Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 22, pp. 1–17.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

LINEARIZATION OF MULTI-FREQUENCY QUASI-PERIODICALLY FORCED CIRCLE FLOWS BEYOND BRJUNO CONDITION

ZIYANG LIANG, TAIAN JIN, JIAYI WANG, YUAN SHAN

Abstract. In this article, we considered the linearization of analytic quasi-periodically forced circle flows. We generalized the rotational linearization of systems with two-dimensional base frequency to systems with any finite dimensional base frequency case. Meanwhile, we relaxed the arithmetical limitations on the base frequencies. Our proof is based on a generalized Kolmogorov–Arnold–Moser (KAM) scheme.

1. Introduction and statement of main results

In this article, we consider the quasi-periodically forced (qpf) circle flow

$$\begin{aligned} \dot{x} &= \rho + f(\phi, x), \\ \dot{\phi} &= \Omega, \end{aligned} \tag{1.1}$$

where $f : \mathbb{T}^d \times \mathbb{T}^1 \to \mathbb{R}$ is a real analytic function with $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$. Here, $\Omega \in \mathbb{T}^d$ is rationally independent. We denote the system (1.1) by $(\Omega, \rho + f)$ for simplicity.

The time discrete counterparts of the qpf circle flows are the qpf circle maps. The research of qpf circle maps is an important topic in mathematical physics and dynamical systems: The qpf circle maps are related to Arnold tongue, more specifically, Arnold circle map, which attempts to capture the motion of the spinning disks at discrete time intervals. The qpf circle maps also provide a simple model of the mode-locked loop in electronics, of mechanically musical instruments and of heart issue. And the qpf circle maps appeared in the study of quasi-periodic crystals and damped pendulum motions too. Moreover, the qpf circle maps can also be used to investigate the quasi-periodic schrödinger operator [4], which is an important mathematical model of the quantum Hall effect and many other quantum physics problems.

In this paper, we mainly focus on the C^∞ rotational linearization of (1.1) with weak Liouvillean frequency, provided the analytic norm of f is small enough. We say the system $(\Omega, \rho + f)$ is C^r $(r = \infty, \omega)$ linearizable or C^r reducible, if there exists a C^r map $H : \mathbb{T}^d \times \mathbb{T}^1 \to \mathbb{T}^d \times \mathbb{T}^1$ and $\tilde{\rho} \in \mathbb{R}$ such that H conjugates the system $(\Omega, \rho + f)$ to $(\Omega, \tilde{\rho})$. And, if the qpf circle flow $(\Omega, \rho + f)$ can be C^r $(r = \infty, \omega)$ conjugate to $(\Omega, g(\phi))$, then we say it is C^r rotation linearizable or

2010 *Mathematics Subject Classification.* 37C15, 34C20.
Key words and phrases. Linearization; quasi-periodically forced circle flow;
Liouvillean frequency.

Submitted September 4, 2019. Published March 12, 2020.

reducible. Meanwhile, using the embedding result of You-Zhou [10], when f is small enough, the discrete system can be embedded into the continuous case, that is the qpf circle flows. This implies the equivalence of the linearization between the discrete case and continuous case for the perturbation f small.

In recent years, the qpf circle flows have been studied extensively by many mathematicians. Herman [5] investigated system (1.1) with (Ω, ρ_f) satisfying the Diophantine condition

$$|\langle k, \Omega\rangle + l\rho_f| \geq \frac{\gamma}{(|k|+|l|)^{\tau}}, \quad \forall (k,l) \in \mathbb{Z}^d \times \mathbb{Z},\ |k|+|l| \neq 0, \tag{1.2}$$

where ρ_f is the fibred rotation number of the system and proved the system is C^ω linearized provided the analytic norm of f is sufficiently small.

For a qpf circle flow (Ω, f), we say $\rho_f = \rho(\Omega, f) = \lim_{t\to\infty} \frac{\hat{\Phi}^t_{\phi}(\hat{x})}{t}$ is the fibred rotation number associated with (Ω, f), where $\hat{\Phi}^t_{\phi}(\hat{x}) : \mathbb{R}_+ \times \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$ via $(t, \phi, \hat{x}) \mapsto \hat{\Phi}^t_{\phi}(\hat{x})$ denotes the lift of the flow of (Ω, f) of the valuable x, and for any $\tilde{\rho} \in \mathbb{R}$ we have $|\rho(\Omega, \tilde{\rho}+f) - \rho(\Omega, \tilde{\rho})| \leq \varepsilon$, provided that $\|f\|_{C^0} \leq \varepsilon$ small enough [6].,

Note that, in (1.2), $l = 0$ implies that Ω is a Diophantine vector (depending on τ and γ). Without the assumption of Ω being Diophantine, the problem is quite different and there is not much work done so far. Recently, using the almost reducibility theory, Krikorian-Wang-You-Zhou[9] managed to relax the Diophantine assumption to non-super Liouvillean frequencies for $d = 2$, and get the rotational linearization. More precisely, let $\Omega = (1, \alpha)$ with $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and $\{\frac{p_n}{q_n}\}$ be the best convergence of α. Assume that α is not super-Liouvillean, that is

$$\sup_{n>0} \frac{\ln\ln q_{n+1}}{\ln q_n} < +\infty. \tag{1.3}$$

Then for f with sufficiently small analytic norm, the system $(\Omega, \rho + f)$ is C^∞ rotation reducible, provided (Ω, ρ_f) satisfying

$$|\langle k, \Omega\rangle + l\rho_f| \geq \frac{\gamma}{(|k|+|l|)^{\tau}}, \quad \forall k \in \mathbb{Z}^2, 0 \neq l \in \mathbb{Z}. \tag{1.4}$$

However, when the base is of higher dimension, there is no result on this issue. Thus, in this paper, we consider the rotation linearization for qpf circle flows with multiple base frequencies satisfying the weak-Liouvillean condition. Furthermore, we managed to relax Krikorian-Wang-You-Zhou's condition (1.3) on one variable of the multi-frequency. More precisely, for the frequency $\Omega = (1, \alpha, \tilde{\omega}) \in \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^{d-2}$, we denote

$$\tilde{U}(\alpha) = \sup_{n>0} \frac{\ln\ln\ln q_{n+1}}{\ln\ln q_n}, \tag{1.5}$$

where $\{p_n/q_n\}$ is the best convergence of α. If $\tilde{U} := \tilde{U}(\alpha) < \infty$, and

$$|\langle k, \omega\rangle + \langle l, \tilde{\omega}\rangle| \geq \frac{\gamma}{(|k|+|l|+1)^{\tau}}, \quad \forall k \in \mathbb{Z}^2,\ l \in \mathbb{Z}^{d-2}\backslash\{0\}, \tag{1.6}$$

with $\omega = (1, \alpha)$, then we say Ω is weak-Liouvillean. We will denote by $WL(\gamma, \tau, \tilde{U})$ the set of all such vectors and

$$WL = \cup_{\gamma,\tau>0, 0<\tilde{U}<+\infty} WL(\gamma, \tau, \tilde{U}).$$

It is obvious that WL is of full Lebesgue measure. Then our main result can be stated as follows.

Theorem 1.1. *Let $\rho \in \mathbb{R}$, $d \geq 2$, γ', τ', γ'', τ'', $r_1, r_2 > 0$, $\Omega = (1, \alpha, \tilde{\omega}) \in \mathbb{R}^d$ with $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\tilde{U} := \tilde{U}(\tilde{\alpha}) < \infty$. If $\Omega \in WL(\gamma', \tau', \tilde{U})$ and $\rho(\Omega, \rho + f) =: \rho_f \in DC_\Omega(\gamma'', \tau'')$ in the sense that*

$$|l\rho_f + \langle k, \Omega\rangle| \geq \frac{\gamma}{(|k| + |l| + 1)^\tau}, \quad \forall k \in \mathbb{Z}^d,\ l \in \mathbb{Z}\backslash\{0\},$$

then there exists $\varepsilon = \varepsilon(\gamma', \gamma'', \tau', \tau'', r_1, r_2, \tilde{U}) > 0$ such that if $\|f\|_{r_1, r_2} \leq \varepsilon$, (see section 2.1 for a precise definition of the norm) then the system $(\Omega, \rho + f)$ is C^∞ rotation linearizable.

We want to point out that, using the method in [9] by Krikorian-Wang-You-Zhou, we can obtain the same result as [9, Corollary 1.1]. That is $(\Omega, \rho + f)$ is C^∞ accumulated by analytic flows $\{(\Omega, \tilde{f}_n)\}$, where the qpf flow $(\Omega, \tilde{f}_n)$ is mode-locked.

In fact, our condition on the base frequency was partially inspired by recent progress of almost reducibility in linear quasi-periodic $SL(2, \mathbb{R})$ cocycles

$$\begin{aligned} (\alpha, A) : \mathbb{T}^{d-1} \times \mathbb{R}^2 &\to \mathbb{T}^{d-1} \times \mathbb{R}^2 \\ (\theta, v) &\mapsto (\theta + \alpha, A(\theta)v), \end{aligned} \tag{1.7}$$

When $d = 2$, as for the reducibility of quasi-periodic $SL(2, \mathbb{R})$ cocycles, which is one-dimensional base frequency case, there are fruitful results. For the local case, meaning the cocycle is close to a constant system, Dinaburg and Sinai [2] first proved the positive measure reducibility with Diophantine frequency α, and it was deepened by Eliasson to full measure reducibility [3]. In these papers, the Diophantine condition on α allows the authors to use a Kolmogorov–Arnold–Moser (KAM) argument. Recently, using generalized KAM schemes, Avila-Fayad-Krikorian [1] and Hou-You [7] proved the local C^ω rotation reducible result for any base forcing frequency $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ in discrete case and continuous case respectively.

Compared the above one frequency case, very little is known for multifrequency case. Recently, Hou-Wang-Zhou [8] considered the reducibility of multi-frequency analytic quasi-periodic $SL(2, \mathbb{R})$-cocycles (1.7) with the frequency $\alpha = (\tilde{\alpha}, \alpha) \in \mathbb{T}^1 \times \mathbb{T}^{d-2}$ satisfying the conditions

$$\tilde{u}(\tilde{\alpha}) := \sup_{n>0} \frac{\ln \ln \tilde{q}_{n+1}}{\ln \tilde{q}_n} < \infty, \tag{1.8}$$

and

$$\|k\tilde{\alpha} + \langle l, \alpha'\rangle\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{\gamma}{(|k| + |l| + 1)^\tau}, \quad \forall k \in \mathbb{Z},\ l \in \mathbb{Z}^{d-2}\backslash\{0\},$$

for some $\gamma > 0$, $\tau > d - 1$, where $\{\tilde{p}_n/\tilde{q}_n\}$ is the best convergence of $\tilde{\alpha}$, and

$$\|a\|_{\mathbb{R}/\mathbb{Z}} = \inf_{p \in \mathbb{Z}} |a - p|.$$

They proved the local positive measure rotation reducibility for the $SL(2, \mathbb{R})$ cocycles provided that the cocycle is C^ω close enough to constant ones.

Note that our system is continuous non-linear system, which is quite different and much more complicated than the above $SL(2, \mathbb{R})$-cocycles (discrete linear system). Moreover, comparing the frequency condition of Theorem 1.1 with condition (1.8), we handled more frequencies Ω including more Liouvillean ones.

2. Preliminaries

2.1. **Norm and Basic definitions.** Denote by $C^{\omega}_{r_1,r_2}(\mathbb{T}^{d+1},\mathbb{R})$ the set of all $\mathbb{R}$-valued functions admitting an analytic extension on

$$\mathbb{T}^{d+1}_{r_1,r_2} := \{(\phi,x)\in\mathbb{T}^{d+1} : |\Im\phi_1|\le r_1,\dots,|\Im\phi_d|\le r_1, |\Im x|\le r_2\},$$

where $r_1,r_2>0$. For any $f\in C^{\omega}_{r_1,r_2}(\mathbb{T}^{d+1},\mathbb{R})$, let

$$\|f\|_{r_1,r_2} := \sup_{(\phi,x)\in\mathbb{T}^{d+1}_{r_1,r_2}} |f(\phi,x)|.$$

In this article, we also frequently consider real-valued functions admitting an analytic extension on

$$\begin{aligned}\mathbb{T}^{d+1}_{r_1,r_2,r_3} :=\big\{&(\phi,x)\in\mathbb{T}^{d+1} : |\Im\phi_1|\le r_1, |\Im\phi_2|\le r_1, |\Im\phi_3|\le r_2,\dots,\\ &|\Im\phi_d|\le r_2, |\Im x|\le r_3\big\},\end{aligned}$$

where $r_1,r_2,r_3>0$, and use $\|f\|_{r_1,r_2,r_3}$ to denote the norm

$$\|f\|_{r_1,r_2,r_3} := \sup_{(\phi,x)\in\mathbb{T}^{d+1}_{r_1,r_2,r_3}} |f(\phi,x)|.$$

We use $C^{\omega}_{r_1,r_2,r_3}(\mathbb{T}^{d+1},\mathbb{R})$ to denote the set of all such functions.

An integrable real-valued function f on $\mathbb{T}^d$ has the Fourier expansion $f=\sum_{k\in\mathbb{Z}^d}\hat f(k)e^{2\pi i\langle k,\phi\rangle}$ with $\hat f(k)=\int_{\mathbb{T}^d} f(\phi)e^{-2\pi i\langle k,\phi\rangle}d\phi$. For any $N>0$, $\mathcal{T}_N$ and $\mathcal{R}_N$ are used to denote the truncation operators:

$$\mathcal{T}_N(f)=\sum_{|k|<N}\hat f(k)e^{2\pi i\langle k,\phi\rangle},\quad \mathcal{R}_N(f)=\sum_{|k|\ge N}\hat f(k)e^{2\pi i\langle k,\phi\rangle}. \tag{2.1}$$

2.2. **Continued fraction expansion.** Let $\alpha\in\mathbb{R}$ be irrational. Define $a_0=[\alpha], \alpha_0=\alpha-a_0$, and for $k\ge 1$,

$$a_k=[\alpha_{k-1}^{-1}],\quad \alpha_k=G(\alpha_{k-1})=\alpha_{k-1}^{-1}-a_k.$$

We define $p_0=0$, $p_1=1$, $q_0=1$, $q_1=a_1$, and inductively,

$$p_k=a_kp_{k-1}+p_{k-2},\quad q_k=a_kq_{k-1}+q_{k-2}.$$

So the sequence $\{p_n/q_n\}_{n\in\mathbb{N}}$ is the best rational approximation for $\alpha\in\mathbb{R}\backslash\mathbb{Q}$, and it satisfies

$$\|k\alpha\|_{\mathbb{R}/\mathbb{Z}}\ge\|q_{n-1}\alpha\|_{\mathbb{R}/\mathbb{Z}},\quad \forall\ 1\le k<q_n,$$

and

$$\frac{1}{q_n+q_{n+1}}\le\|q_n\alpha\|_{\mathbb{R}/\mathbb{Z}}\le\frac{1}{q_{n+1}}.$$

2.3. **CD bridge.** For any $\alpha\in\mathbb{R}\backslash\mathbb{Q}$, we will fix in the sequel a particular subsequence $(q_{n_k})_{k\in\mathbb{N}}$ of the denominators of the continued fraction expansion for α, which is denoted by $(Q_k)_{k\in\mathbb{N}}$, and the subsequence $(q_{n_k+1})_{k\in\mathbb{N}}$ is denoted by $(\bar Q_k)_{k\in\mathbb{N}}$.

Definition 2.1 ([1])**.** Let $0<\mathcal{A}\le\mathcal{B}\le\mathcal{C}$. We say that the pair of denominators (q_l,q_n) $(l<n)$ forms a $CD(\mathcal{A},\mathcal{B},\mathcal{C})$ bridge if

$$\begin{gathered}q_{i+1}\le q_i^{\mathcal{A}},\quad \forall i=l,\dots,n-1\\ q_l^{\mathcal{C}}\ge q_n\ge q_l^{\mathcal{B}}.\end{gathered}$$

Lemma 2.2 ([1]). *For any $\mathcal{A}>0$, there exists a subsequence $(Q_k)_{k\in\mathbb{N}}$ such that $Q_0=1$ and for each $k\geq 0$, $Q_{k+1}\leq \bar{Q}_k^{\mathcal{A}^4}$. Furthermore, either $\bar{Q}_k\geq Q_k^{\mathcal{A}}$, or the pairs $(\bar{Q}_{k-1},Q_k)$ and (Q_k,Q_{k+1}) are both $CD(\mathcal{A},\mathcal{A},\mathcal{A}^3)$ bridges.*

In the sequel, we let $\mathcal{A}\geq 3$ and assume $(Q_n)_{n\in\mathbb{N}}$ is the selected subsequence in above lemma accordingly. As an immediate corollary of the above lemma, we have a corollary.

Corollary 2.3. *If $\tilde{U}(\alpha)<\infty$, then $Q_n\geq Q_{n-1}^{\mathcal{A}}$ for every $n\geq 1$. Furthermore,*

$$\sup_{n>0}\frac{\ln\ln\ln Q_{n+1}}{\ln\ln Q_n}\leq U(\alpha),$$

where $U(\alpha):=\tilde{U}(\alpha)+\frac{\ln\ln\mathcal{A}^4}{\ln\ln 3}+36<\infty$.

Proof. For $n=1$, according to $Q_0=1$, we obviously get $Q_1\geq Q_0^{\mathcal{A}}$.

For $n\geq 2$, there are two cases below. If $\bar{Q}_{n-1}\geq Q_{n-1}^{\mathcal{A}}$, then $Q_n\geq \bar{Q}_{n-1}\geq Q_{n-1}^{\mathcal{A}}$. Otherwise, since (Q_{n-1},Q_n) is a $CD(\mathcal{A},\mathcal{A},\mathcal{A}^3)$ bridge, and then, $Q_n\geq Q_{n-1}^{\mathcal{A}}$. Furthermore, owing to $Q_{n+1}\leq \bar{Q}_n^{\mathcal{A}^4}$, for $n\geq 1$ we obtain

$$\begin{aligned}\frac{\ln\ln\ln Q_{n+1}}{\ln\ln Q_n}&\leq\frac{\ln(\ln\mathcal{A}^4+\ln\ln\bar{Q}_n)}{\ln\ln Q_n}\\&\leq\frac{\ln\ln\ln\bar{Q}_n}{\ln\ln Q_n}+\frac{\ln(27\ln\mathcal{A}^4)}{\ln\ln 3}\\&\leq\tilde{U}+\frac{\ln\ln\mathcal{A}^4}{\ln\ln 3}+36=U.\end{aligned}$$ □

3. The inductive step

For convenience, in the sequel, we will rewrite the system (1.1) as

$$\begin{gathered}\dot{x}=\rho+f(\varphi,\theta,x)\\\dot{\theta}=\tilde{\omega}\\\dot{\varphi}=\omega=(1,\alpha),\end{gathered}\tag{3.1}$$

where $\phi=(\varphi,\theta)\in\mathbb{T}^2\times\mathbb{T}^{d-2}$, $\Omega=(\omega,\tilde{\omega})$. Before the linearization steps, we give a notation for simplicity: For any $r_1,r_2,r_3,\epsilon_g,\epsilon_f>0,\rho\in\mathbb{R}$, we denote

$$\begin{aligned}&\mathcal{F}_{r_1,r_2,r_3}(\rho,\epsilon_g,\epsilon_f)\\&:=\big\{\tilde{\rho}+g(\varphi)+f(\varphi,\theta,x)\in C^{\omega}_{r_1,r_2,r_3}(\mathbb{T}^{d+1},\mathbb{R}):\rho(\Omega,\tilde{\rho}+g+f)=\rho,\\&\quad\|g\|_{r_1}\leq\epsilon_g,\ \|f\|_{r_1,r_2,r_3}\leq\epsilon_f\big\}.\end{aligned}$$

Let $\alpha\in\mathbb{R}\backslash\mathbb{Q}$ with $\tilde{U}(\alpha)<\infty$, and $(Q_n)_{n\in\mathbb{N}}$ be the selected sequence of α in Lemma 2.2 with $\mathcal{A}=9$. Then $U:=\tilde{U}(\alpha)+\frac{\ln\ln\mathcal{A}^4}{\ln\ln 3}+36<\infty$. For $r_{1,0},r_{2,0},r_{3,0},\gamma,\tau$ positive, let $Q_{\min}\geq 3$ be the smallest $Q\in\mathbb{N}$ such that for any $Q\geq Q_{\min}$ we have

$$6(\ln(2Q))^{U+c(\tau+d)}<r_{1,0}Q^{\frac{2}{3}},\tag{3.2}$$

where $c>1$ is a constant with $(\ln 4)^{c(\tau+d)}\cdot\frac{\ln\frac{3}{2}}{64(\tau+d+2)\ln 3}>U+c(\tau+d)$. Meanwhile, let ε_0 small enough such that

$$\begin{aligned}\varepsilon_0<\min\Big\{&\frac{(r_{1,0}r_{2,0}r_{3,0}\gamma)^{12(\tau+d+2)}}{2\tau!e^{(\ln 2Q_1)^{U+c(\tau+d)+1}}},e^{-36(\tau+d+3)^{U+c(\tau+d)}},\\&Q_{\min}^{-6(\ln 2Q_{\min})^{U+c(\tau+d)}-3}\Big\},\end{aligned}\tag{3.3}$$

and

$$\ln \frac{1}{\varepsilon_0} < (\frac{1}{\varepsilon_0})^{\frac{1}{12(\tau+d+2)}}. \tag{3.4}$$

For any given $r_{1,0}, r_{2,0}, r_{3,0}, \varepsilon_0 > 0$, we define the following sequences inductively for $j \geq 1$:

$$\begin{gathered}
\Lambda_1 = \frac{r_{2,0}}{10}, \quad \Lambda_j = \frac{\Lambda_1}{2^{j-1}}, \quad \Delta_1 = \frac{r_{3,0}}{10}, \quad \Delta_j = \frac{\Delta_1}{2^{j-1}}, \\
r_{1,j} = \frac{r_{1,0}}{4Q_j^3}, \quad r_{2,j} = r_{2,j-1} - \Lambda_j, \quad r_{3,j} = r_{3,j-1} - \Delta_j, \\
\varepsilon_j = \frac{\varepsilon_{j-1}}{e^{(\ln 2Q_{j+1})^{U+c(d+\tau)}}}, \quad \tilde{\varepsilon}_j = \sum_{m=0}^{j-1} \varepsilon_m, K^{(j)} = \Big[\Big(\frac{\gamma^2}{2\varepsilon_{j-1}^{1/2}}\Big)^{\frac{1}{\tau+d+2}}\Big].
\end{gathered} \tag{3.5}$$

3.1. **Eliminate the lower-frequency terms.** To minimize the norm of the perturbation f, we have to solve a homological equation involving a function $g(\varphi)$. For solving such an equation, we will use diagonal domination, which demands the norms of g and f are in the same order. Therefore, first, we will do the transformation in the form $x_+ = x - h(\varphi)$ to achieve this.

Lemma 3.1. *For $n \geq 2$, given a qpf circle flow*

$$\begin{aligned}
\dot{x} &= \rho + g(\varphi) + f(\varphi, \theta, x) \\
\dot{\theta} &= \tilde{\omega} \\
\dot{\varphi} &= \omega = (1, \alpha)
\end{aligned} \tag{3.6}$$

with $\rho + g + f \in \mathcal{F}_{r_{1,n-1}, r_{2,n-1}, r_{3,n-1}}(\rho, 4\tilde{\varepsilon}_{n-1}, \varepsilon_{n-1})$, there exists $h \in C^\omega_{r_{1,n-1}}(\mathbb{T}^2, \mathbb{R})$, where $\partial_\omega h(\varphi) = \mathcal{T}_{Q_n} g(\varphi) - \hat{g}(0)$, such that the transformation $\bar{x} = x - h(\varphi) \pmod 1$, conjugates system (3.6) into

$$\begin{aligned}
\dot{\bar{x}} &= \rho + \bar{g}(\varphi) + \bar{f}(\varphi, \theta, \bar{x}) \\
\dot{\theta} &= \tilde{\omega} \\
\dot{\varphi} &= \omega = (1, \alpha)
\end{aligned} \tag{3.7}$$

with $\rho + \bar{g} + \bar{f} \in \mathcal{F}_{\bar{r}_{1,n}, r_{2,n-1}, \bar{r}_{3,n}}(\rho, \varepsilon_{n-1}^{1/2}, \varepsilon_{n-1})$, where $\bar{r}_{1,n} = \frac{r_{1,0}}{Q_n^3}, \bar{r}_{3,n} = r_{3,n-1} - \frac{\Delta_n}{3}$.

Proof. Let $\bar{x} = x - h(\varphi) \pmod 1$, where $\partial_\omega h(\varphi) = \mathcal{T}_{Q_n} g(\varphi) - \hat{g}(0)$. Then the fibred equation becomes

$$\dot{\bar{x}} = \rho + \hat{g}(0) + \mathcal{R}_{Q_n} g(\varphi) + f(\varphi, \theta, \bar{x} + h(\varphi)). \tag{3.8}$$

Because the norm of h is unknown, which will affect the norm of f, we need to estimate $\|h(\varphi)\|$. Since

$$h(\varphi) = \sum_{0<|k|<Q_n} \frac{\hat{g}(k)}{2\pi i \langle k, \omega \rangle} e^{2\pi i \langle k, \varphi \rangle}, \tag{3.9}$$

together with the fact that $|\langle k, \omega \rangle| > \frac{1}{2Q_n}$ for $0 < |k| < Q_n$, we have

$$\begin{aligned}
\|h(\varphi)\|_{\frac{r_{1,n-1}}{2}} &\leq \frac{Q_n}{\pi} \sum_{0<|k|<Q_n} \|g\|_{r_{1,n-1}} e^{-2\pi |k| \frac{r_{1,n-1}}{2}} \\
&\leq \frac{CQ_n \varepsilon_0}{r_{1,n-1}^2} \leq Q_n^{\frac{5}{3}} \varepsilon_0^{1/2}.
\end{aligned}$$

It is clear that, $\|h(\varphi)\|_{\frac{r_{1,n-1}}{2}}$ has an influence on $f(\varphi,\theta,\bar{x}+h(\varphi))$, and in order to indicate f within control, we should estimate the norm of f in a smaller region, which also means to estimate $|\Im h(\varphi)|$. To establish this, we consider the region $|\Im\varphi| \le \bar{r}_{1,n}$.

Let $\varphi = \varphi_1 + i\varphi_2, \varphi_1 \in \mathbb{T}^2, \varphi_2 \in \mathbb{R}^2$, and

$$\begin{gathered} h_1(\varphi_1) = \sum_{0<|k|<Q_n} \frac{\hat{g}(k)}{2\pi i\langle k,\omega\rangle} e^{2\pi i\langle k,\varphi_1\rangle}, \\ h_2(\varphi) = h(\varphi) - h_1(\varphi_1). \end{gathered}$$

It is clear that $h(\varphi)$ is real analytic, since $g(\varphi)$ is real analytic. Therefore, $\Im h_1(\varphi_1) = 0$. Then, owing to $Q_n \ge Q_{n-1}^9$, we have

$$\begin{aligned} \|\Im h(\varphi)\|_{\bar{r}_{1,n}} &= \|\Im h_2(\varphi)\|_{\bar{r}_{1,n}} \le \|h_2(\varphi)\|_{\bar{r}_{1,n}} \\ &\le Q_n \sum_{0<|k|<Q_n} |\hat{g}(k)||e^{-2\pi\langle k,\varphi_2\rangle} - 1| \\ &\le 2\pi Q_n \sum_{0<|k|<Q_n} \|g\|_{r_{1,n-1}} e^{-2\pi|k|(r_{1,n-1}-\bar{r}_{1,n})}|k|\bar{r}_{1,n} \\ &\le \frac{Cr_{1,0}\varepsilon_0}{Q_n^2(r_{1,n-1}-\bar{r}_{1,n})^3} \\ &\le \frac{\Delta_n}{3}, \end{aligned}$$

by the selection of ε_0 and Δ_n. So, we have that

$$\|f(\varphi,\theta,\bar{x}+h(\varphi))\|_{\bar{r}_{1,n},r_{2,n-1},\bar{r}_{3,n}} \le \|f(\varphi,\theta,x)\|_{r_{1,n-1},r_{2,n-1},r_{3,n-1}} \le \varepsilon_{n-1},$$

and

$$\begin{aligned} \|\mathcal{R}_{Q_n} g(\varphi)\|_{\bar{r}_{1,n}} &= \|\sum_{|k|\ge Q_n} \hat{g}(k) e^{2\pi i\langle k,\varphi\rangle}\|_{\bar{r}_{1,n}} \\ &\le \sum_{|k|\ge Q_n} \|g\|_{r_{1,n-1}} e^{-2\pi|k|(r_{1,n-1}-\bar{r}_{1,n})} \\ &\le \frac{C\varepsilon_0 Q_{n-1}^6}{r_{1,0}^2} e^{-2\pi Q_n \frac{r_{1,0}}{Q_{n-1}^3}} \le \frac{\varepsilon_{n-1}^{1/2}}{4}, \end{aligned}$$

by (3.2) and (3.3), and

$$|\hat{g}(0)| \le \|f(\varphi,\theta,\bar{x}+h(\varphi))\|_{\bar{r}_{1,n},r_{2,n-1},\bar{r}_{3,n}} + \|\mathcal{R}_{Q_n} g(\varphi)\|_{\bar{r}_{1,n}} \le \frac{\varepsilon_{n-1}^{1/2}}{2}.$$

Let

$$\begin{gathered} \bar{g}(\varphi) = \hat{g}(0) + \mathcal{R}_{Q_n} g(\varphi) \\ \bar{f}(\varphi,\theta,\bar{x}) = f(\varphi,\theta,\bar{x}+h(\varphi)). \end{gathered}$$

Then the result follows, since the transformation we did is homotopic to the identity, which assures the fibred rotation number remains unchanged. $\square$

3.2. **Solve the homological equation and reduce the perturbation.** Under the assumptions of Lemma 3.1, after the transformation, the norms of $g(\varphi)$ and $f(\varphi,\theta,x)$ are in the same order of magnitude. Then we want to reduce the perturbation. However, we have no Diophantine assumption on ω, which means we need to solve a different and more complicated homological equation in the process of reducing $\|f\|$. Moreover, it also means there might be no analytic solution of the homological equation. Fortunately, it turns out that we can get the approximate solution by using the method of diagonally dominant operators.

Considering the transformation: $(\varphi,\theta,x)=(\varphi,\theta,x_++h(\varphi,\theta,x_+)\pmod 1)$, we obtain

$$\dot{x}=\dot{x}_++\partial_\omega h+\partial_{\tilde{\omega}}h+\frac{\partial h}{\partial x_+}\dot{x}_+.$$

Suppose that the system

$$\begin{gathered}\dot{x}=\rho+g(\varphi)+f(\varphi,\theta,x)\\ \dot{\theta}=\tilde{\omega}\\ \dot{\varphi}=\omega=(1,\alpha)\end{gathered}$$

is conjugate to

$$\begin{gathered}\dot{x}_+=\rho+g_+(\varphi)+f_+(\varphi,\theta,x_+)\\ \dot{\theta}=\tilde{\omega}\\ \dot{\varphi}=\omega=(1,\alpha).\end{gathered}$$

In this case, we have

$$\dot{x}_+=\rho+g+f(\varphi,\theta,x_+)-\partial_\omega h-\partial_{\tilde{\omega}}h-(\rho+g(\varphi))\frac{\partial h}{\partial x_+}+h.o.t.$$

where h.o.t. represents the high order terms of h and f, and for simplicity, we consider it as an error term and neglect it at this moment. Then we have the homological equation

$$f(\varphi,\theta,x)=\partial_\omega h+\partial_{\tilde{\omega}}h+(\rho+g(\varphi))\frac{\partial h}{\partial x}. \tag{3.10}$$

The following lemma gives the method to solve equation (3.10) by diagonally dominant operators. We should point out the solution we obtained in the following lemma is not an exact but approximate solution.

Lemma 3.2. *Let $\gamma',\gamma'',\tau',\tau''>0$, $\gamma:=\min\{\gamma',\gamma''\}$, $\tau:=\max\{\tau',\tau''\}$, $0<\tilde{U}<\infty$, $r_1>\delta_1>0$, $r_2>\delta_2>0$, $r_3>\delta_3>0$ with $\delta_1<\delta_2$, $\delta_1<\delta_3/2$, $(\omega,\tilde{\omega})\in WL(\gamma',\tau',\tilde{U})$, $\rho\in DC_{(\omega,\tilde{\omega})}(\gamma'',\tau'')$, $g\in C^\omega_{r_1}(\mathbb{T}^2,\mathbb{R})$, $f\in C^\omega_{r_1,r_2,r_3}(\mathbb{T}^{d+1},\mathbb{R})$ and $\int_{\mathbb{T}}\int_{\mathbb{T}^{d-2}}f(\varphi,\theta,x)d\theta dx=0$, $0<\epsilon_f\le\epsilon_g\le\varepsilon_0^{1/2}\ll 1$, where ε_0 satisfies (3.3) and (3.4). If*

$$\|g(\varphi)\|_{r_1}\le\epsilon_g,\|f(\varphi,\theta,x)\|_{r_1,r_2,r_3}\le\epsilon_f,$$

and

$$K=[\frac{1}{\pi\delta_1}\ln\frac{1}{\epsilon_f}]+1<\Big(\frac{\gamma^2}{\epsilon_g}\Big)^{\frac{1}{\tau+d+2}},$$

then the homological equation (3.10) has an approximate solution $h(\varphi,\theta,x)$ with the estimate

$$\|h\|_{r_1-\delta_1,r_2-\delta_2,r_3-\delta_3}\le\frac{C}{\gamma\delta_1^{\tau+d+1}}\epsilon_f,$$

where C is a constant, and the error term $P = \mathcal{R}_K(f - (\rho + g)\frac{\partial h}{\partial x})$ with

$$\|P\|_{r_1-\delta_1,r_2-\delta_2,r_3-\delta_3} \le 2(2K)^{d+1}\epsilon_f^2.$$

Proof. First, we consider the truncated equation

$$\partial_\omega h + \partial_{\tilde\omega} h + \rho\frac{\partial h}{\partial x} + \mathcal{T}_K\big(g(\varphi)\frac{\partial h}{\partial x}\big) = \mathcal{T}_K f(\varphi,\theta,x).$$

Let

$$\begin{gathered} f(\varphi,\theta,x) = \sum_l f_l(\varphi,\theta)e^{2\pi i l x}, \quad h(\varphi,\theta,x) = \sum_{|l|<K} h_l(\varphi,\theta)e^{2\pi i l x}, \\ f_l(\varphi,\theta) = \sum_\nu f_{l,\nu}(\varphi)e^{2\pi i\langle\nu,\theta\rangle}, \quad h_l(\varphi,\theta) = \sum_{0<|l|+|\nu|<K} h_{l,\nu}(\varphi)e^{2\pi i\langle\nu,\theta\rangle}, \\ f_{l,\nu}(\varphi) = \sum_k \hat f_{l,\nu}(k)e^{2\pi i\langle k,\varphi\rangle}, \quad h_{l,\nu}(\varphi) = \sum_{\substack{|k|<K-|l|-|\nu| \\ |l|+|\nu|\neq 0}} \hat h_{l,\nu}(k)e^{2\pi i\langle k,\varphi\rangle}. \end{gathered}$$

Then plugging the Fourier expansions into the truncated equation, and simplifying it, for $|l|+|\nu|+|k|<K$, $|l|+|\nu|>0$, we have

$$(\langle k,\omega\rangle + \langle\nu,\tilde\omega\rangle + l\rho)\hat h_{l,\nu}(k) + l\sum_{|k_1|<K-|l|-|\nu|} \hat g(k-k_1)\hat h_{l,\nu}(k_1) = \frac{\hat f_{l,\nu}(k)}{2\pi i}.$$

It equals the matrix equation, for fixed $0<|l|+|\nu|<K$,

$$(A_{l,\nu} + G_{l,\nu})\tilde h_{l,\nu} = \tilde f_{l,\nu},$$

where

$$\begin{gathered} A_{l,\nu} = \operatorname{diag}\{\langle k,\omega\rangle + \langle\nu,\tilde\omega\rangle + l\rho : |k| < K-|l|-|\nu|\}, \\ G_{l,\nu} = (l\hat g(p-q))_{|p|,|q|<K-|l|-|\nu|}, \\ \tilde f_{l,\nu} = (\frac{\hat f_{l,\nu}(k)}{2\pi i})^T_{|k|<K-|l|-|\nu|}, \\ \tilde h_{l,\nu} = (\hat h_{l,\nu}(k))^T_{|k|<K-|l|-|\nu|}. \end{gathered}$$

If we denote $M_{l,\nu} = \operatorname{diag}(\dots, e^{2\pi|k|(r_1-\frac{\delta_1}{2})}, \dots)_{|k|<K-|l|-|\nu|}$, then

$$M_{l,\nu}(A_{l,\nu} + G_{l,\nu})M_{l,\nu}^{-1}M_{l,\nu}\tilde h_{l,\nu} = M_{l,\nu}\tilde f_{l,\nu}.$$

For simplicity, we drop the subscripts l,ν temporarily and denote

$$A = A_{l,\nu}, \quad G = M_{l,\nu}G_{l,,\nu}M_{l,\nu}^{-1}, \quad \bar h = M_{l,\nu}\tilde h_{l,\nu}, \quad \bar f = M_{l,\nu}\tilde f_{l,\nu}.$$

Then, we have

$$A(I + A^{-1}G)\bar h = \bar f.$$

Obviously, to get the norm of $\bar h$, we need to estimate the norm of $(I + A^{-1}G)^{-1}$. We have

$$\|A^{-1}G\| \le \|A^{-1}\|\|G\|,$$

where $\|\cdot\|$ denotes the operator norm associated to the l^1 norm that satisfies $\|u\|_{l^1} = \sum_k |u_k|$ (indeed, if X is a matrix, $\|X\| = \max_j \sum_i |X_{ij}|$). Then by the condition $(\omega,\tilde\omega) \in WL(\gamma',\tau',\tilde U)$, $\rho \in DC_{(\omega,\tilde\omega)}(\gamma'',\tau'')$, we obtain

$$|l\rho + \langle k,\omega\rangle + \langle\nu,\tilde\omega\rangle| \ge \frac{\gamma}{(|k|+|l|+|\nu|+1)^\tau} \ge \frac{\gamma}{K^\tau}$$

for $|l|+|\nu|+|k|<K$ and $|l|+|\nu|\neq 0$. We obtain that

$$\begin{aligned}\|A^{-1}G\| &\le \max_{|k|<K-|l|-|\nu|}\frac{|l|\max_{|q|<K-|l|-|\nu|}\Sigma_{|p|<K-|l|-|\nu|}\|g\|_{r_1}e^{-\pi|p-q|\delta_1}}{|l\rho+\langle k,\omega\rangle+\langle\nu,\tilde\omega\rangle|}\\ &\le \frac{4K^{\tau+3}}{\gamma}\|g\|_{r_1}.\end{aligned}$$

Since $K<(\frac{\gamma^2}{\epsilon_g})^{\frac{1}{\tau+d+2}}$, we know that $\|A^{-1}G\|<\frac12$, and thus

$$\|(I+A^{-1}G)^{-1}\|<2.$$

From the result above, there is no doubt about the existence of $\bar h$. And it satisfies

$$\bar h=(I+A^{-1}G)^{-1}A^{-1}\bar f.$$

As for the estimate of $h(\varphi,\theta,x)$, we have

$$\begin{aligned}&\|h(\varphi,\theta,x)\|_{r_1-\frac{\delta_1}{2},r_2-\frac{\delta_2}{2},r_3-\frac{\delta_3}{4}}\\ &\le \sum_{0<|l|+|\nu|<K}\|\bar h_{l,\nu}\|_{l^1}e^{2\pi|\nu|(r_2-\frac{\delta_2}{2})}e^{2\pi|l|(r_3-\frac{\delta_3}{4})}\\ &= \sum_{0<|l|+|\nu|<K}\|(I+A^{-1}G)^{-1}A^{-1}\bar f_{l,\nu}\|_{l^1}e^{2\pi|\nu|(r_2-\frac{\delta_2}{2})}e^{2\pi|l|(r_3-\frac{\delta_3}{4})}\\ &\le \sum_{0<|l|+|\nu|+|k|<K}\frac{2(|k|+|l|+|\nu|+1)^\tau}{\gamma}\frac{|\hat f_{l,\nu}(k)|}{2\pi}e^{2\pi|k|(r_1-\frac{\delta_1}{2})}e^{2\pi|\nu|(r_2-\frac{\delta_2}{2})}e^{2\pi|l|(r_3-\frac{\delta_3}{4})}\\ &\le \frac{C\epsilon_f}{\gamma\delta_1^{\tau+d+1}},\end{aligned}$$

and the error term satisfies

$$\begin{aligned}&\|P\|_{r_1-\delta_1,r_2-\delta_2,r_3-\delta_3}\\ &\le \|\mathcal{R}_Kf(\varphi,\theta,x)\|_{r_1-\delta_1,r_2-\delta_2,r_3-\delta_3}+\|\mathcal{R}_K(g(\varphi)\frac{\partial h}{\partial x})\|_{r_1-\delta_1,r_2-\delta_2,r_3-\delta_3}\\ &\le (2K)^{d+1}e^{-2\pi K\delta_1/2}(\epsilon_f+\frac{C\epsilon_g}{\gamma\delta_1^{\tau+d+2}}\epsilon_f)\\ &< 2(2K)^{d+1}\epsilon_f^2.\end{aligned}$$

□

Using diagonal domination, we can solve the approximate homological equation (3.10), and the following lemma will apply it to reduce the perturbation.

Lemma 3.3. *Under the conditions of Lemma 3.1, if $(\omega,\tilde\omega)\in WL(\gamma',\tau',\tilde U)$ and $\rho\in DC_{(\omega,\tilde\omega)}(\gamma'',\tau'')$, there exists a transformation $\bar H\in C^\omega_{r^*_{1,n},r^*_{2,n},r^*_{3,n}}(\mathbb{T}^{d+1},\mathbb{T}^{d+1})$ where*

$$(r^*_{1,n},r^*_{2,n},r^*_{3,n})=(\frac{r_{1,0}}{2Q_n^3},r_{2,n-1}-\frac{\Lambda_n}{3},\bar r_{3,n}-\frac{\Delta_n}{3}),$$

with

$$\|\bar H-id\|_{r^*_{1,n},r^*_{2,n},r^*_{3,n}}\le 4\varepsilon_{n-1}^{3/4},$$

$$\|D(\bar H-id)\|_{r^*_{1,n},r^*_{2,n},r^*_{3,n}}\le 4\varepsilon_{n-1}^{5/8},$$

such that $\bar{H}$ conjugates the system (3.7) to

$$\begin{gathered}\dot{\bar{x}}_+ = \rho + g_*(\varphi) + f_*(\varphi,\theta,\bar{x}_+)\\ \dot{\theta} = \tilde{\omega}\\ \dot{\varphi} = \omega = (1,\alpha)\end{gathered} \tag{3.11}$$

where $\rho + g_* + f_* \in \mathcal{F}_{r^*_{1,n},r^*_{2,n},r^*_{3,n}}(\rho, 2\varepsilon_{n-1}^{1/2}, \varepsilon_n)$ and $\|g_* - \bar{g}\|_{r^*_{1,n}} \le 3\varepsilon_{n-1}$.

Proof. For simplicity, we drop the subscript n and denote temporarily $\bar{g}_0(\varphi) = \bar{g}(\varphi)$, $\bar{f}_0(\varphi,\theta,x) = \bar{f}(\varphi,\theta,x)$, $\tilde{\eta} = 2\varepsilon_{n-1}$, $\eta = 2\varepsilon_{n-1}^{1/2}$, $\tilde{r}_1 = \bar{r}_{1,n} = \frac{r_{1,0}}{Q_n^3}$, $\tilde{r}_2 = r_{2,n-1}$ $\tilde{r}_3 = \bar{r}_{3,n} = r_{3,n-1} - \frac{\Delta_n}{3}$. Now we define the following sequences inductively:

$$\begin{gathered}\tilde{r}_{1,0} = \tilde{r}_1, \quad \tilde{r}_{2,0} = \tilde{r}_2, \quad \tilde{r}_{3,0} = \tilde{r}_3,\\ \delta_{1,1} = \frac{\tilde{r}_{1,0}}{4}, \quad \delta_{1,j} = \frac{1}{2^{j-1}}\delta_{1,1}, \quad \tilde{r}_{1,j} = \tilde{r}_{1,j-1} - \delta_{1,j},\\ \delta_{2,1} = \frac{\Lambda_n}{6}, \quad \delta_{2,j} = \frac{1}{2^{j-1}}\delta_{2,1}, \quad \tilde{r}_{2,j} = \tilde{r}_{2,j-1} - \delta_{2,j},\\ \delta_{3,1} = \frac{\Delta_n}{6} \quad \delta_{3,j} = \frac{1}{2^{j-1}}\delta_{3,1}, \quad \tilde{r}_{3,j} = \tilde{r}_{3,j-1} - \delta_{3,j},\\ \tilde{\eta}_j = \tilde{\eta}^{(\frac{3}{2})^j}, \quad K_j = [\frac{1}{\pi\delta_{1,j}} \ln \frac{1}{\tilde{\eta}_{j-1}}] + 1.\end{gathered}$$

From the above definitions, we can assume that $\delta_{1,j} < \min\{\delta_{2,j}, \delta_{3,j}/2\}$.

Let

$$N = [\frac{(\ln 2Q_n)^{U+c(\tau+d)}}{32(\tau+d+2)\ln 3}] + 1.$$

Supposing for $\nu = 1,2,\dots,j-1 < N$, there are $f_\nu, h_\nu \in C^\omega_{\tilde{r}_{1,\nu},\tilde{r}_{2,\nu},\tilde{r}_{3,\nu}}(\mathbb{T}^{d+1},\mathbb{R})$, $g_\nu \in C^\omega_{\tilde{r}_{1,\nu-1}}(\mathbb{T}^2,\mathbb{R})$, such that the transformation $(\varphi,\theta,\bar{x}_{\nu-1}) = (\varphi,\theta,\bar{x}_\nu + h_\nu(\varphi,\theta,\bar{x}_\nu) \pmod 1)$ conjugates the system

$$\begin{gathered}\dot{\bar{x}}_{\nu-1} = \rho + \bar{g}_{\nu-1}(\varphi) + \bar{f}_{\nu-1}(\varphi,\theta,\bar{x}_{\nu-1})\\ \dot{\theta} = \tilde{\omega}\\ \dot{\varphi} = \omega = (1,\alpha)\end{gathered}$$

to

$$\begin{gathered}\dot{\bar{x}}_\nu = \rho + \bar{g}_\nu(\varphi) + \bar{f}_\nu(\varphi,\theta,\bar{x}_\nu)\\ \dot{\theta} = \tilde{\omega}\\ \dot{\varphi} = \omega = (1,\alpha)\end{gathered}$$

satisfying

$$\|\bar{f}_\nu\|_{\tilde{r}_{1,\nu},\tilde{r}_{2,\nu},\tilde{r}_{3,\nu}} \le \tilde{\eta}_\nu, \quad \|h_\nu\|_{\tilde{r}_{1,\nu},\tilde{r}_{2,\nu},\tilde{r}_{3,\nu}} \le \tilde{\eta}_{\nu-1}^{3/4},$$

and $\bar{g}_\nu = \bar{g}_{\nu-1} + \int_{\mathbb{T}^{d-2}\times\mathbb{T}^1} \bar{f}_{\nu-1}(\varphi,\theta,\bar{x}_\nu)d\theta d\bar{x}_\nu$. For $\nu = j$, under the similar transformation as above, $(\varphi,\theta,\bar{x}_{j-1}) = (\varphi,\theta,\bar{x}_j + h_j \pmod 1)$, the fibred equation becomes

$$\dot{\bar{x}}_j = \rho + \bar{g}_{j-1}(\varphi) + \bar{f}_{j-1}(\varphi,\theta,\bar{x}_j) - \partial_\omega h_j - \partial_{\tilde{\omega}} h_j - (\rho + \bar{g}_{j-1}(\varphi))\frac{\partial h_j}{\partial \bar{x}_j} + h.o.t.$$

where $h.o.t.$ is higher order terms about $\bar{f}_{j-1}$ and h_j. Then the homological equation is

$$
\begin{aligned}
&\partial_\omega h_j + \partial_{\tilde{\omega}} h_j + (\rho + \bar{g}_{j-1}(\varphi))\frac{\partial h_j}{\partial \bar{x}_j} \\
&= \bar{f}_{j-1}(\varphi,\theta,\bar{x}_j) - \int_{\mathbb{T}^{d-2}\times\mathbb{T}} \bar{f}_{j-1}(\varphi,\theta,\bar{x}_j)d\theta d\bar{x}_j.
\end{aligned}
\tag{3.12}
$$

First, we know that

$$
\|\bar{g}_{j-1}\|_{\tilde{r}_{1,j-2}} \le \|\bar{g}\|_{\tilde{r}_{1,0}} + \sum_{\nu=0}^{j-2}\tilde{\eta}_\nu < 2\varepsilon_{n-1}^{1/2}.
$$

Furthermore, from (3.2)-(3.4), for $j \le N$, we obtain

$$
\begin{aligned}
K_j &= \big[\frac{1}{\pi\delta_{1,j}}\ln\frac{1}{\tilde{\eta}_{j-1}}\big] + 1 < \frac{2\cdot 3^{j-1}Q_n^3}{r_{1,0}}\ln\frac{1}{\varepsilon_{n-1}} \\
&\le \frac{2Q_n^3}{r_{1,0}}e^{c_1(\ln 2Q_n)^{U+c(\tau+d)}}\ln\frac{1}{\varepsilon_{n-1}} < \frac{e^{2c_1(\ln 2Q_n)^{U+c(\tau+d)}}}{r_{1,0}}\ln\frac{1}{\varepsilon_{n-1}} \\
&< \frac{1}{r_{1,0}}\frac{1}{\varepsilon_{n-1}^{2c_1}}\Big(\frac{1}{\varepsilon_{n-1}}\Big)^{\frac{1}{12(\tau+d+2)}} < \Big(\frac{1}{\varepsilon_{n-1}}\Big)^{\frac{1}{6(\tau+d+2)}}(\frac{1}{\varepsilon_{n-1}})^{2c_1} \\
&< \Big(\frac{\gamma^2}{2\varepsilon_{n-1}^{1/2}}\Big)^{\frac{1}{\tau+d+2}},
\end{aligned}
\tag{3.13}
$$

where $c_1 = 1/\big(32(\tau+d+2)\big)$. Moreover, since $(\omega,\tilde{\omega}) \in WL(\gamma',\tau',\tilde{U})$ and $\rho \in DC_{(\omega,\tilde{\omega})}(\gamma'',\tau'')$, by Lemma 3.2, there exists an approximate solution of (3.12) satisfying

$$
\|h_j\|_{\tilde{r}_{1,j-1}-\delta_{1,j}/2,\tilde{r}_{2,j-1}-\delta_{2,j}/2,\tilde{r}_{3,j-1}-\delta_{3,j}/2} \le \frac{2^{\tau+d+1}C}{\gamma\delta_{1,j}^{\tau+d+1}}\tilde{\eta}_{j-1} < \tilde{\eta}_{j-1}^{3/4} < \frac{\delta_{3,j}}{2},
$$

and the error term satisfies

$$
\|P_j\|_{\tilde{r}_{1,j-1}-\delta_{1,j}/2,\tilde{r}_{2,j-1}-\delta_{2,j}/2,\tilde{r}_{3,j-1}-\delta_{3,j}/2} \le 2(2K_j)^{d+1}\tilde{\eta}_{j-1}^2 < \tilde{\eta}_{j-1}^{\frac{7}{4}}.
$$

As a consequence, the fibred equation becomes

$$
\dot{\bar{x}}_j = \rho + \bar{g}_j(\varphi) + \bar{f}_j(\varphi,\theta,\bar{x}_j),
$$

where $\bar{g}_j = \bar{g}_{j-1} + \int_{\mathbb{T}^{d-2}\times\mathbb{T}^1}\bar{f}_{j-1}(\varphi,\theta,\bar{x}_j)d\theta d\bar{x}_j$ and

$$
\begin{aligned}
&\bar{f}_j(\varphi,\theta,\bar{x}_j) + \bar{f}_j(\varphi,\theta,\bar{x}_j)\frac{\partial h_j}{\partial \bar{x}_j} \\
&= \bar{f}_{j-1}(\varphi,\theta,\bar{x}_j + h_j(\varphi,\theta,\bar{x}_j)) - \bar{f}_{j-1}(\varphi,\theta,\bar{x}_j) \\
&\quad + P_j - \frac{\partial h_j}{\partial \bar{x}_j}\cdot\int_{\mathbb{T}^d\times\mathbb{T}}\bar{f}_{j-1}(\varphi,\theta,\bar{x}_j)d\theta d\bar{x}_j.
\end{aligned}
$$

We have the estimate

$$
\begin{aligned}
&\|\bar{f}_{j-1}(\varphi,\theta,\bar{x}_j + h_j(\varphi,\theta,\bar{x}_j)) - \bar{f}_{j-1}(\varphi,\theta,\bar{x}_j)\|_{\tilde{r}_{1,j},\tilde{r}_{2,j},\tilde{r}_{3,j}} \\
&\le \|\frac{\partial \bar{f}_{j-1}}{\partial \bar{x}_j}\|_{\tilde{r}_{1,j},\tilde{r}_{2,j},\tilde{r}_{3,j}+\delta_{3,j}/2}\|h_j\|_{\tilde{r}_{1,j},\tilde{r}_{2,j},\tilde{r}_{3,j}} < \frac{\tilde{\eta}_{j-1}^{3/2}}{3}.
\end{aligned}
$$

Thus,

$$\|\bar{f}_j\|_{\tilde{r}_{1,j},\tilde{r}_{2,j},\tilde{r}_{3,j}} < 2\big(\frac{\tilde{\eta}_{j-1}^{3/2}}{3} + \tilde{\eta}_{j-1}^{\frac{7}{4}} + \tilde{\eta}_{j-1}\frac{\tilde{\eta}_{j-1}^{3/4}}{\delta_{3,j}/2}\big) < \tilde{\eta}_{j-1}^{3/2} = \tilde{\eta}_j.$$

By estimate (3.13), we know that we can do the above iterations until $j = N$. Now we give the estimation of $\bar{f}_N$ for $|\Im\varphi| \le \tilde{r}_{1,N}$, $|\Im\theta| \le \tilde{r}_{2,N}$, $|\Im\bar{x}_N| \le \tilde{r}_{3,N}$. In fact, under the Corollary 2.3, we have $\ln Q_{n+1} \le e^{(\ln Q_n)^U}$. Then

$$\begin{aligned}
\big(\frac{3}{2}\big)^N - 1 &\ge e^{(\ln 2Q_n)^{U+c(\tau+d)}\frac{c_1\ln\frac{3}{2}}{2\ln 3}} \\
&\ge e^{(\ln Q_n)^U(\ln 2Q_n)^{c(\tau+d)}\frac{c_1\ln\frac{3}{2}}{2\ln 3}} \\
&\ge (\ln Q_{n+1})^{(\ln 2Q_n)^{c(\tau+d)}\frac{c_1\ln\frac{3}{2}}{2\ln 3}},
\end{aligned}$$

and therefore,

$$\begin{aligned}
\|\bar{f}_N\|_{\tilde{r}_{1,N},\tilde{r}_{2,N},\tilde{r}_{3,N}} &\le \tilde{\eta}^{(3/2)^N} \le \tilde{\eta}e^{-((\frac{3}{2})^N-1)\ln\frac{1}{\tilde{\eta}}} \\
&\le \tilde{\eta}e^{-(\ln Q_{n+1})^{(\ln 2Q_n)^{c(\tau+d)}\frac{c_1\ln\frac{3}{2}}{2\ln 3}}\ln\frac{1}{\tilde{\eta}}} \\
&< \tilde{\eta}e^{-2(\ln Q_{n+1})^{U+c(\tau+d)}2^{U+c(\tau+d)}} \\
&< \frac{\tilde{\eta}}{2}\cdot e^{-(\ln 2Q_{n+1})^{U+c(\tau+d)}} = \varepsilon_n.
\end{aligned}$$

There is no harm in denoting $H_j(\varphi,\theta,\bar{x}_j) = (\varphi,\theta,\bar{x}_j + h_j(\varphi,\theta,\bar{x}_j) \pmod 1)$ for $1 \le j \le N$. Then let

$$\bar{H}_j(\varphi,\theta,\bar{x}_j) = H_1 \circ \cdots \circ H_{j-1} \circ H_j(\varphi,\theta,\bar{x}_j),$$

and $\bar{H}_j(\varphi,\theta,\bar{x}_j)$ is analytic for $(\varphi,\theta,\bar{x}_j) \in \mathbb{T}^{d+1}_{\tilde{r}_{1,j},\tilde{r}_{2,j},\tilde{r}_{3,j}+\frac{\delta_{3,j}}{2}}$ with

$$\|\partial_{\bar{x}_j}(\pi_3 \circ \bar{H}_j(\varphi,\theta,\bar{x}_j))\|_{\tilde{r}_{1,j},\tilde{r}_{2,j},\tilde{r}_{3,j}} \le \prod_{\nu=0}^{j-1}(1+\tilde{\eta}_\nu^{3/4}),$$

where $\pi_3 : \mathbb{T}^2 \times \mathbb{T}^{d-2} \times \mathbb{T}^1 \to \mathbb{T}^1$ is the natural projection to the third variable. Rewriting $\bar{x}_N =: \bar{x}_+$ and $\bar{H}_N(\varphi,\theta,\bar{x}_+) =: (\varphi,\theta,\bar{x}_+ + \tilde{h}(\varphi,\theta,\bar{x}_+) \pmod 1)$, we have

$$\begin{aligned}
\|\tilde{h}(\varphi,\theta,\bar{x}_+)\|_{\tilde{r}_{1,N},\tilde{r}_{2,N},\tilde{r}_{3,N}} &= \|\pi_3 \circ \bar{H}_N(\varphi,\theta,\bar{x}_+) - \bar{x}_+\|_{\tilde{r}_{1,N},\tilde{r}_{2,N},\tilde{r}_{3,N}} \\
&\le \|\pi_3 \circ (\bar{H}_N(\varphi,\theta,\bar{x}_+) - \bar{H}_{N-1}(\varphi,\theta,\bar{x}_+))\|_{\tilde{r}_{1,N},\tilde{r}_{2,N},\tilde{r}_{3,N}} \\
&\quad + \cdots + \|\pi_3 \circ \bar{H}_1(\varphi,\theta,\bar{x}_+) - \bar{x}_+\|_{\tilde{r}_{1,1},\tilde{r}_{2,1},\tilde{r}_{3,1}} \\
&\le \sum_{j=1}^{N}\prod_{\nu=0}^{j-2}(1+\tilde{\eta}_\nu^{3/4})\|h_j\|_{\tilde{r}_{1,j},\tilde{r}_{2,j},\tilde{r}_{3,j}} \\
&\le 4\varepsilon_{n-1}^{3/4},
\end{aligned}$$

and

$$\|D\tilde{h}\|_{\tilde{r}_{1,N}-\frac{\delta_{1,N}}{2},\tilde{r}_{2,N}-\frac{\delta_{2,N}}{2},\tilde{r}_{3,N}-\frac{\delta_{3,N}}{2}} \le 4\varepsilon_{n-1}^{\frac{5}{8}}.$$

In conclusion, letting $(g_*, f_*) = (\bar{g}_N, \bar{f}_N), \bar{H}(\varphi,\theta,\bar{x}_+) = (\varphi,\theta,\bar{x}_+ + \tilde{h}(\varphi,\theta,\bar{x}_+)$ (mod 1) together with

$$\|g_*(\varphi) - \bar{g}(\varphi)\|_{r^*_{1,n}} \le \sum_{j=0}^{N-1} \|\bar{f}_j\|_{\tilde{r}_{1,j},\tilde{r}_{2,j},\tilde{r}_{3,j}} < \sum_{j=0}^{N-1} \tilde{\eta}_j < 3\varepsilon_{n-1},$$

we obtain the result. □

3.3. **Do the inverse transformation of the first step.** Via the two transforming steps above, our system has been simplified to some degree. However, the transformation we made in the first step is not close to the identity. So we have to do another transformation in this step, in order to let the ultimate transformation be close to the identity.

Lemma 3.4. *Under the conditions of Lemma 3.3, there exists a transformation $\widetilde{H} \in C^\omega(\mathbb{T}^{d+1}, \mathbb{T}^{d+1})$, with*

$$\|\widetilde{H} - id\|_{r_{1,n},r_{2,n},r_{3,n}} \le 4\varepsilon_{n-1}^{3/4},$$
$$\|D(\widetilde{H} - id)\|_{r_{1,n},r_{2,n},r_{3,n}} \le 4\varepsilon_{n-1}^{1/2},$$

such that $\widetilde{H}$ conjugates the system (3.6) *to*

$$\begin{aligned}\dot{x}_+ &= \rho + g_+(\varphi) + f_+(\varphi,\theta,x_+)\\ \dot{\theta} &= \tilde{\omega}\\ \dot{\varphi} &= \omega\end{aligned}$$

with $\rho + g_+ + f_+ \in \mathcal{F}_{r_{1,n},r_{2,n},r_{3,n}}(\rho, 4\tilde{\varepsilon}_n, \varepsilon_n)$.

Proof. By Lemma 3.1 and 3.3, there exist $h \in C^\omega_{\tilde{r}_{1,n}}(\mathbb{T}^2,\mathbb{R})$ and $\bar{H} \in C^\omega_{r^*_{1,n},r^*_{2,n},r^*_{3,n}}(\mathbb{T}^{d+1},\mathbb{T}^{d+1})$ such that $H \circ \bar{H}$ conjugates the system (3.6) to (3.11), where $\partial_\omega h = \mathcal{T}_{Q_n} g - \hat{g}(0)$ and $H(\varphi,\theta,\bar{x}) = (\varphi,\theta,\bar{x} + h(\varphi) \pmod 1)$, $\bar{H}(\varphi,\theta,\bar{x}_+) = (\varphi,\theta,\bar{x}_+ + \tilde{h}(\varphi,\theta,\bar{x}_+) \pmod 1)$. Now let $(\varphi,\theta,\bar{x}_+) = (\varphi,\theta,x_+ - h(\varphi) \pmod 1) = H^{-1}(\varphi,\theta,x_+)$. Then by the transformation $(\varphi,\theta,x) = H \circ \bar{H} \circ H^{-1}(\varphi,\theta,x_+)$, the fibred equation becomes

$$\begin{aligned}\dot{x}_+ &= \rho + g_*(\varphi) + \partial_\omega h + f_*(\varphi,\theta,x_+ - h(\varphi))\\ &= \rho + g + g_*(\varphi) - \bar{g}(\varphi) + f_*(\varphi,\theta,x_+ - h(\varphi))\\ &= \rho + g_+(\varphi) + f_+(\varphi,\theta,x_+),\end{aligned}$$

where $g_+(\varphi) := g(\varphi) + g_*(\varphi) - \bar{g}(\varphi)$, $f_+(\varphi,\theta,x_+) = f_*(\varphi,\theta,x_+ - h(\varphi))$. Thus, $\|g_+\|_{r_{1,n}} \le 4\tilde{\varepsilon}_{n-1} + 4\varepsilon_{n-1} = 4\tilde{\varepsilon}_n$ and $\|f_+\|_{r_{1,n},r_{2,n},r_{3,n}} \le \|f_*\|_{r^*_{1,n},r^*_{2,n},r^*_{3,n}} \le \varepsilon_n$ by Lemmas 3.1 and 3.3. Since the transformation H^{-1} is homotopic to the identity, the fibred rotation number remains the same. That is,

$$\rho + g_+ + f_+ \in \mathcal{F}_{r_{1,n},r_{2,n},r_{3,n}}(\rho, 4\tilde{\varepsilon}_n, \varepsilon_n).$$

Let $\tilde{H} = H \circ \bar{H} \circ H^{-1}$. Then

$$\tilde{H}(\varphi,\theta,x_+) = (\varphi,\theta,x_+ + \tilde{h}(\varphi,\theta,x_+ - h(\varphi))),$$

and hence

$$\|\tilde{H} - id\|_{r_{1,n},r_{2,n},r_{3,n}} \le \|\tilde{h}(\varphi,\theta,\bar{x}_+)\|_{r^*_{1,n},r^*_{2,n},r^*_{3,n}} < 4\varepsilon_{n-1}^{3/4},$$

and

$$\|D(\tilde{H}-id)\|_{r_{1,n},r_{2,n},r_{3,n}} \le \|D\tilde{h}\|_{r^*_{1,n},r^*_{2,n},r^*_{3,n}} \|D(H^{-1})\|_{r_{1,n},r_{2,n},r_{3,n}} \le 4\varepsilon_{n-1}^{1/2}.$$

$\square$

3.4. **Iterative Lemma.** According to the above three lemmas, we combine them into the following iterative lemma.

Lemma 3.5. *For any $\varepsilon_0, r_{1,0}, r_{2,0}, r_{3,0}, \gamma', \gamma'', \tau', \tau'' > 0$, $\gamma := \min\{\gamma', \gamma'\}, \tau := \max\{\tau', \tau''\}$, $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ with $\tilde{U} = \tilde{U}(\alpha) < \infty$, the sequences $\varepsilon_n, \tilde{\varepsilon}_n, r_{1,n}, r_{2,n}, r_{3,n}$ are defined as in (3.5). If ε_0 satisfies (3.3) and (3.4), and $(\omega, \tilde{\omega}) \in WL(\gamma', \tau', \tilde{U})$, $\rho \in DC_{(\omega,\tilde{\omega})}(\gamma'', \tau'')$, then for all $n \ge 1$ the following holds: If the system*

$$\begin{aligned} \dot{x} &= \rho + g_n(\varphi) + f_n(\varphi, \theta, x) \\ \dot{\theta} &= \tilde{\omega} \\ \dot{\varphi} &= \omega = (1, \alpha) \end{aligned} \tag{3.14}$$

satisfies $\rho + g_n + f_n \in \mathcal{F}_{r_{1,n},r_{2,n},r_{3,n}}(\rho, 4\tilde{\varepsilon}_n, \varepsilon_n)$, then there exists a transformation $H_n : \mathbb{T}^{d+1} \to \mathbb{T}^{d+1}$ with estimates

$$\|H_n - id\|_{r_{1,n+1},r_{2,n+1},r_{3,n+1}} \le 4\varepsilon_n^{3/4},$$
$$\|D(H_n - id)\|_{r_{1,n+1},r_{2,n+1},r_{3,n+1}} \le 4\varepsilon_n^{1/2},$$

such that it transforms system(3.14) into

$$\begin{aligned} \dot{x} &= \rho + g_{n+1}(\varphi) + f_{n+1}(\varphi, \theta, x) \\ \dot{\theta} &= \tilde{\omega} \\ \dot{\varphi} &= \omega \end{aligned}$$

with $\rho + g_{n+1} + f_{n+1} \in \mathcal{F}_{r_{1,n+1},r_{2,n+1},r_{3,n+1}}(\rho, 4\tilde{\varepsilon}_{n+1}, \varepsilon_{n+1})$.

4. Proof of the main theorem

Let ε_0 small enough satisfying (3.3) and (3.4) with $\tau := \max\{\tau', \tau''\}$, $\gamma := \min\{\gamma', \gamma''\}$, $r_{1,0} := r_1$, $r_{2,0} := r_1$, $r_{3,0} := r_2$. For convenience, we rewrite the system $(\Omega, \rho + f)$ as

$$\begin{aligned} \dot{x} &= \rho_f + g(\varphi) + f(\varphi, \theta, x) \\ \dot{\theta} &= \tilde{\omega} \\ \dot{\varphi} &= \omega = (1, \alpha) \end{aligned} \tag{4.1}$$

with $\rho_f + g + f \in \mathcal{F}_{r_{1,0},r_{2,0},r_{3,0}}(\rho_f, \varepsilon_0, \varepsilon_0)$, where $g := \rho - \rho_f$, $\Omega = (\omega, \tilde{\omega})$, and $\phi = (\varphi, \theta) \in \mathbb{T}^2 \times \mathbb{T}^{d-2}$. Owing to the fact that $(\omega, \tilde{\omega}) \in WL(\gamma', \tau', \tilde{U})$ and $\rho_f \in DC_{(\omega,\tilde{\omega})}(\gamma'', \tau'')$, we can apply Lemma 3.3 and obtain the transformation $H_0 \in C^\omega_{r_{1,1},r_{2,1},r_{3,1}}(\mathbb{T}^{d+1}, \mathbb{T}^{d+1})$, such that system (4.1) can be conjugate to

$$\begin{aligned} \dot{x} &= \rho_f + g_1(\varphi) + f_1(\varphi, \theta, x) \\ \dot{\theta} &= \tilde{\omega} \\ \dot{\varphi} &= \omega \end{aligned}$$

with $\rho_f + g_1 + f_1 \in \mathcal{F}_{r_{1,1},r_{2,1},r_{3,1}}(\rho_f, 4\tilde{\varepsilon}_1, \varepsilon_1)$ and $\|H_0 - id\|_{r_{1,1},r_{2,1},r_{3,1}} \le 4\varepsilon_0^{3/4}$, $\|D(H_0 - id)\|_{r_{1,1},r_{2,1},r_{3,1}} \le 4\varepsilon_0^{1/2}$. Now, by Lemma 3.5, we obtain the sequence of transformations $H_j \in C^\omega_{r_{1,j+1},r_{2,j+1},r_{3,j+1}}(\mathbb{T}^{d+1}, \mathbb{T}^{d+1})$ $(j = 1, \dots, n-1)$ such that

$H^{(n)} := H_0 \circ H_1 \circ \cdots \circ H_{n-1}$ conjugates (4.1) to $(\omega, \tilde{\omega}, \rho_f + g_n(\varphi) + f_n(\varphi, \theta, x))$, with $\|H_j - id\|_{r_{1,j+1}, r_{2,j+1}, r_{3,j+1}} \le 4\varepsilon_j^{3/4}$, $\|D(H_j - id)\|_{r_{1,j+1}, r_{2,j+1}, r_{3,j+1}} \le 4\varepsilon_j^{1/2}$. Then for $H^{(n)}$, we have

$$\begin{aligned}
&\|DH^{(n)}\|_{r_{1,n}, r_{2,n}, r_{3,n}} \\
&\le \|DH_0\|_{r_{1,1}, r_{2,1}, r_{3,1}} \|DH_1\|_{r_{1,2}, r_{2,2}, r_{3,2}} \dots \|DH_{n-1}\|_{r_{1,n}, r_{2,n}, r_{3,n}} \\
&\le \Pi_{j=0}^{n-1} (1 + 4\varepsilon_j^{1/2}) < 2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\|H^{(n+1)} - H^{(n)}\|_{r_{1,n+1}, r_{2,n+1}, r_{3,n+1}} \\
&\le \|DH^{(n)}\|_{r_{1,n}, r_{2,n}, r_{3,n}} \|H_n - id\|_{r_{1,n+1}, r_{2,n+1}, r_{3,n+1}} \le 8\varepsilon_n^{3/4},
\end{aligned}$$

which means the limit $H^{(n)}$ exists in C^0, and we denote $H = \lim_{n\to\infty} H^{(n)}$, $g_\infty = \lim_{n\to\infty} g_n$. To prove the transformation H is actually in C^∞, we have to prove $\frac{\partial^{|m|} H}{\partial \zeta^m}$ exists for all $m \in \mathbb{N}^{d+1}$ (denoting $\zeta = (\varphi, \theta, x)$). Let $r_* = \min\{r_{1,0}, r_{2,0}, r_{3,0}\}$. Actually, for any $m \in \mathbb{N}^{d+1}$, there exists $n_m \in \mathbb{N}$, so that if $n \ge n_m$, then we have $(\frac{4Q_n^3}{r_*})^{|m|} < \varepsilon_{n-1}^{-\frac{1}{4}}$, that is

$$(\frac{4Q_n^3}{r_*})^{|m|} \varepsilon_{n-1}^{3/4} < \varepsilon_{n-1}^{1/2}, \quad \forall n \ge n_m.$$

Meanwhile, we have the following inequality, for $n \ge n_m - 1$,

$$\begin{aligned}
|\frac{\partial^{|m|}(H^{(n+1)} - H^{(n)})}{\partial \zeta^m}| &\le \frac{\|H^{(n+1)} - H^{(n)}\|_{r_{1,n+1}, r_{2,n+1}, r_{3,n+1}}}{r_{1,n+1}^{|m|}} \\
&\le 8\Big(\frac{4Q_{n+1}^3}{r_*}\Big)^{|m|} \varepsilon_n^{3/4} < 8\varepsilon_n^{1/2}.
\end{aligned}$$

In conclusion, the transformation H is actually in C^∞, and under this transformation, the system(4.1) is conjugate to

$$\begin{aligned}
\dot{x} &= \rho_f + g_\infty(\varphi), \\
\dot{\theta} &= \tilde{\omega}, \\
\dot{\varphi} &= \omega,
\end{aligned}$$

which can be rewritten as $(\Omega, \rho_f + g_\infty(\varphi))$. The result follows.

Acknowledgements. This work was partially supported by The Fundamental Research Funds for the Central Universities, No. 30918011336. Y. Shan was supported by the Natural Science Foundation of Jiangsu Province(Grant No. BK20161053), and the National Science Foundation of China(Grant No. 11701285). Z. Liang, T. Jin and J.Y. Wang were supported by the National research project at the Nanjing University of Science and Technology. The authors would like to thank Jing Wang for inspiring and stimulating discussions.

References

[1] A. Avila, B. Fayad, R. Krikorian; *A KAM scheme for $SL(2,\mathbb{R})$ cocycles with Liouvillean frequencies*, Geom. Funct. Anal., 21(5) (2011), 1001-1019.

[2] E. I. Dinaburg, Y. G. Sinai; *The one-dimensional Schrödinger equation with quasiperiodic potential*, Funk. Anal. i Prilozen., 9(4) (1975), 279-289.

[3] H. Eliasson; *Floquet solutions for the 1-dimensional quasi-periodic Schrödinger equation*, Comm. Math. Phys., 146(3) (1992), 447-482.

[4] C. Haile, *A second eigenvalue bound for the Dirichlet Schrodinger equation wtih a radially symmetric potential*, Electron. J. Differential Equations, 2000 (2000), no. 10, 1-19.

[5] M. Herman; *Sur la conjugaison diffŕentiable des difféomorphismes du cercle à des rotations*, Publ. Math. Inst. Hautes Etudes Sci., 49 (1979), 5-233.

[6] M. R. Herman; *Une methode pour minorer les exposants de Lyapunov et quelques exemples montrant le caractere local d'un theoreme d'Arnold et de Moser sur le tore de dimension 2*, Comment. Math. Helv., 58(3) (1983), 453-502.

[7] X. Hou, J. You; *Almost reducibility and non-perturbative reducibility of quasi-periodic linear systems*, Invent. Math., 190(1) (2012), 209-260.

[8] X. Hou, J. Wang, Q. Zhou; *Absolutely Continuous Spectrum of Multifrequency Quasiperiodic Schrödinger operator*, preprint.

[9] R. Krikorian, J. Wang, J. You, Q. Zhou; *Linearization of quasiperiodically forced circle flow beyond Brjuno condition*, Comm. Math. Phys., 358(1) (2018), 81-100.

[10] J. You, Q. Zhou; *Embedding of analytic quasi-periodic cocycles into analytic quasi-periodic linear systems and its applications*, Comm. Math. Phys., 323(3) (2013), 975-1005.

Ziyang Liang
Department of Mathematics, School of Science, Nanjing University of Science and Technology, Nanjing, 210094, China
Email address: lianggzy98@163.com

Taian Jin
Department of Mathematics, School of Science, Nanjing University of Science and Technology, Nanjing, 210094, China
Email address: 1056418286@qq.com

Jiayi Wang
Department of Mathematics, School of Science, Nanjing University of Science and Technology, Nanjing, 210094, China
Email address: 13236582927@163.com

Yuan Shan (corresponding author)
School of Statistics and Mathematics, Nanjing Audit University, Nanjing 210029, China
Email address: shannjnu@gmail.com