SUB-SUPER SOLUTION METHOD FOR NONLOCAL SYSTEMS INVOLVING THE $p(x)$-LAPLACIAN OPERATOR

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Abstract. In this article we study the existence of solutions for nonlocal systems involving the $p(x)$-Laplacian operator. The approach is based on a new sub-super solution method.

1. Introduction

In this work we are interested in the nonlocal system

\[
\begin{align*}
-\mathcal{A}(|v|_{L^{r_1}(x)})\Delta p_1(x)u & = f_1(x, u, v)|v|^{\alpha_1(x)}_{L^{\infty}(x)} + g_1(x, u, v)|v|^{\gamma_1(x)}_{L^{\infty}(x)} \quad \text{in } \Omega, \\
-\mathcal{A}(|u|_{L^{r_2}(x)})\Delta p_2(x)v & = f_2(x, u, v)|u|^{\alpha_2(x)}_{L^{\infty}(x)} + g_2(x, u, v)|u|^{\gamma_2(x)}_{L^{\infty}(x)} \quad \text{in } \Omega, \\
\quad u = v = 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

(1.1)

where $\Omega$ is a bounded domain in $\mathbb{R}^N (N > 1)$ with $C^2$ boundary, $|\cdot|_{L^{m(x)}}$ is the norm of the space $L^{m(x)}(\Omega)$, $-\Delta p(x)u := -\text{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacian operator, $r_i, p_i, q_i, s_i, \alpha_i, \gamma_i : \Omega \to [0, \infty), i = 1, 2$ are measurable functions and $\mathcal{A}, f_1, f_2, g_1, g_2 : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying certain conditions.

In the previous decades there have been several works related to the $p$ and $p(x)$-Laplacian operator; see for example [1, 4, 9, 12, 25, 26, 27, 28, 29, 34, 35, 38, 39] and the references therein. Partial differential equations involving the $p(x)$-Laplacian arise in several areas of Science and Technology such as nonlinear elasticity, fluid mechanics, non-Newtonian fluids and image processing. Regarding the mentioned applications we point out [1, 14] [26] [41] [42].

The nonlocal term $|\cdot|_{L^{m(x)}}$ with the condition $p(x) = r(x) \equiv 2$ was considered in the well known Carrier’s equation

\[ \rho u_{tt} - a(x, t, |u|_{L^2}^2)\Delta u = 0 \]

which models the vibrations of a elastic string under certain conditions. See [11] for more details. We also quote the applicability of such nonlocal term in Population Dynamics, see [15] [17]. Several works related to (1.1) in the $p$-Laplacian case, that is, with $p(x) = p$ (a constant) can be found, see [10, 13, 19, 20, 23, 33] and the references provided in such manuscripts. For example Corrêa & Lopes [20] studied the system

\[ -\Delta u^m = a|v|^{p}_{L^p} \quad \text{in } \Omega, \]

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and in [13] a related system was considered using the Galerkin method. In [19] the authors used a theorem due to Rabinowitz [40] to study the problem

\[-\Delta v^n = b|u|^{\frac{\beta}{2}}_{L^p} \quad \text{in } \Omega, \quad u = v = 0 \quad \text{on } \partial \Omega,
\]

The system

\[-A(x, |v|_{L^{\gamma_1}(x)}) \Delta u = f_1(x, u, v)|v|^{\gamma_1(x)}_{L^{\gamma_1}(x)} + g_1(x, u, v)|v|_{L^{\gamma_2}(x)} \quad \text{in } \Omega,
\]

\[-A(x, |u|_{L^{\gamma_2}(x)}) \Delta u = f_2(x, u, v)|u|^{\gamma_2(x)}_{L^{\gamma_2}(x)} + g_2(x, u, v)|u|_{L^{\gamma_2}(x)} \quad \text{in } \Omega,
\]

\[u = v = 0 \quad \text{on } \partial \Omega,
\]

where $A : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function satisfying some conditions, was considered in [43]. The approach in such paper consists in use an abstract result involving sub and supersolutions, whose proof is based on the Schaefer’s fixed point theorem. Specifically, it was considered a sublinear system, a concave-convex problem and a system of logistic equations.

The scalar version of (1.1),

\[-A(x, |u|_{L^{\gamma}(x)}) \Delta p(x) u = f(x, u)|u|^{\alpha(x)}_{L^{\gamma}(x)} + g(x, u)|u|^{\gamma(x)}_{L^{\gamma}(x)} \quad \text{in } \Omega,
\]

\[u = 0 \quad \text{on } \partial \Omega,
\]

was considered in [44]. The authors obtained an abstract result involving sub and super solutions for (1.1) that generalizes [43, Theorem 1]. As an application of such result the authors generalized for the $p(x)$-Laplacian operator the three applications of [43, Theorem 1].

The goal of this work is to prove [43, Theorem 2] for the $p(x)$-Laplacian operator and use it in three applications of the mentioned paper. Thus, we provide a generalization of [43] with respect to systems with variable exponents. Next we describe the main differences and difficulties of this work when compared with [43].

(i) The homogeneity of the Laplacian operator $(-\Delta, H^1_0(\Omega))$ and the eigenfunction associated to the first eigenvalue were used in [43] for constructing a subsolution. Differently from the $p$-Laplacian ($p(x) \equiv p$ constant) the $p(x)$-Laplacian is not homogeneous. Besides that, it can occurs that the first eigenvalue and the first eigenfunction of the $p(x)$-Laplacian operator $(-\Delta_{p(x)}, W^{1,p(x)}_{0}(\Omega))$ do not exist. Even if the first eigenvalue and the associated eigenfunction exist the homogeneity, in general, does not allows to use the first eigenfunction to construct a subsolution. In order to avoid such difficulties we explore some arguments of [43].

(ii) Some arguments of [43] were improved and weaker conditions on $r_1, q_i, s_i, \alpha_i, \gamma_i, i = 1, 2$ are considered here.

(iii) We generalize [43, Theorem 2] and as an application it is considered some nonlocal problems that generalizes the three systems studied in [43].

(iv) As in [43, Theorem 2] and differently from several works that consider the nonlocal term $A(x, |u|_{L^{\gamma}(x)})$ satisfying $A(x, t) \geq a_0 > 0$ (where $a_0$ is a constant), Theorem 1.1 permits us to study (1.1) in the mentioned case and in situations where $A(x, 0) = 0$. 
(v) The abstract result involving sub and super solutions is proved by using a different argument. It is used a theorem due to Rabinowitz that can be found in [43] and some arguments of [43] are improved.

In this work we assume that \( r_i, p_i, q_i, s_i, \alpha_i, \gamma_i \) satisfy

\[
(\text{H1}) \quad p_i \in C^1(\Omega), \quad r_i, q_i, s_i \in L_+^\infty(\Omega), \quad \text{where} \\
L_+^\infty(\Omega) = \{ m \in L^\infty(\Omega) \text{ with } \text{ess inf } m(x) \geq 1 \}
\]

and for \( i = 1, 2, \alpha_i, \gamma_i \in L^\infty(\Omega) \) and satisfy

\[
1 < p_i^- := \inf \frac{1}{p_i} < p_i^+ := \sup \frac{1}{p_i} < N, \quad \alpha_i(x), \gamma_i(x) \geq 0 \quad \text{a.e. in } \Omega.
\]

Some definitions are needed to present the main results. We say that the pair \((u_1, u_2)\) is a weak solution of (1.1), if \( u_i \in W^{1,p_i}(\Omega) \cap L^\infty(\Omega) \) and

\[
\int_\Omega \nabla u_i |p_i(x) - 2 \nabla u_i \nabla \varphi = \int_\Omega \left( \frac{f_i(x, u_1, u_2)|u_j|^{\alpha_i(x)}_{L^\infty(\Omega)}}{A(x, |u_j|_{L^2(\Omega)})} + \frac{g_i(x, u_1, u_2)|u_j|^{\gamma_i(x)}}{A(x, |u_j|_{L^2(\Omega)})} \right) \varphi,
\]

for all \( \varphi \in W^{1,p_i}(\Omega) \) and \( i \neq j \) with \( i, j = 1, 2 \). Given \( u, v \in S(\Omega) \) we write \( u \leq v \) if \( u(x) \leq v(x) \) a.e. in \( \Omega \). If \( u \leq v \) we define

\[
[u, v] := \{ w \in S(\Omega) : u(x) \leq w(x) \leq v(x) \text{ a.e. in } \Omega \}.
\]

To simplify the next definition we denote

\[
\tilde{f}_i(x, t, s) = f_i(x, t, s), \quad \tilde{g}_i(x, t, s) = g_i(x, t, s),
\]

We say that the pairs \((u_i, \overline{u}_i), i = 1, 2\) are a sub-super solutions for (1.1) if \( u_i \in W^{1,p_i}(\Omega) \cap L^\infty(\Omega) \), \( \overline{u}_i \in W^{1,p_i}(\Omega) \cap L^\infty(\Omega) \) with \( u_1 \leq \overline{u}_1, u_2 = 0 \leq \overline{u}_2 \) on \( \partial \Omega \) and for all \( \varphi \in W^{1,p_i}(\Omega) \) with \( \varphi \geq 0 \) the following inequalities hold

\[
\int_\Omega |\nabla u_i|^{p_i(x)} - 2 \nabla u_i \nabla \varphi \leq \int_\Omega \left( \frac{\tilde{f}_i(x, u_1, u_2)|u_j|^{\alpha_i(x)}_{L^\infty(\Omega)}}{A(x, |u_j|_{L^2(\Omega)})} + \frac{\tilde{g}_i(x, u_1, u_2)|u_j|^{\gamma_i(x)}}{A(x, |u_j|_{L^2(\Omega)})} \right) \varphi,
\]

for all \( w \in [u_j, \overline{u}_j] \) where \( i, j = 1, 2 \) with \( i \neq j \). Our main result reads as follows.

**Theorem 1.1.** Suppose that \( r_i, p_i, q_i, s_i, \alpha_i \) and \( \gamma_i \) satisfy (H1), that \((u_i, \overline{u}_i)\) is a sub-super solution for (1.1) with \( u_i > 0 \) a.e. in \( \Omega \), that \( f_i(x, t, s), g_i(x, t, s) \geq 0 \) in \( \overline{\Omega} \times [0, |\overline{u}_i|_{L^\infty}] \times [0, |\overline{u}_i|_{L^\infty}] \) and that \( A : \overline{\Omega} \times (0, \infty) \to \mathbb{R} \) is a continuous function with \( A(x, t) > 0 \) in \( \overline{\Omega} \times \overline{\mathbb{R}} \) where \( \overline{\mathbb{R}} := \min \{ |\overline{u}_i|_{L^\infty}, i = 1, 2 \}, \overline{\mathbb{R}} := \max \{ |\overline{u}_i|_{L^\infty}, i = 1, 2 \}, \overline{\mathbb{R}} := \overline{\min \{ |\overline{u}_i|_{L^\infty}, i = 1, 2 \}} \text{ and } \overline{\mathbb{R}} := \min \{ |\overline{u}_i|_{L^\infty}, i = 1, 2 \}. \) Then (1.1) has a weak positive solution \((u_1, u_2)\) with \( u_i \in [u_j, \overline{u}_j], i = 1, 2 \).

2. Preliminaries

In this section, we present some facts regarding the spaces \( L^{p(x)}(\Omega), W^{1,p(x)}(\Omega) \) and \( W^{1,p(x)}_0(\Omega) \) that will be often used in this work. For more details see Fan-Zhang [27] and the references therein.
Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a bounded domain. Given $p \in L^\infty_+(\Omega)$, we define the generalized Lebesgue space

$$L^{p(x)}(\Omega) = \{ u \in \mathcal{S}(\Omega) : \int_\Omega |u(x)|^{p(x)}\,dx < \infty \},$$

where $\mathcal{S}(\Omega) := \{ u : \Omega \to \mathbb{R} : u$ is measurable $\}$. Then $L^{p(x)}(\Omega)$ is a Banach space with the norm

$$|u|_{p(x)} := \inf \{ \lambda > 0 : \int_\Omega \frac{|u(x)|}{\lambda} d\mu \leq 1 \}.$$

Given $m \in L^\infty(\Omega)$, we define

$$m^+ := \text{ess sup}_\Omega m(x), \quad m^- := \text{ess inf}_\Omega m(x).$$

**Proposition 2.1.** Let $\rho(u) := \int_\Omega |u|^{p(x)}\,dx$. Then for $u, u_n \in L^{p(x)}(\Omega)$, and $n \in \mathbb{N}$, the following assertions hold

(i) Let $u \neq 0$ in $L^{p(x)}(\Omega)$, then $|u|_{L^{p(x)}} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1$.

(ii) If $|u|_{L^{p(x)}} < 1$ ($= 1$, $> 1$), then $\rho(u) < 1$ ($= 1$, $> 1$).

(iii) If $|u|_{L^{p(x)}} > 1$, then $|u|^p_{L^{p(x)}} \leq \rho(u) \leq |u|_{L^{p(x)}}^p$.

(iv) If $|u|_{L^{p(x)}} < 1$, then $|u|^p_{L^{p(x)}} \leq \rho(u) \leq |u|_{L^{p(x)}}^p$.

(v) $|u_n|_{L^{p(x)}} \to 0 \Leftrightarrow \rho(u_n) \to 0$, and $|u_n|_{L^{p(x)}} \to \infty \Leftrightarrow \rho(u_n) \to \infty$.

**Theorem 2.2.** Let $p, q \in L^\infty_+(\Omega)$. Then the following statements hold

(i) If $p^{-} > 1$ and $\frac{1}{q(x)} = \frac{1}{p(x)} = 1$ a.e. in $\Omega$, then

$$\left| \int_\Omega u v dx \right| \leq \left( \frac{1}{p} + \frac{1}{q} \right) |u|_{L^{p(x)}} |v|_{L^{q(x)}}.$$

(ii) If $q(x) \leq p(x)$ a.e. in $\Omega$ and $|\Omega| < \infty$, then $L^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

We define the generalized Sobolev space as

$$W^{1,p(x)}(\Omega) := \{ u \in L^{p(x)}(\Omega) : \frac{\partial u}{\partial x_j} \in L^{p(x)}(\Omega), j = 1, \ldots, N \}$$

with the norm

$$\| u \|_* = |u|_{L^{p(x)}} + \sum_{j=1}^N |\frac{\partial u}{\partial x_j}|_{L^{p(x)}}.$$

The space $W^{1,p(x)}_0(\Omega)$ is defined as the closure of $C^\infty_0(\Omega)$ with respect to the norm $\| \cdot \|_*$.

**Theorem 2.3.** If $p^{-} > 1$, then $W^{1,p(x)}(\Omega)$ is a Banach, separable and reflexive space.

**Proposition 2.4.** Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $p, q \in C(\overline{\Omega})$. Define the function $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ and $p^*(x) = \infty$ if $N \geq p(x)$. Then the following statements hold.

(i) (Poincaré inequality) If $p^{-} > 1$, then there is a constant $C > 0$ such that

$$|u|_{L^{p(x)}} \leq C|\nabla u|_{L^{p(x)}}$$

for all $u \in W^{1,p(x)}_0(\Omega)$.

(ii) If $p^{-}, q^{-} > 1$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding

$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$

is continuous and compact.

From (i) of Proposition 2.4 we have that $\| u \| := |\nabla u|_{L^{p(x)}}$ defines a norm in $W^{1,p(x)}_0(\Omega)$ which is equivalent to the norm $\| \cdot \|_*$. 
Definition 2.5. For \( u, v \in W^{1,p(x)}(\Omega) \), we say that \(-\Delta_{p(x)}u \leq -\Delta_{p(x)}v \), if
\[
\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla \varphi \leq \int_\Omega |\nabla v|^{p(x)-2}\nabla v \nabla \varphi,
\]
for all \( \varphi \in W_{0}^{1,p(x)}(\Omega) \) with \( \varphi \geq 0 \).

The following result appears in [29] Lemma 2.2 and [26] Proposition 2.3.

Proposition 2.6. Let \( u, v \in W^{1,p(x)}(\Omega) \). If \(-\Delta_{p(x)}u \leq -\Delta_{p(x)}v \) and \( u \leq v \) on \( \partial \Omega \), (i.e., \((u-v)^+ \in W_{0}^{1,p(x)}(\Omega)\)) then \( u \leq v \) in \( \Omega \). If \( u, v \in C(\bar{\Omega}) \) and \( S = \{ x \in \Omega : u(x) = v(x) \} \) is a compact set of \( \Omega \), then \( S = \emptyset \).

Lemma 2.7 ([26] Lemma 2.1). Let \( \lambda > 0 \) be the unique solution of the problem
\[
-\Delta_{p(x)}z \lambda = \lambda \quad \text{in } \Omega,
\]
\[
u = 0 \quad \text{on } \partial \Omega.
\]
Define \( \rho_0 = \frac{\rho^-}{2|\Omega| + C_0} \). If \( \lambda \geq \rho_0 \) then \(|z_\lambda|_{L^\infty} \leq C\lambda^{\frac{1}{p^*-1}} \), and \(|z_\lambda|_{L^\infty} \leq C\lambda^{\frac{1}{p^*-1}} \)
if \( \lambda < \rho_0 \). Here \( C^* \) and \( C_* \) are positive constants depending only on \( p^+, p^-, N, |\Omega| \) and \( C_0 \), where \( C_0 \) is the best constant of the embedding \( W_0^{1,1}(\Omega) \hookrightarrow L^{\infty}(\Omega) \).

Regarding the function \( z_\lambda \) of the previous result, it follows from [25] Theorem 1.2 and [29] Theorem 1) that \( z_\lambda \in C^1(\bar{\Omega}) \) with \( z_\lambda > 0 \) in \( \Omega \). The proof of Theorem 1.1 is mainly based on the following result by Rabinowitz:

Theorem 2.8 ([30]). Let \( E \) be a Banach space and \( \Phi : \mathbb{R}^+ \times E \to E \) a compact map such that \( \Phi(0,u) = 0 \) for all \( u \in E \). Then the equation
\[
u = \Phi(\lambda, u)
\]
possesses an unbounded continuum \( C \subset \mathbb{R}^+ \times E \) of solutions with \((0,0) \in C\).

We point out that a mapping \( \Phi : E \to E \) is compact if it is continuous and for each bounded subset \( U \subset E \), the set \( \Phi(U) \) is compact.

3. Proof of Main Results

Proof of Theorem 1.1. For \( i = 1, 2 \) consider the operators \( T_i : L^{p_i}(\Omega) \to L^\infty(\Omega) \) defined by
\[
T_i z(x) = \begin{cases} u_i(x), & \text{if } z(x) \leq u_i(x), \\ z(x), & \text{if } u_i(x) \leq z(x) \leq \bar{u}_i(x), \\ \bar{u}_i(x), & \text{if } z(x) \geq \bar{u}_i(x). \end{cases}
\]
Since \( T_i z \in [u_i, \bar{u}_i] \) and \( u_i, \bar{u}_i \in L^\infty(\Omega) \) it follows that the operators \( T_i \) are well-defined.

We define \( p'_i(x) = p_i(x)/(p_i(x) - 1) \) and consider the operators \( H_i : [u_1, \bar{u}_1] \times [u_2, \bar{u}_2] \to L^{p'_i}(\Omega) \) given by
\[
H_i(u_1, u_2)(x) = \frac{f_i(x, u_1(x), u_2(x))|u_j|_{L^{p'_i}(\Omega)}}{A(x, |u_j|_{L^{p'_i}(\Omega)})} + \frac{g_i(x, u_1(x), u_2(x))|u_j|_{L^{p'_i}(\Omega)}}{A(x, |u_j|_{L^{p'_i}(\Omega)})},
\]
where \( i \neq j \) with \( i, j = 1, 2 \), and \( | \cdot |_{L^{m}(\Omega)} \) denotes the norm of the space \( L^{m}(\Omega) \).

We consider in the space \( L^{p_1}(\Omega) \times L^{p_2}(\Omega) \) with the norm
\[
|\langle u, v \rangle|_{1,2} = |u|_{L^{p_1}} + |v|_{L^{p_2}}.
\]
Since \( f_i, g_i, \mathcal{A} \) are continuous functions, \( \mathcal{A}(x, t) > 0 \) in the compact set \( \overline{\Omega} \times [\sigma, \bar{\sigma}] \), \( T_i z_i \in [\underline{u}_i, \Pi_i] \) for all \( z_i \in L^{p_i}(\Omega) \), \( \underline{u}_i, \Pi_i \in L^\infty(\Omega) \), and \( |u_n^{\beta(x)}| \leq |w_n^{\beta(x)}| + |u_n^{\beta(x)}| \) for all \( w \in L^{m(x)}(\Omega) \) with \( \beta \in L^\infty(\Omega) \), it follows that there are constants \( K_i > 0 \) such that

\[
|H_i(T_1 z_1, T_2 z_2)| \leq K_i
\]

(3.1)

for all \( (z_1, z_2) \in L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega) \).

By the Lebesgue Dominated Convergence Theorem, the mappings \( (z_1, z_2) \mapsto H_i(T_1 z_1, T_2 z_2) \) are continuous from \( L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega) \) to \( L^{p_i(x)}(\Omega) \), \( i = 1, 2 \).

From [27, Theorem 4.1] the operator \( \Phi : \mathbb{R}^+ \times L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega) \mapsto L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega) \) given by

\[
\Phi(\lambda, z_1, z_2) = (u_1, u_2),
\]

where \( (u_1, u_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega) \) is the unique solution of

\[
\begin{align*}
-\Delta_{p_1(x)} u_1 &= \lambda H_i(T_1 z_1, T_2 z_2) \quad \text{in } \Omega, \\
-\Delta_{p_2(x)} u_2 &= \lambda H_i(T_1 z_1, T_2 z_2) \quad \text{in } \Omega, \\
u &= v = 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

(3.2)

is well-defined.

**Claim 1: \( \Phi \) is compact.** Let \( (\lambda_n, z_1^n, z_2^n) \subset \mathbb{R}^+ \times L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega) \) be a bounded sequence and consider \( (u_1^n, u_2^n) = \Phi(\lambda_n, z_1^n, z_2^n) \). The definition of \( \Phi \) imply that

\[
\int_{\Omega} |\nabla u_n^{1,p_i}|^{-2} \nabla u_n \nabla \varphi = \lambda_n \int_{\Omega} H_i(T_1 z_1^n, T_2 z_2^n) \varphi, \quad \forall \varphi \in W_0^{1,p_i}(\Omega),
\]

where \( i, j = 1, 2 \) blue with \( i \neq j \).

Consider the test function \( \varphi = u_n \), the boundness of \( (\lambda_n) \) and inequality (3.1), we obtain

\[
\int_{\Omega} |\nabla u_n^{1,p_i}| \leq \bar{\lambda} K_i \int_{\Omega} |u_n^{1,p_i}|,
\]

for all \( n \in \mathbb{N} \). Here \( \bar{\lambda} \) is a constant that does not depend on \( n \in \mathbb{N} \).

Since \( p_i > 1 \), the embedding \( L^{p_i(x)}(\Omega) \hookrightarrow L^1(\Omega) \) holds. Combining such embedding with the Poincaré inequality we obtain

\[
\int_{\Omega} |\nabla u_n^{1,p_i}| \leq C K_i \|u_n\|,
\]

for all \( n \in \mathbb{N} \). Suppose that \( |\nabla u_n|_{L^{p_i(x)}} > 1 \). Thus by Proposition 2.1 we have \( \|u_n\|_p^{-1} \leq C K_i \) for all \( n \in \mathbb{N} \) where \( C \) is a constant that does not depend on \( n \). Then we conclude that \( (u_n) \) is bounded in \( W_0^{1,p_i}(\Omega) \). The reflexivity of \( W_0^{1,p_i}(\Omega) \) and the compact embedding \( W_0^{1,p_i}(\Omega) \hookrightarrow L^{p_i(x)}(\Omega) \) provides the result.

**Claim 2: \( \Phi \) is continuous.** Consider a sequence \( (\lambda_n, z_1^n, z_2^n) \in \mathbb{R}^+ \times L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega) \) converging to \( (\lambda, z_1, z_2) \in \mathbb{R}^+ \times L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega) \). Define \( (u_n, u_n) = \Phi(\lambda_n, z_1^n, z_2^n) \) and \( (u_1, u_2) = \Phi(\lambda, z_1, z_2) \). Using the definition of \( \Phi \) we obtain

\[
\begin{align*}
\int_{\Omega} |\nabla u_n^{1,p_i}|^{-2} \nabla u_n \nabla \varphi &= \lambda_n \int_{\Omega} H_i(T_1 z_1^n, T_2 z_2^n) \varphi, \quad (3.3) \\
\int_{\Omega} |\nabla u^{1,p_i}|^{-2} \nabla u \nabla \varphi &= \lambda \int_{\Omega} H_i(T_1 z_1, T_2 z_2) \varphi \quad (3.4)
\end{align*}
\]
for all $\varphi \in W_0^{1,p_i}(\Omega)$ where $i, j = 1, 2$ and $i \neq j$.

Considering $\varphi = (u_i^j - u^j)$ in (3.3) and (3.4) and subtracting (3.4) from (3.3) we obtain

$$
\int_{\Omega} \langle |\nabla u_n^i|^{p_i(x)} - 2 \nabla u_n^i - |\nabla u^i|^{p_i(x)} - 2 \nabla u^i, \nabla (u_n^i - u^i) \rangle
= \int_{\Omega} \lambda_n H_i(T_1 z_n^i, T_2 z_n^2)(u_n^i - u^i) - \int_{\Omega} \lambda H_i(T_1 z_n^1, T_2 z_n^2)(u_n^i - u^i).
$$

Using Hölder’s inequality we have

$$
|\int_{\Omega} \langle |\nabla u_n^i|^{p_i(x)} - 2 \nabla u_n^i - |\nabla u^i|^{p_i(x)} - 2 \nabla u^i, \nabla (u_n^i - u^i) \rangle| \\
\leq |u_n^i - u^i|_{p_i(x)} |\lambda_n H_i(T_1 z_n^i, T_2 z_n^2) - \lambda H_i(T_1 z_n^1, T_2 z_n^2)|_{p_i(x)}.
$$

The arguments above ensures that $(u^i_n)$ is bounded in $W_0^{1,p_i(x)}(\Omega)$. Since $\lambda_n \to \lambda$ and $H_i(T_1 z_n^i, T_2 z_n^2) \to H_i(T_1 z^i, T_2 z^2)$ in $L^{p_i}(\Omega)$ for $i = 1, 2$ we have

$$
|\int_{\Omega} \langle |\nabla u_n^i|^{p_i(x)} - 2 \nabla u_n^i - |\nabla u^i|^{p_i(x)} - 2 \nabla u^i, \nabla (u_n^i - u^i) \rangle| \to 0.
$$

Therefore $u_n^i \to u^i$ in $L^{p_i}(\Omega)$ for $i = 1, 2$ which proves the continuity of $\Phi$.

Combining the fact that $\Phi(0, z_1, z_2) = (0, 0, 0)$ for all $(z_1, z_2) \in L^{p_1}(\Omega) \times L^{p_2}(\Omega)$ with the previous claims we have by Theorem 2.8 that the equation $\Phi(\lambda, u, v) = (u, v)$ possesses an unbounded continuum $C \subset \mathbb{R}^+ \times L^{p_1}(\Omega) \times L^{p_2}(\Omega)$ of solutions with $(0, 0, 0) \in C$.

**Claim 3:** $C$ is bounded with respect to the parameter $\lambda$. Suppose that there exists $\lambda^* > 0$ such that $\lambda \leq \lambda^*$ for all $(\lambda, u^1, u^2) \in C$. For $(\lambda, u^1, u^2) \in C$ the definition of $\Phi$ imply that

$$
-\Delta_{p_i(x)} u_1 = \lambda H_i(T_1 u_1, T_2 u_2) \quad \text{in} \quad \Omega, \\
-\Delta_{p_2(x)} u_2 = \lambda H_2(T_1 u_1, T_2 u_2) \quad \text{in} \quad \Omega,
$$

$$
u_1 = u_2 = 0 \quad \text{on} \quad \partial \Omega. \tag{3.5}
$$

Using the test function $u_i$ in (3.5) and considering (3.1) we obtain

$$
\int_{\Omega} |\nabla u_i|^{p_i(x)} \leq \lambda^* C|u_i|_{L^{p_i}(\Omega)}.
$$

Suppose that $|\nabla u_i|_{L^{p_i}(\Omega)} > 1$. Then using Proposition 2.1 and the Poincaré inequality we obtain that

$$
|u_i|^{p_i-1}_{L^{p_i}(\Omega)} \leq \lambda^* C.
$$

Thus $C$ is bounded in $\mathbb{R}^+ \times L^{p_1}(\Omega) \times L^{p_2}(\Omega)$, which is a contradiction.

Considering $\lambda = 1$, by (3.5) we have

$$
\int_{\Omega} |\nabla u_i|^{p_i(x)} - 2 \nabla u_i \nabla \varphi = \int_{\Omega} \left( f_i(x, T_1 u_1, T_2 u_2)|T_j u_j|^{\alpha_i(x)}_{L^{p_i}(\Omega)} \right) \nabla \varphi + \int_{\Omega} \left( g_i(x, T_1 u_1, T_2 u_2)|T_j u_j|^{\gamma_i(x)}_{L^{p_i}(\Omega)} \right) \varphi, \tag{3.6}
$$

for all $\varphi \in W_0^{1,p_i(x)}(\Omega)$ where $i, j = 1, 2$ with $i \neq j$. 

Now we claim that \( u_i \in [\underline{u}_i, \bar{u}_i] \) for \( i = 1, 2 \). To prove the claim we define
\[
L_1(u_1 - u_1)_+ := \int_{\{u_1 \geq u_1\}} \langle |\nabla u_1|^{p_1(x)-2}\nabla u_1 - |\nabla u_1|^{p_1(x)-2}\nabla u_1, \nabla (u_1 - u_1) \rangle.
\]
Using the facts that \( \int_{u_1}^{u_2} \underline{u} \in \underline{u}_i, \bar{u}_i \) > 0 a.e. in \( \Omega \), \( i = 1, j = 2 \), considering \( w = \int_{u_1}^{u_2} \phi \) and \( \varphi = (u_1 - u_1)_+ \) in the first inequality of \([1,3]\) and combining with equation \([4.6]\) we obtain
\[
L_1(u_1 - u_1)_+ \leq \int_{\{u_1 \geq u_1\}} \frac{f_1(x, u_1, T_{2u_2})\langle |u_2|^{\alpha_1(x)_{u_1}} - |T_{2u_2}|^{\alpha_1(x)} \rangle}{A(x, |T_{2u_2}|^{\alpha_1(x)})} (u_1 - u_1) + \int_{\{u_1 \geq u_1\}} \frac{g_1(x, u_1, T_{2u_2})\langle |u_2|^{\gamma_1(x)} - |T_{2u_2}|^{\gamma_1(x)} \rangle}{A(x, |T_{2u_2}|^{\alpha_1(x)})} (u_1 - u_1),
\]
which implies that
\[
\int_{\{u_1 \geq u_1\}} \langle |\nabla u_1|^{p_1(x)-2}\nabla u_1 - |\nabla u_1|^{p_1(x)-2}\nabla u_1, \nabla (u_1 - u_1) \rangle \leq 0.
\]
Therefore \( \underline{u}_1 \leq u_1 \). The same reasoning imply the other inequalities. Since \( u_i \in [\underline{u}_i, \bar{u}_i] \), we have \( T_i u_i = u_i \). Therefore the pair \((u_1, u_2)\) is a weak positive solution of \((S)\). \( \square \)

4. Applications

In this section we apply Theorem \([1,1]\) to some nonlocal problems.

4.1. A sublinear problem: In this section, we use Theorem \([1,1]\) to study the nonlocal problem
\[
-\mathcal{A}(x, |v|^{\alpha_1(x)}, \nabla v) \Delta_{p_1(x)} u = (u^{\beta_1(x)} + v^{\gamma_1(x)}) |u|^{\alpha_1(x)} \text{ in } \Omega, \\
-\mathcal{A}(x, |u|^{\alpha_2(x)}, \nabla u) \Delta_{p_2(x)} v = (u^{\beta_2(x)} + v^{\gamma_2(x)}) |u|^{\alpha_2(x)} \text{ in } \Omega, \tag{4.1}
\]
\[
u = v = 0 \quad \text{on } \partial \Omega.
\]
This problem with \( p_1(x) \equiv p_1(x) = 2 \), was considered in [93]. The result in this section generalizes [93, Theorem 6].

**Theorem 4.1.** Suppose that \( p_i, q_i, r_i, s_i, i = 1, 2 \) satisfy \((H1)\) and \( \alpha_i, \beta_i \in L^\infty(\Omega) \), \( i = 1, 2 \). Assume also that
\[
0 < \alpha_i^+ + \gamma_i^+ < p_i^- - 1, \quad 0 < \frac{\alpha_i^+}{p_2 - 1} + \frac{\beta_i^+}{p_1 - 1} < 1,
\]
\[
0 < \alpha_i^+ + \gamma_i^+ < p_i^- - 1, \quad 0 < \frac{\alpha_i^+}{p_1 - 1} + \frac{\beta_i^+}{p_2 - 1} < 1
\]
for \( i = 1, 2 \). Let \( a_0 > 0 \) be a positive constant. Suppose that one of the following two sets of conditions holds
\[
\mathcal{A}(x, t) \geq a_0 \quad \text{in } \bar{\Omega} \times [0, \infty), \tag{4.2}
\]
or
\[
0 < \mathcal{A}(x, t) \leq a_0 \quad \text{in } \bar{\Omega} \times (0, \infty) \quad \text{and} \quad \lim_{t \to +\infty} \mathcal{A}(x, t) = a_\infty > 0 \quad \text{uniformly in } \Omega. \tag{4.3}
\]

Then \([4.1]\) has a positive solution.
Proof. Suppose that (4.2) holds. We will start by constructing \((\tilde{w}, \tilde{v})\). Let \(\lambda > 0\) be a positive number, which will be chosen later and denote by \(z_\lambda \in W^{1,p_1(x)}_0(\Omega) \cap L^\infty(\Omega)\) and \(y_\lambda \in W^{1,p_2(x)}_0(\Omega) \cap L^\infty(\Omega)\) the unique solutions of (2.1) respectively.

For \(\lambda > 0\) sufficiently large it follows from Lemma 2.7 that there is a constant \(K > 1\) that does not depend on \(\lambda\) such that

\[
0 < z_\lambda(x) \leq K\lambda^{\frac{1}{p_1-1}} \quad \text{in } \Omega, \tag{4.4}
\]

\[
0 < y_\lambda(x) \leq K\lambda^{\frac{1}{p_2-1}} \quad \text{in } \Omega. \tag{4.5}
\]

Since \(\alpha_1^+ + \gamma_1^+ < p_2 - 1\) and \(\frac{\alpha_1^+}{p_2-1} + \frac{\beta_1^+}{p_1-1} < 1\), it is possible to choose \(\lambda > 1\) such that (4.4), (4.5) and

\[
\frac{1}{a_0} (K^{\beta_1} \lambda^{\frac{\alpha_1^+}{p_2-1} + \frac{\alpha_1^+}{p_1-1} + K^{\gamma_1} \lambda^{\frac{\alpha_1^+}{p_2-1}}}) \max\{|K|^\alpha_1^-_{L^1(w)}; |K|^\alpha_1^+_{L^1(w)}\} \leq \lambda \tag{4.6}
\]

hold. By (4.4), (4.5), and (4.6), we obtain

\[
\frac{1}{a_0} (z_\lambda^\beta(x) + w^\gamma_1(x))|y_\lambda|^\alpha_1(x) \leq \lambda, \quad w \in [0, y_\lambda].
\]

Thus for \(w \in [0, z_\lambda]\) we obtain

\[
-\Delta_{p_1(x)} z_\lambda \geq \frac{1}{\mathcal{A}(x, |w|_{L^\gamma_1(x)})} (z_\lambda^\beta(x) + w^\gamma_1(x))|y_\lambda|^\alpha_1(x) \quad \text{in } \Omega,
\]

\[
z_\lambda = 0 \quad \text{on } \partial \Omega.
\]

Considering, if necessary, a larger \(\lambda > 0\), the previous reasoning imply that

\[
-\Delta_{p_2(x)} y_\lambda \geq \frac{1}{\mathcal{A}(x, |w|_{L^\gamma_2(x)})} (w^\delta_2(x) + y^\gamma_2(x))|z_\lambda|^\alpha_2(x) \quad \text{in } \Omega,
\]

\[
y_\lambda = 0 \quad \text{on } \partial \Omega,
\]

for all \(w \in [0, z_\lambda]\).

Now we construct \((\tilde{w}_i, \tilde{v}_i), i = 1, 2\). Since \(\partial \Omega\) is \(C^2\), there is a constant \(\delta > 0\) such that \(d \in C^2(\overline{\Omega}_3)\) and \(|\nabla d(x)| \equiv 1\), where \(d(x) := \text{dist}(x, \partial \Omega)\) and \(\overline{\Omega}_3 := \{x \in \overline{\Omega}; d(x) \leq 3\delta\}\). From [34, Page 12], we have that, for \(\sigma \in (0, \delta)\) sufficiently small, the function \(\phi_i = \phi_i(k, \sigma), i = 1, 2\) defined by

\[
\phi_i(x) = \begin{cases} e^{kd(x)} - 1 & \text{if } d(x) < \sigma, \\ e^{k\sigma} - 1 + \int_0^{d(x)} k e^{k\sigma} \left( \frac{2d - t}{2\sigma - t} \right)^{\frac{2}{p_1-1}} dt & \text{if } \sigma \leq d(x) < 2\delta, \\ e^{k\sigma} - 1 + \int_0^{2\delta} k e^{k\sigma} \left( \frac{2d - t}{2\sigma - t} \right)^{\frac{2}{p_1-1}} dt & \text{if } 2\delta \leq d(x), \end{cases}
\]

belongs to \(C^1_0(\overline{\Omega})\), where \(k > 0\) is an arbitrary number and that

\[
-\Delta_{p_i(x)} (\mu \phi_i)
\]
for all \( \mu > 0 \) and \( i = 1, 2 \).

Define \( \mathcal{A}_\lambda := \max \{ \mathcal{A}(x, t) : (x, t) \in \overline{\Omega} \times \left[ 0, \max\{ |y_\lambda|_{L^1(\Omega)} \} \right] \} \). Then we have

\[
a_0 \leq \mathcal{A}(x, |w|_{L^1(\Omega)}) \leq \mathcal{A}_\lambda \quad \text{in } \Omega
\]

for all \( w \in [0, y_\lambda] \). Let \( \sigma = \frac{1}{k} \ln 2 \) and \( \mu = e^{-ak} \) where

\[
a = \frac{\min\{p_1 - 1, p_2 - 1\}}{\max\{\max_{\overline{\Omega}} |\nabla p_1| + 1, \max_{\overline{\Omega}} |\nabla p_2| + 1\}}.
\]

Then \( e^{k\sigma} = 2 \) and \( k\mu \leq 1 \) if \( k > 0 \) is sufficiently large.

Let \( x \in \Omega \) with \( d(x) < \sigma \). If \( k > 0 \) is large enough we have \( |\nabla d(x)| = 1 \) and then

\[
|d(x) + \frac{\ln(k\mu)}{k}| |\nabla p_1(x)||\nabla d(x)| \leq \left( |d(x)| + \frac{|\ln(k\mu)|}{k} \right) |\nabla p_1(x)|
\]

\[
\leq \left( \sigma - \frac{|\ln(k\mu)|}{k} \right) |\nabla p_1(x)|
\]

\[
eq \left( \frac{\ln 2}{k} - \frac{|\ln(k\mu)|}{k} \right) |\nabla p_1(x)| + a|\nabla p_1(x)|
\]

\[
< p_1^{-1} - 1.
\]

Note also that there exists a constant \( A > 0 \), that does not depend on \( k \), such that \( |\Delta d(x)| < A \) for all \( x \in \partial \Omega_{\delta} \). Using the last inequality and the expression of \( -\Delta d(x)(\mu \phi_1) \), we obtain \( -\Delta d(x)(\mu \phi_1) \leq 0 \) for \( x \in \Omega \) with \( d(x) < \sigma \) or \( d(x) > 2\delta \) for \( k > 0 \) large enough. Therefore

\[
-\Delta p_1(x)(\mu \phi_1) \leq 0 \leq \frac{1}{\mathcal{A}_\lambda} (\mu \phi_1)^{\alpha_1(x)} |\mu \phi_2|_{L^1(\Omega)}^{\alpha_1(x)}
\]

\[
\leq \frac{1}{\mathcal{A}_\lambda} ((\mu \phi_1)^{\alpha_1(x)} + |w|^{\gamma_1(x)}) |\mu \phi_2|_{L^1(\Omega)}^{\alpha_1(x)}
\]

for all \( w \in L^\infty(\Omega) \) with \( w \geq \mu \phi_2 \) and \( d(x) < \sigma \) or \( 2\delta < d(x) \). Using the idea in the proof of [44] estimate (3.10) we obtain

\[
-\Delta p_1(x)(\mu \phi_1) \leq \tilde{C}(k\mu)^{p_1^{-1}} |\ln(k\mu)|
\]

\[
= \frac{\tilde{C}(k\mu)^{p_1^{-1}} |\ln(k\mu)|}{e^{ak}} \quad \text{if } \sigma < d(x) < 2\delta.
\]

From the proof of [44] Theorem 2] and the fact that \( \alpha_1^+ + \gamma_1^+ < p_1^- - 1 \) we obtain

\[
\lim_{k \to +\infty} \frac{\tilde{C}(k\mu)^{p_1^{-1}} |\ln(k\mu)|}{e^{ak}} = 0.
\]

Note that \( \alpha_1(x) \geq 1 \) if \( \sigma < d(x) < 2\delta \) because \( \alpha_1(x) \geq e^{k\sigma} - 1 \) and \( e^{k\sigma} = 2 \) for all \( k > 0 \). Thus, there is a constant \( C_0 > 0 \) that does not depend on \( k \) such that
for all \( \omega \). By \( (4.9) \), we can choose \( k > 0 \) large enough such that
\[
\frac{\mathcal{C}k^{p^*_1-1}}{\epsilon^{ak[(p^*_1-1)-(\alpha^*_1+\beta^*_1)]}} |\ln \frac{k}{\epsilon^{ak}}| \leq \frac{C_0}{\mathcal{A}_\lambda}, \tag{4.10}
\]
Therefore from \( (4.8) \) and \( (4.10) \) we have
\[
-\Delta_{p_1(x)}(\mu \phi_1) \leq \frac{1}{\mathcal{A}_\lambda}((\mu \phi_1)^{\beta_1(x)} + w^{\gamma_1(x)})|\mu \phi_2|^{\alpha_1(x)}_{L^1(x)} \leq \frac{1}{\mathcal{A}_\lambda}|\mu \phi_2|^{\alpha_1(x)}_{L^1(x)},
\]
for all \( w \in L^\infty(\Omega) \) with \( w \geq \mu \phi_2 \) and \( \sigma < d(x) < 2\delta \) for \( k > 0 \) large enough. Thus it is possible to conclude that
\[
-\Delta_{p_1(x)}(\mu \phi_1) \leq \frac{1}{\mathcal{A}_\lambda}((\mu \phi_1)^{\beta_1(x)} + w^{\gamma_1(x)})|\mu \phi_2|^{\alpha_1(x)}_{L^1(x)} \leq \frac{1}{\mathcal{A}_\lambda}|\mu \phi_2|^{\alpha_1(x)}_{L^1(x)} \text{ in } \Omega.
\]
Fix \( k > 0 \) satisfying the above property and \( -\Delta_{p_1(x)}(\mu \phi_1) \leq 1 \). For \( \lambda > 1 \) we have \( -\Delta_{p_1(x)}(\mu \phi_1) \leq -\Delta_{p_1(x)}\cdot \leq \lambda \). Therefore \( \mu \phi_1 \leq \lambda \). Since \( \alpha^*_2 + \gamma^*_2 < p^*_2 - 1 \), a similar reasoning imply that there is \( \mu > 0 \) small enough such that
\[
-\Delta_{p_2(x)}(\mu \phi_2) \leq \frac{1}{\mathcal{A}(\Omega)}(w^{\beta_2} + (\mu \phi_2)^{\gamma_2})|\mu \phi_1|^{\alpha_2(x)}_{L^1(x)} \leq \frac{1}{\mathcal{A}(\Omega)}|\mu \phi_1|^{\alpha_2(x)}_{L^1(x)} \text{ in } \Omega \tag{4.11}
\]
for all \( w \in L^\infty(\Omega) \) with \( w \geq \mu \phi_1 \) and that \( \mu \phi_2 \leq y_\lambda \). The first part of the result is proved.

Now suppose that \( 0 < \mathcal{A}(x, t) \leq a_0 \) in \( \Omega \times (0, \infty) \). Let \( \delta, \sigma, \mu, a, \lambda, z_\lambda, y_\lambda \) and \( \phi_i \) for \( i = 1, 2 \) as before. From the previous arguments there exist \( k > 0 \) large enough and \( \mu > 0 \) small such that
\[
-\Delta_{p_1(x)}(\mu \phi_1) \leq 1, \quad -\Delta_{p_1(x)}(\mu \phi_1) \leq \frac{1}{a_0}((\mu \phi_1)^{\beta_1(x)} + w^{\gamma_1(x)})|\mu \phi_2|^{\alpha_1(x)}_{L^1(x)} \leq \frac{1}{a_0}|\mu \phi_2|^{\alpha_1(x)}_{L^1(x)} \text{ in } \Omega \tag{4.11}
\]
in \( \Omega \) for all \( w \in [\mu \phi_2, y_\lambda] \), and
\[
-\Delta_{p_2(x)}(\mu \phi_2) \leq 1, \quad -\Delta_{p_2(x)}(\mu \phi_2) \leq \frac{1}{a_0}(w^{\beta_2} + (\mu \phi_2)^{\gamma_2})|\mu \phi_1|^{\alpha_2(x)}_{L^2(x)} \leq \frac{1}{a_0}|\mu \phi_1|^{\alpha_2(x)}_{L^2(x)} \text{ in } \Omega \tag{4.12}
\]
in \( \Omega \) for all \( w \in [\mu \phi_1, y_\lambda] \).

Since \( \lim_{t \to \infty} \mathcal{A}(x, t) = a_\infty > 0 \) uniformly in \( \Omega \) there is a large constant \( a_1 > 0 \) such that \( \mathcal{A}(x, t) \geq \frac{a_\infty}{2} \) on \( \Omega \times (a_1, \infty) \). Let
\[
m_k := \min \left\{ \mathcal{A}(x, t) : (x, t) \in \Omega \times [\min\{|\mu \phi_1|_{L^1(x)}, |\mu \phi_2|_{L^2(x)}\}, a_1]\right\} > 0
\]
and \( \mathcal{A}_k := \min \left\{ m_k, \frac{a_\infty}{2}\right\} \). Then we have
\[
\mathcal{A}(x, t) \geq \mathcal{A}_k \text{ in } \Omega \times [\min\{|\mu \phi_1|_{L^1(x)}, |\mu \phi_2|_{L^2(x)}\}, \infty).
\]
Fix \( k > 0 \) satisfying \( (4.11) \) and \( (4.12) \). Consider \( \lambda > 1 \) such that \( (4.4), (4.5) \) and
\[
\frac{1}{\mathcal{A}_k} \left( K^{\beta_1^*} \frac{\alpha^*_1}{p^*_1 - 1} + \frac{\alpha^*_1}{p^*_2 - 1} + K^{\gamma_1^*} \frac{\alpha^*_1 + \gamma^*_1}{p^*_2 - 1} \right) \max\{|K|^{\alpha^*_1_{L^1(x)}}, |K|^{\alpha^*_1_{L^2(x)}}\} \leq \lambda,
\]
\[
\frac{1}{\mathcal{A}_k} \left( K^{\beta_2^*} \frac{\alpha^*_2}{p^*_1 - 1} + K^{\gamma_2^*} \frac{\alpha^*_2 + \gamma^*_2}{p^*_2 - 1} \right) \max\{|K|^{\alpha^*_2_{L^1(x)}}, |K|^{\alpha^*_2_{L^2(x)}}\} \leq \lambda,
\]
where \( K > 1 \) is a constant that does not depend on \( k \) or \( \lambda \) (see Lemma 2.7). Therefore,
\[
-\Delta_{p_1(x)} z_\lambda \leq \frac{1}{\mathcal{A}(x, |w|_{L^1(x)})}(z_\lambda^{\beta_1(x)} + w^{\gamma_1(x)})|y_\lambda|^{\alpha_1(x)}_{L^1(x)} \text{ in } \Omega, \quad w \in [\mu \phi_2, y_\lambda].
\]
Arguing as before and considering a suitable choice for \( \lambda \) and \( k \) we obtain

\[
-\Delta_{p_2}(x)y_\lambda \leq \frac{1}{A(x,|w|_{L^{r_1(x)}}^r)} (w^{\beta_2(x)} + \beta_\lambda^2(x)) |z_\lambda|_{L^{r_2(x)}}^2 \quad \text{in } \Omega, \quad w \in [\mu \phi_1, z_\lambda].
\]

The comparison principle implies that \( \mu \phi_1 \leq z_\lambda \) and \( \mu \phi_2 \leq y_\lambda \) if \( \mu \) is small. The proof is complete. \( \square \)

4.2. A concave-convex problem. In this section we consider the following non-local problem with concave-convex nonlinearities

\[
\begin{align*}
-\mathcal{A}(x,|v|_{L^{r_1(x)}}^r)u &= \lambda |v|^{\beta_1(x)-1}u|v|^{\alpha_1(x)}_{L^{r_1(x)}} + \theta |v|^{\eta(x)-1}u|v|^{\gamma_1(x)}_{L^{r_1(x)}} \quad \text{in } \Omega, \\
-\mathcal{A}(x,|u|_{L^{r_2(x)}}^r)v &= \lambda |v|^{\beta_2(x)-1}u|v|^{\alpha_2(x)}_{L^{r_2(x)}} + \theta |v|^{\eta(x)-1}u|v|^{\gamma_2(x)}_{L^{r_2(x)}} \quad \text{in } \Omega,
\end{align*}
\]

\( u = v = 0 \) on \( \partial \Omega. \) \hspace{1cm} (4.13)

The scalar and local version of (4.13) with \( p(x) \equiv 2 \) and constant exponents was considered in the famous paper by Ambrosetti-Brezis-Cerami \[43\] in which a supersolution argument is used. In \[43\], problem (4.13) was studied with \( p(x) \equiv 2. \) The following result generalizes \[43\, Theorem 7].

**Theorem 4.2.** Suppose that \( r_i, p_i, q_i, s_i, \alpha_i, \eta_i \) satisfy (H1) for \( i = 1, 2 \) and that \( \beta_i \in L^\infty(\Omega), i = 1, 2 \) are nonnegative functions with \( 0 < \alpha_i^- + \beta_i^+ \leq \alpha_i^+ + \beta_i^- < p_i^- - 1, i = 1, 2. \) Let \( a_0, b_0 > 0 \) be positive numbers. Then the following assertions hold.

1. If \( p_2^- - 1 < \eta_2^- + \gamma_2^- \), \( p_1^- - 1 < \eta_2^- + \gamma_2^- \) and \( \mathcal{A}(x, s) \geq a_0 \) in \( \overline{\Omega} \times [0, b_0], \) then for each \( \theta > 0 \) there exists \( \lambda_0 > 0 \) such that for each \( \lambda \in (0, \lambda_0), \) problem (4.13) has a positive solution \( u_{\lambda, \theta}. \)

2. \( p_2^- - 1 < \eta_2^- + \gamma_2^- \) and

\[
\frac{\beta_1^+}{p_1^- - 1} + \frac{\alpha_1^+}{p_2^- - 1} < 1, \quad \frac{\beta_2^+}{p_2^- - 1} + \frac{\alpha_2^+}{p_1^- - 1} < 1.
\]

Suppose that \( 0 < \mathcal{A}(x, s) \leq a_0 \) in \( \overline{\Omega} \times (0, \infty) \) and \( \lim_{s \to \infty} \mathcal{A}(x, s) = b_0 \) uniformly in \( \overline{\Omega}. \) Then given a \( \lambda > 0, \) there exists \( \theta_0 > 0 \) such that for each \( \theta \in (0, \theta_0), \) problem (4.13) has a positive solution \( u_{\lambda, \theta}. \)

**Proof.** Suppose that (1) occurs. Consider \( z_\lambda \in W^{1,p_1(x)}_0(\Omega) \cap L^\infty(\Omega) \) and \( y_\lambda \in W^{1,p_2(x)}_0(\Omega) \cap L^\infty(\Omega) \) the unique solutions of (2.1) respectively, where \( \lambda \in (0, 1) \) will be chosen later.

Lemma 2.7 imply that for \( \lambda > 0 \) small enough there exists a constant \( K > 1 \) that does not depend on \( \lambda \) such that

\[
0 < z_\lambda(x) \leq K \lambda^{\frac{1}{r_1^- - 1}} \quad \text{in } \Omega, \quad \mu > 0 \lambda \leq K \lambda^{\frac{1}{r_1^- - 1}} \quad \text{in } \Omega.
\]

To construct \( \pi_i \) we will prove, for each \( \theta > 0, \) that there exists \( \lambda_0 > 0 \) such that

\[
\frac{1}{a_0} \left( \lambda |z_\lambda|^{\beta_1(x)-1}z_\lambda|y_\lambda|^{\alpha_1(x)}_{L^{r_1(x)}} + \theta |w|^{\eta(x)-1}w|y_\lambda|^{\gamma_1(x)}_{L^{r_1(x)}} \right) \leq \lambda, \quad \forall w \in [0, y_\lambda], \quad (4.16)
\]

\[
\frac{1}{a_0} \left( \lambda |y_\lambda|^{\beta_2(x)-1}y_\lambda|z_\lambda|^{\alpha_2(x)}_{L^{r_2(x)}} + \theta |w|^{\eta(x)-1}w|z_\lambda|^{\gamma_2(x)}_{L^{r_2(x)}} \right) \leq \lambda, \quad \forall w \in [0, z_\lambda]. \quad (4.17)
\]
Let
\[ K := \max_{i=1,2} \{ K^{\alpha_i^+} | K |_{L^1(x)}, K^{\alpha_i^-} | K |_{L^1(x)}, K^{\alpha_i^+} | K |_{L^1(x)}, K^{\alpha_i^-} | K |_{L^1(x)} \}. \tag{4.18} \]
Since \( 0 < \alpha_1^+ + \beta_i^+ \) and \( p_i^+ - 1 < 1 \), there exists \( \lambda_0 > 0 \) such that
\[ \frac{1}{a_0} \left( \lambda^{\frac{1}{p_i^+ - 1} + \frac{\alpha_i^-}{p_i^- - 1}} + \frac{\alpha_i^+}{p_i^- - 1} K + \theta \lambda^{\frac{1}{p_i^- - 1}} K \right) \leq \lambda, \tag{4.19} \]
for all \( \lambda \in (0, \lambda_0) \).

If necessary, we consider small \( \lambda_0 > 0 \) such that \( |y_\lambda| |x|_{L^1(x)} \leq |K| |x|_{L^1(x)} \lambda^{\frac{1}{p_i^- - 1}} \leq b_0 \) for all \( \lambda \in (0, \lambda_0) \). Therefore \( A(x, |w| |x|_{L^1(x)}) \geq a_0, w \in [0, y_\lambda] \). It follows from (4.14), (4.15) and (4.19) that (4.16) holds. Then we can conclude that
\[ -\Delta_{p_i(x)} z_\lambda \geq \frac{1}{A(x, |w| |x|_{L^1(x)})} \left( \lambda \beta_i(x) |y_\lambda| + \theta w |x|_{L^1(x)} \right), \tag{4.20} \]
for all \( w \in [0, y_\lambda] \). Assume also that \( \lambda_0 \) satisfies
\[ \frac{1}{a_0} \left( \lambda^{\frac{1}{p_i^+ - 1} + \frac{\alpha_i^-}{p_i^- - 1}} + \frac{\alpha_i^+}{p_i^- - 1} K + \theta \lambda^{\frac{1}{p_i^- - 1}} K \right) \leq \lambda \tag{4.21} \]
and \( |z_\lambda|_{L^1(x)} \leq |K| |x|_{L^1(x)} \lambda^{\frac{1}{p_i^- - 1}} \leq b_0 \) for all \( \lambda \in (0, \lambda_0) \). Therefore \( A(x, |w| |x|_{L^1(x)}) \geq a_0, w \in [0, z_\lambda] \). Thus from (4.14), (4.15) and (4.21) we have that (4.17) holds. Then we can conclude that
\[ -\Delta_{p_2(x)} y_\lambda \geq \frac{1}{A(x, |w| |x|_{L^1(x)})} \left( \lambda \beta_2(x) |z_\lambda| + \theta w |x|_{L^1(x)} \right), \tag{4.22} \]
for all \( w \in [0, z_\lambda] \).

To construct \( u_i \) consider \( \phi_i, \delta, \sigma, \mu \) as in the proof of Theorem 4.1. Using the inequalities \( \alpha_i^+ + \beta_i^+ < p_i^+ - 1, i = 1, 2 \) and repeating the arguments of Theorem 4.1 we have that exists a number \( \mu > 0 \) such that
\[ \mu \phi_1 \leq z_\lambda, \quad \mu \phi_2 \leq y_\lambda, \quad -\Delta_{p_1(x)} (\mu \phi_1) \leq \lambda, \]
\[ -\Delta_{p_1(x)} (\mu \phi_1) \leq \frac{1}{A(x, |w| |x|_{L^1(x)})} \left( \lambda \beta_1(x) |\mu \phi_1| + \theta w |x|_{L^1(x)} \right), \]
for all \( w \in [\mu \phi_2, y_\lambda] \) and
\[ -\Delta_{p_2(x)} (\mu \phi_2) \leq \lambda, \]
\[ -\Delta_{p_2(x)} (\mu \phi_2) \leq \frac{1}{A(x, |w| |x|_{L^1(x)})} \left( \lambda \beta_2(x) |\mu \phi_2| + \theta w |x|_{L^1(x)} \right), \]
for all \( w \in [\mu \phi_2, z_\lambda] \). Then by Theorem 1.1 we have the desired result.

Now we consider the condition (2). Let \( \phi_i, \delta \) and \( \sigma_i, i = 1, 2 \) as in the first part of the result and let \( \lambda > 0 \) fixed. Since \( \alpha_i^+ + \beta_i^+ < p_i^+ - 1, i = 1, 2 \) there exists \( \mu > 0 \) depending only on \( \lambda \) such that
\[ -\Delta_{p_1(x)} (\mu \phi_1) \leq 1, \quad -\Delta_{p_1(x)} (\mu \phi_2) \leq \frac{1}{a_0} \lambda (\mu \phi_i) |\mu \phi_j| |\phi_i|, \]
for \( w \in L^\infty(\Omega) \) with \( w \geq \mu \phi_j, i \neq j \) and \( i, j = 1, 2 \).

Let \( M > 0 \) that will be chosen later and assume \( z_M \in W_0^{1,p_1(\Omega)}(\Omega) \cap L^\infty(\Omega) \) is a solution of
\[ -\Delta_{p_1(x)} z_M = M \text{ in } \Omega, \]
\[ z_M = 0 \quad \text{on } \partial \Omega, \]
and \( y_M \in W_0^{1,p_2(x)}(\Omega) \cap L^\infty(\Omega) \) is a solution of
\[
-\Delta_{p_2(x)} y_M = M \quad \text{in } \Omega, \\
y_M = 0 \quad \text{on } \partial \Omega.
\]

For \( M \) large enough from Lemma 2.7 there exists a constant \( K > 1 \) that does not depend on \( M \) such that
\[
0 < z_M(x) \leq K M^{\frac{1}{\rho_1} - 1} \quad \text{in } \Omega, \tag{4.23}
\]
\[
0 < y_M(x) \leq K M^{\frac{1}{\rho_2} - 1} \quad \text{in } \Omega. \tag{4.24}
\]

To construct \( \pi_\varepsilon \) we will show that exist \( \theta_0 > 0 \) depending on \( \lambda \) with the following property: if we assume \( \theta \in (0, \theta_0) \) then there is a constant \( M \) depending only on \( \lambda \) and \( \theta \) satisfying
\[
M \geq \frac{1}{\mathcal{A}(x, |w|_{L^\infty(\varepsilon)})} \left( \lambda z_M^{\beta_1(x)} |y_M|_{L^{\rho_1}(\varepsilon)}^{\alpha_1(x)} + \theta w^{\eta_1(x)} |y_M|_{L^{\rho_1}(\varepsilon)}^{\gamma_1(x)} \right), \tag{4.25}
\]
for \( w \in [\mu \phi_2, y_M] \), and
\[
M \geq \frac{1}{\mathcal{A}(x, |w|_{L^{\rho_2}(\varepsilon)})} \left( \lambda y_M^{\beta_2(x)} |z_M|_{L^{\rho_2}(\varepsilon)}^{\alpha_2(x)} + \theta w^{\eta_2(x)} |z_M|_{L^{\rho_2}(\varepsilon)}^{\gamma_2(x)} \right), \tag{4.26}
\]
for \( w \in [\mu \phi_2, z_M] \).

Since \( \mathcal{A} \) is continuous and \( \lim_{t \to +\infty} \mathcal{A}(x, t) = b_0 > 0 \) uniformly in \( \Omega \), there exists \( a_1 > 0 \) large enough such that \( \mathcal{A}(x, t) \geq \frac{b_0}{2} \) in \( \overline{\Omega} \times (a_1, +\infty) \). Define
\[
m_\lambda := \{ \mathcal{A}(x, t) : (x, t) \in \overline{\Omega} \times [\min\{ |\mu \phi_1|_{L^{\rho_1}(\varepsilon)}, |\mu \phi_2|_{L^{\rho_2}(\varepsilon)} \}, a_1] \}
\]
and \( A_\lambda := \min\{ m_\lambda, \frac{b_0}{2} \} \). Then \( \mathcal{A}(x, t) \geq A_\lambda \) in \( \overline{\Omega} \times [\min\{ |\mu \phi_1|_{L^{\rho_1}(\varepsilon)}, |\mu \phi_2|_{L^{\rho_2}(\varepsilon)} \}, \infty) \). Thus \( A_\lambda \leq \mathcal{A}(x, |w|_{L^{\rho_2}(\varepsilon)}) \leq a_0 \) for all \( w \in L^{\infty}(\Omega) \) with \( |\mu \phi_1| \leq w \) or \( |\mu \phi_2| \leq w \). Note that from (4.23) and (4.24) the inequalities (4.25) and (4.26) hold if we have simultaneously the inequalities
\[
\frac{1}{A_\lambda} \left( \Lambda K M^{\rho_1} + \theta K M^{\rho_2} \right) \leq M,
\]
\[
\frac{1}{A_\lambda} \left( \Lambda K M^{\rho_2} + \theta K M^{\rho_1} \right) \leq M,
\]
where \( \Lambda \) is given by (4.18). To obtain such inequalities we will study the inequality
\[
\frac{1}{A_\lambda} \left( \Lambda K M^{\rho_1} + \theta K M^{\rho_2} \right) \leq 1 \tag{4.27}
\]
where
\[
\rho := \max \left\{ \frac{\beta_1^+}{p_1 - 1} + \frac{\alpha_1^+}{p_1 - 1}, \frac{\beta_2^+}{p_2 - 1} + \frac{\alpha_2^+}{p_1 - 1} \right\},
\]
\[
\tau := \max \left\{ \eta_1^+, \eta_2^+, \eta_1^+, \eta_2^+ \right\}.
\]

Define
\[
\Psi_{\lambda, \theta}(M) := \frac{\Lambda K}{A_\lambda} M^{\rho_1} + \frac{\theta K}{A_\lambda} M^{\rho_2}, \quad M > 0.
\]
Since $0 < \rho < 1$ and $\tau > 1$ we have $\lim_{M \to 0^+} \Psi_{\lambda, \theta}(M) = \lim_{M \to +\infty} \Psi_{\lambda, \theta}(M) = +\infty$. Note that $\Psi_{\lambda, \theta}'(M) = 0$ if, and only if

$$M = M_{\lambda, \theta} := \left(\frac{\lambda}{\hat{\theta}}\right) \frac{1}{\tau - \rho} c, \quad c := \left(\frac{1 - \rho}{\tau - 1}\right) \frac{1}{\tau - \rho}.$$  \hspace{1cm} (4.28)

From the above properties of $\Psi_{\lambda, \mu}$ we have that the global minimum of $\Psi_{\lambda, \theta}$ is attained at $M_{\lambda, \theta}$. The inequality (4.27) is equivalent to finding $M_{\lambda, \theta} > 0$ such that $\Psi_{\lambda, \theta}(M_{\lambda, \theta}) \leq 1$. By (4.28), we have that $\Psi_{\lambda, \theta}(M_{\lambda, \theta}) \leq 1$, if and only if

$$\frac{\lambda \bar{K}}{\lambda} \left(\frac{\lambda}{\hat{\theta}}\right)^{\frac{\rho - 1}{\tau - \rho}} c^{\rho - 1} + \theta^{1 - (\frac{\rho - 1}{\tau - \rho})} \frac{K}{\lambda} \left(\frac{\lambda}{\hat{\theta}}\right)^{\frac{\rho - 1}{\tau - \rho}} c^{\rho - 1} \leq 1.$$  \hspace{1cm} (4.29)

Thus from (4.28) and (4.29), we have that given $\lambda > 0$ there exists $\theta_0 > 0$ such that for each $\theta \in (0, \theta_0)$ there exists $M_{\lambda, \theta}$ satisfying

$$M_{\lambda, \theta} \geq 1 \quad \text{and} \quad \frac{1}{A_\lambda} \left(\frac{\lambda \bar{K}}{\lambda} \left(\frac{\lambda}{\hat{\theta}}\right)^{\frac{\rho - 1}{\tau - \rho}} c^{\rho - 1} + \theta^{1 - (\frac{\rho - 1}{\tau - \rho})} \frac{K}{\lambda} \left(\frac{\lambda}{\hat{\theta}}\right)^{\frac{\rho - 1}{\tau - \rho}} c^{\rho - 1}\right) \leq 1.$$  

Therefore,

$$-\Delta_{p_1(x)} z_M \geq \frac{1}{A_\lambda} \left(\lambda z_M \beta_1(x) |y_M|^{\alpha_1(x)} + \theta w \gamma_1(x) |y_M|^{\gamma_1(x)}\right) \in \Omega,$$

for all $w \in [\mu \phi_2, y_M]$, and

$$-\Delta_{p_2(x)} y_M \geq \frac{1}{A_\lambda} \left(\lambda y_M \beta_2(x) |z_M|^{\alpha_2(x)} + \mu w \gamma_2(x) |z_M|^{\gamma_2(x)}\right) \in \Omega,$$

for all $w \in [\mu \phi_1, z_M]$. Since $M_{\lambda, \theta} \to +\infty$ as $\theta \to 0^+$ and the map $\theta \mapsto M_{\lambda, \theta}$ is decreasing we have

$$-\Delta_{p_1(x)} (\mu \phi_1) \leq 1 \leq M_{\lambda, \theta_0} \leq M_{\lambda, \theta}, \quad \theta \in (0, \theta_0)$$

for $\theta_0$ small enough. Similarly, we have $-\Delta_{p_2(x)} (\mu \phi_2) \leq M_{\lambda, \theta_0} \leq M_{\lambda, \theta}$ for all $\theta \in (0, \theta_0)$, for $\theta_0$ small. The weak maximum principle imply that $\mu \phi_1 \leq z_M$ and $\mu \phi_2 \leq y_M$. The proof is complete. \hfill $\square$

### 4.3. A generalization of the logistic equation.

In the previous sections, we considered at least one of the conditions $A(x, t) \geq a_0 > 0$ or $0 < A(x, t) \leq a_\infty$, $t > 0$. In this section we study a generalization of the classic logistic equation where the function $A(x, t)$ satisfies

$$A(x, 0) \geq 0, \quad \lim_{t \to 0^+} A(x, t) = \infty, \quad \text{and} \quad \lim_{t \to +\infty} A(x, t) = \pm \infty.$$  

We consider the problem

$$-A(x, |v|_{L^{q_1}(x)}) \Delta_{p_1(x)} u = \lambda f_1(u)|v|_{L^{q_1}(x)}^{\alpha_1(x)} \quad \text{in} \ \Omega,$$

$$-A(x, |u|_{L^{q_2}(x)}) \Delta_{p_2(x)} v = \lambda f_2(v)u|v|_{L^{q_2}(x)}^{\alpha_2(x)} \quad \text{in} \ \Omega,$$

$$u = v = 0 \quad \text{on} \ \partial \Omega.$$  \hspace{1cm} (4.30)

We suppose that there are numbers $\theta_i > 0$, $i = 1, 2$ such that the functions $f_i : \ [0, \infty) \to \mathbb{R}$ satisfy the following conditions:

(H2) $f_i \in C^0([0, \theta_i], \mathbb{R})$, $i = 1, 2$;

(H3) $f_i(0) = f_i(\theta_i) = 0$, $f_i(t) > 0$ in $(0, \theta_i)$ for $i = 1, 2$.

Problem (4.30) is a generalization of the problemes studied in [16, 18, 43]. The next result generalizes [43, Theorem 8].
Theorem 4.3. Suppose that $r_i, p_i, q_i, \alpha_i$ satisfy (H1). Also that $f_i, i = 1, 2$ satisfies (H2), (H3) and that $A(x, t) > 0$ in $\overline{\Omega} \times (0, \max\{|\theta_1|_{L^{r_2}(\Omega)}^1, |\theta_2|_{L^{r_1}(\Omega)}^1\}]$. Then there exists $\lambda_0 > 0$ such that \(\ref{4.30}\) has a positive solution for $\lambda \geq \lambda_0$.

Proof. Consider the functions $\tilde{f}_i(t) = f_i(t)$ for $t \in [0, \theta_i]$, and $\tilde{f}_i(t) = 0$ for $t \in \mathbb{R} \setminus [0, \theta_i]$, $i = 1, 2$. The functional

$$J_\lambda(u, v) = \int_\Omega \frac{1}{p_1(x)}|\nabla u|^{p_1(x)}dx - \lambda \int_\Omega \tilde{F}_1(u)dx + \int_\Omega \frac{1}{p_2(x)}|\nabla v|^{p_2(x)}dx - \lambda \int_\Omega \tilde{F}_2(v)dx,$$

where $\tilde{F}_i(t) = \int_0^t \tilde{f}_i(s)ds$ is of class $C^1(W_0^{1, p_1(x)} \times W_0^{1, p_2(x)}(\Omega), \mathbb{R})$ and $W_0^{1, p_1(x)}(\Omega) \times W_0^{1, p_2(x)}(\Omega)$ is a Banach space endowed with the norm

$$|(u, v)| := \max\{||\nabla u|_{p_1(x)}, |\nabla v|_{p_2(x)}\}.$$

Since $|\tilde{f}_i(t)| \leq C$, $t \in \mathbb{R}$ for some constant which does not depends on $i = 1, 2$ we have that $J$ is coercive. Thus $J$ has a minimum $(z_\lambda, w_\lambda) \in W_0^{1, p_1(x)}(\Omega) \times W_0^{1, p_2(x)}(\Omega)$ with

$$-\Delta_{p_1(x)} z_\lambda = \lambda \tilde{f}_1(z_\lambda) \quad \text{in } \Omega, \quad z_\lambda = 0 \quad \text{on } \partial \Omega,$$

and

$$-\Delta_{p_2(x)} w_\lambda = \lambda \tilde{f}_2(w_\lambda) \quad \text{in } \Omega, \quad w_\lambda = 0 \quad \text{on } \partial \Omega.$$  \hspace{1cm} (4.31)

(4.32)

Note that the unique solutions of \(\ref{4.31}\) and \(\ref{4.32}\) are given by the minimizers of functionals $J_{1, \lambda}$ and $J_{2, \lambda}$ respectively.

Consider a function $\varphi_0 \in W_0^{1, p_1(x)}(\Omega), i = 1, 2$ with $\tilde{F}_i(\varphi_0) > 0$, $i = 1, 2$. Define $(z_0, w_0) := (z_{\bar{\lambda}_0}, w_{\bar{\lambda}_0})$, where $\tilde{\lambda}_0$ satisfies

$$\int_\Omega \frac{1}{p_1(x)}|\nabla \varphi_0|^{p_1(x)}dx < \tilde{\lambda}_0 \int_\Omega \tilde{F}_i(\varphi_0)dx, \quad i = 1, 2.$$

We have $J_{1, \tilde{\lambda}_0}(z_0) \leq J_{1, \tilde{\lambda}_0}(\varphi_0) < 0$ and that $J_{2, \tilde{\lambda}_0}(z_0) < 0$. Therefore $z_0 \neq 0$ and $w_0 \neq 0$. Since $-\Delta_{p_1(x)} z_0$ and $-\Delta_{p_2(x)} w_0$ are nonnegative, we have $z_0, w_0 > 0$ in $\Omega$. Note that by \cite{28} Theorem 4.1 and \cite{25} Theorem 1.2, we obtain that $z_0, w_0 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1]$.

Using the test function $\varphi = (z_0 - \theta_1)^+ \in W_0^{1, p_1(x)}(\Omega)$ in \(\ref{4.31}\) we obtain

$$\int_{\{z_0 > \theta_1\}} |\nabla z_0|^{p_1(x)} - 2\nabla z_0 \nabla (z_0 - \theta_1)^+ dx = \tilde{\lambda}_0 \int_{\{z_0 > \theta_1\}} \tilde{f}_1(z_0)(z_0 - \theta_1)dx = 0.$$

Therefore,

$$\int_{\{z_0 > \theta_1\}} \langle |\nabla z_0|^{p_1(x)} - 2\nabla z_0 - |\nabla \theta_1|^{p_1(x)} - 2\nabla \theta_1, \nabla (z_0 - \theta_1)\rangle dx = 0,$$

which imply $(z_0 - \theta_1)^+ = 0$ in $\Omega$. Thus $0 < z_0 \leq \theta_1$. A similar reasoning provides $0 < w_0 \leq \theta_2$.

Note that there is a constant $C > 0$ such that $|z_0|_{L^{r_2}(\Omega)}^{\alpha_1(\varepsilon)}$, $|w_0|_{L^{r_2}(\Omega)}^{\alpha_2(\varepsilon)} \geq C$. We define

$$A_0 = \max \{A(x, t) : (x, t) \in \overline{\Omega} \times [\min\{|z_0|_{L^{r_2}(\Omega)}^1, |w_0|_{L^{r_2}(\Omega)}^1\}],$$
max\{\|\theta_1\|_{L^{r_2}(x)}, \|\theta_2\|_{L^{r_1}(x)}\}\}
and \mu_0 = \frac{A_0}{C_0}. Then, we have
\begin{align*}
-\Delta p_1(x)z_0 &= \tilde{\lambda}_0 f_1(z_0) \\
&= \frac{1}{A_0} \tilde{\lambda}_0 \mu_0 f_1(z_0)\|w_0\|_{L^{r_1}(x)}^{\alpha_1(x)} \frac{A_0}{\mu_0} \|z_0\|_{L^{r_1}(x)}^{\alpha_2(x)} \\
&\leq \frac{1}{A_0} \tilde{\lambda}_0 \mu_0 f_1(z_0)\|w_0\|_{L^{r_1}(x)}^{\alpha_1(x)}.
\end{align*}
Thus for each \(\lambda \geq \lambda_0 : = \tilde{\lambda}_0 \mu_0\) and \(w \in [w_0, \theta_2]\), we obtain
\begin{align*}
-\Delta p_1(x)z_0 \leq \frac{1}{A(x,|w|_{L^{r_1}(x)})} \lambda f_1(z_0)\|w_0\|_{L^{r_1}(x)}^{\alpha_1(x)}.
\end{align*}
If necessary, we can consider a larger \(\lambda_0 > 0\) such that
\begin{align*}
-\Delta p_2(x)w_0 \leq \frac{1}{A(x,|w|_{L^{r_2}(x)})} \lambda f_2(w_0)\|z_0\|_{L^{r_2}(x)}^{\alpha_2(x)},
\end{align*}
for all \(\lambda \geq \lambda_0\) and \(w \in [z_0, \theta_1]\).
Since \(f_i(\theta_i) = 0\), \(i = 1, 2\), we have that \((z_0, \theta_1)\) and \((w_0, \theta_2)\) are sub-super solutions pairs for \((4.30)\). The proof is complete. \(\square\)

We remark that is possible to use the functions \(\phi_i\) from the proof of Theorem 4.1 for problem \((4.30)\). However, more restrictions on the functions \(p_i, f_i, i = 1, 2\) are needed.

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