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PERIODICITY OF NON-HOMOGENEOUS TRAJECTORIES FOR NON-INSTANTANEOUS IMPULSIVE HEAT EQUATIONS

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Abstract. In this article, we introduce a non-instantaneous impulsive operator associated with the heat semigroup and give some basic properties. We derive an abstract formula for the solutions to non-instantaneous impulsive heat equations. Also we show the existence and uniqueness of the non-homogeneous periodic trajectory.

1. Introduction

Non-instantaneous differential equations are used to characterize evolution processes in pharmacotherapy and ecological systems. This type of impulsive equations was introduced in [4] their basic theory can be found in [1, 2, 3, 4, 6, 7, 8, 9, 10]. Motivated by [4, 5, 8], we study periodicity of non-homogeneous trajectories for the non-instantaneous impulsive heat equation with Dirichlet boundary conditions

$$\begin{gathered}
u_t(t,y) = \Delta u(t,y) + f(t,y), \quad y \in \Omega,\ t \in [s_{i-1}, t_i], \\
\delta u(t_i,y) = I_i u(t_i,y) + c_i(y), \quad y \in \Omega, \\
u(t,y) = B_i(t) u(t_i^+,y), \quad y \in \Omega,\ t \in (t_i, s_i], \\
u(0,y) = \xi(y), \quad y \in \Omega,
\end{gathered} \tag{1.1}$$

where $i \in \mathbb{N}^+$, $\delta u(t_i,y) := u(t_i^+,y) - u(t_i,y)$, $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial y_i^2}$ denotes the Laplace operator and $\Omega \subseteq \mathbb{R}^n$ is an open set. The sequences $\{s_i\}_{i\in\mathbb{N}^+}$ and $\{t_i\}_{i\in\mathbb{N}^+}$ satisfy $s_0 = 0$ and $s_{i-1} < t_i < s_i < t_{i+1} < \cdots$ for any $i \in \mathbb{N}^+$, and $\lim_{i\to+\infty} t_i = +\infty$.

Let $\mathbb{I} = \cup_{i=1}^\infty [s_{i-1}, t_i]$ and $\mathbb{J} = \cup_{i=1}^\infty (t_i, s_i]$. Assume that $X = L^1(\mathbb{R}^n)$, $I_i, B_i(\cdot) \in \mathcal{L}(X)$, $c_i(y), \xi(y) \in X$, and $f \in C(\mathbb{I}, X)$; here $\mathcal{L}(X)$ is the set of bounded linear operators on X. In addition, we suppose $B_i(t_i^+) = E$, where E is the identity map. Let $z(t)(y) := u(t,y)$, $g(t)(y) := f(t,y)$, $\kappa_i(y) := c_i(y)$, and then we may transform the non-instantaneous impulsive heat equation (1.1) into the abstract

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non-instantaneous impulsive evolution equation

$$\begin{gathered} z'(t)=\Delta z(t)+g(t), \quad t\in[s_{i-1},t_i],\\ \delta z(t_i)=I_iz(t_i)+\kappa_i,\\ z(t)=B_i(t)z(t_i^+), \quad t\in(t_i,s_i],\\ z(0)=z_0. \end{gathered} \tag{1.2}$$

Thus, it is sufficient to show the existence and uniqueness of the inhomogeneous periodic trajectory of (1.2) to study the same problem for (1.1).

2. Preliminaries

Let $\Xi:=\{t_k;k\in\mathbb{N}^+\}$, $\mathbb{R}_+=\mathbb{I}\cup\mathbb{J}$,

$$PC(\mathbb{R}_+,X):=\big\{z:\mathbb{R}_+\setminus\Xi\to X \text{ is continuous}, z(t_i)=z(t_i^-) \text{ and } z(t_i)\neq z(t_i^+)\big\}.$$

The bounded piecewise continuous function space with values in a Banach space X is defined as

$$BPC(\mathbb{R}_+,X):=\big\{z\in PC(\mathbb{R}_+,X), \sup_{t\in\mathbb{R}_+}\|z(t)\|<\infty\big\}$$

endowed with the norm $\|z\|_{BPC}:=\sup_{t\in\mathbb{R}_+}\|z(t)\|$.

Recall that the fundamental solution of the heat equation is

$$\Phi(x,t)=\begin{cases}\frac{1}{(4\pi t)^{n/2}}\exp\big(-|x|^2/(4t)\big), & x\in\mathbb{R}^n,\ t>0,\\ 0, & x\in\mathbb{R}^n,\ t<0.\end{cases}$$

Note that Φ is singular at the point $(0,0)$. For each $t>0$,

$$\int_{\mathbb{R}^n}\Phi(x,t)dx=1.$$

A semigroup of bounded linear operators $(H(t))_{t\geq0}$ on X defined by

$$(H(t)\xi)(y)=\frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n}e^{-\frac{|y-s|^2}{4t}}\xi(s)ds, \quad t>0; \quad H(0)=E$$

is called the heat semigroup generated by Δ.

Lemma 2.1. *For each $t\geq0$,*

$$\|H(t)\|_{\mathcal{L}(X)}\leq1.$$

Proof. For $t=0$, the conclusion is obvious. For each $t>0$, we have

$$\begin{aligned}\|H(t)\|_{L(X)}&=\sup_{\|\xi\|\leq1}\frac{\|\frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n}e^{-\frac{|y-s|^2}{4t}}\xi(s)ds\|}{\|\xi\|}\\ &\leq\sup_{\|\xi\|\leq1}\frac{\frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n}e^{-\frac{|y-s|^2}{4t}}ds\|\xi\|}{\|\xi\|}=1.\end{aligned}$$

$\square$

It is well known that the solution of $z_t(t)=\Delta z(t),\ t>\tau$ with $z(\tau)=z_\tau$, is $z(t)=S(t,\tau)z_\tau$, where $S(t,\tau)=H(t-\tau)$.

Definition 2.2. A non-instantaneous impulsive operator $G(\cdot,\cdot) : \Pi := \{(t,s) \in \mathbb{R}_+ \times \mathbb{I} : s \leq t\} \to \mathcal{L}(X)$ is defined as

$$G(t,s) = \begin{cases} S_i(t,s), \quad \text{if } t,s \in [s_{i-1},t_i], \\ S_k(t,s_{k-1})B_{k-1}(s_{k-1})(E+I_{k-1}) \\ \times \prod_{j=i+1}^{k-1}\{S_j(t_j,s_{j-1})B_{j-1}(s_{j-1})(E+I_{j-1})\}S_i(t_i,s), \\ \quad \text{if } s_{i-1} \leq s \leq t_i < \dots < s_{k-1} \leq t \leq t_k, \\ B_k(t)(E+I_k)\prod_{j=i+1}^{k}\{S_j(t_j,s_{j-1})B_{j-1}(s_{j-1})(E+I_{j-1})\}U_i(t_i,s), \\ \quad \text{if } s_{i-1} \leq s \leq t_i < \dots < t_k < t \leq s_k, \end{cases}$$

where $S_i(t,\tau) := S(t,\tau)_{|t,\tau\in[s_{i-1},t_i]}$.

Note that $G(t,s) = E$ if $t = s$ and $G(t_i^+,s) = (E+I_i)G(t_i,s)$ and $B_i(s_i)G(t_i^+,s) = G(s_i,s)$.

Clearly, any solution of

$$\begin{gathered} z'(t) = \Delta z(t),\ t \in [s_{i-1},t_i], \\ \delta z(t_i) = I_i z(t_i) + \kappa_i, \\ z(t) = B_i(t)z(t_i^+), \quad t \in (t_i,s_i], \\ z(0) = z_0, \end{gathered}$$

has the form $z(t) = G(t,0)z_0$ for $t \geq 0$.

A function $z(t)$ is called a mild solution of (1.2), if it satisfies the integral equation

$$z(t) = G(t,0)z_0 + \int_0^t G(t,\omega)\tilde{g}(\omega)d\omega + \sum_{j=1}^{r(0,t)} G(t,s_j)B_j(s_j)\kappa_j, \tag{2.1}$$

where

$$\tilde{g}(t) = \begin{cases} g(t), & t \in \mathbb{I}, \\ 0, & t \in \mathbb{J}. \end{cases}$$

The function $z(\cdot)$ is also called the inhomogeneous trajectory of equation (1.1).

Now we present the periodic conditions that will be used in the rest of the paper.

- (A1) There exists a $m \in \mathbb{N}^+$ such that $B_{i+m}(t+T) = B_i(t)$ for $t \in (t_i,s_i]$ and $i \in \mathbb{N}^+$.
- (A2) $I_{i+m} = I_i$ for $i \in \mathbb{N}^+$.
- (A3) $s_{i+m} = s_i + T$ for $i \in N$ and $t_{i+m} = t_i + T$ for $i \in \mathbb{N}^+$.
- (A4) $c_{i+m}(y) = c_i(y)$ for $i \in \mathbb{N}^+$ and every $y \in \Omega$.
- (A5) $f(t+T,y) = f(t,y)$ for $t \in \mathbb{I}$ and every $y \in \Omega$.

3. Basic properties for group G

Let $r(s,t)$ be the number of impulsive points in the interval (s,t). Note $r(0,T) = m$.

Theorem 3.1. *For any $s \in \mathbb{I}$ and $t \in \mathbb{R}_+$, we have*

$$\|G(t,s)\| \leq (\beta\gamma)^{r(s,t)},$$

where $\beta = \sup_{i\geq 1}\sup_{t\in(t_i,s_i]}\|B_i(t)\|$ and $\gamma = \sup_{i\geq 1}\|E+I_i\|$.

Proof. Using Definition 2.2 and $\|H(t)\|_{\mathcal{L}(X)} \leq 1$, Following a process similar to that in [9, Theorem 3.1] we obtain the desired result. □

Theorem 3.2 ([9, Theorem 3.3])**.** *If $s \le u \le t$ and $u, s \in \mathbb{I}$, then $G(t,s) = G(t,u)G(u,s)$.*

Theorem 3.3 ([9, Theorem 3.2])**.** *If* (A1)–(A4) *are satisfied, then $G(\cdot+T,\cdot+T) = G(\cdot,\cdot)$.*

From Theorems 3.2 and 3.3, we have the following result.

Corollary 3.4. *For any $t \in \mathbb{R}_+$ and $p \in N$, $G(t+pT,0) = [G(t,0)][G(T,0)]^p$.*

4. Inhomogeneous periodic trajectory

In this section, we establish the existence and uniqueness of the inhomogeneous periodic trajectory for (1.1).

Theorem 4.1 (see [9, Theorem 4.3])**.** *If* (A3) *holds, then*

$$\lim_{t-s\to\infty} \frac{r(s,t)}{t-s} = \frac{m}{T}.$$

Remark 4.2. Theorem 4.1 shows that for an arbitrary ε, with $0 < \varepsilon < \frac{m}{T}$, there exists $J > 0$, and for $t - s > J$,

$$\Big|\frac{r(s,t)}{t-s} - \frac{m}{T}\Big| < \varepsilon.$$

To guarantee the boundedness of the solution, we introduce the following assumption:

(A6) $\beta\gamma < 1$.

Then we set

$$M := \frac{(\beta\gamma)^{(\frac{m}{T}-\varepsilon)J}}{\ln\beta\gamma}\|g\|_{BPC} + \beta c \sum_{s_j\in\Omega_4} (\beta\gamma)^{(\frac{m}{T}-\varepsilon)(t-s_j)},$$

$$\Omega_1 := \{\omega \mid t-\omega \le J\}, \quad \Omega_2 := \{\omega \mid t-\omega > J\},$$

$$\Omega_3 := \{s_j \mid t-s_j \le J\}, \quad \Omega_4 := \{s_j \mid t-s_j > J\}.$$

Clearly, for any fixed point t, the function M is bounded.

Theorem 4.3. *Suppose* (A1)–(A5) *hold. For any $p \in \mathbb{N}^+$, the solution of* (1.2) *satisfies*

$$z((p+1)T) = G(T,0)z(pT) + b_m,$$

where

$$b_m := \int_0^T G(T,\omega)\tilde{g}(\omega)d\omega + \sum_{j=1}^{r(0,t)} G(t,s_j)B_j(s_j)\kappa_j.$$

Proof. From (2.1), and Theorems 3.2 and 3.3, and Corollary 3.4 one has

$$\begin{aligned}
z((p+1)T) &= G((p+1)T,0)\xi(y) + \int_0^{(p+1)T} G((p+1)T,\omega)\tilde{g}(\omega)d\omega \\
&\quad + \sum_{j=1}^{(p+1)m} G((p+1)T,s_j)B_j((p+1)T)c_j \\
&= G((p+1)T,pT)\Big[G(pT,0)z_0 + \int_0^{pT} G(pT,\omega)\tilde{g}(\omega)d\omega
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j=1}^{pm} G(pT, s_j)B_j(s_j)c_j\Big] + \int_{pT}^{(p+1)T} G((p+1)T,\omega)\tilde{g}(\omega)d\omega \\
&+ \sum_{j=pm+1}^{(p+1)m} G((p+1)T, s_j)B_j((p+1)T)c_j \\
&= G(T,0)z(pT) + \int_0^T G((p+1)T, \omega + pT)\tilde{g}(\omega)d\omega \\
&+ \sum_{j=1}^{m} G((p+1)T, s_{j+pm})B_{j+pm}((p+1)T)c_{j+pm} \\
&= G(T,0)z(pT) + \int_0^T G(T,\omega)\tilde{g}(\omega)d\omega + \sum_{j=1}^{m} G(T,s_j)B_j(T)c_j \\
&= G(T,0)z(pT) + b_m.
\end{aligned}
$$

The proof is complete. □

Corollary 4.4. *For $p \in \mathbb{N}^+$, we have*

$$
z(pT) = [G(T,0)]^p z_0 + \sum_{i=0}^{p-1} [G(T,0)]^i b_m.
$$

The above corollary follows directly from Theorem 4.3.

Theorem 4.5. *Suppose* (A1)–(A6) *hold. Then* (1.2) *has a unique T-periodic inhomogeneous trajectory belonging to $BPC(\mathbb{R}_+, L^1(\Omega))$.*

Proof. Using Theorems 3.1 and 4.1, we obtain

$$
\begin{aligned}
&\|z\|_{BPC} \\
&= \sup_{t\in R^+} \|G(t,0)z_0 + \int_0^t G(t,\omega)\tilde{g}(\omega)d\omega + \sum_{j=1}^{r(0,t)} G(t,s_j)B_j(s_j)\kappa_j\| \\
&\le \sup_{t\in\mathbb{R}_+} \|G(t,0)\|\|z_0\| + \sup_{t\in\mathbb{R}_+} \int_0^t \|G(t,\omega)\|d\omega\|g\|_{BPC} \\
&\quad + \sup_{t\in\mathbb{R}_+} \sum_{j=1}^{r(0,t)} \|G(t,s_j)\|\|B_j(s_j)\|\|\kappa_j\| \\
&\le \sup_{t\in\mathbb{R}_+} (\beta\gamma)^{r(0,t)}\|z_0\| + \sup_{t\in\mathbb{R}_+} \int_0^t (\beta\gamma)^{r(\omega,t)}d\omega\|g\|_{BPC} + \sup_{t\in\mathbb{R}_+} \beta c \sum_{j=1}^{r(0,t)} (\beta\gamma)^{r(s_j,t)} \\
&\le \sup_{t\in\mathbb{R}_+} (\beta\gamma)^{r(0,t)}\|z_0\| + \int_{\Omega_1} (\beta\gamma)^{r(\omega,t)}d\omega\|g\|_{BPC} + \int_{\Omega_2} (\beta\gamma)^{r(\omega,t)}d\omega\|g\|_{BPC} \\
&\quad + \beta c \sum_{s_j\in\Omega_3} (\beta\gamma)^{r(s_j,t)} + \beta c \sum_{s_j\in\Omega_4} (\beta\gamma)^{r(s_j,t)} \\
&\le \|z_0\| + J\|g\|_{BPC} + \int_{\Omega_2} (\beta\gamma)^{(\frac{m}{T}-\varepsilon)(t-\omega)}d\omega\|g\|_{BPC} + r(0,J)\beta c \\
&\quad + \beta c \sum_{s_j\in\Omega_4} (\beta\gamma)^{(\frac{m}{T}-\varepsilon)(t-s_j)}
\end{aligned}
$$

$$
\begin{aligned}
&\le \|z_0\| + J\|g\|_{BPC} + \frac{(\beta\gamma)^{(\frac{m}{T}-\varepsilon)J}}{\ln\beta\gamma}\|g\|_{BPC} - \frac{(\beta\gamma)^{(\frac{m}{T}-\varepsilon)t}}{\ln\beta\gamma}\|g\|_{BPC} \\
&\quad + r(0,J)\beta c + \beta c\sum_{s_j\in\Omega_4}(\beta\gamma)^{(\frac{m}{T}-\varepsilon)(t-s_j)} \\
&\le \|z_0\| + J\|g\|_{BPC} - \frac{1}{\ln\beta\gamma}\|g\|_{BPC} + r(0,J)\beta c + M \\
&= \|z_0\| + (J - \frac{1}{\ln\beta\gamma})\|g\|_{BPC} + r(0,J)\beta c + M.
\end{aligned}
$$

We now prove that $\{z(aT)\}_{a\in N}$ is a Cauchy sequence in $L^1(\Omega)$. Indeed, for any fixed natural numbers $a > b$, using Corollary 4.4, we obtain

$$
\begin{aligned}
&\|z(aT) - z(bT)\| \\
&= \|([G(T,0)]^a - [G(T,0)]^b)z_0 + \sum_{i=b}^{a-1}[G(T,0)]^i b_m\| \\
&\le [(\beta\gamma)^{ar(0,T)} + (\beta\gamma)^{br(0,T)}]\|z_0\| + \sum_{i=b}^{a-1}(\beta\gamma)^{ir(0,T)}\|b_m\| \\
&\le [(\beta\gamma)^{am} + (\beta\gamma)^{bm}]\|z_0\| + \sum_{i=b}^{a-1}(\beta\gamma)^{im}(\|g\|_{BPC} + m\beta c) \\
&= [(\beta\gamma)^{am} + (\beta\gamma)^{bm}]\|z_0\| + (\|g\|_{BPC} + m\beta c)\frac{(\beta\gamma)^{bm}(1-(\beta\gamma)^{a-b})}{1-\beta\gamma}.
\end{aligned}
$$

When a and b are large enough, we have $\|z(aT) - z(bT)\| \to 0$. Therefore, $\{z(aT)\}_{a\in N}$ is a Cauchy sequence in $L^1(\Omega)$, so the sequence $\{z(aT)\}_{a\in N}$ is convergent in $L^1(\Omega)$, and we put

$$
z^* := \lim_{a\to+\infty} z(aT) \in L^1(\Omega).
$$

Take now z^* as the initial value, and we will prove that the inhomogeneous trajectory

$$
\hat{z}(t) = G(t,0)z^* + \int_0^t G(t,\omega)\tilde{g}(\omega)d\omega + \sum_{j=1}^{r(0,t)} G(t,s_j)B_j(s_j)\kappa_j
$$

is T-periodic. Using Theorem 4.3, we obtain

$$
\begin{aligned}
\|\hat{z}(T) - z((a+1)T)\| &= \|G(T,0)(z^* - z(aT))\| \\
&\le (\beta\gamma)^{r(0,T)}\|z^* - z(aT)\| \\
&= (\beta\gamma)^m\|z^* - z(aT)\|.
\end{aligned}
$$

Let $a \to +\infty$ and using the fact that $\lim_{a\to+\infty} z(aT) = z^* = \hat{z}(0)$, we obtain

$$
\hat{z}(T) = \hat{z}(0).
$$

Therefore, $\hat{z}(t)$ is T-periodic.

Next, we prove the uniqueness of the inhomogeneous T-periodic trajectory. Let $\hat{z}_1$ and $\hat{z}_2$ be two T-periodic trajectories of (1.1) with initial values $\hat{z}_{10}$ and $\hat{z}_{20}$, and we obtain

$$
\|\hat{z}_1 - \hat{z}_2\| = \|G(t,0)(\hat{z}_{10} - \hat{z}_{20})\| \le (\beta\gamma)^{r(0,t)}\|\hat{z}_{10} - \hat{z}_{20}\|.
$$

Then, using Theorem 4.1 and (A6), we have

$$\lim_{t\to+\infty}\|\hat{z}_1-\hat{z}_2\|\leq\lim_{t\to+\infty}(\beta\gamma)^{(\frac{m}{T}-\varepsilon)t}\|\hat{z}_{10}-\hat{z}_{20}\|=0.$$

From the periodicity of $\hat{z}_1$ and $\hat{z}_2$, we obtain $\hat{z}_1-\hat{z}_2=0$. That is $\hat{z}_1(t)=\hat{z}_2(t)$ for $t\in\mathbb{R}_+$. $\square$

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