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EXISTENCE OF SOLUTIONS TO NONLOCAL BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH IMPULSES

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Abstract. In this work, through the application of fixed point theory, we consider the properties of the solutions to a nonlocal boundary value problem for fractional differential equations subject to impulses at fixed times. We compute the Green's function related to the problem, which allows us to obtain an integral representation of the solution. This representation gives an explicit description of the solution when the source term does not depend on the solution. Nevertheless, when the description of the source term is implicit, we can not ensure the existence of a solution. In this case, we prove the existence of a solution for the integral problem via fixed point techniques. To do this, we develop a slight generalization of Arzelà-Ascoli theorem that makes it suitable for piecewise uniformly continuous functions.

1. Introduction

The investigation on fractional differential equations has experimented a huge expansion in the previous decades, and new applications have been proposed since then. Some examples of applications of fractional order equations can be found, for instance, in [1], where models of viscoplasticity are considered. The work [3] is focused on protein dynamics; [7] is devoted to continuum and statistical mechanics; and [8] in relaxation in filled polymers.

On the other hand, the physical interpretation for fractional differential equations has been considered in [4] from the point of view of Riemann-Liouville derivatives, also in [11].

In this article, we consider a boundary value problem with integral conditions for a class of fractional differential equations subject to impulses. Some existence results for higher-order fractional differential equations with integral conditions can be found in [2], while some other results related to Pettis integral are included in [13].

We represent by D^δ the fractional derivative of Riemann-Liouville type and consider the following impulsive fractional differential equation with nonlocal boundary conditions:

$$D^{\delta}_{0^+}u(t) + f(t,u(t)) = 0, \quad t \in (0,t_1), \tag{1.1}$$

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$$D^{\delta}_{t_1^+}u(t) + f(t,u(t)) = 0, \quad t \in (t_1, 1), \tag{1.2}$$

$$u(0) = 0, \quad u(t_1^+) = 0, \quad \alpha_0 u(\xi_0) + \alpha_1 u(\xi_1) = u(1), \quad \beta \int_0^1 u(s)\,ds = u(1), \tag{1.3}$$

where $1 < \delta \le 2$, $0 = t_0 < t_1 < 1$, $\xi_i \in (t_i, t_{i+1})$, $i = 0, 1$, $\alpha_0, \alpha_1, \beta \in \mathbb{R} \setminus \{0\}$, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ continuous on $[0, t_1] \times \mathbb{R}$. We also require f to be continuous on $(t_1, 1] \times \mathbb{R}$ and that $f|_{(t_1,1]\times\mathbb{R}}$ can be extended continuously to $[t_1, 1] \times \mathbb{R}$, that is, the discontinuity at t_1 is of finite jump.

The previous problem is a particular case of the following problem

$$D^{\delta}_{t_k^+}u(t) + f(t,u(t)) = 0, \quad t_k < t < t_{k+1}, \quad k = 0, \dots, m, \tag{1.4}$$

$$u(t_k^+) = 0,\ k = 0, \dots, m, \quad \sum_{k=0}^{m} \alpha_k u(\xi_k) = u(T), \quad \beta \int_0^T u(s)\,ds = u(T), \tag{1.5}$$

where $1 < \delta \le 2$, $T > 0$, $0 = t_0 < t_1 < \dots < t_k < t_{k+1} < \dots < t_{m+1} = T$, $\xi_i \in (t_i, t_{i+1})$, $i = 0, \dots, m$, $\alpha_0, \alpha_1, \dots, \alpha_m, \beta \in \mathbb{R}$, and $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ continuous on $[0, t_1]\times\mathbb{R}$ and on $(t_k, t_{k+1}]\times\mathbb{R}$, for $k = 1, \dots, m$, in such a way that $f|_{(t_k,t_{k+1}]\times\mathbb{R}}$ can be extended continuously to $[t_k, t_{k+1}]\times\mathbb{R}$, for $k = 1, \dots, m$. Here, if we consider $T = 1$ and $m = 1$, that is, $0 = t_0 < t_1 < t_2 = 1$, then we obtain problem (1.1)–(1.3).

In Section 2, we present some basic definitions and results. In Section 3, we consider a linear problem and obtain the corresponding Green's function, and in Section 4, we give some existence results for the general nonlinear problem.

2. Basic definitions

We introduce some basic concepts about fractional integrals and derivatives. Some relevant monographs on fractional calculus are [5, 6, 9, 12, 14].

Definition 2.1. The Riemann-Liouville fractional integral of order $\delta > 0$ of a function $f : (a,b] \to \mathbb{R}$ is given by

$$I^{\delta}_{a^+} f(t) = \frac{1}{\Gamma(\delta)} \int_a^t (t-\tau)^{\delta-1} f(\tau)\,d\tau,$$

provided that the right-hand side is pointwise defined on $(a,b]$, and where Γ denotes the Gamma function.

Definition 2.2. For a continuous function $f : (a,b] \to \mathbb{R}$, the Riemann-Liouville derivative of fractional order $\delta > 0$ is given by

$$D^{\delta}_{a^+} f(t) = \frac{1}{\Gamma(n-\delta)} \Big(\frac{d}{dt}\Big)^n \int_a^t (t-\tau)^{n-\delta-1} f(\tau)\,d\tau,$$

where $n = \lfloor \delta \rfloor + 1$, being $\lfloor \delta \rfloor$ the integer part of the real number δ.

Lemma 2.3 ([14]). *Given $\delta > 0$, the solutions to the fractional differential equation*

$$D^{\delta}_{0^+} u(t) = 0$$

are the functions

$$u(t) = c_1 t^{\delta-1} + c_2 t^{\delta-2} + \dots + c_n t^{\delta-n}, \quad c_i \in \mathbb{R},\ i = 1, \dots, n,$$

where $n = \lceil \delta \rceil$, the largest integer less than equal to δ.

Lemma 2.4. *Given $\delta > 0$, the solutions to the fractional differential equation*

$$D^{\delta}_{a^+}u(t) = 0$$

are the functions

$$u(t) = c_1(t-a)^{\delta-1} + c_2(t-a)^{\delta-2} + \dots + c_n(t-a)^{\delta-n}, \quad c_i \in \mathbb{R},\ i = 1, \dots, n,$$

where $n = \lceil \delta \rceil$.

Proof. The result is well known for $\delta \in \mathbb{Z}^+$. Consequently, we can assume that $\delta \notin \mathbb{Z}^+$. Then, the equation

$$D^{\delta}_{a^+}u(t) = \frac{1}{\Gamma(n-\delta)}\Big(\frac{d}{dt}\Big)^n \int_a^t (t-\tau)^{n-\delta-1}u(\tau)\,d\tau = 0$$

can be written, via the change of variable $\tau = a + s$, as

$$\frac{1}{\Gamma(n-\delta)}\Big(\frac{d}{dt}\Big)^n \int_0^{t-a} (t-a-s)^{n-\delta-1}u(a+s)\,ds = 0,$$

or

$$\frac{1}{\Gamma(n-\delta)}\Big(\frac{d}{dt}\Big)^n \int_0^{t-a} (t-a-s)^{n-\delta-1}v(s)\,ds = 0,$$

where $v(s) := u(a+s)$. Now, if we consider $z = t - a$, we obtain that

$$\frac{1}{\Gamma(n-\delta)}\Big(\frac{d}{dz}\Big)^n \int_0^{z} (z-s)^{n-\delta-1}v(s)\,ds = 0.$$

Hence, the function v satisfies the equation $D^{\delta}_{0^+}v(z) = 0$. From Lemma 2.3, we know that

$$u(a+z) = v(z) = c_1 z^{\delta-1} + c_2 z^{\delta-2} + \dots + c_n z^{\delta-n}, \quad c_i \in \mathbb{R},\ i = 1, \dots, n,$$

or, equivalently,

$$u(t) = v(t-a) = c_1\,(t-a)^{\delta-1} + c_2\,(t-a)^{\delta-2} + \dots + c_n\,(t-a)^{\delta-n}, c_i \in \mathbb{R}, i = 1, \dots, n.$$

□

It is straightforward to derive the following result, after a direct application of Lemma 2.4.

Lemma 2.5. *Given $\delta > 0$, we have*

$$I^{\delta}_{a^+}(D^{\delta}_{a^+}u(t)) = u(t) + c_1(t-a)^{\delta-1} + c_2(t-a)^{\delta-2} + \dots + c_n(t-a)^{\delta-n},$$

for $c_i \in \mathbb{R}$, $i = 1, \dots, n$.

Proof. We write $I^{\delta}_{a^+}(D^{\delta}_{a^+}u(t)) = u(t) + f(t)$ and we apply $D^{\delta}_{a^+}$ to both sides of this expression. Since $D^{\delta}_{a} \circ I^{\delta}_{a} = \mathrm{Id}$, we have that $D^{\delta}_{a^+}f = 0$ and the conclusion follows from Lemma 2.4. □

Remark 2.6. From Lemma 2.5, for $a = t_k$, $k = 0, 1, \dots, m$, and $1 < \delta \le 2$, we obtain

$$I^{\delta}_{t_k^+}(D^{\delta}_{t_k^+}u(t)) = u(t) + c_{1,k}(t-t_k)^{\delta-1} + c_{2,k}(t-t_k)^{\delta-2}, \quad c_{1,k},\, c_{2,k} \in \mathbb{R}.$$

3. Green's function for a linear fractional differential equation

In this section, we consider a related linear fractional differential equation with the same boundary conditions, to obtain the explicit expression of the Green's function. First, we need to define the following space of functions.

Definition 3.1. Given a partition $0 = t_0 < t_1 < t_2 = 1$, the space of piecewise continuous functions from $[0, 1]$ to $\mathbb{R}$, is defined as

$$PC([0,1],\mathbb{R}) = \Big\{u : [0,1] \to \mathbb{R} : u \in C([0,t_1],\mathbb{R}) \text{ and } u \in C((t_1,1],\mathbb{R}) \\ \text{with } \lim_{t\to t_1^+} u(t) \text{ finite}\Big\}.$$

Lemma 3.2. *Consider $1 < \delta \le 2$, $\alpha_0, \alpha_1, \beta \in \mathbb{R} \setminus \{0\}$, $0 = t_0 < t_1 < t_2 = 1$, and $\xi_i \in (t_i, t_{i+1})$ for $i = 0, 1$. Assume also that $\sigma \in PC([0, 1], \mathbb{R})$. A function $u \in PC([0, 1], \mathbb{R})$ is a solution to the boundary value problem*

$$D^{\delta}_{0+}u(t) + \sigma(t) = 0, \quad t \in (0, t_1), \tag{3.1}$$

$$D^{\delta}_{t_1+}u(t) + \sigma(t) = 0, \quad t \in (t_1, 1), \tag{3.2}$$

$$u(0) = 0,\ u(t_1^+) = 0, \quad \alpha_0 u(\xi_0) + \alpha_1 u(\xi_1) = u(1), \quad \beta \int_0^1 u(s)\,ds = u(1) \tag{3.3}$$

if and only if it satisfies the integral equation

$$u(t) = \int_0^1 H(t,s)\sigma(s)\,ds,$$

where $H(t, s)$ is the Green's function given in the proof, provided that

$$\varphi(1) - \sum_{k=0}^{1} \alpha_k \varphi(\xi_k) = \frac{t_1^{\delta}\beta(1-t_1)^{1-\delta}(1-t_1)^{\delta-1}}{\delta - \beta(1-t_1)} - \alpha_0 \xi_0^{\delta-1} \\ - \alpha_1 \frac{t_1^{\delta}\beta(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}}{\delta-\beta(1-t_1)} \neq 0,$$

where

$$\varphi(t) = \begin{cases} t^{\delta-1}, & t \in (0, t_1), \\ \frac{t_1^{\delta}\beta(1-t_1)^{1-\delta}(t-t_1)^{\delta-1}}{\delta-\beta(1-t_1)}, & t \in (t_1, 1]. \end{cases}$$

Proof. Integrating (3.1) and (3.2), we obtain

$$I^{\delta}_{t_k^+} D^{\delta}_{t_k^+} u(t) = -I^{\delta}_{t_k^+}\sigma(t), \quad t \in (t_k, t_{k+1}), \quad k = 0, 1.$$

From Remark 2.6, equations (3.1) and (3.2) can be rewritten as

$$u(t) + c_{1,k}(t-t_k)^{\delta-1} + c_{2,k}(t-t_k)^{\delta-2} = -\frac{1}{\Gamma(\delta)}\int_{t_k}^{t} (t-\tau)^{\delta-1}\sigma(\tau)\,d\tau,$$

for $t \in (t_k, t_{k+1})$, where $c_{1,k}, c_{2,k} \in \mathbb{R}$, $k = 0, 1$, or equivalently, by renaming the constants used,

$$u(t) = -\frac{1}{\Gamma(\delta)}\int_{t_k}^{t} (t-\tau)^{\delta-1}\sigma(\tau)\,d\tau + c_{1,k}(t-t_k)^{\delta-1} + c_{2,k}(t-t_k)^{\delta-2},$$

for $t \in (t_k, t_{k+1})$, where $c_{1,k}, c_{2,k} \in \mathbb{R}$, for $k = 0, 1$.

Note that the conditions $u(t_k^+) = 0$, for $k = 0, 1$, imply that $c_{2,k} = 0$, for $k = 0, 1$, so that

$$u(t) = -\frac{1}{\Gamma(\delta)} \int_{t_k}^{t} (t-\tau)^{\delta-1} \sigma(\tau)\, d\tau + c_{1,k}(t-t_k)^{\delta-1}, \quad t \in (t_k, t_{k+1}),$$

where $c_{1,k} \in \mathbb{R}$, for $k = 0, 1$.

Using the integral condition $\beta \int_0^1 u(s) ds = u(1)$, we obtain

$$c_{1,1}(1-t_1)^{\delta-1} = \beta \int_0^1 u(s)\, ds + \frac{1}{\Gamma(\delta)} \int_{t_1}^{1} (1-s)^{\delta-1} \sigma(s)\, ds.$$

The previous calculations lead, for $k = 0$, to

$$u(t) = -\frac{1}{\Gamma(\delta)} \int_0^t (t-\tau)^{\delta-1} \sigma(\tau)\, d\tau + c_{1,0}\, t^{\delta-1}, \quad t \in (0, t_1), \tag{3.4}$$

and, for $k = 1$, to

$$\begin{aligned} u(t) = &- \frac{1}{\Gamma(\delta)} \int_{t_1}^{t} (t-\tau)^{\delta-1} \sigma(\tau)\, d\tau + \beta(1-t_1)^{1-\delta}(t-t_1)^{\delta-1} \int_0^1 u(s)\, ds \\ &+ \frac{(1-t_1)^{1-\delta}}{\Gamma(\delta)} (t-t_1)^{\delta-1} \int_{t_1}^{1} (1-s)^{\delta-1} \sigma(s)\, ds, \quad t \in (t_1, 1). \end{aligned} \tag{3.5}$$

From these expressions, we obtain

$$\begin{aligned} \int_0^1 u(s)\, ds &= \int_0^{t_1} u(s)\, ds + \int_{t_1}^{1} u(s)\, ds \\ &= -\frac{1}{\Gamma(\delta)} \int_0^{t_1} \int_0^t (t-s)^{\delta-1} \sigma(s)\, ds\, dt + \int_0^{t_1} c_{1,0}\, t^{\delta-1}\, dt \\ &\quad - \frac{1}{\Gamma(\delta)} \int_{t_1}^{1} \int_{t_1}^{t} (t-s)^{\delta-1} \sigma(s)\, ds\, dt \\ &\quad + \beta(1-t_1)^{1-\delta} \int_{t_1}^{1} (t-t_1)^{\delta-1}\, dt \int_0^1 u(s)\, ds \\ &\quad + \frac{(1-t_1)^{1-\delta}}{\Gamma(\delta)} \int_{t_1}^{1} (t-t_1)^{\delta-1}\, dt \int_{t_1}^{1} (1-s)^{\delta-1} \sigma(s)\, ds \\ &= -\frac{1}{\delta\Gamma(\delta)} \int_0^{t_1} (t_1-s)^{\delta} \sigma(s)\, ds + \frac{c_{1,0}}{\delta}\, t_1^{\delta} \\ &\quad - \frac{1}{\delta\Gamma(\delta)} \int_{t_1}^{1} (1-s)^{\delta} \sigma(s)\, ds + \beta \frac{1-t_1}{\delta} \int_0^1 u(s)\, ds \\ &\quad + \frac{1-t_1}{\delta\Gamma(\delta)} \int_{t_1}^{1} (1-s)^{\delta-1} \sigma(s)\, ds. \end{aligned}$$

Then

$$\begin{aligned} \int_0^1 u(s)\, ds = &- \frac{1}{(\delta-\beta(1-t_1))\Gamma(\delta)} \int_0^{t_1} (t_1-s)^{\delta} \sigma(s)\, ds + \frac{c_{1,0}}{(\delta-\beta(1-t_1))} t_1^{\delta} \\ &- \frac{1}{(\delta-\beta(1-t_1))\Gamma(\delta)} \int_{t_1}^{1} (1-s)^{\delta} \sigma(s)\, ds \\ &+ \frac{(1-t_1)}{(\delta-\beta(1-t_1))\Gamma(\delta)} \int_{t_1}^{1} (1-s)^{\delta-1} \sigma(s)\, ds. \end{aligned} \tag{3.6}$$

Therefore, we can find a piecewise defined integral kernel $G(t,s)$ that allows us to express u in a simple way. On the one hand, for $t \in (0,t_1)$, it follows from (3.4) that

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1}\sigma(s)\,ds + c_{1,0}t^{\delta-1} \\ &= \int_0^1 G(t,s)\sigma(s)\,ds + c_{1,0}t^{\delta-1} \\ &= \int_0^1 G(t,s)\sigma(s)\,ds + c_{1,0}\varphi(t), \quad t \in (0,t_1). \end{aligned}$$

On the other hand, if we replace expression (3.6) in (3.5), we can describe $u(t)$, for $t \in (t_1,1)$, as

$$\begin{aligned} u(t) =& -\frac{1}{\Gamma(\delta)} \int_{t_1}^t (t-s)^{\delta-1}\sigma(s)\,ds \\ &- \frac{\beta(1-t_1)^{1-\delta}(t-t_1)^{\delta-1}}{(\delta-\beta(1-t_1))\Gamma(\delta)} \int_0^{t_1} (t_1-s)^{\delta}\sigma(s)\,ds \\ &+ \frac{c_{1,0}}{\delta-\beta(1-t_1)} t_1^{\delta}\beta(1-t_1)^{1-\delta}(t-t_1)^{\delta-1} \\ &- \frac{\beta(1-t_1)^{1-\delta}(t-t_1)^{\delta-1}}{(\delta-\beta(1-t_1))\Gamma(\delta)} \int_{t_1}^1 (1-s)^{\delta}\sigma(s)\,ds \\ &+ \frac{\beta(1-t_1)^{2-\delta}(t-t_1)^{\delta-1}}{(\delta-\beta(1-t_1))\Gamma(\delta)} \int_{t_1}^1 (1-s)^{\delta-1}\sigma(s)\,ds \\ &+ \frac{(1-t_1)^{1-\delta}}{\Gamma(\delta)}(t-t_1)^{\delta-1} \int_{t_1}^1 (1-s)^{\delta-1}\sigma(s)\,ds \\ =& -\frac{1}{\Gamma(\delta)} \int_0^{t_1} \frac{\beta(1-t_1)^{1-\delta}(t-t_1)^{\delta-1}}{\delta-\beta(1-t_1)}(t_1-s)^{\delta}\sigma(s)\,ds \\ &+ \frac{1}{\Gamma(\delta)} \int_{t_1}^t \left[-(t-s)^{\delta-1} + (1-t_1)^{1-\delta}(t-t_1)^{\delta-1}(1-s)^{\delta-1}\right]\sigma(s)\,ds \\ &- \frac{1}{\Gamma(\delta)} \int_{t_1}^t \frac{\beta(1-t_1)^{1-\delta}(t-t_1)^{\delta-1}}{\delta-\beta(1-t_1)}(1-s)^{\delta}\sigma(s)\,ds \\ &+ \frac{1}{\Gamma(\delta)} \int_{t_1}^t \frac{\beta(1-t_1)^{2-\delta}(t-t_1)^{\delta-1}}{\delta-\beta(1-t_1)}(1-s)^{\delta-1}\sigma(s)\,ds \\ &- \frac{1}{\Gamma(\delta)} \int_t^1 \frac{\beta(1-t_1)^{1-\delta}(t-t_1)^{\delta-1}}{\delta-\beta(1-t_1)}(1-s)^{\delta}\sigma(s)\,ds \\ &+ \frac{1}{\Gamma(\delta)} \int_t^1 \frac{\beta(1-t_1)^{2-\delta}(t-t_1)^{\delta-1}}{\delta-\beta(1-t_1)}(1-s)^{\delta-1}\sigma(s)\,ds \\ &+ \frac{1}{\Gamma(\delta)} \int_t^1 (1-t_1)^{1-\delta}(t-t_1)^{\delta-1}(1-s)^{\delta-1}\sigma(s)\,ds \\ &+ \frac{c_{1,0}}{\delta-\beta(1-t_1)} t_1^{\delta}\beta(1-t_1)^{1-\delta}(t-t_1)^{\delta-1} \\ =& \int_0^1 G(t,s)\sigma(s)\,ds + \frac{c_{1,0}}{\delta-\beta(1-t_1)} t_1^{\delta}\,\beta(1-t_1)^{1-\delta}(t-t_1)^{\delta-1} \end{aligned}$$

$$= \int_0^1 G(t,s)\sigma(s)\,ds + c_{1,0}\varphi(t), \quad t \in (t_1, 1].$$

In summary,

$$u(t) = \int_0^1 G(t,s)\sigma(s)\,ds + c_{1,0}\varphi(t), \quad t \in (0,1], \tag{3.7}$$

where

$$G(t,s) = \frac{1}{\Gamma(\delta)} \begin{cases} -(t-s)^{\delta-1} \quad \text{if } 0 \le s \le t < t_1, \\ 0 \quad \text{if } 0 \le t \le s \le 1,\ 0 \le t < t_1, \\ -\frac{\beta(1-t_1)^{1-\delta}(t-t_1)^{\delta-1}}{\delta-\beta(1-t_1)}(t_1-s)^{\delta} \quad \text{if } 0 \le s \le t_1 \le t, \\ -(t-s)^{\delta-1} + (1-t_1)^{1-\delta}(t-t_1)^{\delta-1}(1-s)^{\delta-1} \\ -\frac{\beta(1-t_1)^{1-\delta}(t-t_1)^{\delta-1}}{\delta-\beta(1-t_1)}(1-s)^{\delta} \\ +\frac{\beta(1-t_1)^{2-\delta}(t-t_1)^{\delta-1}}{\delta-\beta(1-t_1)}(1-s)^{\delta-1} \quad \text{if } t_1 \le s \le t \le 1, \\ -\frac{\beta(1-t_1)^{1-\delta}(t-t_1)^{\delta-1}}{\delta-\beta(1-t_1)}(1-s)^{\delta} \\ +\frac{\beta(1-t_1)^{2-\delta}(t-t_1)^{\delta-1}}{\delta-\beta(1-t_1)}(1-s)^{\delta-1} \\ +(1-t_1)^{1-\delta}(t-t_1)^{\delta-1}(1-s)^{\delta-1} \quad \text{if } t_1 \le t \le s \le 1; \end{cases}$$

that is,

$$G(t,s) = \frac{1}{\Gamma(\delta)} \begin{cases} -(t-s)^{\delta-1} \quad \text{if } 0 \le s \le t < t_1, \\ 0 \quad \text{if } 0 \le t \le s \le 1, 0 \le t < t_1, \\ -\frac{\beta(1-t_1)^{1-\delta}(t-t_1)^{\delta-1}}{\delta-\beta(1-t_1)}(t_1-s)^{\delta} \quad \text{if } 0 \le s \le t_1 \le t, \\ -(t-s)^{\delta-1} + (1-t_1)^{1-\delta}(t-t_1)^{\delta-1}(1-s)^{\delta-1} \\ \times\big[1 - \frac{\beta(1-s)}{\delta-\beta(1-t_1)} + \frac{\beta(1-t_1)}{\delta-\beta(1-t_1)}\big] \quad \text{if } t_1 \le s \le t \le 1, \\ (1-t_1)^{1-\delta}(t-t_1)^{\delta-1}(1-s)^{\delta-1} \\ \times\big[1 - \frac{\beta(1-s)}{\delta-\beta(1-t_1)} + \frac{\beta(1-t_1)}{\delta-\beta(1-t_1)}\big] \quad \text{if } t_1 \le t \le s \le 1, \end{cases}$$

or, equivalently,

$$\begin{aligned} &G(t,s) \\ &= \frac{1}{\Gamma(\delta)(\delta-\beta(1-t_1))} \\ &\times \begin{cases} -(t-s)^{\delta-1}(\delta-\beta(1-t_1)) \quad \text{if } 0 \le s \le t < t_1, \\ 0 \quad \text{if } 0 \le t \le s \le 1,\ 0 \le t < t_1, \\ -\beta(1-t_1)^{1-\delta}(t-t_1)^{\delta-1}(t_1-s)^{\delta} \quad \text{if } 0 \le s \le t_1 \le t, \\ -(t-s)^{\delta-1}(\delta-\beta(1-t_1)) \\ +(1-t_1)^{1-\delta}(t-t_1)^{\delta-1}(1-s)^{\delta-1}(\delta-\beta(1-s)) \quad \text{if } t_1 \le s \le t \le 1, \\ (1-t_1)^{1-\delta}(t-t_1)^{\delta-1}(1-s)^{\delta-1}(\delta-\beta(1-s)) \quad \text{if } t_1 \le t \le s \le 1. \end{cases} \end{aligned} \tag{3.8}$$

Finally, if we consider the condition $\alpha_0 u(\xi_0) + \alpha_1 u(\xi_1) = u(1)$, we have

$$u(1) = \int_0^1 G(1,s)\sigma(s)\,ds + c_{1,0}\varphi(1)$$

$$
\begin{aligned}
&= \alpha_0 \int_0^1 G(\xi_0, s)\sigma(s)\,ds + \alpha_0\, c_{1,0}\varphi(\xi_0) + \alpha_1 \int_0^1 G(\xi_1, s)\sigma(s)\,ds + \alpha_1\, c_{1,0}\varphi(\xi_1) \\
&= \int_0^1 [\alpha_0 G(\xi_0, s) + \alpha_1 G(\xi_1, s)]\sigma(s)\,ds + (\alpha_0\varphi(\xi_0) + \alpha_1\varphi(\xi_1))c_{1,0} \\
&= \int_0^1 \sum_{k=0}^{1} \alpha_k\, G(\xi_k, s)\sigma(s)\,ds + c_{1,0} \sum_{k=0}^{1} \alpha_k\varphi(\xi_k);
\end{aligned}
$$

that is,

$$
\int_0^1 \Big[\sum_{k=0}^{1} \alpha_k G(\xi_k, s) - G(1, s)\Big]\sigma(s)\,ds = c_{1,0}\Big[\varphi(1) - \sum_{k=0}^{1} \alpha_k\varphi(\xi_k)\Big].
$$

Since, by hypothesis, $\varphi(1) - \sum_{k=0}^{1} \alpha_k\varphi(\xi_k) \neq 0$, we have

$$
c_{1,0} = \frac{1}{\varphi(1) - \sum_{k=0}^{1} \alpha_k\varphi(\xi_k)} \int_0^1 \Big[\sum_{k=0}^{1} \alpha_k\, G(\xi_k, s) - G(1, s)\Big]\sigma(s)\,ds.
$$

We know that

$$
G(1, s) = \frac{1}{\Gamma(\delta)(\delta - \beta(1 - t_1))} \begin{cases} -\beta(t_1 - s)^{\delta}, & 0 \le s \le t_1, \\ (1 - s)^{\delta - 1}\beta(s - t_1), & t_1 \le s \le 1. \end{cases}
$$

Moreover, for $\xi_0 \in (0, t_1)$, we have

$$
G(\xi_0, s) = \frac{1}{\Gamma(\delta)(\delta - \beta(1 - t_1))} \begin{cases} -(\xi_0 - s)^{\delta - 1}(\delta - \beta(1 - t_1)), & 0 \le s \le \xi_0, \\ 0, & \xi_0 \le s \le 1, \end{cases}
$$

and, for $\xi_1 \in (t_1, 1)$,

$$
\begin{aligned}
G(\xi_1, s) = &\frac{1}{\Gamma(\delta)(\delta - \beta(1 - t_1))} \\
&\times \begin{cases} -\beta(1 - t_1)^{1-\delta}(\xi_1 - t_1)^{\delta - 1}(t_1 - s)^{\delta}, & 0 \le s \le t_1, \\ -(\xi_1 - s)^{\delta - 1}(\delta - \beta(1 - t_1)) \\ +(1 - t_1)^{1-\delta}(\xi_1 - t_1)^{\delta - 1}(1 - s)^{\delta - 1}(\delta - \beta(1 - s)), & t_1 \le s \le \xi_1, \\ (1 - t_1)^{1-\delta}(\xi_1 - t_1)^{\delta - 1}(1 - s)^{\delta - 1}(\delta - \beta(1 - s)), & \xi_1 \le s \le 1. \end{cases}
\end{aligned}
$$

If we take $A = \frac{1}{\varphi(1) - \sum_{k=0}^{1} \alpha_k\varphi(\xi_k)}$ and $B = \frac{A}{\Gamma(\delta)(\delta - \beta(1 - t_1))}$, we have

$$
\begin{aligned}
c_{1,0} =& A \int_0^1 \Big[\sum_{k=0}^{1} \alpha_k G(\xi_k, s) - G(1, s)\Big]\sigma(s)\,ds \\
=& A \int_0^1 [\alpha_0 G(\xi_0, s) + \alpha_1\, G(\xi_1, s) - G(1, s)]\sigma(s)\,ds \\
=& B \int_0^{\xi_0} \big[-\alpha_0(\xi_0 - s)^{\delta - 1}(\delta - \beta(1 - t_1)) \\
&- \alpha_1\beta(1 - t_1)^{1-\delta}(\xi_1 - t_1)^{\delta - 1}(t_1 - s)^{\delta} + \beta(t_1 - s)^{\delta}\big]\sigma(s)\,ds \\
&+ B \int_{\xi_0}^{t_1} [-\alpha_1\beta(1 - t_1)^{1-\delta}(\xi_1 - t_1)^{\delta - 1}(t_1 - s)^{\delta} + \beta(t_1 - s)^{\delta}]\sigma(s)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ B\int_{t_1}^{\xi_1} -\alpha_1(\xi_1-s)^{\delta-1}(\delta-\beta(1-t_1))\sigma(s)\,ds\\
&+ B\int_{t_1}^{\xi_1} \alpha_1(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(1-s)^{\delta-1}(\delta-\beta(1-s))\sigma(s)\,ds\\
&+ B\int_{t_1}^{\xi_1} [-(1-s)^{\delta-1}\beta(s-t_1)]\sigma(s)\,ds\\
&+ B\int_{\xi_1}^{1} \big[\alpha_1(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(1-s)^{\delta-1}(\delta-\beta(1-s))\\
&- (1-s)^{\delta-1}\beta(s-t_1)\big]\sigma(s)\,ds\\
={}&\int_0^1 K(s)\,ds,
\end{aligned}
$$

where

$$
K(s)=\begin{cases}
B\big[-\alpha_0(\xi_0-s)^{\delta-1}(\delta-\beta(1-t_1))\\
-\alpha_1\beta(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(t_1-s)^{\delta}\big]+B\beta(t_1-s)^{\delta}, & 0\le s\le \xi_0,\\
B[-\alpha_1\beta(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(t_1-s)^{\delta}+\beta(t_1-s)^{\delta}], & 0\le\xi_0\le s\le t_1,\\
-B\alpha_1(\xi_1-s)^{\delta-1}(\delta-\beta(1-t_1))\\
+B\alpha_1(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(1-s)^{\delta-1}(\delta-\beta(1-s))\\
+B[-(1-s)^{\delta-1}\beta(s-t_1)], & t_1\le s\le\xi_1,\\
B\alpha_1(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(1-s)^{\delta-1}(\delta-\beta(1-s))\\
-B(1-s)^{\delta-1}\beta(s-t_1), & \xi_1<s\le 1.
\end{cases}
$$

Then, for $t\in(0,t_1]$, we can write

$$
u(t)=-\frac{1}{\Gamma(\delta)}\int_0^t(t-s)^{\delta-1}\sigma(s)\,ds+c_{1,0}\,t^{\delta-1}=\int_0^1 H(t,s)\sigma(s)\,ds,
$$

where H is defined in the following way. Denoting $C=\frac{A}{\delta-\beta(1-t_1)}$, for $t\in(0,\xi_0]$, we have

$$
H(t,s)=\frac{t^{\delta-1}}{\Gamma(\delta)}\begin{cases}
-(t-s)^{\delta-1}t^{1-\delta}+C\big[-\alpha_0(\xi_0-s)^{\delta-1}\\
-\alpha_1\beta(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(t_1-s)^{\delta}\big]+C\beta(t_1-s)^{\delta}\\
\quad \text{if } 0\le s\le t\le\xi_0,\\
C\big[-\alpha_0(\xi_0-s)^{\delta-1}-\alpha_1\beta(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(t_1-s)^{\delta}\big]\\
+C\beta(t_1-s)^{\delta}\quad \text{if } 0\le t<s\le\xi_0,\\
C\big[-\alpha_1\beta(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(t_1-s)^{\delta}+\beta(t_1-s)^{\delta}\big]\\
\quad \text{if } 0\le t\le\xi_0<s\le t_1,\\
-C\alpha_1(\xi_1-s)^{\delta-1}(\delta-\beta(1-t_1))\\
+C\alpha_1(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(1-s)^{\delta-1}(\delta-\beta(1-s))\\
+C\big[-(1-s)^{\delta-1}\beta(s-t_1)\big]\\
\quad \text{if } 0\le t\le\xi_0,\ t_1\le s\le\xi_1,\\
C\alpha_1(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(1-s)^{\delta-1}(\delta-\beta(1-s))\\
-C(1-s)^{\delta-1}\beta(s-t_1)\quad \text{if } 0\le t\le\xi_0,\ \xi_1<s\le 1,
\end{cases}
$$

and, for $t \in (\xi_0, t_1]$,

$$
H(t,s) = \frac{t^{\delta-1}}{\Gamma(\delta)} \begin{cases} -(t-s)^{\delta-1} t^{1-\delta} + C\big[-\alpha_0(\xi_0-s)^{\delta-1} \\ -\alpha_1\beta(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(t_1-s)^{\delta}\big] + C\beta(t_1-s)^{\delta} \\ \quad \text{if } 0 \le s \le \xi_0 < t \le t_1, \\ -(t-s)^{\delta-1}t^{1-\delta} \\ +C\left[-\alpha_1\beta(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(t_1-s)^{\delta} + \beta(t_1-s)^{\delta}\right] \\ \quad \text{if } 0 \le \xi_0 \le s \le t \le t_1, \\ C[-\alpha_1\beta(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(t_1-s)^{\delta} + \beta(t_1-s)^{\delta}] \\ \quad \text{if } 0 \le \xi_0 \le t < s \le t_1 \\ -C\alpha_1(\xi_1-s)^{\delta-1}(\delta-\beta(1-t_1)) \\ +C\alpha_1(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(1-s)^{\delta-1}(\delta-\beta(1-s)) \\ +C[-(1-s)^{\delta-1}\beta(s-t_1)] \quad \text{if } \xi_0 < t \le t_1,\ t_1 \le s \le \xi_1, \\ C\alpha_1(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(1-s)^{\delta-1}(\delta-\beta(1-s)) \\ -C(1-s)^{\delta-1}\beta(s-t_1) \quad \text{if } \xi_0 < t \le t_1,\ \xi_1 < s \le 1. \end{cases}
$$

If $t \in (t_1, 1]$, we have

$$
\begin{aligned} u(t) &= \int_0^1 G(t,s)\sigma(s)\,ds + \frac{t_1^{\delta}\beta(1-t_1)^{1-\delta}(t-t_1)^{\delta-1}}{\delta-\beta(1-t_1)}\, c_{1,0} \\ &= \int_0^1 H(t,s)\sigma(s)\,ds, \end{aligned}
$$

where G is given by (3.8). To find an explicit expression for H when $t \in (t_1, 1]$, we distinguish two cases. When $t \in (t_1, \xi_1]$, we have

$$
H(t,s) = \frac{t_1^{\delta}\beta(1-t_1)^{1-\delta}(t-t_1)^{\delta-1}}{\Gamma(\delta)(\delta-\beta(1-t_1))}
$$

$$
\times \begin{cases}
-\frac{(t_1-s)^\delta}{t_1^\delta} + C\Big[-\alpha_0(\xi_0-s)^{\delta-1}(\delta-\beta(1-t_1)) \\
-\alpha_1\beta(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(t_1-s)^\delta\Big] \\
+C\beta(t_1-s)^\delta \\
\quad \text{if } 0\le s\le \xi_0,\ t_1<t\le \xi_1, \\
-\frac{(t_1-s)^\delta}{t_1^\delta} + C[-\alpha_1\beta(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(t_1-s)^\delta + \beta(t_1-s)^\delta] \\
\quad \text{if } 0\le \xi_0\le s\le t_1<t\le \xi_1, \\
-\frac{(\delta-\beta(1-t_1))}{t_1^\delta\beta(1-t_1)^{1-\delta}(t-t_1)^{\delta-1}}(t-s)^{\delta-1} + \frac{1}{t_1^\delta\beta}(1-s)^{\delta-1}(\delta-\beta(1-s)) \\
-C\alpha_1(\xi_1-s)^{\delta-1}(\delta-\beta(1-t_1)) \\
+C\alpha_1(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(1-s)^{\delta-1}(\delta-\beta(1-s)) \\
+C[-(1-s)^{\delta-1}\beta(s-t_1)] \quad \text{if } t_1\le s\le t\le \xi_1, \\
\frac{(\delta-\beta(1-s))}{t_1^\delta\beta}(1-s)^{\delta-1} - C\alpha_1(\xi_1-s)^{\delta-1}(\delta-\beta(1-t_1)) \\
+C\alpha_1(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(1-s)^{\delta-1}(\delta-\beta(1-s)) \\
+C[-(1-s)^{\delta-1}\beta(s-t_1)] \quad \text{if } t_1\le t<s\le \xi_1, \\
\frac{(\delta-\beta(1-s))}{t_1^\delta\beta}(1-s)^{\delta-1} \\
+C\alpha_1(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(1-s)^{\delta-1}(\delta-\beta(1-s)) \\
-C(1-s)^{\delta-1}\beta(s-t_1) \quad \text{if } t_1<t\le \xi_1<s\le 1,
\end{cases}
$$

and, for $t\in(\xi_1,1]$,

$$
H(t,s) = \frac{t_1^\delta\beta(1-t_1)^{1-\delta}(t-t_1)^{\delta-1}}{\Gamma(\delta)(\delta-\beta(1-t_1))}
$$
$$
\begin{cases}
-\frac{(t_1-s)^\delta}{t_1^\delta} + C\big[-\alpha_0(\xi_0-s)^{\delta-1}(\delta-\beta(1-t_1)) \\
-\alpha_1\beta(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(t_1-s)^\delta\big] + C\beta(t_1-s)^\delta \\
\quad \text{if } 0\le s\le \xi_0,\ \xi_1<t\le 1, \\
-\frac{(t_1-s)^\delta}{t_1^\delta} + C[-\alpha_1\beta(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(t_1-s)^\delta + \beta(t_1-s)^\delta] \\
\quad \text{if } 0\le \xi_0\le s\le t_1,\ \xi_1<t\le 1, \\
-\frac{(\delta-\beta(1-t_1))}{t_1^\delta\beta(1-t_1)^{1-\delta}(t-t_1)^{\delta-1}}(t-s)^{\delta-1} + \frac{1}{t_1^\delta\beta}(1-s)^{\delta-1}(\delta-\beta(1-s)) \\
-C\alpha_1(\xi_1-s)^{\delta-1}(\delta-\beta(1-t_1)) \\
+C\alpha_1(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(1-s)^{\delta-1}(\delta-\beta(1-s)) \\
+C\left[-(1-s)^{\delta-1}\beta(s-t_1)\right] \quad \text{if } t_1\le s\le \xi_1<t\le 1, \\
-\frac{(\delta-\beta(1-t_1))}{t_1^\delta\beta(1-t_1)^{1-\delta}(t-t_1)^{\delta-1}}(t-s)^{\delta-1} + \frac{1}{t_1^\delta\beta}(1-s)^{\delta-1}(\delta-\beta(1-s)) \\
C\alpha_1(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(1-s)^{\delta-1}(\delta-\beta(1-s)) \\
-C(1-s)^{\delta-1}\beta(s-t_1) \quad \text{if } \xi_1<s\le t<1, \\
\frac{(\delta-\beta(1-s))}{t_1^\delta\beta}(1-s)^{\delta-1} \\
+C\alpha_1(1-t_1)^{1-\delta}(\xi_1-t_1)^{\delta-1}(1-s)^{\delta-1}(\delta-\beta(1-s)) \\
-C(1-s)^{\delta-1}\beta(s-t_1) \quad \text{if } \xi_1<t<s<1.
\end{cases}
$$

The proof is complete. $\square$

4. Existence results

We denote by X the set of piecewise uniformly continuous functions on $[0,1]$. Let t_j for $j \in \{0,1,\dots,n\}$, with $0 = t_0 < t_1 < \dots < t_{n-1} < t_n = 1$ be a fixed collection of finitely many points in $[0,1]$. We will say that $u \in X$ if and only if $u_{|(t_{j-1},t_j)}$ is uniformly continuous for each j from 1 to n. We endow the set X with the supremum norm

$$\|u\|_\infty := \sup_{t\in[0,1]} |u(t)|, \quad u \in X,$$

which makes X a Banach space.

The main goal of this section is to show that the integral equation

$$u(t) = \int_0^1 H(t,s)\, f(s,u(s))\, ds \tag{4.1}$$

has a solution in X.

The kernel H associated to the problem of interest has been already computed, and it is trivial to check that there exists a finite collection of (sixteen) open rectangular regions such that H is uniformly continuous on each of them, with their closures covering $[0,1]\times[0,1]$. In particular, we can conclude that H is bounded, in absolute value, by some constant $A \in \mathbb{R}^+$.

Observe that, until now, we have not imposed hypotheses over the function f, since we were focused on the study of linear problems. Now, we assume the following conditions:

- (H1) There exists $\alpha \in [0,1)$ and $B \in \mathbb{R}^+$ such that $|f(t,x)t^\alpha| \le B$ for all $(t,x) \in (0,1]\times\mathbb{R}$.
- (H2) H is uniformly continuous on each $(t_i,t_{i+1})\times(t_j,t_{j+1})$, for $i,j \in \{0,\dots,n-1\}$.
- (H3) There exists $A \in \mathbb{R}^+$ such that $|H(t,s)| \le A$, for all $(t,s) \in [0,1]\times[0,1]$.
- (H4) $f(t,x)t^\alpha$ is uniformly continuous on each $(t_i,t_{i+1}) \times [-\frac{AB}{1-\alpha}, \frac{AB}{1-\alpha}]$, for $i \in \{0,\dots,n-1\}$.

Equation (4.1) is of Hammerstein-type, and it is not very difficult to prove (in fact, it is well known) that it has continuous solutions under the assumption of continuity on the functions H and f. In this case, the essential tool to prove the existence of continuous solutions is Schauder's fixed point theorem. In our case, we will reproduce a similar proof. However, since we only have hypotheses concerning piecewise uniform continuity, we will only seek for piecewise uniformly continuous solutions.

To apply Schauder's theorem to our problem, we need to find a closed and convex set $S \subset X$ such that $T : S \to S$ is continuous, and $T(S)$ is a relatively compact subset of X, where

$$[Tu](t) = \int_0^1 H(t,s)\, f(s,u(s))\, ds, \quad t \in [0,1].$$

Lemma 4.1. *Under the hypotheses* (H1) *and* (H2)*, the map* $T : X \to X$ *is well defined.*

Proof. From (H1), we know that there exist $B > 0$ and $\alpha \in [0,1)$ such that

$$|f(s,u)| \le B s^{-\alpha}, \quad \text{for all } (s,u) \in (0,1]\times\mathbb{R}.$$

Furthermore, we know that $(t_i, t_{i+1}) \times (t_j, t_{j+1})$, for $i, j \in \{0, \dots, n-1\}$, is a finite collection of open rectangles in $[0,1] \times [0,1]$ such that H is uniformly continuous on each of them, and the union of their closures covers $[0,1] \times [0,1]$. Thus, we can ensure that, for each $\varepsilon > 0$ and each $i \in \{0, \dots, n-1\}$ fixed, there exists $\delta > 0$ such that

$$|H(t,s) - H(\hat{t}, s)| < \frac{1-\alpha}{B}\varepsilon, \quad \text{for every } t, \hat{t} \in (t_i, t_{i+1}) \text{ with } |t - \hat{t}| < \delta.$$

Note that the pair of points (t,s) and $(\hat{t}, s)$ lie in the same rectangle for each $s \in [0,1]$, but that common rectangle changes with the value of s. However, the conclusion holds because of the finiteness of the number of rectangles.

The argument to check that T maps X into itself can be developed in the following way. We consider a fixed $u \in X$ and a fixed $i \in \{0, \dots, n-1\}$. Then, we deduce that

$$\begin{aligned} |[Tu](t) - [Tu](\hat{t})| &= \Big| \int_0^1 (H(t,s) - H(\hat{t}, s)) f(s, u(s)) ds \Big| \\ &\leq \int_0^1 |H(t,s) - H(\hat{t}, s)||f(s,u(s))| ds \\ &\leq B \frac{1-\alpha}{B} \varepsilon \int_0^1 s^{-\alpha} ds = \varepsilon, \end{aligned} \tag{4.2}$$

for t, $\hat{t} \in (t_i, t_{i+1})$ such that $|t - \hat{t}| < \delta$. Hence, we have shown that Tu is uniformly continuous on any interval (t_i, t_{i+1}), for $i \in \{0, \dots, n-1\}$. By definition, this implies that $Tu \in X$. □

As we mentioned above, the idea is to show the existence of a fixed point for the equation $u = Tu$, with a suitable application of the following theorem.

Theorem 4.2 (Schauder's Theorem [15]). *Let S be a closed and convex set in a Banach space and assume that $T : S \to S$ is a continuous mapping such that $T(S)$ is a relatively compact set of S. Then T has, at least, one fixed point.*

The first step in our study is to make an adequate choice for S, the set in which we will look for a fixed point of the mapping T. Under conditions (H1) and (H3), we consider

$$S = \Big\{ u \in X : \|u\|_\infty \leq \frac{AB}{1-\alpha} \Big\},$$

where A, B are bounds for $H(t,s)$ and $f(t,x)t^\alpha$, respectively, with α given by (H1).

Lemma 4.3. *Under hypotheses* (H1) *and* (H3), *$T(S) \subseteq S$.*

Proof. Certainly, T maps the set S into itself since, for every $t \in [0,1]$, we have

$$|[Tu](t)| = \Big| \int_0^1 H(t,s)\, f(s,u(s))\, ds \Big| \leq \int_0^1 |H(t,s)|\, |f(s,u(s))|\, ds \leq \frac{AB}{1-\alpha}\,.$$

Furthermore, S is closed since it is the pre-image of a closed interval via a continuous function (the norm function). Note that, by the triangle inequality, S is also convex. □

By Theorem 4.2, we only need to prove that $T(S)$ is relatively compact, and that T is continuous on S. The relative compactness will be an immediate consequence of previous checks and a general version of Arzelà-Ascoli theorem. First, we state the original version of Arzelà-Ascoli theorem with a trivial remark.

Theorem 4.4 (Arzelà-Ascoli Theorem [16]). *Consider a subset U.of the set of continuous functions on a compact interval I. Then U is relatively compact if and only if the functions in U are uniformly bounded and they are equicontinuous.*

Remark 4.5. The assumption concerning the compactness of the interval I can be replaced by relative compactness, provided that the functions in U are uniformly continuous. The reason is the well known theorem of extension of uniformly continuous functions on a relatively compact set.

Recall that, given the sequence of points $0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = 1$, S consists of the functions u such that the restrictions $u_{|(0,t_1)}, u_{|(t_1,t_2)}, \dots, u_{|(t_{n-1},t_n)}$ are uniformly continuous, and such that $\|u\|_\infty \leq \frac{AB}{1-\alpha}$. Define X_j as the set of uniformly continuous functions on (t_{j-1}, t_j), for j from 1 to n. Then, there is a natural isometry between X and

$$P := \mathbb{R} \times X_1 \times \mathbb{R} \times X_2 \times \cdots \times X_{n-1} \times \mathbb{R} \times X_n \times \mathbb{R},$$

where each copy of $\mathbb{R}$ reflects the value of $u \in X$ at t_i, for $i = 0, 1, \dots, n$. The distance between two elements in the previous cartesian product is the maximum distance between their respective coordinates, and the distance between each pair of respective coordinates is obtained with $\|\cdot\|_\infty$.

Theorem 4.6 (General version of Arzelà-Ascoli Theorem). *Consider a subset $R \subset X$ that is isometric to a cartesian product*

$$C_0 \times R_1 \times C_1 \times R_2 \times \cdots \times R_{n-1} \times C_{n-1} \times R_n \times C_n \subset P. \tag{4.3}$$

Then R is relatively compact if and only if the functions in the subset R are uniformly bounded and piecewise uniformly equicontinuous.

Proof. Assume that R is relatively compact, that is, $\bar{R}$ is compact. By the isometry between X and P, we deduce that

$$\overline{C_0 \times R_1 \times C_1 \times R_2 \times \cdots \times R_{n-1} \times C_{n-1} \times R_n \times C_n}$$

is compact. However, the product of the closures is the closure of the product, so

$$\overline{C_0} \times \overline{R_1} \times \overline{C_1} \times \overline{R_2} \times \cdots \times \overline{R_{n-1}} \times \overline{C_{n-1}} \times \overline{R_n} \times \overline{C_n}$$

is compact [10]. We also know that a cartesian product is compact if and only if each factor is compact, so each $\overline{R_i}$ and each $\overline{C_j}$ is compact, meaning that each R_i and each C_j is relatively compact. By the original version of Arzelà-Ascoli theorem and Remark 4.5, we know that each R_i and each C_j is uniformly bounded and equicontinuous. However, this is clearly equivalent to say that

$$C_0 \times R_1 \times C_1 \times R_2 \times \cdots \times R_{n-1} \times C_{n-1} \times R_n \times C_n$$

is uniformly bounded and equicontinuous in each coordinate. By the isometry, we conclude that R is uniformly bounded and piecewise uniformly equicontinuous. Note that every argument used above is reversible, so the converse deduction also holds. $\square$

Remark 4.7. In our case, the condition of equicontinuity of C_j is trivial, since C_j consists of a family of functions whose domain is $\{t_j\}$.

We will use Theorem 4.6 to deduce that S is relatively compact. Note that S is isometric to a cartesian product, like the one stated in (4.3), after choosing $C_0 = \dots = C_n = [-\frac{AB}{1-\alpha}, \frac{AB}{1-\alpha}]$, and R_i, for each $i \in \{0, \dots, n-1\}$, as the set of uniformly continuous functions on (t_{i-1}, t_i) that are bounded, in absolute value, by $\frac{AB}{1-\alpha}$.

Lemma 4.8. *Operator $T : S \to S$ is continuous and $T(S)$ is relatively compact, provided that* (H1)–(H4) *hold.*

Proof. By the construction of S and Lemma 4.3, it is trivial to show that $T(S) \subset S$ is uniformly bounded. Furthermore, in (4.2), we have proved that the functions in $T(S)$ are piecewise uniformly equicontinuous. Hence, Theorem 4.6 implies that $T(S)$ is relatively compact.

To conclude, we only need to prove that $T : S \to S$ is continuous. If $u \in S$, we know that $|u(t)| \le \frac{AB}{1-\alpha}$ for every $t \in [0, 1]$. Moreover, because of *(H4)*, we know that $f(t, x)t^\alpha$ is uniformly continuous on each $(t_i, t_{i+1}) \times [-\frac{AB}{1-\alpha}, \frac{AB}{1-\alpha}]$. Hence, if we fix $\varepsilon > 0$ and an index $i \in \{0, \dots, n-1\}$, we can find $\delta_i > 0$ ensuring that, if $t_i < t < t_{i+1}$ and $x, \hat{x} \in \left[-\frac{AB}{1-\alpha}, \frac{AB}{1-\alpha}\right]$ are such that $|x - \hat{x}| < \delta_i$, we have

$$|f(t, x) - f(t, \hat{x})| < \frac{1-\alpha}{A} \varepsilon t^{-\alpha}.$$

Hence, after fixing $u_0 \in S$ and $\varepsilon > 0$, given any $u \in S$ such that

$$\|u - u_0\| < \delta := \min\{\delta_i \ : \ i = 0, \dots n-1\},$$

we have $|u(t) - u_0(t)| < \delta$, for all $t \in [0, 1]$. Thus,

$$\begin{aligned}
\|Tu - Tu_0\| &= \sup_{t \in [0,1]} \Big| \int_0^1 H(t, s)(f(s, u(s)) - f(s, u_0(s)))\, ds \Big| \\
&\le \sup_{t \in [0,1]} \int_0^1 |H(t, s)|\, |f(s, u(s)) - f(s, u_0(s))|\, ds \\
&\le A \frac{1-\alpha}{A} \varepsilon \int_0^1 s^{-\alpha} ds = \varepsilon.
\end{aligned}$$

This proves the continuity of T on S so, due to Schauder's fixed point theorem, we conclude the existence of at least one solution u for the equation $u = Tu$, with $u \in S$. $\square$

Remark 4.9. Note that, in the previous results, the term t^α can be replaced by any integrable function g on $[0, 1]$, replacing $1 - \alpha$ by the value $\int_0^1 g(s)ds$.

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