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ORLICZ ESTIMATES FOR GENERAL PARABOLIC OBSTACLE PROBLEMS WITH $p(t,x)$-GROWTH IN REIFENBERG DOMAINS

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Abstract. This article shows a global gradient estimate in the framework of Orlicz spaces for general parabolic obstacle problems with $p(t,x)$-Laplacian in a bounded rough domain. It is assumed that the variable exponent $p(t,x)$ satisfies a strong log-Hölder continuity, that the associated nonlinearity is measurable in the time variable and have small BMO semi-norms in the space variables, and that the boundary of the domain has Reifenberg flatness.

1. Introduction

We devote this article to obtaining a nonlinear Calderón-Zygmund type estimate in the framework of Orlicz spaces for general parabolic obstacle problems of nonstandard growths with weaker regularity assumptions imposed on given datum. First, let us review recent studies on the related topic. The Calderón-Zygmund estimate for elliptic p-Laplacian in the scalar setting $N=1$ had been first obtained by Iwaniec [17], while the vectorial setting $N>1$ was treated by DiBenedetto and Manfredi [14]. An extension to general elliptic equations with VMO leading coefficients was achieved by Kinnunen and Zhou [22]. Recently, a nonlinear Calderón-Zygmund estimate for parabolic obstacle problems involving possibly degenerate operators of p-growth was obtained by Bögelein, Duzaar and Mingione [6]. Byun and Cho [8] also established a local Calderón-Zygmund estimate for parabolic variational inequalities of general type degenerate and singular operators in divergence form, and they proved that for any $q\in(1,\infty)$ it holds

$$|\psi_t|^{p'},\ |D\psi|^p,\ |F|^p\in L^q_{\rm loc}(\Omega_T)\Longrightarrow |Du|^p\in L^q_{\rm loc}(\Omega_T).$$

A local regularity version in Lorentz spaces for the gradients of weak solution to parabolic obstacle problems has been also achieved by Baroni [3]. Later, Byun and Cho in [9] showed a global regularity in Orlicz spaces for the gradients of weak solution to parabolic variational inequalities of p-Laplacian type under weak assumptions that the nonlinearities are merely measurable in the time-variable and have small BMO semi-norms in the spatial variables, while the underlying domain is a Reifenberg flatness. Tian and Zheng [27] also derived a global weighted Lorentz estimate to nonlinear parabolic equations with partial regular nonlinearity in a nonsmooth domain. On the other hand, we would like to mention that Zhang and

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Zheng [28] got Lorentz estimates for asymptotically regular elliptic equations in quasiconvex domains, and Liang and Zheng [19] established the gradient estimate in Orlicz spaces for elliptic obstacle problems with partially BMO nonlinearities. Very recently, Liang, Zheng and Feng [20] showed a global Calderón-Zygmund type estimate in Lorentz spaces for a variable power of the gradients of weak solution pair (u, P) to generalized steady Stokes system in a bounded Reifenberg domain.

Nonlinear elliptic and parabolic problem under consideration with a variable growth naturally originates from some mathematical modeling of fluid dynamics, such as certain models for non-Newtonian fluids and electrorheological fluids. Indeed, there are also various phenomena involved some energy functionals, for example, elastic mechanics, porous media problems, and thermistor problems (cf. [26]). Therefore, it is a rather interesting topic in the fields of analysis and PDEs to get nonlinear Calderón-Zygmund theory for general elliptic and parabolic equations with variable growths. In recent decades, a lot of attention has been paid to a systematic study on the Calderón-Zygmund theory for nonlinear elliptic and parabolic problems with nonstandard growths. For instance, some regularities regarding general elliptic equations of $p(x)$-growth have been treated by Acerbi and Mingione [1]. Naturally, there also have been many interesting theoretic developments involving more general obstacle problems since this kind of problems of variable growths always appeared in various phenomena of physical applications. It was observed by Bögelein and Duzaar in [5] that it holds a higher integrability for the gradients of weak solutions to possibly degenerate parabolic systems with nonstandard growth. Later Baroni and Bögelein in [4] showed nonlinear Calderón-Zygmund estimate for evolutionary $p(t, x)$-Laplacian system in requiring the variable exponent $p(t, x)$ being a logarithmic Hölder continuity and the coefficients $a(t, x)$ satisfying VMO condition in the spatial variables. Erhardt [16] considered an interior L^q-estimate of $|Du|^{p(t,x)}$ for general parabolic variational inequality in the weak form as

$$\begin{aligned} &\langle \phi_t, \phi - u\rangle_{\Omega_T} + \int_{\Omega_T} a(t,x)|Du|^{p(t,x)-2} Du \cdot D(\phi - u)\,dx\,dt \\ &+ \frac{1}{2}\|\phi(a,\cdot) - u_a\|^2_{L^2(\Omega)} \\ &\geq \int_{\Omega_T} |\mathbf{f}|^{p(t,x)-2}\mathbf{f} \cdot D(\phi - u)\,dx\,dt, \end{aligned} \tag{1.1}$$

and he showed that $|Du|^{p(t,x)}$ belongs to a local integrability with the same index as an assigned obstacle $|D\psi|^{p(t,x)}$, $|\psi_t|^{\gamma_1'}$ as well as $|\mathbf{f}|^{p(t,x)}$, which implies that

$$|\psi_t|^{\gamma_1'},\ |D\psi|^{p(t,x)},\ |\mathbf{f}|^{p(t,x)} \in L^q_{\rm loc}(\Omega_T) \Longrightarrow |Du|^{p(t,x)} \in L^q_{\rm loc}(\Omega_T)$$

for any $q \in (1, \infty)$. On the other hand, Li [24] handled a higher integrability for the derivatives of very weak solutions to parabolic systems of $p(t, x)$-Laplacian type with the inhomogeneity being different growths, respectively. Furthermore, Bui and Duong [7] derived global weighted estimate in Lorentz spaces for nonlinear parabolic equations of $p(t, x)$-growth in a Reifenberg flat domain with the nonlinearities $\mathbf{a}(t, x; \xi)$ being small BMO in the spatial variables, while the variable growth $p(t, x)$ satisfying a strong log-Hölder continuity. Byun and Ok [10] reached a global $L^{s(t,x)}$-integrability with $s(t, x) > p(t, x)$ for the gradients of weak solution to general parabolic equations of $p(t, x)$-growth in Reifenberg flat domains by imposing the same weak regular assumptions as shown in [7] on $\mathbf{a}(t, x; \xi)$, $p(t, x)$ and the boundary of the underlying domains. Li, Zhang and Zheng [18] established a local Orlicz estimate for nondivergence linear elliptic equations with partially BMO coefficients, and Chlebicka in [12] provided the Lorentz and Morrey estimates for the gradients of solution to general nonlinear elliptic equations with the datum of Orlicz growths. Byun and Park [11] considered global weighted Orlicz estimate to nonlinear parabolic equation with measurable nonlinearity in a bounded nonsmooth domain while the right-hand side is of finite signed Radon measure.

This article is inspired by these above-mentioned recent progresses. The aim of this article is to show a global Calderón-Zygmund type estimate in Orlicz spaces for nonlinear parabolic obstacle problems of nonstandard growth with weaker regularity assumptions on the given datum, which means an implication that

$$|\psi_t|^{\gamma_1'},\ |D\psi|^{p(t,x)},\ |\mathbf{f}|^{p(t,x)} \in L^{\phi}(\Omega_T) \Longrightarrow |Du|^{p(t,x)} \in L^{\phi}(\Omega_T) \tag{1.2}$$

for Young's function $\phi \in \Delta_2 \cap \nabla_2$ defined below. As we know, the Orlicz space is a generalization of Lebesgue spaces. Jia, Li and Wang [21] recently obtained a global Orlicz estimate to linear elliptic equations of divergence form with small BMO coefficients in Reifenberg flat domains. Byun and Cho [9] obtained Orlicz estimates to parabolic obstacles problems of p-Laplacian type for $\frac{2d}{d+2} < p < \infty$, they proved that

$$|\psi_t|^{p'},\ |D\psi|^{p},\ |F|^{p} \in L^{\phi}(\Omega_T) \Longrightarrow |Du|^{p} \in L^{\phi}(\Omega_T)$$

for $\phi \in \Delta_2 \cap \nabla_2$ while the nonlinearity is small BMO in spatial variables and the domain is Reifenberg flatness.

A key ingredient under consideration is the power $p(t,x)$ being a variable function with respect to the independent variables (t,x). In this way, the Hardy-Littlewood maximal operators technique does not work well for parabolic equations of $p(t,x)$-growth since the usual scaling arguments used for $p = 2$ do not work smoothly. The main difficulty for parabolic setting comes from the nonhomogeneous scaling behavior for variational inequalities so that any solution multiplied by a constant is in general no longer a solution of original problem. We here employ the technique of the so-called intrinsic parabolic cylinder first introduced by DiBenedetto and Friedman [13], which applies the time-space scaling dependent on a local behavior of the solution itself to re-balance the nonhomogeneous scaling for parabolic problem of p-Laplacian. Another point is that we adapt the so-called large-M-inequality principle from Acerbi and Mingione's work [2] to our situation with non-trivial modifications and significant improvements. In order to get a suitable power decay for the following upper level we set

$$\left\{(t,x) \in \Omega_T : |Du|^{p(t,x)} > \kappa\right\}$$

with the scaling parameter $\kappa > 0$ sufficiently large, we make use of the so-called stop-time argument and the modified Vitali type covering with a countable covering by the intrinsic parabolic cylinder $\{Q^{\kappa}_{r_i}(\tau_i, y_i)\}_{i=1}^{\infty}$ satisfying

$$\fint_{Q^{\kappa}_{r_i}(\tau_i,y_i)} |Du|^{p(t,x)}\,dx\,dt \approx \kappa,$$

which will be discussed in Section 3.

The rest of this article is organized as follows. In the next section we present the weaker regular assumptions on the datum, and state our main result. Section 3 is to give necessary preliminary lemmas, in which shows various comparison estimates to the reference problems. Finally, we devote Section 4 to the proof of main Theorem 2.5.

2. Minimal assumptions on the datum and main result

Let Ω be a bounded domain in $\mathbb{R}^d$ for $d \geq 2$ with its rough boundary $\partial\Omega$ specified later. For a fixed $a \in \mathbb{R}$ and $0 < T < \infty$, let $\Omega_T = (a, a+T) \times \Omega$ denote the parabolic cylinder in $\mathbb{R} \times \mathbb{R}^d$, and the typical parabolic boundary $\partial\Omega_T = \big((a, a+T) \times \partial\Omega\big) \cup \big(\{t = a\} \times \overline{\Omega}\big)$ be the typical parabolic boundary of Ω_T. We suppose that the main nonlinearity

$$\mathbf{a}(t,x;\xi) = \Big(a^1(t,x;\xi), a^2(t,x;\xi), \dots, a^d(t,x;\xi)\Big) : \Omega_T \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$$

is a Carathéodory vectorial-valued function with the following basic structural conditions: for a.a. $(t,x) \in \Omega_T$ and all $\xi, \eta \in \mathbb{R}^d$, there exist constants $0 < \lambda \leq 1 \leq \Lambda$ and $0 \leq \mu \leq 1$ such that

$$\begin{gathered}\lambda\Big(\mu^2 + |\xi|^2\Big)^{\frac{p(t,x)-2}{2}} |\eta|^2 \leq D_\xi \mathbf{a}(t,x;\xi)\eta \cdot \eta, \\ |\mathbf{a}(t,x;\xi)| + \Big(\mu^2 + |\xi|^2\Big)^{1/2} |D_\xi \mathbf{a}(t,x;\xi)| \leq \Lambda \left(\mu^2 + |\xi|^2\right)^{\frac{p(t,x)-1}{2}}.\end{gathered} \tag{2.1}$$

Let the given obstacle function $\psi : \Omega_T \to \mathbb{R}$ satisfy

$$\begin{gathered}\psi \in C^0([a, a+T]; L^2(\Omega)) \cap W^{p(t,x)}(\Omega_T), \quad \psi_t \in L^{\gamma_1'}(a, a+T; W^{-1,\gamma_1'}(\Omega)), \\ \psi \leq 0 \text{ a.e. on } (a, a+T) \times \partial\Omega, \quad \psi(a, \cdot) \leq 0 \text{ a.e. on } \Omega;\end{gathered} \tag{2.2}$$

and let an initial value u_a be such that

$$u_a = u(a, \cdot) \in L^2(\Omega) \quad \text{and} \quad u_a \geq \psi(a, \cdot) \quad \text{a.e. on } \Omega.$$

We introduce an admissible set defined by

$$\mathcal{A}(\Omega_T) = \big\{\phi \in C^0([a, a+T]; L^2(\Omega)) \cap W_0^{p(t,x)}(\Omega_T) : \phi \geq \psi \text{ a.e. on } \Omega_T\big\}. \tag{2.3}$$

Note that a minimizing the energy functional with certain constraint in $\mathcal{A}(\Omega_T)$ immediately leads to the following form: for $u = u(t,x) \in \mathcal{A}(\Omega_T)$ it holds in the weak form of the parabolic variational inequality

$$\begin{aligned}&\langle \phi_t, \phi - u\rangle_{\Omega_T} + \int_{\Omega_T} \mathbf{a}(t,x,Du) \cdot D(\phi - u)\,dx\,dt + \frac{1}{2}\|\phi(a, \cdot) - u_a\|^2_{L^2(\Omega)} \\ &\geq \int_{\Omega_T} |\mathbf{f}|^{p(t,x)-2}\mathbf{f} \cdot D(\phi - u)\,dx\,dt\end{aligned} \tag{2.4}$$

for all test functions $\phi \in \mathcal{A}'(\Omega_T) = \big\{\phi \in \mathcal{A}(\Omega_T) : \phi_t \in (W^{p(t,x)}(\Omega_T))'\big\}$, where $\mathbf{f} \in L^{p(t,x)}(\Omega_T)$ is a given inhomogeneous term.

For convenience, throughout this paper we assume that $R \leq 1$ is an arbitrary given positive number, while $\delta \in (0, 1/8)$ is to be determined later. Let us now endow the variable exponent $p(t,x) : \Omega_T \to \mathbb{R}$ with the regularity of the so-called strong log-Hölder continuity. We write $d_p(z_1, z_2)$ the parabolic distance by $d_p(z_1, z_2) := \max\big\{|x - y|, \sqrt{|\tau - t|}\big\}$ for any $z_1 = (t,x), z_2 = (\tau, y) \in \mathbb{R}^{d+1}$. We say that $p(t,x)$ is *locally strong log-Hölder continuous*, denoted by $p(t,x) \in SLH(\Omega_T)$, if for some constant $\bar{\rho} > 0$ such that for all $z_1 = (t,x), z_2 = (\tau, y) \in \Omega_T$ with

$0 < d_p(z_1, z_2) < \bar{\rho}$, one has that there exists a nondecreasing continuous function $\omega(\cdot) : [0,\infty) \longrightarrow [0,1]$ satisfying $\omega(0) = 0$ such that

$$|p(z_1) - p(z_2)| \le \omega(d_p(z_1, z_2)) \tag{2.5}$$

with

$$\limsup_{\rho\to 0} \omega(\rho) \log\Big(\frac{1}{\rho}\Big) = 0.$$

It is easy to check that if $p(t,x)$ is a strong logarithmic Hölder continuity, then for any given $\delta \in (0, 1/8)$ there exists a small $R > 0$ such that

$$\sup_{0<\rho<R} \omega(\rho) \log\Big(\frac{1}{\rho}\Big) \le \delta. \tag{2.6}$$

Regarding the parabolic problems with variable exponent growth in the context, the exponent $p(t,x) : \Omega_T \to \mathbb{R}$ is supposed to be a strong log-Hölder continuity (2.5) with the constraint (2.6); moreover, there exist constants γ_1 and γ_2 such that the range distribution by

$$\frac{2d}{d+2} < \gamma_1 := \inf_{\Omega_T} p(t,x) \le \gamma_2 := \sup_{\Omega_T} p(t,x) < \infty. \tag{2.7}$$

Indeed, to ensure the solvability for nonlinear parabolic problems of p-Laplacian type, the lower bound $\gamma_1 > \frac{2d}{d+2}$ is unavoidable even in the constant exponent setting $p(z) \equiv p$, for more details see [10, Section 2]. With the assumptions (2.1) (2.2) (2.6) and (2.7) in hand, the existence of such weak solution is ensured by the result from Erhardt [16], which leads to that there exists a unique weak solution $u \in \mathcal{A}(\Omega_T)$ to the parabolic variational inequality (2.4) with the estimate

$$\begin{aligned} &\sup_{t\in[a,a+T]} \int_\Omega |u(t,x)|^2\,dx + \int_{\Omega_T} |Du|^{p(t,x)}\,dx\,dt \\ &\le C\Big(\int_{\Omega_T} \Big(|\psi_t|^{\gamma_1'} + |D\psi|^{p(t,x)} + |\mathbf{f}|^{p(t,x)} + 1\Big)\,dx\,dt\Big), \end{aligned} \tag{2.8}$$

where C is a positive constant depending only on $d, \gamma_1, \gamma_2, \lambda, \Lambda$ and $\|u_a\|_{L^2(\Omega)}$, see also [16, Theorem 7.1].

We now recall that the space $L^{p(t,x)}(\Omega_T)$ is defined to be the set of these measurable functions $g(t,x) : \Omega_T \to \mathbb{R}^k$ for $k \in \mathbb{N}$, which satisfies $|g|^{p(t,x)} \in L^1(\Omega_T)$, i.e.

$$L^{p(t,x)}(\Omega_T) := \big\{g(t,x) : \Omega_T \to \mathbb{R}^k \text{ is measurable in } \Omega_T : \int_{\Omega_T} |g|^{p(t,x)}\,dx\,dt < +\infty\big\},$$

which is a Banach space equipped with the Luxemburg norm

$$\|g\|_{L^{p(t,x)}(\Omega_T)} := \inf\Big\{\lambda > 0 : \int_{\Omega_T} \big|\frac{g}{\lambda}\big|^{p(t,x)}\,dx\,dt \le 1\Big\}. \tag{2.9}$$

The Sobolev spaces $W^{p(t,x)}(\Omega_T)$ is defined by

$$W^{p(t,x)}(\Omega_T) := \big\{g \in L^{p(t,x)}(\Omega_T) : Dg \in L^{p(t,x)}(\Omega_T)\big\}$$

endowed with the norm

$$\|g\|_{W^{p(t,x)}(\Omega_T)} := \|g\|_{L^{p(t,x)}(\Omega_T)} + \|Dg\|_{L^{p(t,x)}(\Omega_T)}. \tag{2.10}$$

It would be worthwhile to mention that for $g \in W_0^{p(t,x)}(\Omega_T)$ it indicates that $g(t,x) = 0$ in the sense of trace on the boundary of Ω. For $1 < p(t,x) < \infty$, we also denote the dual space of $W_0^{p(t,x)}(\Omega_T)$ by $(W^{p(t,x)}(\Omega_T))'$, which means that for $g \in (W^{p(t,x)}(\Omega_T))'$ there exist functions $g_i \in L^{p'(t,x)}(\Omega_T)$ with $p'(t,x) = \frac{p(t,x)}{p(t,x)-1}$ for $i = 0, 1, \dots, d$ such that the dual parting

$$\langle g, w\rangle_{\Omega_T} = \int_{\Omega_T} \Big(g_0 w + \sum_{i=1}^d g_i D_i w \Big)\, dx\, dt$$

for all $w \in W_0^{p(t,x)}(\Omega_T)$. In particular, if $p(t,x) = \gamma_1$ it yields that

$$W^{\gamma_1}(\Omega_T) = L^{\gamma_1}\big(a, a+T; W^{1,\gamma_1}(\Omega)\big).$$

Consequently, the dual space of $W_0^{\gamma_1}(\Omega_T)$ is given by

$$\big(W_0^{\gamma_1}(\Omega_T)\big)' = \Big(L^{\gamma_1}(a, a+T; W_0^{1,\gamma_1}(\Omega))\Big)' = L^{\gamma_1'}(a, a+T; W^{-1,\gamma_1'}(\Omega)),$$

where $\frac{1}{\gamma_1} + \frac{1}{\gamma_1'} = 1$.

Now we impose some regularity assumptions on the nonlinearities $\mathbf{a}(t,x;\xi)$ and on the boundary $\partial\Omega$ of domain. For this, let $\rho, \theta > 0$, $B_\rho(y) = \{x \in \mathbb{R}^d : |x-y| < \rho\}$, and the local parabolic cylinders

$$Q_{(\theta,\rho)}(\tau, y) = (\tau - \theta, \tau + \theta) \times B_\rho(y)$$

with any $(\tau, y) \in \mathbb{R} \times \mathbb{R}^d$. For the abbreviations, $B_\rho = B_\rho(0)$, $Q_{(\theta,\rho)} = Q_{(\theta,\rho)}(0,0)$ and $Q_\rho = Q_{(\rho^2,\rho)}$, we measure the oscillation of $\mathbf{a}(t,x;\xi)/(\mu^2 + |\xi|^2)^{\frac{p(t,x)-1}{2}}$ in the x-variables over the ball $B_\rho(y)$ by

$$\Theta[\mathbf{a}; B_\rho(y)](t,x) := \sup_{\xi\in\mathbb{R}^d} \Big| \frac{\mathbf{a}(t,x;\xi)}{(\mu^2 + |\xi|^2)^{\frac{p(t,x)-1}{2}}} - \Big(\frac{\mathbf{a}(t,\cdot,\xi)}{(\mu^2 + |\xi|^2)^{\frac{p(t,\cdot)-1}{2}}} \Big)_{B_\rho(y)} \Big|,$$

where

$$\Big(\frac{\mathbf{a}(t,\cdot,\xi)}{(\mu^2 + |\xi|^2)^{\frac{p(t,\cdot)-1}{2}}} \Big)_{B_\rho(y)} := \frac{1}{|B_\rho(y)|} \int_{B_\rho(y)} \frac{\mathbf{a}(t,x;\xi)}{(\mu^2 + |\xi|^2)^{\frac{p(t,x)-1}{2}}}\, dx$$

represents an integral average of $\mathbf{a}(t,x;\xi)/(\mu^2 + |\xi|^2)^{\frac{p(t,x)-1}{2}}$ in the x-variables over $B_\rho(y)$ for any fixed $\xi \in \mathbb{R}^d$ and $t \in \mathbb{R}$.

Assumption 2.1. Let $\delta \in (0, 1/8)$ to be specified later. We say that $(\mathbf{a}, \Omega_T)$ is a (δ, R)-vanishing in the spatial variables, if for every point $(\tau, y) \in \Omega_T$ there exists a constant $0 < R \le 1$ such that for any $\rho \in (0, R)$ the following relation holds: (i) If

$$\operatorname{dist}(y, \partial\Omega) = \min_{x\in\partial\Omega} \operatorname{dist}(y, x) > \sqrt{2}\rho,$$

then there exists a coordinate system depending on (τ, y) and ρ, whose variables are still denoted by (t, x) such that in this new coordinate system (τ, y) is the origin, and for every $\theta \in (0, \rho^2)$ one has

$$\fint_{Q_{(\theta,\rho)}(\tau,y)} \big|\Theta[\mathbf{a}; B_\rho(y)](t,x)\big|^2\, dx\, dt \le \delta^2;$$

(ii) while

$$\operatorname{dist}(y, \partial\Omega) = \min_{x\in\partial\Omega} \operatorname{dist}(y, x) = \operatorname{dist}(y, \bar{y}) \le \sqrt{2}\rho$$

for some $\bar{y} \in \partial\Omega$, there exists a new coordinate system depending on (τ, y) and ρ, whose variables are denoted by (t, x), such that in this new coordinate system $(\tau, \bar{y})$ is the origin, and for any $\theta \in \big(0, (3\rho)^2\big)$ it holds

$$B_{3\rho}(\bar{y}) \cap \{x_1 > 3\delta\rho\} \subset B_{3\rho}(\bar{y}) \cap \Omega \subset B_{3\rho}(\bar{y}) \cap \{x_1 > -3\delta\rho\} \tag{2.11}$$

and

$$\fint_{Q_{(\theta,3\rho)}(\tau,\bar{y})} \big|\Theta[\mathbf{a}; B_{3\rho}(\bar{y})](t,x)\big|^2 \, dx\, dt \le \delta^2.$$

Remark 2.2. Roughly speaking, the nonlinearity $\mathbf{a}(t, x; \xi)/(\mu^2 + |\xi|^2)^{\frac{p(t,x)-1}{2}}$ is assumed to be a small BMO semi-norm in the x-variables, but there is no regular requirement in the t-variable, uniformly in $\xi \in \mathbb{R}^d$; while the domain Ω is assumed to be the (δ, R)-Reifenberg flatness as a necessary geometric condition if (2.11) holds, which leads to the following measure density conditions:

$$\sup_{0<r\le R_2} \sup_{x_0\in\partial\Omega} \frac{|B_r(x_0)|}{|\Omega \cap B_r(x_0)|} \le \Big(\frac{2}{1-\delta}\Big)^d \tag{2.12}$$

and

$$\inf_{0<r\le R_2} \inf_{x_0\in\partial\Omega} \frac{|\Omega^c \cap B_r(x_0)|}{|B_r(x_0)|} \ge \Big(\frac{1-\delta}{2}\Big)^d, \tag{2.13}$$

which actually guarantees a local reverse Hölder inequality automatically holds on the boundary.

It is our aim to obtain global Calderón-Zygmund type estimate in Orlicz spaces for nonlinear parabolic obstacle problems. For this, let Φ consist of all functions $\phi : \mathbb{R} \to [0, \infty)$ which are nonnegative, even, nondecreasing on $[0, \infty)$ and $\phi(0^+) = 0$, $\lim_{\nu\to\infty} \phi(\nu) = \infty$. We say that ϕ is Young function, if $\phi \in \Phi$ is convex and $\lim_{\nu\to 0^+} \frac{\phi(\nu)}{\nu} = \lim_{\nu\to\infty} \frac{\nu}{\phi(\nu)} = 0$. To make the function ϕ grow moderately near 0 and ∞, the Young function ϕ is said to be global Δ_2-condition, denoted by $\phi \in \Delta_2$, if there exists a positive constant $\bar{K}$ such that for every $\nu > 0$ with

$$\phi(2\nu) \le \bar{K}\phi(\nu). \tag{2.14}$$

On the other hand, the Young function ϕ is said to be global ∇_2-condition, denoted by $\phi \in \nabla_2$, if there exists a constant $\bar{a} > 1$ such that for every $\nu > 0$ one has

$$\phi(\nu) \le \frac{\phi(\bar{a}\nu)}{2\bar{a}}. \tag{2.15}$$

Remark 2.3. Actually, $\phi \in \Delta_2$ implies that for any $\beta_1 > 1$ there exists $\alpha_1 = log_2\bar{K}$ such that $\phi(\beta_1\nu) \le \bar{K}\beta_1^{\alpha_1}\phi(\nu)$, which describes the growth for $\phi(\nu)$ near $\nu = \infty$. Meanwhile, the condition $\phi \in \nabla_2$ means that for any $0 < \beta_2 < 1$, there exists $\alpha_2 = log_{\bar{a}}2 + 1$ such that $\phi(\beta_2\nu) \le 2\bar{a}\beta_2^{\alpha_2}\phi(\nu)$, and it describes the growth for $\phi(\nu)$ near $\nu = 0$. The simplest example for $\phi(\nu)$ satisfying the $\Delta_2 \cap \nabla_2$ condition is the power function $\phi(\nu) = \nu^p$ with $p > 1$. Moreover, we also remark that for $p > 1$, $\phi(\nu) = |\nu|^p(1 + |log|\nu||) \in \Delta_2 \cap \nabla_2$.

Definition 2.4. Let $\mathcal{D}$ be an open subset in $\mathbb{R}^{d+1}$ and ϕ be a Young function. The Orlicz class $K^\phi(\mathcal{D})$ is called to be the set of all measurable functions $g : \mathcal{D} \to \mathbb{R}$ satisfying

$$\int_{\mathcal{D}} \phi\left(|g|\right) \, dx\, dt < \infty.$$

Orlicz space $L^\phi(\mathcal{D})$ is just a linear hull of $K^\phi(\mathcal{D})$. It consists of all measurable functions f such that $\hat{\eta} f \in K^\phi(\mathcal{D})$ for some $\hat{\eta} > 0$. Moreover, the norm $\|\cdot\|_{L^\phi(\mathcal{D})}$ is denoted by

$$\|g\|_{L^\phi(\mathcal{D})} = \inf\big\{\lambda > 0 : \int_{\mathcal{D}} \phi\big(\frac{|g|}{\lambda}\big)\,dx\,dt \le 1\big\}.$$

If $\mathcal{D}$ is bounded, then

$$L^{\alpha_1}(\mathcal{D}) \subset L^\phi(\mathcal{D}) \subset L^{\alpha_2}(\mathcal{D}) \subset L^1(\mathcal{D})$$

with the constants α_1 and α_2 as Remark 2.3, for more details see [25]. We are now in a position to state the main result of this paper.

Theorem 2.5. *Let the Young function $\phi \in \Delta_2 \cap \nabla_2$, and $p(t,x) \in SLH(\Omega_T)$ with its range in $[\gamma_1, \gamma_2]$ shown as (2.7). Assume that $u \in \mathcal{A}(\Omega_T)$ is a weak solution of the variational inequality (2.4) with the given datum*

$$|\psi_t|^{\gamma_1'},\ |D\psi|^{p(t,x)},\ \ |\boldsymbol{f}|^{p(t,x)} \in L^\phi(\Omega_T).$$

Then, there exists a small positive constant $\delta = \delta(d, \lambda, \Lambda, \gamma_1, \gamma_2, \partial\Omega)$ such that if $(\mathbf{a}, \Omega_T)$ satisfies (δ, R)-vanishing as Assumption 2.1, then we have $|Du|^{p(t,x)} \in L^\phi(\Omega_T)$ with the estimate

$$\int_{\Omega_T} \phi\big(|Du|^{p(t,x)}\big)\,dx\,dt \le C\Big[\phi\Big(\Big(\fint_{\Omega_T} \Psi(t,x)\,dx\,dt\Big)^m\Big) + \int_{\Omega_T} \phi(\Psi(t,x))\,dx\,dt\Big],$$

where

$$C = C(d, \gamma_1, \gamma_2, \lambda, \Lambda, \alpha_1, \alpha_2, \delta, R, T, |\Omega|, \|u_a\|_{L^2(\Omega)}),$$
$$\Psi(t,x) = |\psi_t|^{\gamma_1'} + |D\psi|^{p(t,x)} + |\boldsymbol{f}|^{p(t,x)} + 1,$$

and $m \ge 1$ with

$$m = \sup_{(\tau,y)\in\Omega_T} m(\tau, y), \tag{2.16}$$

$$m(\tau,y) = \begin{cases} \frac{p(\tau,y)}{2} & \text{if } p(\tau,y) \ge 2,\\ \frac{2p(\tau,y)}{p(\tau,y)(d+2)-2d} & \text{if } \frac{2d}{d+2} < p(\tau,y) < 2. \end{cases} \tag{2.17}$$

3. Comparison estimates to the reference problems

We start this section with introducing some related notations and basic facts which will be useful in the paper. Throughout the paper, we always use C_i and c_i for $i = 1, 2, \dots$, to denote positive constants that only depend on $d, \lambda, \Lambda, \gamma_1, \gamma_2, \dots$, but whose values may differ from line to line. For any fixed point $z = (\tau, y) \in \mathbb{R}^{d+1}$ with $\tau \in \mathbb{R}$ and $y \in \mathbb{R}^d$, we denote the spatial open ball $B_\rho(y) \subset \mathbb{R}^d$ with center y and the radius $\rho > 0$. For any $\kappa > 1$ we write the *intrinsic parabolic cylinder* by

$$Q_\rho^\kappa(z) = Q_\rho^\kappa(\tau, y) = \Big(\tau - \kappa^{\frac{2-p(z)}{p(z)}}\rho^2, \tau + \kappa^{\frac{2-p(z)}{p(z)}}\rho^2\Big) B_\rho(y). \tag{3.1}$$

We also set

$$\Omega_\rho = \Omega \cap Q_\rho, \quad K_\rho^\kappa(z) = Q_\rho^\kappa(z) \cap \Omega_T,$$
$$\partial Q_\rho^\kappa(z) = \Big(\tau - \kappa^{\frac{2-p(z)}{p(z)}}\rho^2, \tau + \kappa^{\frac{2-p(z)}{p(z)}}\rho^2\Big)\partial B_\rho(y),$$
$$\partial K_\rho^\kappa(z) = \Big(Q_\rho^\kappa(z) \cap \big((a, a+T) \times \partial\Omega\big)\Big) \cup \Big(\partial Q_\rho^\kappa(z) \cap \Omega_T\Big)$$

for $a \in \mathbb{R}$ and $T > 0$. For the sake of convenience, while $z = (\tau, y) = (0,0)$ we simply write $Q_\rho^\kappa = Q_\rho^\kappa(0)$, $K_\rho^\kappa = K_\rho^\kappa(0)$ and $\partial K_\rho^\kappa = \partial K_\rho^\kappa(0)$. In the following we write

$$B_\rho^+(y) = B_\rho(y) \cap \{x_1 > 0\}, \quad Q_\rho^{\kappa +}(z) = \Big(\tau - \kappa^{\frac{2-p(z)}{p(z)}} \rho^2, \tau + \kappa^{\frac{2-p(z)}{p(z)}} \rho^2\Big) B_\rho^+(y),$$
$$T_\rho^\kappa(z) = \Big(\tau - \kappa^{\frac{2-p(z)}{p(z)}} \rho^2, \tau + \kappa^{\frac{2-p(z)}{p(z)}} \rho^2\Big)\Big(B_\rho(y) \cap \{x_1 = 0\}\Big).$$

Also, we briefly denote $B_\rho^+ = B_\rho^+(0)$, $Q_\rho^{\kappa+} = Q_\rho^{\kappa+}(0)$ and $T_\rho^\kappa = T_\rho^\kappa(0)$.

We use the following localizing technique, which is first used by Bögelein and Duzaar in [5]. As we know, an interior estimate for parabolic obstacle problems with nonstandard growth had been obtained by Erhardt in [16]. Owing to the measure density (2.12), this readily allows an obvious extension to the Reifenberg flat domain. More precisely, we state the following boundary estimate by setting $K_\rho^\kappa(z) = Q_\rho^\kappa(z) \cap \Omega_T$ for a fixed $z = (\tau, y) \in (a, a+T) \times \partial\Omega$.

Lemma 3.1. *Suppose that $p(t,x) \in SLH(\Omega_T)$ with its range in $[\gamma_1, \gamma_2]$ shown as (2.7), and*

$$\overline{M} := \int_{\Omega_T} |Du|^{p(t,x)}\, dx\, dt + \int_{\Omega_T} \Psi(t,x)\, dx\, dt \tag{3.2}$$

with

$$\Psi(t,x) := |\psi_t|^{\gamma_1'} + |D\psi|^{p(t,x)} + |\boldsymbol{f}|^{p(t,x)} + 1. \tag{3.3}$$

For any fixed $\delta \in (0, 1/8)$, $M > 1$ and $\alpha := \min\Big\{1, \gamma_1 \frac{d+2}{4} - \frac{d}{2}\Big\} \in (0,1]$, let $\rho_1 = \Gamma^{-\frac{2}{\alpha}}$ with

$$\Gamma := 4\left(\Big(\frac{2}{1-\delta}\Big)^d \frac{M\overline{M}}{2\delta\omega_d} + 1\right)^{1/2} \geq 4, \tag{3.4}$$

where ω_d denotes the measure of the unit ball of $\mathbb{R}^d$. If Ω is a (δ, R)-Reifenberg flat domain; moreover, for any fixed $\kappa > 1$, $z = (\tau, y) \in (a, a+T) \times \partial\Omega$ and for any $0 < \rho < \rho_1$ we have

$$\kappa \leq M\Big(\fint_{K_\rho^\kappa(z)} |Du|^{p(t,x)}\, dx\, dt + \frac{1}{\delta} \fint_{K_\rho^\kappa(z)} \Psi(t,x)\, dx\, dt\Big), \tag{3.5}$$

then there exists $c_a := \exp\Big(\gamma_2\Big(\delta + \frac{\delta(d+2)}{\alpha}\Big)\Big) > 1$ such that

$$p_2 - p_1 \leq \omega(\Gamma\, \rho^\alpha), \quad \kappa^{\frac{2}{p(z)}} \leq \Gamma^2 \rho^{-(d+2)}, \quad \kappa^{p_2 - p_1} \leq c_a, \tag{3.6}$$

where

$$p_1 = p(z_1) = \inf_{K_\rho^\kappa(z)} p(t,x), \quad p_2 = p(z_2) = \sup_{K_\rho^\kappa(z)} p(t,x). \tag{3.7}$$

Proof. For a fixed point $z = (\tau, y) \in (a, a+T) \times \partial\Omega$, it suffices to prove our estimate in the setting $\big(\tau - \kappa^{\frac{2-p(z)}{p(z)}} \rho^2, \tau + \kappa^{\frac{2-p(z)}{p(z)}} \rho^2\big) \subset (a, a+T)$. Otherwise, if $Q_\rho^\kappa(z)$ touches the bottom or the top of Ω_T, i.e. $\big(\tau - \kappa^{\frac{2-p(z)}{p(z)}} \rho^2, \tau + \kappa^{\frac{2-p(z)}{p(z)}} \rho^2\big) \not\subset (a, a+T)$, then we may consider an extended variational inequality (2.4) in $(a-T, a+2T) \times \Omega$ in terms of an argument from [10, Remark 2.6], which results in that

$$\Big(\tau - \kappa^{\frac{2-p(z)}{p(z)}} \rho^2, \tau + \kappa^{\frac{2-p(z)}{p(z)}} \rho^2\Big) \subset (a-T, a+2T).$$

Consequently, it yields the same process as follows.

Now, by the measure density (2.12) we know that

$$\begin{aligned}\frac{1}{|K^\kappa_\rho(z)|} &= \frac{1}{|Q^\kappa_\rho(z)|}\frac{|Q^\kappa_\rho(z)|}{|K^\kappa_\rho(z)|}\\ &= \frac{1}{2\omega_d\rho^{d+2}\kappa^{\frac{2-p(z)}{p(z)}}}\frac{|B_\rho(y)|}{|B_\rho(y)\cap\Omega|}\\ &\le \frac{1}{2\omega_d\,\rho^{d+2}\kappa^{\frac{2-p(z)}{p(z)}}}\Big(\frac{2}{1-\delta}\Big)^d.\end{aligned}$$

Hence, from (3.5) it follows that

$$\begin{aligned}\kappa &\le \frac{M}{|K^\kappa_\rho(z)|}\Big(\int_{K^\kappa_\rho(z)}|Du|^{p(t,x)}\,dx\,dt + \frac{1}{\delta}\int_{K^\kappa_\rho(z)}\Psi(t,x)\,dx\,dt\Big)\\ &\le \frac{M}{2\omega_d\rho^{d+2}\kappa^{\frac{2-p(z)}{p(z)}}}\Big(\frac{2}{1-\delta}\Big)^d\Big(\int_{\Omega_T}|Du|^{p(t,x)}\,dx\,dt + \frac{1}{\delta}\int_{\Omega_T}\Psi(t,x)\,dx\,dt\Big),\end{aligned}$$

which implies that

$$\begin{aligned}\kappa^{\frac{2}{p(z)}} &\le \frac{M}{2\omega_d\rho^{d+2}}\Big(\frac{2}{1-\delta}\Big)^d\Big(\int_{\Omega_T}|Du|^{p(t,x)}\,dx\,dt + \frac{1}{\delta}\int_{\Omega_T}\Psi(t,x)\,dx\,dt\Big)\\ &\le \frac{M\overline{M}}{2\omega_d\rho^{d+2}}\Big(\frac{2}{1-\delta}\Big)^d,\end{aligned}\tag{3.8}$$

where we have used (3.2) in the last inequality. Recalling the definitions of p_1 and p_2, by (2.5) it yields

$$p_2 - p_1 \le |p_2 - p_1| \le \omega(d_p(z_1,z_2)) \le \omega\Big(2\rho + \sqrt{2\kappa^{\frac{2-p(z)}{p(z)}}}\rho\Big).$$

So, if $2 \le p(z) \le \gamma_2 < \infty$, then

$$p_2 - p_1 \le \omega(4\rho);\tag{3.9}$$

if $\frac{2d}{d+2} < \gamma_1 \le p(z) < 2$ then by (3.8) and (3.4) we obtain

$$p_2 - p_1 \le \omega(\Gamma\rho^{\gamma_1\frac{d+2}{4}-\frac{d}{2}}).\tag{3.10}$$

Combining (3.9) and (3.10), we obtain the first estimate of (3.6). Further, putting (3.8) and (3.4) together, we also obtain the second estimate of (3.6). Finally, recalling $p(t,x) \in SLH(\Omega_T)$ and $0 < \rho < \rho_1 = \Gamma^{-\frac{2}{\alpha}}$, we have

$$\Gamma^{p_2-p_1} \le \exp(\delta), \quad \rho^{-(p_2-p_1)} \le \exp\Big(\frac{2\delta}{\alpha}\Big),$$

which implies

$$\kappa^{p_2-p_1} \le \Big(\Gamma\,\rho^{-\frac{d+2}{2}}\Big)^{(p_2-p_1)\gamma_2} \le \exp\Big(\gamma_2\Big(\delta + \frac{\delta(d+2)}{\alpha}\Big)\Big) = c_a.\tag{3.11}$$

This concludes the proof. □

Let us recall the modified Vitali type covering lemma with a covering of intrinsic parabolic cylinders, see [10, Lemma 3.5].

Lemma 3.2. *For $\kappa > 1$, we set that $\mathcal{F} = \{Q^{\kappa}_{\rho_i}(z_i)\}_{i\in\mathcal{J}}$ is a family of intrinsic parabolic cylinders with $z_i = (\tau_i, y_i) \in \mathbb{R}^{d+1}$ and $\rho_i > 0$, which satisfy that $\cup_{i\in\mathcal{J}} Q^{\kappa}_{\rho_i}(z_i)$ is bounded in $\mathbb{R}^{d+1}$ and*

$$\kappa^{p_i^+ - p_i^-} \le c_a \quad \text{for all } i \in \mathcal{J},$$

where $c_a > 1$ is the same as Lemma 3.1. Let

$$p_i^+ = \sup_{Q^{\kappa}_{\rho_i}(z_i)} p(t,x) \quad \text{and} \quad p_i^- = \inf_{Q^{\kappa}_{\rho_i}(z_i)} p(t,x),$$

then there exists a countable sub-collection $\mathcal{G} \subset \mathcal{F}$ of disjoint parabolic cylinders such that

$$\cup_{Q^{\kappa}_{\rho_i}(z_i)\in\mathcal{F}} Q^{\kappa}_{\rho_i}(z_i) \subset \cup_{Q^{\kappa}_{\rho_i}(z_i)\in\mathcal{G}} \chi Q^{\kappa}_{\rho_i}(z_i),$$

where $\chi \ge \{5, (8c_a^{\frac{4}{\gamma_1^2}} + 1)^{1/2}\}$, and χQ_i denotes the χ-time enlarged cylinder Q_i.

To obtain the interior and boundary comparison estimates with the reference problems on the intrinsic parabolic cylinders, respectively, we suppose that $u \in \mathcal{A}(\Omega_T)$ is a weak solution of (2.4) under the regularity assumptions that $p(t,x) \in SLH(\Omega_T)$ with its range (2.7), and $(\mathbf{a}, \mathbb{R}\times\Omega)$ is (δ, R)-vanishing with the specified $\delta \in (0, 1/8)$ and $R \in (0,1)$. It is clearly checked that the condition (2.1) easily leads to the following monotonicity

$$\begin{gathered}\Big(\mathbf{a}(t,x;\xi) - \mathbf{a}(t,x,\eta)\Big)(\xi-\eta) \ge C_1\Big(|\mu|^2 + |\xi|^2 + |\eta|^2\Big)^{\frac{p(t,x)-2}{2}} |\xi-\eta|^2 \\ \text{if } \frac{2d}{d+2} < p(t,x) < 2, \\ \Big(\mathbf{a}(t,x;\xi) - \mathbf{a}(t,x,\eta)\Big)(\xi-\eta) \ge C_2|\xi-\eta|^{p(t,x)} \quad \text{if } p(t,x) \ge 2\end{gathered} \tag{3.12}$$

for all $\xi,\ \eta \in \mathbb{R}^d$ and a.a. $(t,x) \in \Omega_T$, where C_1 and C_2 are positive constants depending only on $d, \gamma_1, \gamma_2, \lambda$ and Λ, see [16, Section 2] or [7, Formula (10)]. Setting

$$W(\Omega_T) := \big\{ g \in W^{p(t,x)}(\Omega_T) : g_t \in \big(W^{p(t,x)}(\Omega_T)\big)' \big\}.$$

We recall the following comparison principle, which is useful to construct a comparison that it almost everywhere satisfies an obstacle constrain $\psi \le k$, see [16, Lemma 3.15].

Lemma 3.3. *Let Ω_T be an open subset of $\mathbb{R}^{d+1}$. Assume that $p(t,x) \in SLH(\Omega_T)$ satisfying (2.7), and $\psi, k \in W(\Omega_T)$ satisfy the following relations with $\mathbf{a}(t,x;\xi)$ such that (3.12) holds,*

$$\begin{gathered}\psi_t - \operatorname{div}(\mathbf{a}(t,x,D\psi)) \le k_t - \operatorname{div}(\mathbf{a}(t,x,Dk)) \quad \text{in } \Omega_T, \\ \psi \le k \quad \text{on } \partial\Omega_T.\end{gathered} \tag{3.13}$$

Then $\psi \le k$ a.e. on Ω_T.

We set a fixed point $z = (\tau, y) \in \Omega_T$, $\kappa > 1$ and a sufficiently small $r > 0$ specified later. Without loss of generality, we assume that $y = 0$, i.e., $z = (\tau, 0)$. We only consider the boundary case of $B^+_{6r} \subset \Omega_{6r} := B_{6r} \cap \Omega \subset \{x_1 > -12r\delta\}$ and $\Big(\tau - \kappa^{\frac{2-p_z}{p_z}}(6r)^2, \tau + \kappa^{\frac{2-p_z}{p_z}}(6r)^2\Big) \subset (a, a+T)$ with $p_z = p(z)$ since the interior case is simpler for $Q^{\kappa}_{6r}(z) = K^{\kappa}_{6r}(z) \subset \Omega_T$. By an argument of normalization we can assume that for suitable $r > 0$ such that

$$\fint_{K^{\kappa}_{6r}(z)} |Du|^{p(t,x)}\,dx\,dt + \frac{1}{\delta}\fint_{K^{\kappa}_{6r}(z)} \Psi(t,x)\,dx\,dt \le c_*\kappa \tag{3.14}$$

for some $c_* > 1$, where $\Psi(t,x)$ is as (3.3). Let $k \in W(K^{\kappa}_{6r}(z))$ be any weak solution of the following local initial-boundary problem

$$\begin{gathered} k_t - \operatorname{div}(\mathbf{a}(t,x,Dk)) = \psi_t - \operatorname{div}(\mathbf{a}(t,x,D\psi)) \quad \text{in } K^{\kappa}_{6r}(z), \\ k = u \quad \text{on } \partial K^{\kappa}_{6r}(z). \end{gathered} \tag{3.15}$$

Then, by Lemma 3.3 we immediately conclude the following, cf. [16, Lemma 8.2].

Lemma 3.4. *Under the normalization assumption of* (3.14)*, for any* $\varepsilon_1 \in (0,1)$ *there exists a small constant* $\delta = \delta(d,\lambda,\Lambda,\gamma_1,\gamma_2,\varepsilon_1) > 0$ *such that*

$$\fint_{K^{\kappa}_{4r}(z)} |Du - Dk|^{p(t,x)}\,dx\,dt \le \varepsilon_1\kappa \quad \textit{and} \quad \fint_{K^{\kappa}_{4r}(z)} |Dk|^{p(t,x)}\,dx\,dt \le c_1\kappa \tag{3.16}$$

for some $c_1 = c_1(d,\lambda,\Lambda,\gamma_1,\gamma_2,\partial\Omega) > 1$.

Let $w \in W(K^{\kappa}_{4r}(z))$ be the weak solution of

$$\begin{gathered} w_t - \operatorname{div}(\mathbf{a}(t,x,Dw)) = 0 \quad \text{in } K^{\kappa}_{4r}(z), \\ w = k \quad \text{on } \partial K^{\kappa}_{4r}(z). \end{gathered} \tag{3.17}$$

Lemma 3.5. *Under the normalization assumption of* (3.14)*, for any* $\varepsilon_2 \in (0,1)$ *there exists a small* $\delta = \delta(d,\lambda,\Lambda,\gamma_1,\gamma_2,\varepsilon_2) > 0$ *such that*

$$\fint_{K^{\kappa}_{4r}(z)} |Dk - Dw|^{p(t,x)}\,dx\,dt \le \varepsilon_2\kappa \quad \textit{and} \quad \fint_{K^{\kappa}_{4r}(z)} |Dw|^{p(t,x)}\,dx\,dt \le c_2\kappa \tag{3.18}$$

for some $c_2 = c_2(d,\lambda,\Lambda,\gamma_1,\gamma_2,\partial\Omega) > 1$*, see* [10, Lemma 4.1].

Now let us recall a self-improving integrability of Dw to (3.18). For $0 < \rho = 6r < \rho_1$, p_1 and p_2 shown as in (3.7), we assume that

$$p_2 - p_1 \le \omega(\Gamma\,(6r)^{\alpha}), \quad \kappa^{\frac{2}{p(z)}} \le \Gamma^2(6r)^{-(d+2)}, \quad \kappa^{p_2-p_1} \le c_a \tag{3.19}$$

for some $\alpha \in (0,1)$, $\Gamma \ge 4$ and $c_a > 1$ defined by Lemma 3.1. By Lemma 3.5 it holds

$$\fint_{K^{\kappa}_{4r}(z)} |Dw|^{p(t,x)}\,dx\,dt \le c_2\kappa$$

with $c_2 > 1$. Then, thanks to [4, Corollary 5.2] we conclude that there exist $\varepsilon_0 > 0$ and $\rho_2 > 0$ such that for $0 < 4r < \rho_2$ it holds

$$\fint_{K^{\kappa}_{2r}(z)} |Dw|^{p(t,x)(1+\varepsilon_0)}\,dx\,dt \le c\kappa^{1+\varepsilon_0}, \tag{3.20}$$

where c is a positive constant depending only on $d,\lambda,\Lambda,\mu,\gamma_1,\gamma_2,\delta,R,\omega(\cdot)$.

As in [10, 7], let

$$p_2 - p_1 \le \min\Big\{\frac{\lambda}{4\Lambda},\, 1,\, \frac{\varepsilon_0(\gamma_1-1)}{4}\Big\} \tag{3.21}$$

and the vector-valued function $\mathbf{b}(t,x;\xi) : K^{\kappa}_{2r}(z) \times \mathbb{R}^d \to \mathbb{R}^d$ is introduced by

$$\mathbf{b}(t,x;\xi) = \mathbf{a}(t,x;\xi)\left(\mu^2 + |\xi|^2\right)^{\frac{p_z - p(t,x)}{2}}.$$

By using (2.1) and (3.21), we obtain

$$\begin{gathered}(\lambda/2)\Big(\mu^2+|\xi|^2\Big)^{\frac{p_z-2}{2}}|\eta|^2 \le D_\xi \mathbf{b}(t,x;\xi)\,\eta\cdot\eta,\\ |\mathbf{b}(t,x;\xi)|+\Big(\mu^2+|\xi|^2\Big)^{1/2}|D_\xi\mathbf{b}(t,x;\xi)| \le 3\Lambda\Big(\mu^2+|\xi|^2\Big)^{\frac{p_z-1}{2}}\end{gathered} \tag{3.22}$$

for a.a. $(t,x)\in K^\kappa_{2r}(z)$ and all $\xi,\eta\in\mathbb{R}^d$, see [10, Eq. (4.18)] or [7, Lemma 3.6]. For the interior case, we define $\overline{\mathbf{b}}(t,\xi):\big(\tau-\kappa^{\frac{2-p_z}{p_z}}(2r)^2,\tau+\kappa^{\frac{2-p_z}{p_z}}(2r)^2\big)\times\mathbb{R}^d\to\mathbb{R}^d$ by

$$\overline{\mathbf{b}}(t,\xi)=\fint_{B_{2r}(y)}\mathbf{b}(t,x;\xi)\,dx.$$

Then, by 2.1 it yields

$$\fint_{Q^\kappa_{2r}(z)}\sup_{\xi\in\mathbb{R}}\frac{|\overline{\mathbf{b}}(t,\xi)-\mathbf{b}(t,x,\xi)|}{(\mu^2+|\xi|^2)^{\frac{p_z-1}{2}}}\,dx\,dt=\fint_{Q^\kappa_{2r}(z)}\Theta[\mathbf{a};B_{2r}(y)](t,x)\,dx\,dt\le\delta.$$

For the boundary case, we define $\widetilde{\mathbf{b}}(t,\xi):\big(\tau-\kappa^{\frac{2-p_z}{p_z}}(2r)^2,\tau+\kappa^{\frac{2-p_z}{p_z}}(2r)^2\big)\times\mathbb{R}^d\to\mathbb{R}^d$ by

$$\widetilde{\mathbf{b}}(t,\xi):=\begin{cases}\overline{\mathbf{b}}(t,\xi)=\fint_{B^+_{2r}(y)}\mathbf{b}(t,x;\xi)\,dx & (t,x)\in B^+_{2r}(y),\\ \overline{\mathbf{b}}(t,\xi) & (t,x)\in\Omega_{2r}(y)\backslash B^+_{2r}(y).\end{cases}$$

Again by Assumption 2.1 we see that

$$\fint_{Q^{\kappa+}_{2r}(z)}\sup_{\xi\in\mathbb{R}}\frac{|\overline{\mathbf{b}}(t,\xi)-\mathbf{b}(t,x,\xi)|}{(\mu^2+|\xi|^2)^{\frac{p_z-1}{2}}}\,dx\,dt=\fint_{Q^{\kappa+}_{2r}(z)}\Theta[\mathbf{a};B^+_{2r}(y)](t,x)\,dx\,dt\le4\delta.$$

Moreover, for both cases we see that $\overline{\mathbf{b}}(t,\xi)$ satisfies (3.22) with $\mathbf{b}(t,x;\xi)$ replaced by $\overline{\mathbf{b}}(t,\xi)$.

With $\widetilde{\mathbf{b}}(t,\xi)$ in hand, we further recall the following two comparisons with the so-called limiting problems. Let $h\in W^{1,p_z}(K^\kappa_{2r}(z))$ be a weak solution of

$$\begin{gathered}h_t-\operatorname{div}(\widetilde{\mathbf{b}}(t,Dh))=0\quad\text{in } K^\kappa_{2r}(z),\\ h=w\quad\text{on }\partial K^\kappa_{2r}(z).\end{gathered}\tag{3.23}$$

Lemma 3.6. *Let*

$$0<r\le\min\Big\{\frac{\rho_1}{6},\frac{\rho_2}{4},(4e)^{-1}\Gamma^{-(\frac{d+3}{\alpha}+2)},(\Gamma^{-1}R)^{\frac{1}{\alpha}}\Big\},\tag{3.24}$$

where ρ_1,ρ_2 are the radi appearing in (3.7) *and* (3.20)*, respectively. For any given $\varepsilon_3\in(0,1)$ there exists a small constant $\delta=\delta(d,\lambda,\Lambda,\gamma_1,\gamma_2,\partial\Omega,\varepsilon_3)>0$ such that*

$$\fint_{K^\kappa_{2r}(z)}|Dw-Dh|^{p_z}\,dx\,dt\le\varepsilon_3\kappa\quad\text{and}\quad\fint_{K^\kappa_{2r}(z)}|Dh|^{p_z}\,dx\,dt\le c_3\kappa\tag{3.25}$$

for some $c_3=c_3(d,\lambda,\Lambda,\gamma_1,\gamma_2,\partial\Omega)>0$, see [10, Lemma 4.2].

Lemma 3.7. *For each $\varepsilon_4\in(0,1)$, there exists a small constant $\delta>0$, $\delta=\delta(d,\lambda,\Lambda,\gamma_1,\gamma_2,\varepsilon_4)$, such that for the weak solution $v\in W^{1,p_z}(Q^{\kappa+}_{2r}(z))$ of*

$$\begin{gathered}v_t-\operatorname{div}(\overline{\mathbf{b}}(t,Dv))=0\quad\text{in } Q^{\kappa+}_{2r}(z),\\ v=0\quad\text{on } T^\kappa_{2r}(z),\end{gathered}\tag{3.26}$$

it holds

$$\fint_{Q^{\kappa+}_{2r}(z)}|Dv|^{p_z}\,dx\,dt\le c_3\kappa\quad\text{and}\quad\fint_{K^\kappa_r(z)}|Dh-D\bar{v}|^{p_z}\,dx\,dt\le\varepsilon_4\kappa,$$

where c_3 is defined in Lemma 3.6. Here, we extend v from $Q_{2r}^{\kappa+}(z)$ to $K_{2r}^{\kappa}(z)$ by zero-extension denoted it by $\bar{v}$, see [10, Lemma 4.3].

We also recall the L^∞-estimate for the gradients of weak solution to the limiting problem of general p-Laplacian type with the nonlinearity independent of the spatial variable. Indeed, DiBenedetto showed an interior gradient bound for parabolic systems, see [15, Theorems 5.1 and 5.2], and Lieberman [23] extended it up to the boundary case for parabolic equations.

Lemma 3.8. *(i) (interior case) For a fixed $\kappa > 1$ and $r > 0$, we suppose that $v \in W^{1,p_z}(Q_{2r}^{\kappa}(z))$ is any weak solution of*

$$v_t - \operatorname{div}(\overline{\mathbf{b}}(t, Dv)) = 0 \quad \text{in } Q_{2r}^{\kappa}(z) \subset \Omega_T$$

with

$$\fint_{Q_{2r}^{\kappa}(z)} |Dv|^{p_z}\, dx\, dt \le c_*\kappa$$

for some $c_* > 1$. Then

$$\|Dv\|_{L^\infty(Q_r^{\kappa}(z))}^{p_z} \le C\kappa, \tag{3.27}$$

where $C = C(d, \lambda, \Lambda, \gamma_1, \gamma_2, c_*) > 0$.

(ii) (boundary case) Let $\kappa > 1$ and $r > 0$, we suppose that $v \in W^{1,p_z}(Q_{2r}^{\kappa+}(z))$ is a weak solution of

$$\begin{gathered} v_t - \operatorname{div}(\overline{\mathbf{b}}(t, Dv)) = 0 \quad \text{in } Q_{2r}^{\kappa+}(z), \\ v = 0 \quad \text{on } T_{2r}^{\kappa}(z) \end{gathered} \tag{3.28}$$

with

$$\fint_{Q_{2r}^{\kappa+}(z)} |Dv|^{p_z}\, dx\, dt \le c_*\kappa$$

for some $c_* > 1$, then

$$\|Dv\|_{L^\infty(Q_r^{\kappa+}(z))}^{p_z} \le C\kappa, \tag{3.29}$$

where $C = C(d, \lambda, \Lambda, \gamma_1, \gamma_2, c_*, \partial\Omega) > 0$.

We finish this section by recalling the following two lemmas.

Lemma 3.9. *Let $\phi \in \Phi$ be a Young function with $\phi \in \Delta_2 \cap \nabla_2$ and $g \in L^\phi(\Omega_T)$. Then*

$$\int_{\Omega_T} \phi(|g|)\, dx\, dt = \int_0^\infty \big|\{(t,x) \in \Omega_T : |g| > k\}\big|\, d\phi(k).$$

Lemma 3.10. *Let $\phi \in \Phi$ be a Young function as shown in Lemma 3.9. Then, for any $\hat{a}, \hat{b} > 0$ one has*

$$I = \int_0^\infty \frac{1}{\kappa}\Big(\int_{\{(t,x)\in\Omega_T : |g| > \hat{a}\kappa\}} |g|\, dx\, dt\Big) d\phi(\hat{b}\kappa) \le C \int_{\Omega_T} \phi(|g|)\, dx\, dt,$$

where $C = C(\hat{a}, \hat{b}, \phi)$, see [8, Lemma 3.4].

4. Proof of Theorem 2.5

Let us assume that $p(t,x) \in SLH(\Omega_T)$ with its range $[\gamma_1,\gamma_2]$ shown as (2.7), $(\mathbf{a},\Omega_T)$ is (δ,R)-vanishing for $R \in (0,1)$ with a small $\delta \in (0,1/8)$ such that the validity of Lemmas 3.4–3.7. Let the given datum

$$|\psi_t|^{\gamma_1'},\ |D\psi|^{p(t,x)},\ |\mathbf{f}|^{p(t,x)} \in L^{\phi}(\Omega_T)$$

for Young's function $\phi \in \Delta_2 \cap \nabla_2$, and $u \in \mathcal{A}(\Omega_T)$ be the weak solution of variational inequality (2.4) with the constants $\overline{M}$, α, Γ, c_a as in Lemma 3.1, $m := \sup_{(\tau,y)\in\Omega_T} m(\tau,y)$ as (2.16), and $R_0 > 0$ chosen as

$$0 < 2R_0 \le \min\big\{\frac{\rho_1}{6}, \frac{\rho_2}{4}, (4e)^{-1}\Gamma^{-(\frac{d+3}{\alpha}+2)}, (\Gamma^{-1}R)^{\frac{1}{\alpha}}\big\}, \tag{4.1}$$

$$\omega(4R_0) \le \min\Big\{\frac{\lambda}{4\Lambda}, 1, \frac{\varepsilon_0(\gamma_1-1)}{4}\Big\}, \tag{4.2}$$

where ρ_1, ρ_2 are shown in Lemma 3.1 and (3.20), $\varepsilon_0 > 0$ as in (3.20). For any $\kappa > 0$, we set

$$\kappa_0 = \Big(\fint_{\Omega_T} |Du|^{p(t,x)}\,dx\,dt + \frac{1}{\delta}\fint_{\Omega_T} \Psi(t,x)\,dx\,dt\Big)^m,$$

the upper-level set

$$E(\kappa) = \big\{(t,x) \in \Omega_T : |Du|^{p(t,x)} > \kappa\big\},$$

and for fixed $(\tau,y) \in \Omega_T$ and $\rho > 0$,

$$J\big(K_\rho^\kappa(\tau,y)\big) = \fint_{K_\rho^\kappa(\tau,y)} |Du|^{p(t,x)}\,dx\,dt + \frac{1}{\delta}\fint_{K_\rho^\kappa(\tau,y)} \Psi(t,x)\,dx\,dt.$$

Without loss of generality, we take a suitable positive constant K such that

$$|\Omega_T| < |Q_{KR_0}|,$$

where $R_0 > 0$ is defined by (4.1) and (4.2).

Step 1. We prove the modified Vitali covering for the major portion of $E(\kappa)$ by a family of countably many disjoint cylinders. To this end,we have the following.

Lemma 4.1. *For $\kappa \ge \kappa_1 := \Big(\big(\frac{2}{1-\delta}\big)^d (48\chi K)^{d+2}\Big)^m \kappa_0$, there exists a family of disjoint cylinders $\{K_{\rho_i}^\kappa(\tau_i,y_i)\}_{i\ge1}$ with $(\tau_i,y_i) \in E(\kappa)$ and*

$$0 < \rho_i < \frac{\min\big\{\kappa^{\frac{p_i-2}{2p_i}}, 1\big\}R_0}{48\chi}$$

such that

$$E(\kappa) \subset \Big(\cup_{i\ge1} \chi K_{\rho_i}^\kappa(\tau_i,y_i)\Big) \cup \text{a negligible set},$$

where the constant χ is shown as in Lemma 3.2, $p_i = p(\tau_i,y_i)$, and for each $i \ge 1$ it holds

$$J\big(K_{\rho_i}^\kappa(\tau_i,y_i)\big) = \kappa, \quad J\big(K_\rho^\kappa(\tau_i,y_i)\big) < \kappa \quad \text{for all } \rho \in \Big(\rho_i, \frac{\min\{\kappa^{\frac{p_i-2}{2p_i}}, 1\}R_0}{2}\Big].$$

Proof. For every fixed point $z_0 = (\tau_0,y_0) \in E(\kappa)$, we consider the radius ρ with

$$\frac{\min\{\kappa^{\frac{p_0-2}{2p_0}}, 1\}R_0}{48\chi} \le \rho \le \frac{\min\{\kappa^{\frac{p_0-2}{2p_0}}, 1\}R_0}{2}, \tag{4.3}$$

where χ is as in Lemma 3.2 and $p_0 = p(z_0)$. It is clear that for any $z_0 \in E(\kappa)$ it holds $J(\rho) < \kappa$. Indeed, it follows from the measure density conditions (2.12) that

$$\begin{aligned}
J(K_\rho^\kappa) &= \frac{1}{|K_\rho^\kappa|}\Big(\int_{K_\rho^\kappa} |Du|^{p(t,x)}\,dx\,dt + \frac{1}{\delta}\int_{K_\rho^\kappa} \Psi(t,x)\,dx\,dt\Big)\\
&\le \frac{|Q_\rho^\kappa|}{|Q_\rho^\kappa \cap \Omega_T|}\frac{|\Omega_T|}{|Q_\rho^\kappa|}\Big(\fint_{\Omega_T} |Du|^{p(t,x)}\,dx\,dt + \frac{1}{\delta}\fint_{\Omega_T} \Psi(t,x)\,dx\,dt\Big)\\
&< \Big(\frac{2}{1-\delta}\Big)^d \frac{|Q_{KR_0}|}{|Q_\rho^\kappa|}\kappa_0^{\frac{1}{m}}\\
&= \Big(\frac{2}{1-\delta}\Big)^d \Big(\frac{KR_0}{\rho}\Big)^{d+2}\kappa^{\frac{p_0-2}{p_0}}\kappa_0^{\frac{1}{m}}.
\end{aligned}$$

We now divide it into the cases $2 \le p_0 < \gamma_2$ and $\gamma_1 \le p_0 < 2$. If $2 \le p_0 \le \gamma_2$, we obtain that $m = \sup_{(\tau,y)\in\Omega_T} m(\tau,y) \ge m(\tau_0,y_0) = \frac{p_0}{2}$ by (2.16) and $\min\{\kappa^{\frac{p_0-2}{2p_0}},1\} = 1$. Therefore,

$$J(K_\rho^\kappa) < \Big(\frac{2}{1-\delta}\Big)^d (48\chi K)^{d+2}\kappa^{\frac{p_0-2}{p_0}}\kappa_0^{\frac{1}{m}} \le \kappa^{\frac{p_0-2}{p_0}}\kappa^{\frac{2}{p_0}} = \kappa;$$

If $\gamma_1 \le p_0 < 2$, one gets that $m = \sup_{(\tau,y)\in\Omega_T} m(\tau,y) \ge m(\tau_0,y_0) = \frac{2p_0}{p_0(d+2)-2d}$ by (2.16) and $\min\{\kappa^{\frac{p_0-2}{2p_0}},1\} = \kappa^{\frac{p_0-2}{2p_0}}$. This implies that

$$J(K_\rho^\kappa) < \Big(\frac{2}{1-\delta}\Big)^d (48\chi K)^{d+2}\kappa^{\frac{2-p_0}{2p_0}(d+2)}\kappa^{\frac{p_0-2}{p_0}}\kappa_0^{\frac{1}{m}} \le \kappa^{\frac{(2-p_0)d}{2p_0}}\kappa^{\frac{p_0(d+2)-2d}{2p_0}} = \kappa.$$

In summary,

$$J\big(K_\rho^\kappa\big) < \kappa \quad \text{for all } \rho \in \Big[\min\{\kappa^{\frac{p_0-2}{2p_0}},1\}R_0/(48\chi), \min\{\kappa^{\frac{p_0-2}{2p_0}},1\}R_0/2\Big]. \tag{4.4}$$

On the other hand, by the Lebesgue differentiation theorem we infer that

$$\lim_{\rho\to 0} J\big(K_\rho^\kappa\big) \ge |Du(z_0)|^{p_0} > \kappa.$$

Consequently, one can select a maximal radius $\rho_0 \in \big(0, \min\{\kappa^{\frac{p_0-2}{2p_0}},1\}R_0/(48\chi)\big]$ by the intermediate value theorem such that

$$J\big(K_{\rho_0}^\kappa\big) = \kappa \quad \text{and} \quad J\big(K_\rho^\kappa\big) < \kappa \quad \text{for all } \rho \in \Big(\rho_0, \min\{\kappa^{\frac{p_0-2}{2p_0}},1\}R_0/2\Big].$$

Now, let us take $\{K_{\rho_z}^\kappa(z) : z = (\tau,y) \in E(\kappa)\}$ as a covering of $E(\kappa)$, and note that

$$\frac{\kappa}{(48\chi)^{d+2}} \le \fint_{48\chi K_{\rho_z}^\kappa(z)} |Du|^{p(t,x)}\,dx\,dt + \frac{1}{\delta}\fint_{48\chi K_{\rho_z}^\kappa(z)} \Psi(t,x)\,dx\,dt \le \kappa. \tag{4.5}$$

Therefore, by taking $M_1 = (48\chi)^{d+2} > 1$ as in Lemma 3.1, we have

$$\kappa^{p_z^+ - p_z^-} \le c_a \quad \text{for all } z \in E(\kappa),$$

where $c_a > 1$ is as in Lemma 3.1, $p_z^+ = \sup_{K_{\rho_z}^\kappa(z)} p(t,x)$ and $p_z^- = \inf_{K_{\rho_z}^\kappa(z)} p(t,x)$. Finally, by employing the Vitali's covering lemma 3.2 we can find a family of disjoint cylinders $\{K_{\rho_i}^\kappa(\tau_i,y_i)\}_{i\ge1}$ with $(\tau_i,y_i) \in E(\kappa)$ and $\rho_i \in \Big(0, \min\{\kappa^{\frac{p_i-2}{2p_i}},1\}R_0/(48\chi)\Big]$, which reached the desired result. □

Step 2. We are now in a position to show a suitable decay estimate to each of the above-mentioned covering $\{K_{\rho_i}^\kappa(\tau_i,y_i)\}_{i\ge1}$.

Lemma 4.2. *Under the same hypotheses as in Lemma 4.1, we have*

$$|K^{\kappa}_{\rho_i}(z_i)| \le \frac{2}{\kappa}\Big(\int_{K^{\kappa}_{\rho_i}(z_i)\cap\{z\in\Omega_T:|Du|^{p(z)}>\frac{\kappa}{4}\}} |Du|^{p(z)}\,dx\,dt + \frac{1}{\delta}\int_{K^{\kappa}_{\rho_i}(z_i)\cap\{z\in\Omega_T:\Psi(t,x)>\frac{\delta\kappa}{4}\}} \Psi(t,x)\,dx\,dt\Big).$$

Proof. By Lemma 4.1 we have

$$\fint_{K^{\kappa}_{\rho_i}(\tau_i,y_i)} |Du|^{p(t,x)}\,dx\,dt + \frac{1}{\delta}\fint_{K^{\kappa}_{\rho_i}(\tau_i,y_i)} \Psi(t,x)\,dx\,dt = \kappa,$$

which implies that

$$\kappa|K^{\kappa}_{\rho_i}(\tau_i,y_i)| = \int_{K^{\kappa}_{\rho_i}(\tau_i,y_i)} |Du|^{p(t,x)}\,dx\,dt + \frac{1}{\delta}\int_{K^{\kappa}_{\rho_i}(\tau_i,y_i)} \Psi(t,x)\,dx\,dt.$$

Now we split the two integrals above in two parts,

$$\begin{aligned}
&\kappa|K^{\kappa}_{\rho_i}(\tau_i,y_i)| \\
&\le \int_{K^{\kappa}_{\rho_i}(\tau_i,y_i)\cap\{(t,x)\in\Omega_T:|Du|^{p(t,x)}>\frac{\kappa}{4}\}} |Du|^{p(t,x)}\,dx\,dt + \frac{\kappa}{4}|K^{\kappa}_{\rho_i}(\tau_i,y_i)| \\
&\quad + \frac{1}{\delta}\int_{K^{\kappa}_{\rho_i}(\tau_i,y_i)\cap\{(t,x)\in\Omega_T:\Psi(t,x)>\frac{\delta\kappa}{4}\}} \Psi(t,x)\,dx\,dt + \frac{\kappa}{4}|K^{\kappa}_{\rho_i}(\tau_i,y_i)|,
\end{aligned}$$

which yields the desired result. □

Based on Lemma 4.1, we constructed a family of disjoint cylinders $\{K^{\kappa}_{\rho_i}(z_i)\}_{i\ge1}$ for $z_i=(\tau_i,y_i)\in E(\kappa)$ with $0<\rho_i<\frac{\min\{\kappa^{\frac{p_i-2}{2p_i}},1\}R_0}{48\chi}$. We denote

$$\begin{gathered}
K^0_{z_i}=K^{\kappa}_{\rho_{z_i}}(z_i),\quad K^1_{z_i}=\chi K^{\kappa}_{\rho_i}(z_i),\quad K^2_{z_i}=2\chi K^{\kappa}_{\rho_i}(z_i),\\
K^3_{z_i}=4\chi K^{\kappa}_{\rho_i}(z_i),\quad K^4_{z_i}=6\chi K^{\kappa}_{\rho_i}(z_i),\quad Q^4_{z_i}=6\chi Q^{\kappa}_{\rho_i}(z_i);
\end{gathered}$$

and consider the following estimates by parting the settings of $Q^4_{z_i}\subset\Omega_T$ and $Q^4_{z_i}\not\subset\Omega_T$.

Case 1. For the interior case $Q^4_{z_i}\subset\Omega_T$, let k, w and h be the unique solution to the initial-boundary value problems (3.15), (3.17) and (3.23) with $Q^4_{z_i}$, $Q^3_{z_i}$ and $Q^2_{z_i}$ instead of K^{κ}_{6r}, K^{κ}_{4r} and K^{κ}_{2r}, respectively. With the same argument as the estimate (4.5), it holds

$$\frac{\kappa}{(6\chi)^{d+2}} \le \fint_{Q^j_{z_i}} |Du|^{p(t,x)}\,dx\,dt + \frac{1}{\delta}\fint_{Q^j_{z_i}} \Psi(t,x)\,dx\,dt \le \kappa \tag{4.6}$$

for $j\in\{0,1,2,3,4\}$. Note that

$$0<6\chi\rho_i \le \frac{\min\{\kappa^{\frac{p_i-2}{2p_i}},1\}R_0}{8} \le R_0, \tag{4.7}$$

we take $M_2=(6\chi)^{d+2}>1$ in Lemma 3.1, and obtain

$$p_i^+-p_i^- \le \omega(\Gamma(6\chi\rho_i)^{\alpha}),\quad \kappa^{\frac{2}{p_i}} \le \Gamma^2(6\chi\rho_i)^{-(d+2)} \quad\text{and}\quad \kappa^{p_i^+-p_i^-}\le c_a \tag{4.8}$$

for $z_i=(\tau_i,y_i)\in E(\kappa)$ and $0<\rho_i<\frac{\min\{\kappa^{\frac{p_i-2}{2p_i}},1\}R_0}{48\chi}$, where $p_i=p(z_i)=p(\tau_i,y_i), p_i^-=\inf_{Q^4_{z_i}} p(t,x), p_i^+=\sup_{Q^4_{z_i}} p(t,x)$ with $\Gamma\ge4, \alpha\in(0,1)$ and $c_a>1$

being the same as in Lemma 3.1. Let us now replace (3.14) by (4.6), and make use of the argument of Proposition 3.10 in [7] by Bui and Duong for general parabolic equation of $p(t,x)$-growth. Then we only make very slightly modifications to our problem, and immediately conclude

Corollary 4.3. *Assume that $u \in \mathcal{A}(\Omega_T)$ is a weak solution of (2.4), and $h_i \in W^{1,p_i}(Q^2_{z_i})$ is a weak solution of (3.23). Then, for any $\varepsilon > 0$ there is a small $\delta = \delta(d,\lambda,\Lambda,\gamma_1,\gamma_2,\varepsilon) > 0$ with (4.6) such that*

$$\fint_{Q^1_{z_i}} |Du - Dh_i|^{p(t,x)}\,dx\,dt \le \varepsilon\kappa \quad \text{and} \quad \|Dh_i\|^{p_i}_{L^\infty(Q^1_{z_i})} \le N_1\kappa, \tag{4.9}$$

where $N_1 = N_1(d,\lambda,\Lambda,\gamma_1,\gamma_2,c_*) > 1$ is the positive constant independent of the index i.

We omit its proof, which is very similar to that of following boundary setting, but a simple process.

Case 2. For the boundary case $Q^4_{z_i} \not\subset \Omega_T$, we suppose that $\operatorname{dist}\{y_i,\partial\Omega\} \le 6\chi\rho_i$, and take $y_i' \in \partial\Omega$ with $|y_i - y_i'| \le 8\chi\rho_i$. Since $0 < 48\chi\rho_i < \min\{\kappa^{\frac{p_i-2}{2p_i}},1\}R_0 \le R_0$, by Assumption 2.1 there exists a new spatial coordinate system, still denoting $x = \{x_1,\dots,x_d\}$-coordinate, with the origin at $y_i' = 0'$ such that

$$B_{48\chi\rho_i}(0') \cap \{x_1 > 48\chi\rho_i\delta\} \subset B_{48\chi\rho_i}(0') \cap \Omega \subset B_{48\chi\rho_i}(0') \cap \{x_1 > -48\chi\rho_i\delta\}.$$

Since $0 < \delta < 1/8$, it leads to $B_{40\chi\rho_i}(48\chi\rho_i\delta e_1) \subset B_{48\chi\rho_i}(0')$ for $e_1 = \{1,0,\dots,0\}$. We then translate the spatial coordinate system to the x_1-direction by $48\chi\rho_i\delta$, and denote the new origin by $48\chi\rho_i\delta e_1 = 0$, so that it yields

$$B^+_{40\chi\rho_i}(0) \subset B_{40\chi\rho_i}(0) \cap \Omega \subset B_{40\chi\rho_i}(0) \cap \{x_1 > -96\chi\rho_i\delta\}.$$

By considering this transformation is composed of only the translation and the rotation, it leads to that the basic structure of the problems (2.4) and the main assumptions are invariant. Here, we will continuously use the original symbols and notations in this new coordinate system. Since $|y_i| \le |y_i - y_i'| + |y_i'| \le 6\chi\rho_i + 48\chi\rho_i\delta \le 12\chi\rho_i$, for $z_i = (\tau_i, 0)$ we obtain

$$K^1_{z_i} \subset K^5_{z_i} \subset K^6_{z_i} \subset K^7_{z_i} \subset K^8_{z_i}, \tag{4.10}$$

where

$$K^5_{z_i} = 15\chi K^\kappa_{\rho_i}(z_i), \quad K^6_{z_i} = 28\chi K^\kappa_{\rho_i}(z_i), \quad K^7_{z_i} = 38\chi K^\kappa_{\rho_i}(z_i), \quad K^8_{z_i} = 48\chi K^\kappa_{\rho_i}(z_i).$$

Similarly, we now let that k, w, h and v is a unique solution to the initial-boundary value problems (3.15), (3.17), (3.23) and (3.26) with $K^8_{z_i}$, $K^7_{z_i}$, $K^6_{z_i}$ and $Q^{6+}_{z_i}$ instead of K^κ_{6r}, K^κ_{4r}, K^κ_{2r} and $Q^{\kappa+}_{2r}$, respectively, where $Q^{6+}_{z_i} = 28\chi Q^{\kappa+}_{\rho_i}(z_i)$. With the same argument as for (4.5), it holds

$$\frac{\kappa}{(48\chi)^{d+2}} \le \fint_{K^j_{z_i}} |Du|^{p(t,x)}\,dx\,dt + \frac{1}{\delta}\fint_{K^j_{z_i}} \Psi(t,x)\,dx\,dt \le \kappa \tag{4.11}$$

for each $j = 0,1,\dots,8$. By taking $M = (48\chi)^{d+2} > 1$ in Lemma 3.1, we obtain

$$p_i^+ - p_i^- \le \omega(\Gamma\,(48\chi\rho_i)^\alpha), \quad \kappa^{\frac{2}{p_i}} \le \Gamma^2\,(48\chi\rho_i)^{-(d+2)}, \quad \kappa^{p_i^+ - p_i^-} \le c_a \tag{4.12}$$

for every $z_i \in E(\kappa)$, $0 < \rho_i < \frac{\min\{\kappa^{\frac{p_i-2}{2p_i}},1\}R_0}{48\chi}$, $p_i = p(z_i)$, $p_i^- = \inf_{K^8_{z_i}} p(t,x)$, $p_i^+ = \sup_{K^8_{z_i}} p(t,x)$; where $\Gamma \geq 4$, $\alpha \in (0,1)$ and $c_a > 1$ are as in Lemma 3.1. Similar to the argument in [7, Corollary 3.9], we have the following result.

Corollary 4.4. *Assume that $u \in \mathcal{A}(\Omega_T)$ is the weak solution of (2.4). For any $\varepsilon > 0$, there exist small constant $\delta = \delta(d,\lambda,\Lambda,\gamma_1,\gamma_2,\varepsilon) > 0$ with (4.11) and a weak solution $v_i \in W^{1,p_i}(Q^{6+}_{z_i})$ of (3.26) such that*

$$\fint_{K^1_{z_i}} |Du - D\bar{v}_i|^{p(t,x)}\,dx\,dt \leq \varepsilon\kappa \quad \text{and} \quad \|D\bar{v}_i\|^{p_i}_{L^\infty(K^1_{z_i})} \leq N_2\kappa, \tag{4.13}$$

where $N_2 = N_2(d,\lambda,\Lambda,\gamma_1,\gamma_2,c_*,\delta,R) > 1$ is the constant independent of i. Here, we extend v_i from $Q^{6+}_{z_i}$ to $Q^6_{z_i}$ by zero extension, and denote it by $\bar{v}_i$.

Proof. Let us begin with the fact that

$$|Dw - D\bar{v}_i|^{p_i} \leq 2^{\gamma_2-1}\left(|Dw - Dh|^{p_i} + |Dh - D\bar{v}_i|^{p_i}\right),$$

then, by Lemmas 3.6 and 3.7 we obtain

$$\begin{aligned}\fint_{K^5_{z_i}} |Dw - D\bar{v}_i|^{p_i}\,dx\,dt &\leq C_1\Big(\fint_{K^5_{z_i}} |Dw - Dh|^{p_i}\,dx\,dt + \fint_{K^5_{z_i}} |Dh - D\bar{v}_i|^{p_i}\,dx\,dt\Big)\\ &\leq C_1\left(\varepsilon_3 + \varepsilon_4\right)\kappa\end{aligned}$$

with ε_3 and ε_4 being the same as in Lemmas 3.6 and 3.7, respectively, where C_1 is a positive constant depending only on $d,\lambda,\Lambda,\gamma_1$ and γ_2. This together with Hölder's inequality implies

$$\begin{aligned}&\fint_{K^5_{z_i}} |Dw - D\bar{v}_i|^{p(t,x)}\,dx\,dt\\ &= \fint_{K^5_{z_i}} |Dw - D\bar{v}_i|^{\frac{p_i}{2}}|Dw - D\bar{v}_i|^{p(t,x)-\frac{p_i}{2}}\,dx\,dt\\ &\leq \Big(\fint_{K^5_{z_i}} |Dw - D\bar{v}_i|^{p_i}\,dx\,dt\Big)^{1/2}\Big(\fint_{K^5_{z_i}} |Dw - D\bar{v}_i|^{2p(t,x)-p_i}\,dx\,dt\Big)^{1/2}\\ &\leq C_2\Big((\varepsilon_3+\varepsilon_4)\kappa\Big)^{1/2}\Big(\fint_{K^5_{z_i}} |Dw|^{2p(t,x)-p_i}\,dx\,dt\\ &\quad + \fint_{K^5_{z_i}} |D\bar{v}_i|^{2p(t,x)-p_i}\,dx\,dt\Big)^{1/2}.\end{aligned} \tag{4.14}$$

Next, by using (4.2) and (4.12) we have

$$\begin{aligned}2p(t,x) - p_i &\leq p(t,x)\Big(1 + p_i^+ - p_i^-\Big)\\ &\leq p(t,x)\Big(1 + \omega\big(\Gamma\,(48\chi\rho_i)^\alpha\big)\Big)\\ &\leq p(t,x)\,(1+\varepsilon_0) \qquad \text{in } K^5_{z_i}\end{aligned}$$

where ε_0 is the same as in the inequality (3.20). Therefore, from (3.20) it follows that

$$\begin{aligned}\fint_{K^5_{z_i}} |Dw|^{2p(t,x)-p_i}\,dx\,dt &\leq \fint_{K^5_{z_i}} |Dw|^{p(t,x)(1+\omega(\Gamma(48\chi\rho_i)^\alpha))}\,dx\,dt + 1\\ &\leq \kappa^{1+\omega(\Gamma\,(48\chi\rho_i)^\alpha)} + 1.\end{aligned} \tag{4.15}$$

By (4.12), $\kappa^{\omega(\Gamma(48\chi\rho_i)^\alpha)} \le c_a$ for $c_a > 1$, which leads to

$$\fint_{K^5_{z_i}} |Dw|^{2p(t,x)-p_i}\,dx\,dt \le C_3\kappa. \tag{4.16}$$

Using Lemma 3.8-(ii) for the weak solution $v_i \in W^{1,p_i}(Q^{6+}_{z_i})$ mentioned in Corollary 4.4, and formula (4.12) for $\kappa > 1$ we obtain

$$\begin{aligned}
\fint_{K^5_{z_i}} |D\bar v_i|^{2p(t,x)-p_i}\,dx\,dt &= \frac{1}{|K^5_{z_i}|}\int_{K^5_{z_i}} |D\bar v_i|^{2p(t,x)-p_i}\,dx\,dt \\
&= \frac{1}{|K^5_{z_i}|}\Big(\int_{Q^{5+}_{z_i}} |Dv_i|^{2p(t,x)-p_i}\,dx\,dt + |K^5_{z_i}\backslash Q^{5+}_{z_i}|\Big) \\
&\le \frac{C_4}{|K^5_{z_i}|}\Big(\sup_{K^5_{z_i}} \kappa^{\frac{2p(t,x)-p_i}{p_i}}|Q^{5+}_{z_i}| + |K^5_{z_i}\backslash Q^{5+}_{z_i}|\Big) \\
&\le \frac{C_5}{|K^5_{z_i}|}\Big(\kappa\kappa^{\frac{2(p_i^+-p_i^-)}{\gamma_1}}|Q^{5+}_{z_i}| + |K^5_{z_i}\backslash Q^{5+}_{z_i}|\Big) \\
&\le C_6\kappa.
\end{aligned} \tag{4.17}$$

Now, combining (4.16) and (4.17) with (4.14) yields

$$\fint_{K^5_{z_i}} |Dw - D\bar v_i|^{p(t,x)}\,dx\,dt \le C_7\big(\varepsilon_3 + \varepsilon_4\big)^{1/2}\kappa. \tag{4.18}$$

Then, it follows from Lemmas 3.4 and 3.5 that

$$\begin{aligned}
&\fint_{K^5_{z_i}} |Du - D\bar v_i|^{p(t,x)}\,dx\,dt \\
&\le 3^{\gamma_2-1}\Big(\fint_{K^5_{z_i}} |Du - Dk|^{p(t,x)}\,dx\,dt + \fint_{K^5_{z_i}} |Dk - Dw|^{p(t,x)}\,dx\,dt \\
&\quad + \fint_{K^5_{z_i}} |Dw - D\bar v_i|^{p(t,x)}\,dx\,dt\Big) \\
&\le C\big(\varepsilon_1 + \varepsilon_2 + (\varepsilon_3 + \varepsilon_4)^{1/2}\big)\kappa
\end{aligned}$$

with $\varepsilon_i >$ for $i = 1, 2, 3, 4$ being the same as in Lemmas 3.4–3.7, respectively, which means that

$$\fint_{K^1_{z_i}} |Du - D\bar v_i|^{p(t,x)}\,dx\,dt \le C_{10}\big(\varepsilon_1 + \varepsilon_2 + (\varepsilon_3 + \varepsilon_4)^{1/2}\big)\kappa.$$

Therefore, by taking $\varepsilon_i > 0$, $i = 1, 2, 3, 4$ sufficiently small we ensure the validity of first inequality of (4.13). The second inequality of (4.13) is proved by following from Lemma 3.8 (ii). □

Step 3. We now prove a decay estimate of the upper-level set for a variable power of the gradients of weak solution to (2.4). To this end, let $N = \max\{N_1, N_2\} > 1$ with N_1, N_2 as shown in Corollary 4.3 and Corollary 4.4, and let $A = N^{\frac{\gamma_2}{\gamma_1}} c_a^{\frac{1}{\gamma_1}} 2^{\gamma_2} > 1$.

Lemma 4.5. *Let $\kappa \ge \kappa_1$ be as in Lemma 4.1. For any $\varepsilon \in (0,1)$, there exists a small constant $\delta = \delta(d, \lambda, \Lambda, \gamma_1, \gamma_2) > 0$ with $\delta \in (0, 1/8)$ such that if $(\mathbf{a}, \Omega_T)$*

satisfies (δ, R)-vanishing for some $0 < R < 1$, $p(t,x) \in SLH$ with the range $[\gamma_1, \gamma_2]$, and $u \in \mathcal{A}(\Omega_T)$ is the weak solution of (2.4). Then we have the estimate

$$
\begin{aligned}
|E(A\kappa)| \le C\varepsilon\Big\{\frac{2}{\kappa}\Big(&\int_{\{(t,x)\in\Omega_T:|Du|^{p(t,x)}>\frac{\kappa}{4}\}} |Du|^{p(t,x)}\,dx\,dt \\
&+ \frac{1}{\delta}\int_{\{(t,x)\in\Omega_T:\Psi(t,x)>\frac{\delta\kappa}{4}\}} \Psi(t,x)\,dx\,dt\Big)\Big\},
\end{aligned}
$$

where $C = C(d, \lambda, \Lambda, \gamma_1, \gamma_2, R, c_a)$.

Proof. Considering the fact $E(A\kappa) \subset E(\kappa)$ for $A > 1$, by Lemma 4.1 we obtain that the family $\{K^1_{z_i}\}_{i\ge1}$ can cover almost all $E(A\kappa)$, which implies that

$$
\begin{aligned}
|E(A\kappa)| &= \big|\big\{(t,x)\in\Omega_T : |Du|^{p(t,x)} > A\kappa\big\}\big| \\
&\le \sum_{i=1}^{\infty} |\{(t,x)\in K^1_{z_i} : |Du|^{p(t,x)} > A\kappa\}| \\
&= \sum_{i:\text{interior case}} |\{(t,x)\in K^1_{z_i} : |Du|^{p(t,x)} > A\kappa\}| \\
&\quad + \sum_{i:\text{boundary case}} |\{(t,x)\in K^1_{z_i} : |Du|^{p(t,x)} > A\kappa\}|.
\end{aligned}
\tag{4.19}
$$

For the interior estimate, by Corollary 4.3 it yields

$$
\sup_{K^1_{z_i}} |Dh_i|^{p(t,x)} \le \sup_{K^1_{z_i}} N_1^{\frac{p(t,x)}{p_{z_i}}} \kappa^{\frac{p(t,x)}{p_{z_i}}} \le N_1^{\frac{\gamma_2}{\gamma_1}} \kappa^{\frac{p_1^+ - p_1^-}{\gamma_1}} \kappa \le N_1^{\frac{\gamma_2}{\gamma_1}} c_a^{\frac{1}{\gamma_1}} \kappa.
$$

Recalling $\frac{2d}{d+2} < \gamma_1 \le p(t,x) \le \gamma_2 < \infty$ in Ω_T leads to

$$
|Du|^{p(t,x)} \le 2^{\gamma_2 - 1}\big(|Du - Dh_i|^{p(t,x)} + |Dh_i|^{p(t,x)}\big)
$$

for all $(t,x) \in K^1_{z_i} \subset \Omega_T$. Therefore,

$$
\begin{aligned}
&|\{(t,x)\in K^1_{z_i} : |Du|^{p(t,x)} > A\kappa\}| \\
&\le |\{(t,x)\in K^1_{z_i} : \Big(|Du - Dh_i|^{p(t,x)} + |Dh_i|^{p(t,x)}\Big) > 2^{1-\gamma_2}A\kappa\}| \\
&\le |\{(t,x)\in K^1_{z_i} : |Du - Dh_i|^{p(t,x)} > N_1^{\frac{\gamma_2}{\gamma_1}} c_a^{\frac{1}{\gamma_1}} \kappa\}| \\
&\le \frac{1}{N_1^{\frac{\gamma_2}{\gamma_1}} c_a^{\frac{1}{\gamma_1}} \kappa} \int_{K^1_{z_i}} |Du - Dh_i|^{p(t,x)}\,dx\,dt \\
&\le \frac{|K^1_{z_i}|\varepsilon}{N_1^{\frac{\gamma_2}{\gamma_1}} c_a^{\frac{1}{\gamma_1}}},
\end{aligned}
$$

which implies

$$
|\{(t,x)\in K^1_{z_i} : |Du|^{p(t,x)} > A\kappa\}| \le C_1\varepsilon|K^\kappa_{\rho_i}(z_i)|. \tag{4.20}
$$

For the boundary case, we carry out the same procedure as the estimate of (4.20), and use Corollary 4.4 to discover that

$$
|\{(t,x)\in K^1_{z_i} : |Du|^{p(t,x)} > A\kappa\}| \le C_2\varepsilon|K^\kappa_{\rho_i}(z_i)|. \tag{4.21}
$$

Putting (4.20) and (4.21) into (4.19) yields

$$|E(A\kappa)| \le C_3\varepsilon \sum_{i=1}^{\infty} |K_{\rho_i}^{\kappa}(z_i)|.$$

Thanks to Lemma 4.2,

$$\begin{aligned}|E(A\kappa)| \le C_3\varepsilon \sum_{i=1}^{\infty} \Big\{ \frac{2}{\kappa}\Big(\int_{K_{\rho_i}^{\kappa}(z_i)\cap\{z\in\Omega_T:|Du|^{p(t,x)}>\frac{\kappa}{4}\}} |Du|^{p(t,x)}\,dx\,dt \\ + \frac{1}{\delta}\int_{K_{\rho_i}^{\kappa}(z_i)\cap\{z\in\Omega_T:\Psi(t,x)>\frac{\delta\kappa}{4}\}} \Psi(t,x)\,dx\,dt\Big)\Big\}.\end{aligned}$$

Note that $\{K_{\rho_i}^{\kappa}(z_i)\}$ are non-overlapping in Ω_T, then the required result follows.

Step 4. The step is devoted to Orlicz estimate for the derivatives of weak solution. Using Lemma 3.9, we have

$$\begin{aligned}\int_{\Omega_T} \phi(|Du|^{p(t,x)})\,dx\,dt &= \int_0^{\infty} |\{(t,x)\in\Omega_T : |Du|^{p(t,x)} > A\kappa\}|\,d\phi(A\kappa) \\ &= \int_0^{\kappa_1} |\{(t,x)\in\Omega_T : |Du|^{p(t,x)} > A\kappa\}|\,d\phi(A\kappa) \\ &\quad + \int_{\kappa_1}^{\infty} |\{(t,x)\in\Omega_T : |Du|^{p(t,x)} > A\kappa\}|\,d\phi(A\kappa) \\ &:= J_1 + J_2.\end{aligned} \tag{4.22}$$

Frist we estimate of J_1. Recalling the above definitions of κ_0 and κ_1, and using that $\phi\in\Delta_2\cap\nabla_2$, we have

$$\begin{aligned}J_1 &\le |\Omega_T|\phi(A\kappa_1) \\ &= |\Omega_T|\phi\Big(N^{\frac{\gamma_2}{\gamma_1}} c_a^{\frac{1}{\gamma_1}} 2^{\gamma_2}\Big(\Big(\frac{2}{1-\delta}\Big)^d (48\chi K)^{d+2}\Big)^m \kappa_0\Big) \\ &\le C_1|\Omega_T|\phi(\kappa_0) \\ &= C_1|\Omega_T|\phi\Big(\Big(\fint_{\Omega_T} |Du|^{p(t,x)}\,dx\,dt + \frac{1}{\delta}\fint_{\Omega_T} \Psi(t,x)\,dx\,dt\Big)^m\Big).\end{aligned}$$

Then by (2.8) it follows that

$$J_1 \le C_2\phi\Big(\Big(\fint_{\Omega_T} \Psi(t,x)\,dx\,dt\Big)^m\Big),$$

where $C_2 = C_2(d,\gamma_1,\gamma_2,\lambda,\Lambda,\alpha_1,\alpha_2,\delta,R,T,|\Omega|)$, and $\Psi(t,x)$ is defined by (3.3).

Now we estimate J_2. From Lemmas 4.5 and 3.10 we observe that

$$\begin{aligned}J_2 &\le C_3\varepsilon\int_0^{\infty} \Big(\frac{2}{\kappa}\Big(\int_{\{z\in\Omega_T:|Du|^{p(z)}>\frac{\kappa}{4}\}} |Du|^{p(z)}\,dx\,dt \\ &\quad + \frac{1}{\delta}\int_{\{z\in\Omega_T:\Psi(z)>\frac{\delta\kappa}{4}\}} \Psi(z)\,dx\,dt\Big)\Big)\,d\phi(A\kappa) \\ &\le C_4\varepsilon\Big(\int_{\Omega_T} \phi(|Du|^{p(t,x)})\,dx\,dt + \int_{\Omega_T} \phi(\Psi(t,x))\,dx\,dt\Big).\end{aligned}$$

Inserting the estimates for J_1 and J_2 into (4.22),

$$\int_{\Omega_T} \phi(|Du|^{p(t,x)})\,dx\,dt$$

$$
\begin{aligned}
&\le C_4\varepsilon \int_{\Omega_T} \phi(|Du|^{p(t,x)})\,dx\,dt + C_2\phi\Big(\Big(\fint_{\Omega_T} \Psi(t,x)\,dx\,dt\Big)^m\Big)\\
&\quad + C_5 \int_{\Omega_T} \phi(\Psi(t,x))\,dx\,dt.
\end{aligned}
$$

If $\int_{\Omega_T} \phi(|Du|^{p(t,x)})\,dx\,dt < \infty$, then we can select $\varepsilon > 0$ sufficiently small such that $0 < C_4\varepsilon < 1/2$, then

$$
\begin{aligned}
&\int_{\Omega_T} \phi(|Du|^{p(t,x)})\,dx\,dt\\
&\le C_6\Big(\phi\Big(\Big(\fint_{\Omega_T} \Psi(t,x)\,dx\,dt\Big)^m\Big) + \int_{\Omega_T} \phi(\Psi(t,x))\,dx\,dt\Big),
\end{aligned}
\tag{4.23}
$$

where $C_6 = C_6(d, \gamma_1, \gamma_2, \lambda, \Lambda, \alpha_1, \alpha_2, \delta, R, T, |\Omega|)$. Otherwise, the integral on the left may be $+\infty$, and wee need to refine the estimate for $|Du|$. Let us consider the truncated gradients

$$
|Du(t,x)|_{\bar\lambda}^{p(t,x)} := \min\big\{|Du(t,x)|^{p(t,x)}, \bar\lambda\big\} \quad \text{for } (t,x) \in \Omega_T \text{ and } \bar\lambda \in [\kappa_1, \infty),
$$

and set

$$
E_{\bar\lambda}(\kappa) = \{(t,x) \in \Omega_T : |Du|_{\bar\lambda}^{p(t,x)} > \kappa\}.
$$

By Lemma 4.5 we obtain

$$
\begin{aligned}
|E_{\bar\lambda}(A\kappa)| \le C\varepsilon\Big\{\frac{2}{\kappa}\Big(&\int_{\{(t,x)\in\Omega_T : |Du|_{\bar\lambda}^{p(t,x)} > \frac{\kappa}{4}\}} |Du|_{\bar\lambda}^{p(t,x)}\,dx\,dt\\
&+ \frac{1}{\delta}\int_{\{(t,x)\in\Omega_T : \Psi(t,x) > \frac{\delta\kappa}{4}\}} \Psi(t,x)\,dx\,dt\Big)\Big\}.
\end{aligned}
$$

In the case $\bar\lambda \le A\kappa$, we have $|E_{\bar\lambda}(A\kappa)| = 0$ so that (4.23) holds trivially. Otherwise, while $\bar\lambda > A\kappa$, working exactly as in the previous lines, we obtain the inequality

$$
\begin{aligned}
&\int_{\Omega_T} \phi(|Du|_{\bar\lambda}^{p(t,x)})\,dx\,dt\\
&\le C_6\Big\{\phi\Big(\Big(\fint_{\Omega_T} \Psi(t,x)\,dx\,dt\Big)^m\Big) + \int_{\Omega_T} \phi(\Psi(t,x))\,dx\,dt\Big\},
\end{aligned}
\tag{4.24}
$$

instead of (4.23). Taking $\bar\lambda \to \infty$ and using the lower semi-continuity of Orlicz norm with respect to almost everywhere convergence, we obtain (4.23). This completes the proof. □

As a direct consequence of Theorem 2.5, by taking $\phi(\nu) = \nu^q$ for $q \in (1, \infty)$ we conclude the classical Calderón-Zygmund theory for parabolic obstacle problems with $p(t,x)$-growth.

Corollary 4.6. *Let $q \in (1, \infty)$. Assume that $p(t,x) \in SLH$ with the range $[\gamma_1, \gamma_2]$, and $(\mathbf{a}, \mathbb{R} \times \Omega)$ satisfies (δ, R)-vanishing. If $u \in \mathcal{A}(\Omega_T)$ is a weak solution of the variational inequality* (2.4) *with*

$$
|\psi_t|^{\gamma_1'},\ |D\psi|^{p(t,x)},\ |\boldsymbol{f}|^{p(t,x)} \in L^q(\Omega_T),
$$

then we have $|Du|^{p(t,x)} \in L^q(\Omega_T)$ with the estimate

$$
\int_{\Omega_T} |Du|^{p(t,x)q}\,dx\,dt \le C\Big(\int_{\Omega_T} \Big(\Psi(t,x)\Big)^q\,dx\,dt + 1\Big)^m,
$$

where $C = C(d, \gamma_1, \gamma_2, \lambda, \Lambda, \delta, R, T, |\Omega|, \|u_a\|_{L^2(\Omega)})$*,* $\Psi(t,x)$ *and* $m \geq 1$ *are the same as in Theorem 2.5.*

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