilization et perturbations de systèmes distribués, *Tome 1. Contrôlabilité exacte, Masson, RMA8*, (1988).

[13] M. Najafi; The study of stabilizability of the coupled wave equations under various end conditions, *The 3 IEEE Conference on Decision and Control,* (1992), 374–379.

[14] L. R. Tcheugoué Tébou, E. Zuazua; Uniform exponential long time decay for the space semi-discretization of a locally damped wave equation via an artificial numerical viscosity, *Numerische Mathematik*, **95** (2003), 563–598.

[15] L. R. T. Tebou; Sharp Observability Estimates for a System of Two Coupled Nonconservative Hyperbolic Equations, *Applied Mathematics & Optimization*, **66** (2012), 2, 175–207.

[16] R. Temam; Infinity-Dimensional Dynamical Systems in Mechanics and Physics, *New York: Springer.*

[17] D. Xu; Observability inequality for piecewise Hermite cubic orthogonal spline collocation semidiscretization of the wave's Petrovsky system with memory, *ZAMM Journal of applied mathematics and mechanics: Zeitschrift für angewandte Mathematik und Mechanik*, **100(2)** (2020), 1–34.

[18] E. Zuazua; Propagation, Observation, Control and Numerical Approximation of Waves, *SIAM Rev.*, **47** (2005), 197–243.

[19] E. Zuazua; Contrôlabilité exacte en un temps arbitrairement petit de quelques modèles de plaques, *Contrôlabilité exacte, stabilization et perturbations de systèmes distribués. Tome 1. Contrôlabilité exacte, Masson, RMA8*, (1988), 465–491.

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