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BOUNDEDNESS AND GLOBAL SOLVABILITY FOR A CHEMOTAXIS-HAPTOTAXIS MODEL WITH p-LAPLACIAN DIFFUSION

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Abstract. We consider a chemotaxis-haptotaxis system with p-Laplacian diffusion in three dimensional bounded domains. It is asserted that for any $p > 2$, under the appropriate assumptions, the chemotaxis-haptotaxis system admits a global bounded weak solution if for initial data satisfies certain conditions.

1. Introduction

In this article, we study the problem

$$\begin{gathered} u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) - \chi\nabla\cdot\Big(\frac{u}{(1+u)^{\alpha}}\nabla v\Big) - \xi\nabla\cdot\Big(\frac{u}{(1+u)^{\beta}}\nabla w\Big) \\ + \mu u(1-u-w), \quad x\in\Omega, t>0, \\ v_t - \Delta v + v = f(u)g(w), \quad x\in\Omega,\ t>0, \\ w_t = -vw, \quad x\in\Omega, t>0, \\ (|\nabla u|^{p-2}\nabla u - \chi\frac{u}{(1+u)^{\alpha}}\nabla v - \xi\frac{u}{(1+u)^{\beta}}\nabla w)\cdot\vec{n}|_{\partial\Omega} = \frac{\partial v}{\partial\vec{n}}\big|_{\partial\Omega} = 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), w(x,0) = w_0(x), \quad x\in\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with the smooth boundary, $p > 2$, $\chi, \xi \geq 0$, $\mu > 0$, $\alpha > 0$, $\beta \geq 0$, f and g satisfy

$$|f(s)| \leq K_1|s|^{q_1} + K_2, \quad |g(s)| \leq K_3|s|^{q_2} + K_4, \quad 0 < q_1 \leq 1,\ 0 < q_2 < \infty, \tag{1.2}$$

with some positive constants K_i. u represent the cancer cell density, v is the matrix degrading enzyme concentration and w stands for the extracellular matrix density. A chemotaxis model was first introduced by Keller and Segel [4] in 1970, later, many modified chemotaxis models have been widely studied by many researchers. Recently, Chen and Tao [1] considered the chemotaxis-haptotaxis system

$$\begin{gathered} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) - \xi\nabla\cdot(v\nabla w) + \mu u(1-u-w), \quad x\in\Omega,\ t>0, \\ v_t = \Delta v - v + f(u)g(w), \quad x\in\Omega,\ t>0, \\ w_t = -vw + \eta w(1-u-w), \quad x\in\Omega, t>0. \end{gathered}$$

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They showed that for any given suitably regular initial data the problem posses a unique global-in-time classical solution which is uniformly bounded (see also Tao and Winkler [11]).

Xu, Zhang and Jin [13] studied the problem

$$\begin{aligned} u_t &= \Delta u^m - \chi\nabla\cdot\Big(\frac{u}{(1+u)^\alpha}\cdot\nabla v\Big) - \xi\nabla\cdot\Big(\frac{u}{(1+u)^\beta}\cdot\nabla w\Big) \\ &\quad + \mu u(1-u-w), \quad x\in\Omega,\ t>0, \\ v_t &- \Delta v + v = u, \quad x\in\Omega,\ t>0, \\ w_t &= -vw, \quad x\in\Omega,\ t>0. \end{aligned}$$

It is shown that under zero-flux boundary conditions, for any $m>0$, the above problem admits a global bounded weak solution. They also discussed the large time behavior of solutions for the fast diffusion case, and showed that if $0<m\le 1$, for appropriately large μ, for any initial datum, the solution (u,v,w) goes to a steady $(1,1,0)$ as $t\to\infty$.

Zheng [15] investigated the Keller-Segel-Stokes system with nonlinear diffusion

$$\begin{gathered} n_t + u\cdot\nabla n = \Delta n^m - \nabla\cdot(n\nabla c), \\ c_t + u\cdot\nabla c = \Delta c - c + n, \\ u_t + \nabla P = \Delta u + n\nabla\phi, \\ \nabla u = 0. \end{gathered}$$

He proved that if $m>4/3$, then for any sufficiently regular nonnegative initial data there exists at least one global bounded solution for system, which in view of the known results for the fluid-free system is an optimal restriction on m.

Tao and Li [7] studied the chemotaxis-Navier-Stokes system

$$\begin{gathered} n_t + u\cdot\nabla n = \nabla\cdot(|\nabla n|^{p-2}\nabla n) - \nabla\cdot(n\chi(c)\nabla c)\,, \\ c_t + u\cdot\nabla c = \Delta c - nf(c), \\ u_t + (u\cdot\nabla)u = \Delta u + \nabla P + n\nabla\phi, \\ \nabla u = 0. \end{gathered}$$

They show that if $p>2$, under appropriate assumptions on f and χ, for all sufficiently smooth initial data (n_0,c_0,u_0), the system has at least one global weak solution. The relevant equations have also been studied in [8, 14].

Liu and Li [6] studied the problem

$$\begin{aligned} u_t &= \nabla\cdot(D(u,\nabla u)\nabla u) - \chi\nabla\cdot(u\cdot\nabla v) - \xi\nabla\cdot(u\cdot\nabla w) \\ &\quad + \mu u(1-u-w), \quad x\in\Omega,\ t>0, \\ v_t &- \Delta v + v = u, \quad x\in\Omega,\ t>0, \\ w_t &= -vw, \quad x\in\Omega,\ t>0. \end{aligned}$$

They proved that the problem admits a global bounded weak solution.

Li [5] considered an attraction-repulsion chemotaxis system with p-Laplacian diffusion

$$\begin{gathered} u_t = \nabla\cdot(|\nabla u|^{p-2}\nabla u) - \chi\nabla\cdot(u\cdot\nabla v) - \xi\nabla\cdot(u\cdot\nabla w), \quad x\in\Omega,\ t>0, \\ 0 = \Delta v + \alpha u - \beta v, \quad x\in\Omega,\ t>0, \\ 0 = \Delta w + \gamma u - \delta w, \quad x\in\Omega,\ t>0. \end{gathered}$$

Now we state our assumptions:

$$
\begin{gathered}
u_0, \Delta\sqrt{w_0} \in L^\infty(\Omega), \quad |\nabla u_0|^{p-2}\nabla u_0 \in L^2(\Omega), \quad v_0, w_0 \in W^{2,\infty}(\Omega),\\
u_0, v_0, w_0 \ge 0, \quad |\nabla u_0|^{p-2}\nabla u_0 \cdot \vec{n} = \frac{\partial v_0}{\partial \vec{n}}\big|_{\partial\Omega} = \frac{\partial w_0}{\partial \vec{n}}\big|_{\partial\Omega} = 0,\\
\partial\Omega \in C^{2,\gamma} \quad \text{with } 0 < \gamma < 1.
\end{gathered} \tag{1.3}
$$

In section 2, the boundary conditions become equivalent to

$$
|\nabla u|^{p-2}\nabla u \cdot \vec{n}|_{\partial\Omega} = \frac{\partial v}{\partial \vec{n}}\big|_{\partial\Omega} = \frac{\partial w}{\partial \vec{n}}\big|_{\partial\Omega} = 0. \tag{1.4}
$$

This article is organized as follows: in Section 2, we prove some lemmas on the regularized problem of the system (1.1). In Section 3, the main result on the existence of a weak solution.

2. Regularized problem

We consider a regularized problem for solving system (1.1),

$$
\begin{gathered}
u_{\varepsilon t} = \operatorname{div}((|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon) - \chi\nabla\cdot\Big(\frac{u_\varepsilon}{(1+u_\varepsilon)^\alpha}\nabla v_\varepsilon\Big)\\
- \xi\nabla\cdot\Big(\frac{u_\varepsilon}{(1+u_\varepsilon)^\beta}\nabla w_\varepsilon\Big) + \mu u_\varepsilon(1-u_\varepsilon-w_\varepsilon), \quad x\in\Omega, t>0,\\
v_{\varepsilon t} - \Delta v_\varepsilon + v_\varepsilon = f(u_\varepsilon)g(w_\varepsilon), \quad x\in\Omega,\ t>0,\\
w_{\varepsilon t} = -v_\varepsilon w_\varepsilon, \quad x\in\Omega,\ t>0,\\
\Big((|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon - \chi\frac{u_\varepsilon}{(1+u_\varepsilon)^\alpha}\nabla v_\varepsilon - \xi\frac{u_\varepsilon}{(1+u_\varepsilon)^\beta}\nabla w_\varepsilon\Big)\cdot\vec{n}|_{\partial\Omega}\\
= \frac{\partial v_\varepsilon}{\partial\vec{n}}\Big|_{\partial\Omega} = 0,\\
u_\varepsilon(x,0) = u_{\varepsilon 0}(x), \quad v_\varepsilon(x,0) = v_{\varepsilon 0}(x), \quad w_\varepsilon(x,0) = w_{\varepsilon 0}(x), \quad x\in\Omega,
\end{gathered} \tag{2.1}
$$

where

$$
\begin{gathered}
u_{\varepsilon 0}, v_{\varepsilon 0}, w_{\varepsilon 0} \in C^{2+\gamma}(\overline{\Omega}), \quad u_{\varepsilon 0}, v_{\varepsilon 0}, w_{\varepsilon 0} \ge 0,\\
\frac{\partial w_{\varepsilon 0}}{\partial\vec{n}}\Big|_{\partial\Omega} = \frac{\partial u_{\varepsilon 0}}{\partial\vec{n}}\Big|_{\partial\Omega} = \frac{\partial v_{\varepsilon 0}}{\partial\vec{n}}\Big|_{\partial\Omega} = 0,\\
u_{\varepsilon 0}\to u_0, \quad v_{\varepsilon 0}\to v_0, \quad w_{\varepsilon 0}\to w_0, \quad \text{uniformly},\\
\|\nabla^i u_{\varepsilon 0}\|_{L^\infty} \le 2\|\nabla^i u_0\|_{L^\infty}, \quad \||\nabla u_{\varepsilon 0}|^{p-2}\nabla u_{\varepsilon 0}\|_{L^2} \le 2\||\nabla u_0|^{p-2}\nabla u_0\|_{L^2},\\
\|\nabla^i v_{\varepsilon 0}\|_{L^\infty} \le 2\|\nabla^i v_0\|_{L^\infty}, \quad \|\nabla^i w_{\varepsilon 0}\|_{L^\infty} \le 2\|\nabla^i w_0\|_{L^\infty}, \quad (i=0,1,2),\\
\|\nabla\sqrt{w_{\varepsilon 0}}\|_{L^\infty} \le 2\|\nabla\sqrt{w_0}\|_{L^\infty}.
\end{gathered} \tag{2.2}
$$

From ODE theory,

$$
w_\varepsilon = w_{\varepsilon 0}\exp\Big(-\int_0^t v_\varepsilon(x,s)\,ds\Big). \tag{2.3}
$$

Through the direct calculation,

$$
\nabla w_\varepsilon = e^{-\int_0^t v_\varepsilon(x,s)\,ds}\Big[\nabla w_{\varepsilon 0} - w_{\varepsilon 0}\int_0^t \nabla v_\varepsilon(x,s)\,ds\Big], \tag{2.4}
$$

and

$$\begin{aligned}\Delta w_\varepsilon &= \Delta w_{\varepsilon 0} e^{-\int_0^t v_\varepsilon(x,s)\,ds} - 2e^{-\int_0^t v_\varepsilon(x,s)\,ds}\nabla w_{\varepsilon 0}\int_0^t \nabla v_\varepsilon(x,s)\,ds \\ &\quad + w_{\varepsilon 0} e^{-\int_0^t v_\varepsilon(x,s)\,ds}\Big|\int_0^t \nabla v_\varepsilon(x,s)\,ds\Big|^2 \\ &\quad - w_{\varepsilon 0} e^{-\int_0^t v_\varepsilon(x,s)\,ds}\int_0^t \Delta v_\varepsilon(x,s)\,ds.\end{aligned} \tag{2.5}$$

Tao and Winkler [10] stated that

$$-\Delta w_\varepsilon(x,t) \le \|w_{\varepsilon 0}\|_{L^\infty} v_\varepsilon(x,t) + \|\Delta w_{\varepsilon 0}\|_{L^\infty} + 4\|\nabla\sqrt{w_{\varepsilon 0}}\|_{L^\infty}^2 + \frac{1}{e}\|w_{\varepsilon 0}\|_{L^\infty}. \tag{2.6}$$

Thanks to $\frac{\partial w_{\varepsilon 0}}{\partial \vec{n}}\big|_{\partial\Omega} = 0$ and $\frac{\partial v_\varepsilon}{\partial \vec{n}}\big|_{\partial\Omega} = 0$ and using (2.4), we posses $\frac{\partial w_\varepsilon}{\partial \vec{n}}\big|_{\partial\Omega} = 0$. Considering the zero-flux boundary conditions of the system (2.1), the boundary conditions are equivalent to

$$\frac{\partial u_\varepsilon}{\partial \vec{n}}\Big|_{\partial\Omega} = \frac{\partial v_\varepsilon}{\partial \vec{n}}\Big|_{\partial\Omega} = \frac{\partial w_\varepsilon}{\partial \vec{n}}\Big|_{\partial\Omega} = 0. \tag{2.7}$$

Based on a fixed point argument similar the one in [9] or [7], the local classical solution existence result of problem (2.1) can be proved.

Lemma 2.1. *Let $p > 2$. Under assumption (2.2), then there exists $T_{\max} \in (0, +\infty]$ such that (2.1) admits a unique classical solution $(u_\varepsilon, v_\varepsilon, w_\varepsilon) \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\Omega \times (0, T_{\max}))$ and for all $(x,t) \in \Omega \times (0, T_{\max})$,*

$$u_\varepsilon > 0, \quad v_\varepsilon > 0, \quad w_\varepsilon \ge 0. \tag{2.8}$$

Moreover, either $T_{\max} = \infty$ or

$$\limsup_{t\to T_{\max}} (\|u_\varepsilon(\cdot,t)\|_{L^\infty} + \|v_\varepsilon\|_{W^{1,\infty}}) = \infty\,.$$

Take

$$\tau = \min\{1, \frac{T_{\max}}{2}\} \le 1. \tag{2.9}$$

Note that if $T_{\max} \ge 2$ then if $\tau = 1$, and if $T_{\max} \le 2$ then $\tau \le 1$.

Lemma 2.2. *Let $p > 2$ and $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be the classical solution of problem (2.1) in $(0, T_{\max})$. Under assumptions (2.2), one obtains*

$$\sup_{t\in(0,T_{\max})} \|w_\varepsilon(\cdot,t)\|_{L^\infty} \le \|w_{\varepsilon 0}\|_{L^\infty} \le 2\|w_0\|_{L^\infty}, \tag{2.10}$$

$$\sup_{t\in(0,T_{\max})} \|u_\varepsilon(\cdot,t)\|_{L^1} + \frac{\mu}{2}\sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^t\int_\Omega u_\varepsilon^2\,dx\,ds \le C, \tag{2.11}$$

$$\sup_{t\in(0,T_{\max})} \|v_\varepsilon\|_{W^{1,2}}^2 + \sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^t (\|v_\varepsilon\|_{W^{2,2}}^2 + \|v_{\varepsilon t}\|_{L^2}^2)\,ds \le C, \tag{2.12}$$

where C is independent of ε.

Proof. According to (2.3),

$$w_\varepsilon = w_{\varepsilon 0} e^{-\int_0^t v(x,s)\,ds} \le |w_{\varepsilon 0}| \le \|w_{\varepsilon 0}\|_{L^\infty} \le 2\|w_0\|_{L^\infty},$$

which implies that (2.10) holds. Integrating the first equation in (2.1) over Ω and combining (2.7) with (2.8), we obtain

$$\frac{d}{dt}\int_\Omega u_\varepsilon dx = \mu\int_\Omega u_\varepsilon(1-u_\varepsilon-w_\varepsilon)dx \le \mu\int_\Omega u_\varepsilon dx - \mu\int_\Omega u_\varepsilon^2 dx.$$

By the above inequality and the Young inequality, we have

$$\frac{d}{dt}\int_\Omega u_\varepsilon dx + \frac{\mu}{2}\int_\Omega u_\varepsilon^2 dx \le \frac{\mu}{2}|\Omega|. \tag{2.13}$$

Utilizing [2, Lemma 2.4] and thanks to (2.13), one obtains

$$\sup_{t\in(0,T_{\max})}\|u_\varepsilon(\cdot,t)\|_{L^1} + \frac{\mu}{2}\sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^t\int_\Omega u_\varepsilon^2\,dx\,ds \le C.$$

Multiplying the second equation in (2.1) by v_ε, integrating over Ω, and using the Young inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v_\varepsilon^2 dx + \int_\Omega|\nabla v_\varepsilon|^2dx + \int_\Omega|v_\varepsilon|^2dx = \int_\Omega f(u_\varepsilon)g(w_\varepsilon)v_\varepsilon dx. \tag{2.14}$$

By (1.2) and (2.10), we obtain

$$\frac{d}{dt}\int_\Omega v_\varepsilon^2dx + \int_\Omega|\nabla v_\varepsilon|^2dx + \int_\Omega v_\varepsilon^2dx \le C((\|w_\varepsilon\|_\infty^{q_2}+1)\int_\Omega|f(u_\varepsilon)v_\varepsilon|dx. \tag{2.15}$$

Using the Hölder inequality, we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega v_\varepsilon^2dx + \int_\Omega|\nabla v_\varepsilon|^2dx + \int_\Omega v_\varepsilon^2dx\\ &\le C\Big(\int_\Omega|f(u_\varepsilon)|^{6/5}dx\Big)^{5/6}\Big(\int_\Omega|v_\varepsilon|^6dx\Big)^{1/6}.\end{aligned} \tag{2.16}$$

From (1.2), the Sobolev imbedding theorem, and the Young inequality, we have

$$\frac{d}{dt}\int_\Omega v_\varepsilon^2dx + \frac{1}{2}\int_\Omega|\nabla v_\varepsilon|^2dx + \frac{1}{2}\int_\Omega v_\varepsilon^2dx \le C\int_\Omega u_\varepsilon^2dx + C. \tag{2.17}$$

It follows from (2.11), (2.15) and [2, Lemma 2.4] that

$$\sup_{t\in(0,T_{\max})}\|v_\varepsilon\|_{L^2}^2 + \sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^t\|v_\varepsilon\|_{W^{1,2}}^2ds \le C. \tag{2.18}$$

Multiplying the second equation in (2.1) by $v_{\varepsilon t}$, integrating it over Ω, applying the Young inequality, and combining with (1.2), we obtain

$$\begin{aligned}\int_\Omega v_{\varepsilon t}^2dx + \frac{1}{2}\frac{d}{dt}\int_\Omega(v_\varepsilon^2+|\nabla v_\varepsilon|^2)dx &= \int_\Omega f(u_\varepsilon)g(w_\varepsilon)v_{\varepsilon t}dx\\ &\le C\int_\Omega u_\varepsilon^2dx + \frac{1}{2}\int_\Omega v_{\varepsilon t}^2dx + C.\end{aligned}$$

Therefore,

$$\sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^t\|v_{\varepsilon t}\|_{L^2}^2ds \le C. \tag{2.19}$$

Similarly, multiplying the second equation in (2.1) by Δv_ε, integrating it over Ω, and using the Young inequality, we obtain

$$\frac{d}{dt}\int_\Omega|\nabla v_\varepsilon|^2dx + \int_\Omega|\Delta v_\varepsilon|^2dx + \int_\Omega|\nabla v_\varepsilon|^2dx \le C\int_\Omega u_\varepsilon^2dx + C. \tag{2.20}$$

By means of (2.11), (2.20) and [2, Lemma 2.4],

$$\sup_{t\in(0,T_{\max})}\|\nabla v_\varepsilon\|_{L^2}^2+\sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^t\|\nabla v_\varepsilon\|_{W^{1,2}}^2ds\le C. \tag{2.21}$$

Thanks to (2.18), (2.19) and (2.21), we obtain

$$\sup_{t\in(0,T_{\max})}\|v_\varepsilon\|_{W^{1,2}}^2+\sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^t(\|v_\varepsilon\|_{W^{2,2}}^2+\|v_{\varepsilon t}\|_{L^2}^2)\,ds\le C.$$

The proof is complete. $\square$

Lemma 2.3. *Let $p>2$ and $(u_\varepsilon,v_\varepsilon,w_\varepsilon)$ be the classical solution of problem (2.1) in $(0,T_{\max})$. Under assumptions (2.2), we have*

$$\sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^t\int_\Omega|\nabla u_\varepsilon^{\frac{p-1}{p}}|^p\,dx\,ds\le C, \tag{2.22}$$

where C is independent of ε.

Proof. Multiplying the first equation in (2.1) by $1+\ln u_\varepsilon$, integrating over Ω, and using (2.7), we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon\ln u_\varepsilon dx\\
&=\int_\Omega(1+\ln u_\varepsilon)div((|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon)dx\\
&\quad-\chi\int_\Omega(1+\ln u_\varepsilon)\nabla\Big(\frac{u_\varepsilon}{(1+u_\varepsilon)^\alpha}\nabla v_\varepsilon\Big)dx\\
&\quad-\xi\int_\Omega(1+\ln u_\varepsilon)\nabla\Big(\frac{u_\varepsilon}{(1+u_\varepsilon)^\beta}\nabla w_\varepsilon\Big)dx\\
&\quad+\mu\int_\Omega(1+\ln u_\varepsilon)u_\varepsilon(1-u_\varepsilon-w_\varepsilon)dx\\
&=-\int_\Omega\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon}\cdot(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}dx+\chi\int_\Omega(1+u_\varepsilon)^{-\alpha}\nabla u_\varepsilon\nabla v_\varepsilon dx\\
&\quad+\xi\int_\Omega(1+u_\varepsilon)^{-\beta}\nabla u_\varepsilon\nabla w_\varepsilon dx+\mu\int_\Omega(1+\ln u_\varepsilon)u_\varepsilon(1-u_\varepsilon-w_\varepsilon)dx\\
&=-\int_\Omega\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon}\cdot(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}dx+\chi\int_\Omega\Big(\nabla\int_0^{u_\varepsilon}(1+s)^{-\alpha}ds\Big)\nabla v_\varepsilon dx\\
&\quad+\xi\int_\Omega\Big(\nabla\int_0^{u_\varepsilon}(1+s)^{-\beta}ds\Big)\nabla w_\varepsilon dx+\mu\int_\Omega(1+\ln u_\varepsilon)u_\varepsilon(1-u_\varepsilon-w_\varepsilon)dx\\
&=-\int_\Omega\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon}\cdot(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}dx-\chi\int_\Omega\Big(\int_0^{u_\varepsilon}(1+s)^{-\alpha}ds\Big)\Delta v_\varepsilon dx\\
&\quad-\xi\int_\Omega\Big(\int_0^{u_\varepsilon}(1+s)^{-\beta}ds\Big)\Delta w_\varepsilon dx+\mu\int_\Omega(1+\ln u_\varepsilon)u_\varepsilon(1-u_\varepsilon-w_\varepsilon)dx.
\end{aligned} \tag{2.23}$$

By (2.8) and $p>2$,

$$\begin{aligned}
\Big(\frac{p}{p-1}\Big)^p\int_\Omega\big|\nabla u_\varepsilon^{\frac{p-1}{p}}\big|^pdx&=\int_\Omega\frac{|\nabla u_\varepsilon|^p}{u_\varepsilon}dx\\
&\le\int_\Omega\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon}(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}dx.
\end{aligned} \tag{2.24}$$

It is not difficult to show that

$$\mu\int_\Omega (1+\ln u_\varepsilon)u_\varepsilon(1-u_\varepsilon-w_\varepsilon)dx \le C. \tag{2.25}$$

Note that

$$w_1 \equiv 2\|\Delta w_0\|_{L^\infty} + 16\|\nabla\sqrt{w_0}\|_{L^\infty}^2 + \frac{2}{e}\|w_0\|_{L^\infty}.$$

Applying (2.6), (2.23), (2.24) and (2.25), considering the Poincaré inequality and the inequalities $\int_0^{u_\varepsilon}(1+s)^{-\alpha}ds \le u_\varepsilon$ and $\int_0^{u_\varepsilon}(1+s)^{-\beta}ds \le u_\varepsilon$, one obtains

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon \ln u_\varepsilon dx + \Big(\frac{p}{p-1}\Big)^p \int_\Omega \big|\nabla u_\varepsilon^{\frac{p-1}{p}}\big|^p dx \\
&\le -\chi\int_\Omega \Big(\int_0^{u_\varepsilon}(1+s)^{-\alpha}ds\Big)\Delta v_\varepsilon dx - \xi\int_\Omega\Big(\int_0^{u_\varepsilon}(1+s)^{-\beta}ds\Big)\Delta w_\varepsilon dx \\
&\quad + \mu\int_\Omega (1+\ln u_\varepsilon)u_\varepsilon(1-u_\varepsilon-w_\varepsilon)dx \\
&\le \chi\int_\Omega u_\varepsilon|\Delta v_\varepsilon|dx + \xi\int_\Omega u_\varepsilon\Big(\|w_{\varepsilon 0}\|_{L^\infty}v_\varepsilon + \|\Delta w_{\varepsilon 0}\|_{L^\infty} \\
&\quad + 4\|\nabla\sqrt{w_{\varepsilon 0}}\|_{L^\infty}^2 + \frac{1}{e}\|w_{\varepsilon 0}\|_{L^\infty}\Big)dx + C \\
&\le \frac{1}{2}\chi\int_\Omega u_\varepsilon^2 dx + \frac{1}{2}\chi\int_\Omega |\Delta v_\varepsilon|^2 dx + \xi\|w_0\|_{L^\infty}\int_\Omega u_\varepsilon^2 dx + \xi\|w_0\|_{L^\infty}\int_\Omega v_\varepsilon^2 dx \\
&\quad + \xi w_1\int_\Omega u_\varepsilon dx + C \\
&\le \Big(\frac{1}{2}\chi + \xi\|w_0\|_{L^\infty}\Big)\int_\Omega u_\varepsilon^2 dx + \frac{1}{2}\chi\int_\Omega|\Delta v_\varepsilon|^2 dx + \xi\|w_0\|_{L^\infty}\int_\Omega v_\varepsilon^2 dx + C \\
&\le \Big(\frac{1}{2}\chi + \xi\|w_0\|_{L^\infty}\Big)\int_\Omega u_\varepsilon^2 dx + \max\big\{\frac{1}{2}\chi, \xi\|w_0\|_{L^\infty}\big\} \\
&\quad \times\int_\Omega (v_\varepsilon^2 + |\Delta v_\varepsilon|^2 + |\nabla v_\varepsilon|^2)dx + C.
\end{aligned} \tag{2.26}$$

Utilizing (2.11), (2.12), (2.26) and [2, Lemma 2.4], we have

$$\sup_{t\in(0,T_{\max})}\int_\Omega u_\varepsilon\ln u_\varepsilon dx + \Big(\frac{p}{p-1}\Big)^p \sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^t\int_\Omega \big|\nabla u_\varepsilon^{\frac{p-1}{p}}\big|^p dx\, ds \le C. \tag{2.27}$$

If $u_\varepsilon \ge 1$, it follows from (2.27) that

$$\sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^t\int_\Omega \big|\nabla u_\varepsilon^{\frac{p-1}{p}}\big|^p dx\, ds \le C\Big(\frac{p}{p-1}\Big)^{-p}. \tag{2.28}$$

If $0 < u_\varepsilon < 1$, using $|x^{q-1}\log x| \le (e(q-1))^{-1}$, inequality (2.27) becomes

$$\sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^t\int_\Omega \big|\nabla u_\varepsilon^{\frac{p-1}{p}}\big|^p dx\, ds \le \Big(C + \frac{|\Omega|}{e}\Big)\Big(\frac{p}{p-1}\Big)^{-p}. \tag{2.29}$$

The proof is complete. □

Lemma 2.4. *Let $p > 2$ and $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be the classical solution of problem (2.1) in $(0, T_{\max})$. Under assumptions (2.2), if $\frac{\chi}{\mu} < \frac{d+1}{6d}(\frac{3}{M})^{\frac{1}{d+2}}$, for any $0 < d \le 4$, we* *have*

$$
\begin{aligned}
&\sup_{t\in(0,T_{\max})}\int_\Omega u_\varepsilon^{d+1}dx\\
&+\Big(\frac{\mu}{2}(d+1)-\frac{6^{d+1}d^{d+2}}{(d+1)^{d+1}}\cdot\frac{\chi^{d+2}}{\mu^{d+1}}M\Big)\sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^t\int_\Omega u_\varepsilon^{d+2}\,dx\,ds\le C,
\end{aligned}
\tag{2.30}
$$

where M satisfies

$$
\int_{t-\tau}^t\int_\Omega|\Delta v_\varepsilon|^{d+2}\,dx\,ds\le M\Big(\sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^t\int_\Omega u_\varepsilon^{d+2}\,dx\,ds+\|v_{\varepsilon0}\|_{W^{2,d+2}}^{d+2}\Big).
$$

Proof. Multiplying the first equation in (2.1) by u_ε^d for any $d>0$ and integrating over Ω, we have

$$
\begin{aligned}
&\frac{1}{d+1}\frac{d}{dt}\int_\Omega u_\varepsilon^{d+1}dx\\
&=\int_\Omega u_\varepsilon^d\operatorname{div}((|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon)dx-\chi\int_\Omega u_\varepsilon^d\nabla\cdot\Big(\frac{u_\varepsilon}{(1+u_\varepsilon)^\alpha}\nabla v_\varepsilon\Big)dx\\
&\quad-\xi\int_\Omega u_\varepsilon^d\nabla\cdot\Big(\frac{u_\varepsilon}{(1+u_\varepsilon)^\beta}\nabla w_\varepsilon\Big)dx+\mu\int_\Omega u_\varepsilon^{d+1}(1-u_\varepsilon-w_\varepsilon)dx\\
&=-d\int_\Omega(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}u_\varepsilon^{d-1}|\nabla u_\varepsilon|^2dx-\chi\int_\Omega u_\varepsilon^d\nabla\cdot\Big(\frac{u_\varepsilon}{(1+u_\varepsilon)^\alpha}\nabla v_\varepsilon\Big)dx\\
&\quad-\xi\int_\Omega u_\varepsilon^d\nabla\cdot\Big(\frac{u_\varepsilon}{(1+u_\varepsilon)^\beta}\nabla w_\varepsilon\Big)dx+\mu\int_\Omega u_\varepsilon^{d+1}(1-u_\varepsilon-w_\varepsilon)dx.
\end{aligned}
\tag{2.31}
$$

By the Sobolev embedding and (2.12),

$$
\sup_{t\in(0,T_{\max})}\|v_\varepsilon\|_{L^{d+2}}\le M_1\sup_{t\in(0,T_{\max})}\|v_\varepsilon\|_{H^1}\le C,
\tag{2.32}
$$

where M_1 is the Sobolev embedding constant. Using (2.6), (2.31), (2.32), the inequalities $\int_0^{u_\varepsilon}s^d(1+s)^{-\alpha}ds\le\int_0^{u_\varepsilon}s^dds\le\frac{1}{d+1}u_\varepsilon^{d+1}$ and $\int_0^{u_\varepsilon}s^d(1+s)^{-\beta}ds\le\int_0^{u_\varepsilon}s^dds\le\frac{1}{d+1}u_\varepsilon^{d+1}$, and the Young inequality, we have

$$
\begin{aligned}
&\frac{1}{d+1}\frac{d}{dt}\int_\Omega u_\varepsilon^{d+1}dx+d\int_\Omega u_\varepsilon^{d-1}|\nabla u_\varepsilon|^pdx+\mu\int_\Omega u_\varepsilon^{d+2}dx\\
&\le-\chi\int_\Omega u_\varepsilon^d\nabla\cdot\Big(\frac{u_\varepsilon}{(1+u_\varepsilon)^\alpha}\nabla v_\varepsilon\Big)dx\\
&\quad-\xi\int_\Omega u_\varepsilon^d\nabla\cdot\Big(\frac{u_\varepsilon}{(1+u_\varepsilon)^\beta}\nabla w_\varepsilon\Big)dx+\mu\int_\Omega u_\varepsilon^{d+1}dx\\
&=-\chi\int_\Omega\Big(d\int_0^{u_\varepsilon}s^d(1+s)^{-\alpha}ds\Big)\Delta v_\varepsilon dx\\
&\quad-\xi\int_\Omega\Big(d\int_0^{u_\varepsilon}s^d(1+s)^{-\beta}ds\Big)\Delta w_\varepsilon dx+\mu\int_\Omega u_\varepsilon^{d+1}dx\\
&\le\frac{\chi d}{d+1}\int_\Omega u_\varepsilon^{d+1}|\Delta v_\varepsilon|dx+\frac{\xi d}{d+1}\int_\Omega u_\varepsilon^{d+1}\Big(\|w_{\varepsilon0}\|_{L^\infty}v_\varepsilon+\|\Delta w_{\varepsilon0}\|_{L^\infty}\\
&\quad+4\|\nabla\sqrt{w_{\varepsilon0}}\|_{L^\infty}^2+\frac{1}{e}\|w_{\varepsilon0}\|_{L^\infty}\Big)dx+\mu\int_\Omega u_\varepsilon^{d+1}dx\\
&\le\frac{\chi d}{d+1}\int_\Omega u_\varepsilon^{d+1}|\Delta v_\varepsilon|dx+\frac{2\xi d\|w_0\|_{L^\infty}}{d+1}\int_\Omega u_\varepsilon^{d+1}v_\varepsilon dx
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(\mu+\frac{\xi d w_1}{d+1}\Big)\int_\Omega u_\varepsilon^{d+1}dx\\
&\leq \frac{\chi d}{d+1}\Big(\frac{6\chi d}{\mu(d+1)}\Big)^{d+1}\int_\Omega |\Delta v_\varepsilon|^{d+2}dx+\frac{\mu}{2}\int_\Omega u_\varepsilon^{d+2}dx+C.
\end{aligned}
$$

The above inequality implies

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^{d+1}dx+d(d+1)\int_\Omega u_\varepsilon^{d-1}|\nabla u_\varepsilon|^p dx+\frac{\mu}{2}(d+1)\int_\Omega u_\varepsilon^{d+2}dx\\
&\leq \frac{6^{d+1}d^{d+2}}{(d+1)^{d+1}}\frac{\chi^{d+2}}{\mu^{d+1}}\int_\Omega |\Delta v_\varepsilon|^{d+2}dx+C.
\end{aligned}
\tag{2.33}
$$

According to [3, Lemma 2.4], there exists a constant M such that

$$
\int_{t-\tau}^t\int_\Omega |\Delta v_\varepsilon|^{d+2}\,dx\,ds\leq M\Big(\sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^t\int_\Omega u_\varepsilon^{d+2}\,dx\,ds+\|v_{\varepsilon0}\|_{W^{2,d+2}}^{d+2}\Big). \tag{2.34}
$$

On the basis of [2, Lemma 2.4], (2.33) and (2.34),

$$
\begin{aligned}
&\sup_{t\in(0,T_{\max})}\int_\Omega u_\varepsilon^{d+1}dx+d(d+1)\sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^t\int_\Omega u_\varepsilon^{d-1}|\nabla u_\varepsilon|^p\,dx\,ds\\
&+\Big(\frac{\mu}{2}(d+1)-\frac{6^{d+1}d^{d+2}}{(d+1)^{d+1}}\cdot\frac{\chi^{d+2}}{\mu^{d+1}}M\Big)\sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^t\int_\Omega u_\varepsilon^{d+2}\,dx\,ds\leq C.
\end{aligned}
\tag{2.35}
$$

The proof is complete. □

Lemma 2.5. *Let $p>2$ and $(u_\varepsilon,v_\varepsilon,w_\varepsilon)$ be the classical solution of problem (2.1) in $(0,T_{\max})$. Under assumption (2.2), if $\frac{\chi}{\mu}<\frac{2}{9}(\frac{3}{M})^{2/5}$, we have*

$$
\sup_{t\in(0,T_{\max})}(\|v_\varepsilon\|_{W^{1,\infty}}+\|w_{\varepsilon t}\|_{L^\infty})\leq C, \tag{2.36}
$$

where C is independent of ε.

Proof. According to Duhamel's principle,

$$
v_{\varepsilon t}=e^{-t}e^{t\Delta}v_{\varepsilon0}+\int_0^t e^{-(t-s)}e^{(t-s)\Delta}f(u_\varepsilon)g(w_\varepsilon)ds,
$$

where $\{e^{t\Delta}\}_{t\geq0}$ is the Neumann heat semigroup (see for [12]) in Ω. Taking $d=4$ in Lemma 2.4, noticing (1.2), and applying the fact $\int_0^\infty e^{-s}s^{-3/10}ds$ convergences to a constant, we have for any $t\in(0,T_{\max})$,

$$
\begin{aligned}
&\|v_\varepsilon(\cdot,t)\|_{L^\infty}\\
&\leq e^{-t}\|v_{\varepsilon0}\|_{L^\infty}+\int_0^t e^{-(t-s)}(t-s)^{-\frac{3}{2}\cdot\frac{1}{5}}\|f(u_\varepsilon)g(w_\varepsilon)\|_{L^5}ds\\
&\leq e^{-t}\|v_{\varepsilon0}\|_{L^\infty}+\sup_{s\in(0,T_{\max})}\|f(u_\varepsilon(\cdot,s))g(w_\varepsilon(\cdot,s))\|_{L^5}\int_0^\infty e^{-s}s^{-3/10}ds\\
&\leq 2\|v_0\|_{L^\infty}+C\Big(\int_\Omega u_\varepsilon^5+C|\Omega|\Big)^{1/5}\int_0^\infty e^{-s}s^{-3/10}ds\leq C.
\end{aligned}
\tag{2.37}
$$

Similarly we have

$$
\|\nabla v_\varepsilon(\cdot,t)\|_{L^\infty}\leq C. \tag{2.38}
$$

Considering the third equation of the problem and utilizing (2.1), (2.10) and (2.37), one has

$$\|w_{\varepsilon t}\|_{L^\infty} = \| - v_\varepsilon w_\varepsilon\|_{L^\infty} \le C. \tag{2.39}$$

From (2.37), (2.38) and (2.39), we have

$$\|v_\varepsilon\|_{W^{1,\infty}} + \|w_{\varepsilon t}\|_{L^\infty} \le C.$$

The proof is complete. □

Lemma 2.6. *Let $p > 2$ and $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be the classical solution of problem (2.1) in $(0, T_{\max})$. Under assumptions (2.2), if $\frac{\chi}{\mu} < \frac{2}{9}(\frac{3}{M})^{2/5}$, then*

$$\sup_{t\in(0,T_{\max})} \|u_\varepsilon\|_{L^\infty} \le C, \tag{2.40}$$

where C is independent of ε.

Proof. we define

$$m_k = 2m_{k-1} + 2 - p, \quad k \in \mathbb{N} = \{1, 2, 3, \dots\}, \tag{2.41}$$

with

$$m_0 > p - 2. \tag{2.42}$$

Utilizing (2.41) and (2.42), one posses a nonnegative strictly increasing sequence $\{m_k\}_{k\in\mathbb{N}}$ such that

$$m_k \to \infty \quad \text{as } k \to \infty, \tag{2.43}$$

$$C_1 2^k \le m_k \le C_2 2^k, \quad \text{for all } k \in \mathbb{N}, \tag{2.44}$$

where C_1 and C_2 are constants independent of k. Let

$$\theta_k = 2\frac{m_k + p - 2}{m_k + p' - 2} > 2, \tag{2.45}$$

where $p' = \frac{p}{p-1}$. It is obvious that

$$\theta_k' = \frac{\theta_k}{\theta_k - 1} \in (1, 2) \quad \text{and} \quad \frac{1}{\theta_k'} > \frac{1}{2} > \frac{p'}{4}. \tag{2.46}$$

Note that

$$M_k = \sup_{t\in(0,\infty)} \int_\Omega \widehat{u_\varepsilon}^{m_k}(x,t)dx, \quad k \in \mathbb{N}, \tag{2.47}$$

where $\widehat{u_\varepsilon}(x,t) = \max\{u_\varepsilon(x,t), 1\}$ for $x \in \overline{\Omega}$ and $t \in [0, T_{\max})$. Multiplying the first equation in (2.1) by $m_k u_\varepsilon^{m_k-1}$ and integrating over Ω, we obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega u_\varepsilon^{m_k}dx &= -m_k(m_k-1)\int_\Omega u_\varepsilon^{m_k-2}(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u_\varepsilon|^2dx \\ &\quad + \chi m_k(m_k-1)\int_\Omega u_\varepsilon^{m_k-1}(1+u_\varepsilon)^{-\alpha}\nabla u_\varepsilon\nabla v_\varepsilon dx \\ &\quad + \xi m_k(m_k-1)\int_\Omega \nabla\Big(\int_0^{u_\varepsilon} s^{m_k-1}(1+s)^{-\beta}ds\Big)\cdot\nabla w_\varepsilon dx \\ &\quad + \int_\Omega \mu m_k u_\varepsilon^{m_k}(1-u_\varepsilon-w_\varepsilon)dx. \end{aligned} \tag{2.48}$$

It is not difficult to check that $m_k - 1 \geq 1$ if $k \geq 1$ according to (2.41), (2.42) and (2.43). Using (2.6), (2.7), (2.8) and (2.48), noticing that $\int_0^{u_\varepsilon} s^{m_k-1}(1+s)^{-\beta} ds \leq \int_0^{u_\varepsilon} s^{m_k-1} ds = \frac{1}{m_k} u_\varepsilon^{m_k}$, and combining the Young inequality, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^{m_k} dx + m_k(m_k-1)\int_\Omega u_\varepsilon^{m_k-2}|\nabla u_\varepsilon|^p dx \\
&\leq \frac{d}{dt}\int_\Omega u_\varepsilon^{m_k} dx + m_k(m_k-1)\int_\Omega u_\varepsilon^{m_k-2}(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u_\varepsilon|^2 dx \\
&\leq \chi m_k(m_k-1)\int_\Omega u_\varepsilon^{m_k-1}|\nabla u_\varepsilon|\cdot|\nabla v_\varepsilon| dx \\
&\quad - \xi m_k(m_k-1)\int_\Omega \Big(\int_0^{u_\varepsilon} s^{m_k-1}(1+s)^{-\beta} ds\Big)\Delta w_\varepsilon dx + \int_\Omega \mu m_k u_\varepsilon^{m_k} dx \\
&\leq \frac{m_k(m_k-1)}{2}\int_\Omega u_\varepsilon^{m_k-2}|\nabla u_\varepsilon|^p dx + 2^{\frac{1}{p-1}}\chi^{\frac{p}{p-1}} m_k(m_k-1) \\
&\quad \times \int_\Omega \Big(u_\varepsilon^{\frac{2-m_k}{p}+m_k-1}|\nabla v_\varepsilon|\Big)^{p'} dx \\
&\quad + \big(\xi\,(2\|w_0\|_{L^\infty} C + w_1) + \mu\big) m_k(m_k-1)\int_\Omega u_\varepsilon^{m_k} dx.
\end{aligned}
$$

The above inequality indicates that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^{m_k} dx + \frac{m_k(m_k-1)}{2}\Big(\frac{p}{m_k+p-2}\Big)^p \int_\Omega |\nabla u_\varepsilon^{\frac{m_k-2}{p}+1}|^p dx \\
&= \frac{d}{dt}\int_\Omega u_\varepsilon^{m_k} dx + \frac{m_k(m_k-1)}{2}\int_\Omega u_\varepsilon^{m_k-2}|\nabla u_\varepsilon|^p dx \\
&\leq 2^{\frac{1}{p-1}}\chi^{\frac{p}{p-1}} m_k(m_k-1)\int_\Omega \Big(u_\varepsilon^{\frac{2-m_k}{p}+m_k-1}|\nabla v_\varepsilon|\Big)^{p'} dx \\
&\quad + (\xi\,(2\|w_0\|_{L^\infty} C + w_1) + \mu)\, m_k(m_k-1)\int_\Omega u_\varepsilon^{m_k} dx.
\end{aligned}
\tag{2.49}
$$

The remaining part of the proof can be done in the same way as that in the proof of [6, Lemma 2.6], we omit the details. □

Lemma 2.7. *Let $p > 2$ and $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be the classical solution of problem (2.1) in $(0, T_{\max})$. Under assumptions (2.2), if $\frac{\chi}{\mu} < \frac{2}{9}(\frac{3}{M})^{2/5}$, s for all $T > 0$, we have*

$$
\int_0^T \|u_{\varepsilon t}(\cdot,t)\|_{(W^{1,p}(\Omega))^*}^{p'} ds \leq CT + CT^{p'+1}, \tag{2.50}
$$

where C is independent of ε.

Proof. For $t \in (0,\infty)$ and $\varphi \in C^\infty(\overline{\Omega})$, multiplying the first equation in (2.1) by φ, integrating over Ω, applying (2.8), (2.36), (2.40), considering the Hölder inequality and using the inequalities $|\nabla\varphi| \le C$ and $|\varphi| \le C$, we have

$$\begin{aligned}
&\Big|\int_\Omega u_{\varepsilon t}\varphi dx\Big| \\
&= \Big| -\int_\Omega (|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon \nabla\varphi dx + \chi\int_\Omega \frac{u_\varepsilon}{(1+u_\varepsilon)^\alpha}\nabla v_\varepsilon\nabla\varphi dx \\
&\quad + \xi\int_\Omega \frac{u_\varepsilon}{(1+u_\varepsilon)^\beta}\nabla w_\varepsilon\nabla\varphi dx + \mu\int_\Omega u_\varepsilon(1-u_\varepsilon-w_\varepsilon)\varphi dx\Big| \\
&\le \int_\Omega |(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon|\cdot|\nabla\varphi| dx + \chi\int_\Omega |u_\varepsilon\nabla v_\varepsilon|\cdot|\nabla\varphi|dx \\
&\quad + \xi\int_\Omega |u_\varepsilon\nabla w_\varepsilon|\,|\nabla\varphi|dx + \mu\int_\Omega u_\varepsilon|\varphi|dx + \mu\int_\Omega u_\varepsilon^2|\varphi|dx \\
&\quad + \mu\int_\Omega u_\varepsilon w_\varepsilon|\varphi|dx \\
&\le C\|(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon\|_{L^{p'}} + C\|\nabla w_\varepsilon\|_{L^{p'}} + C.
\end{aligned} \tag{2.51}$$

Using (2.4) and (2.36), we have

$$\int_0^T\int_\Omega |\nabla w_\varepsilon|^{p'}\,dx\,ds \le CT + CT^{p'+1}. \tag{2.52}$$

It follows from (2.22) and (2.40) that

$$\int_0^T\int_\Omega |\nabla u_\varepsilon|^p\,dx\,ds = \int_0^T\int_\Omega \Big|\frac{p}{p-1}u_\varepsilon^{\frac{1}{p}}\nabla u_\varepsilon^{\frac{p-1}{p}}\Big|^p\,dx\,ds \le CT. \tag{2.53}$$

Hence, by (2.51), (2.52), (2.53) and that

$$\begin{aligned}
|(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon|^{p'} &\le |(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-1}{2}}|^{p'} \\
&\le |(|\nabla u_\varepsilon|^2+1)^{\frac{p-1}{2}}|^{p'} \\
&= (|\nabla u_\varepsilon|^2+1)^{\frac{p}{2}} \le 2^{\frac{p}{2}}(|\nabla u_\varepsilon|^p+1),
\end{aligned}$$

we have

$$\int_0^T \|u_{\varepsilon t}(\cdot,t)\|^{p'}_{(W^{1,p}(\Omega))^*}ds \le CT + CT^{p'+1}.$$

The proof is complete. □

3. Existence of a global bounded weak solution

By Lemmas 2.1, 2.5 and 2.6, we have $T_{\max}=\infty, \tau=1$. We know that problem (2.1) admits a unique classical global solution $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ which belongs to $C^{2+\gamma,1+\frac{\gamma}{2}}(\Omega\times(0,\infty))$ with $u_\varepsilon\ge 0$,$v_\varepsilon \ge 0$ and $w_\varepsilon\ge 0$.

Lemma 3.1. *Let $p>2$, under assumptions (2.2) and $\frac{\chi}{\mu} < \frac{2}{9}(\frac{3}{M})^{2/5}$, we assert there exists a nonnegative triplete (u,v,w) such that for any $T>0$,*

$$u_\varepsilon(\cdot,t)\to u(\cdot,t), \quad a.e.\ in\ Q_T, \tag{3.1}$$

$$u_\varepsilon \overset{\star}{\rightharpoonup} u, \quad in\ L^\infty(Q_T), \tag{3.2}$$

$$u_\varepsilon \to u, \quad in\ L^p_{\rm loc}(Q_T), \tag{3.3}$$

$$\nabla u_\varepsilon \rightharpoonup \nabla u, \quad in\ L^p_{\rm loc}(Q_T), \tag{3.4}$$

$$(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon \rightharpoonup |\nabla u|^{p-2}\nabla u, \quad in\ L^{p'}_{\rm loc}(Q_T), \tag{3.5}$$

$$v_\varepsilon \to v, \quad \textit{uniformly}, \tag{3.6}$$

$$v_\varepsilon \rightharpoonup v, \quad \textit{in } W_r^{2,1}(Q_T) \textit{ for any } r > 1, \tag{3.7}$$

$$w_\varepsilon \to w, \quad \textit{uniformly}, \tag{3.8}$$

$$\nabla w_\varepsilon \to \nabla w, \quad \textit{uniformly}, \tag{3.9}$$

$$\Delta w_\varepsilon \overset{\star}{\rightharpoonup} \Delta w, \quad \textit{in } L^\infty(Q_T), \tag{3.10}$$

as $\varepsilon \to 0$.

Proof. Utilizing (2.53), Lemma 2.6, Lemma 2.7 and Aubin-Lions lemma, there exists a nonnegative function u such that, as $\varepsilon \to 0$,

$$u_\varepsilon \to u, \quad \text{in } L^p_{\rm loc}(\Omega \times [0,T)) \text{ and a.e. in } Q_T. \tag{3.11}$$

By the Fubin-Tonelli theorem and (3.11), we have (3.1) and (3.3). Then (3.11) and Lemma 2.6 yield (3.2). Equations (2.53) and (3.11) imply (3.4). Combining (3.2) and (3.11), applying Lemma 2.6, as in [7], one obtains that

$$|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \rightharpoonup |\nabla u|^{p-2}\nabla u, \quad \text{in } L^{p'}_{\rm loc}(\overline{\Omega} \times [0,T)),$$

which implies (3.5), and

$$(|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon \rightharpoonup |\nabla u|^{p-2}\nabla u, \quad \text{in } L^{p'}_{\rm loc}(\overline{\Omega} \times [0,T)).$$

So (3.5) holds. It follows from Lemma 2.6 and (1.2) that for any $r > 0$,

$$\sup_{t\in(1,\infty)} \int_{t-1}^t \|f(u_\varepsilon)g(w_\varepsilon)\|_{L^r}^r ds \le C. \tag{3.12}$$

Then, by [3, Lemma 2.4] and (3.12), we have

$$\sup_{t\in(1,\infty)} \int_{t-1}^t (\|v_\varepsilon\|_{W^{2,r}}^r + \|v_{\varepsilon t}\|_{L^r}^r)\, ds \le C$$

which means

$$\|v_\varepsilon\|_{W_r^{2,1}(Q_{1(t)})} \le C, \tag{3.13}$$

where $Q_{1(t)} = \Omega \times (t-1, t)$. Then there exists a nonnegative v such that (3.7) holds according to (3.13). Because of (3.13) and $W_r^{2,1}(Q_T) \hookrightarrow C^{2-\frac{5}{r}, 1-\frac{5}{2r}}(Q_T)$, for any $r > 5/2$, one obtains $v_\varepsilon \to v$ uniformly. This shows (3.6). Equations (2.3) and (3.6) yield (3.8). Using $W_r^{1,1}(Q_T) \hookrightarrow C^{1-\frac{5}{r}, 1-\frac{5}{2r}}(Q_T)$ for any $r > 5$ and (3.13), we obtain

$$\nabla v_\varepsilon \to \nabla v, \quad \text{uniformly}. \tag{3.14}$$

Thus, by (2.4), (3.6) and (3.14), (3.9) holds. It follows from the second equation in (2.1), Lemma 2.5, and Lemma 2.6 that

$$\begin{aligned} \int_0^t \Delta v_\varepsilon(x,s)\, ds &= \int_0^t (v_{\varepsilon t} + v_\varepsilon - f(u_\varepsilon)g(w_\varepsilon))\, ds \\ &= v_\varepsilon(x,t) - v_{\varepsilon 0} + \int_0^t (v_\varepsilon - f(u_\varepsilon)g(w_\varepsilon))\, ds. \\ &\le C + Ct. \end{aligned} \tag{3.15}$$

Equations (2.5) and (3.15), and Lemma 2.5 guarantee that

$$|\Delta w_\varepsilon| \le Ct + Ct^2 + C. \tag{3.16}$$

Using (2.4) and Lemma 2.5, one obtains

$$|\nabla w_\varepsilon| \le C + Ct. \tag{3.17}$$

Thanks to (3.16) and (3.17), we find that for any $T > 0$,

$$\sup_{t\in(0,T)} \int_\Omega \|\nabla w_\varepsilon\|_{W^{1,\infty}(Q_T)} dx \le C + CT + CT^2. \tag{3.18}$$

Hence, (3.8), (3.9) and (3.18) indicate that (3.10) holds. □

Let

$$\begin{gathered}
H_1 = \{u \in L^\infty((0,\infty); L^1_{\rm loc}(\Omega\times[0,\infty)); u_t \in L^{p'}([0,\infty); (W^{1,p}(\Omega))^*)\};\\
H_2 = \{v \in L^\infty((0,\infty); W^{1,\infty}(\Omega)); v_t, \Delta v \in L^r_{\rm loc}([0,\infty); L^r(\Omega)) \text{ for any } r > 1\};\\
H_3 = \{w \in L^\infty(\Omega\times(0,\infty)); w_t \in L^\infty(\Omega\times(0,\infty)); w \in L^\infty_{\rm loc}([0,\infty); W^{2,\infty}(\Omega))\}.
\end{gathered}$$

Definition 3.2. A nonnegative triplet (u, v, w) is a weak solution of system (1.1), if $(u, v, w) \in H_1 \times H_2 \times H_3$ and satisfies

$$\begin{aligned}
&\iint_{Q_T} u\varphi_t \,dx\,ds + \int_\Omega u(x,0)\varphi(x,0)dx\\
&= \iint_{Q_T} |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi \,dx\,ds + \chi \iint_{Q_T} \frac{u}{(1+u)^\alpha}\nabla v\cdot\nabla\varphi\,dx\,ds\\
&\quad + \xi \iint_{Q_T} \frac{u}{(1+u)^\beta}\nabla w\cdot\nabla\varphi\,dx\,ds + \mu\iint_{Q_T} u(1-u-w)\varphi\,dx\,ds\,,\\
&\quad \iint_{Q_T} v\varphi_t\,dx\,ds + \int_\Omega v(x,0)\varphi(x,0)dx\\
&\quad = \iint_{Q_T} (\nabla v\nabla\varphi + (v - f(u)g(w))\varphi)\,dx\,ds, -\iint_{Q_T} w\varphi_t\,dx\,ds\\
&\quad\quad - \int_\Omega w(x,0)\varphi(x,0)dx + \iint_{Q_T} vw\varphi\,dx\,ds = 0,
\end{aligned}$$

for any given $T > 0$, for any $\varphi \in C^\infty(\overline{Q}_T)$ with $\frac{\partial\varphi}{\partial\vec n}\big|_{\partial\Omega} = 0$ and $\varphi(x,T) = 0$.

Theorem 3.3. *Let $p > 2$. Under assumptions* (1.2) *for a sufficiently small χ/μ, system* (1.1) *admits a nonnegative weak solution (u, v, w) with $u \in H_1$, $v \in H_2$, $w \in H_3$.*

Proof. For any $\varphi \in C^\infty(\overline{Q}_T)$ with $\frac{\partial\varphi}{\partial\vec n}\big|_{\partial\Omega} = 0$ and $\varphi(x,T) = 0$, we obtain

$$\begin{aligned}
&\iint_{Q_T} u_\varepsilon\varphi_t\,dx\,ds + \int_\Omega u_\varepsilon(x,0)\varphi(x,0)dx\\
&= \iint_{Q_T} (|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon\cdot\nabla\varphi\,dx\,ds + \chi\iint_{Q_T}\frac{u_\varepsilon}{(1+u_\varepsilon)^\alpha}\nabla v_\varepsilon\cdot\nabla\varphi\,dx\,ds\\
&\quad + \xi\iint_{Q_T}\frac{u_\varepsilon}{(1+u_\varepsilon)^\beta}\nabla w_\varepsilon\cdot\nabla\varphi\,dx\,ds + \mu\iint_{Q_T} u_\varepsilon(1-u_\varepsilon-w_\varepsilon)\varphi\,dx\,ds\,,
\end{aligned}$$

$$
\begin{aligned}
&\iint_{Q_T} v_\varepsilon \varphi_t \, dx\, ds + \int_\Omega v_\varepsilon(x,0)\varphi(x,0) dx \\
&= \iint_{Q_T} (\nabla v_\varepsilon \nabla \varphi + (v_\varepsilon - f(u_\varepsilon) g(w_\varepsilon))\varphi) \, dx\, ds, - \iint_{Q_T} w_\varepsilon \varphi_t \, dx\, ds \\
&\quad - \int_\Omega w_\varepsilon(x,0)\varphi(x,0) dx + \iint_{Q_T} v_\varepsilon w_\varepsilon \varphi \, dx\, ds = 0,
\end{aligned}
$$

according to the fact that $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ is the classical solution of the problem (2.1). Applying Lemma 3.1, one has

$$
\begin{aligned}
&\iint_{Q_T} u \varphi_t \, dx\, ds + \int_\Omega u(x,0)\varphi(x,0) dx \\
&= \iint_{Q_T} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx\, ds + \chi \iint_{Q_T} \frac{u}{(1+u)^\alpha} \nabla v \cdot \nabla \varphi \, dx\, ds \\
&\quad + \xi \iint_{Q_T} \frac{u}{(1+u)^\beta} \nabla w \cdot \nabla \varphi \, dx\, ds + \mu \iint_{Q_T} u(1-u-w)\varphi \, dx\, ds, \\
&\quad \iint_{Q_T} v \varphi_t \, dx\, ds + \int_\Omega v(x,0)\varphi(x,0) dx \\
&\quad = \iint_{Q_T} (\nabla v \nabla \varphi + (v - f(u) g(w))\varphi) \, dx\, ds, - \iint_{Q_T} w \varphi_t \, dx\, ds \\
&\quad\quad - \int_\Omega w(x,0)\varphi(x,0) dx + \iint_{Q_T} v w \varphi \, dx\, ds = 0,
\end{aligned}
$$

as $\varepsilon \to 0$. Then (u, v, w) in Lemma 3.1 is the desired global weak solution of (1.1). From the previous parts it follows that $u \in H_1$, $v \in H_2$, and $w \in H_3$. □

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