NONLOCAL PROBLEMS FOR HYPERBOLIC EQUATIONS
FROM THE VIEWPOINT OF STRONGLY REGULAR
BOUNDARY CONDITIONS

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Abstract. In this article, we consider a nonlocal problem for hyperbolic equation with integral conditions and show their close connection with the notion of strongly regular boundary conditions. This has an important bearing on the method of the study of solvability. We propose also a new approach which enables us to prove a unique solvability of the nonlocal problem with integral condition.

1. Introduction

In this article, we consider the nonlocal problem for hyperbolic equations
\[ Lu \equiv u_{tt} - (a(x,t)u_x)_x + c(x,t)u = f(x,t). \] (1.1)
The question is to find a solution of (1.1) in \( Q_T = (0,l) \times (0,T) \), with \( l, T < \infty \), satisfying the initial conditions
\[ u(x,0) = 0, \quad u_t(x,0) = 0 \] (1.2)
and nonlocal conditions
\[ \int_0^l K_i(x)u(x,t)\,dx = 0, \quad i = 1, 2. \] (1.3)

Various phenomena of modern natural science often lead to nonlocal problems on mathematical modeling, and nonlocal models turn out to be often more precise than local conditions; see [5]. Nonlocal problems form a relatively new division of differential equations theory and generate a need in developing some new methods of research [30]. Nowadays various nonlocal problems for partial differential equations are actively studied and one can find a lot of papers dealing with them; see [2, 9, 14, 13, 18] and references therein. We focus our attention on nonlocal problems with integral conditions for hyperbolic equations which have been studied in [1, 8, 4, 6, 12, 9, 25, 17, 19, 23, 27, 28]. Systematic studies of nonlocal problems with integral conditions originated with the papers by Cannon [10] and Kamynin [16]. These and further investigations of nonlocal problems show that classical methods most widely used to prove solvability of initial-boundary problems break down when applied to
nonlocal problems. Nowadays several methods have been devised for overcoming the difficulties arising because of nonlocal conditions.

It appears that conditions for the existence and uniqueness of a solution to the nonlocal problem are closely related to the notion of regular boundary conditions. It is known that the system of root functions of an ordinary differential operator with strongly regular boundary conditions form a Riesz basis in $L^2(0,1)$. This property is particularly useful for obtaining results on solvability of boundary problems. For convenience we state here a criterium for strong regularity of boundary-value conditions for $n=2$ in an easy-to-use form [24 pp. 72-73].

*Sturm-Liouville problem:* Consider

$$y'' + \lambda y = 0$$

with the conditions

$$a_1 y'(0) + b_1 y'(l) + a_0 y(0) + b_0 y(l) = 0,$$

$$c_1 y'(0) + d_1 y'(l) + c_0 y(0) + d_0 y(l) = 0.$$ (1.4)

If the coefficients in (1.4) satisfy one of the following sets of conditions

(I) $a_1 d_1 - b_1 c_1 \neq 0$;

(II) $a_1 d_1 - b_1 c_1 = 0, |a_1| + |b_1| > 0, b_1 c_0 + a_1 d_0 \neq 0$;

(III) $a_1 = b_1 = c_1 = d_1 = 0, a_0 d_0 - b_0 c_0 \neq 0$,

then (1.4) are strongly regular.

Before we return to the main problem (1.1)–(1.3), we mention one of initial works dealing with nonlocal problems. In 1897 Steclov [31] considered the problem for the heat equation with nonlocal boundary conditions

$$a_1 u_x(0,t) + b_1 u_x(l,t) + a_0 u(0,t) + b_0 u(l,t) = 0,$$

$$c_1 u_x(0,t) + d_1 u_x(l,t) + c_0 u(0,t) + d_0 u(l,t) = 0.$$ (1.5)

It is obvious that separating of variables in (1.5) leads to (1.4). Much later nonlocal problems with conditions of the form (1.5) were studied in [15, 14, 22] and other papers. A feature of the problems with nonlocal conditions is that operator generated by such conditions, in particular (1.5), is not self-adjoint. But if nonlocal conditions of the form (1.4) (as a result of the separation of variables in (1.5)) are strongly regular then there exists a unique solution to the nonlocal problem (see [15]).

Now let us return to problem (1.1)–(1.3). Note that (1.3) are first-kind integral conditions as both of them include only integral terms. (The kind of a nonlocal integral condition depends on presence or lack of terms containing a trace of the required solution or its derivative outside the integral). Such conditions cause a considerable difficulties when we try to show that (1.1)–(1.3) is solvable. One method has been advanced for overcoming this difficulty [27]. Its essential idea is as follows. We reduce the first-kind integral conditions to the second-kind ones. To do this, we suppose that $u(x,t)$ is a solution to (1.1)–(1.3), multiply (1.1) by $K_i(x)$
and integrate over \((0, l)\). As a result we obtain
\[
K_i(0)a(0, t)u_x(0, t) - K_i(l)a(l, t)u_x(l, t) - K'_i(0,a)(0, t)u(0, t) \\
+ K'_i(l)a(l, t)u(l, t) - \int_0^l [(K'_i(x)a(x, t))]x - K_i(x)c(x, t)]u(x, t)dx \\
= \int_0^l K_i(x)f(x, t)dx.
\] (1.6)

Let us denote
\[
a_1(t) = K_1(0)a(0, t), \quad b_1(t) = -K_1(l)a(l, t), \\
a_0(t) = -K'_1(0)a(0, t), \quad b_0(t) = K'_1(l)a(l, t), \\
c_1(t) = K_2(0)a(0, t), \quad d_1(t) = -K_2(l)a(l, t), \\
c_0(t) = -K'_2(0)a(0, t), \quad d_0(t) = K'_2(l)a(l, t),
\]
and write now (1.6) (omitting the arguments of coefficients) as
\[
a_1u_x(0, t) + b_1u_x(l, t) + a_0u(0, t) + b_0u(l, t) - \int_0^l H_1(x, t)u(x, t)dx = g_1(t), \\
c_1u_x(0, t) + d_1u_x(l, t) + c_0u(0, t) + d_0u(l, t) - \int_0^l H_2(x, t)u(x, t)dx = g_2(t).
\] (1.7)

This system may be interpreted as perturbed Steklov conditions [1.5] (see [31, 29]). Thus, we establish certain formal connections between (1.7) and (1.4). We will consider it essentially in the next section and show that the nonlocal problem has a unique solution if coefficients of non-perturbed part satisfy one of conditions (I)–(III). The choice of a method depends on a particular criterion.

2. Solvability of nonlocal problems

2.1. Formulation of the problem. It was mentioned in the introduction that integral conditions [1.3] can be reduced to second-kind integral conditions [1.7]. As problems [1.1]–[1.3] and [1.1]–[1.2], and [1.7] are equivalent [27], we will consider the problem with integral conditions [1.7]. For convenience we formulate this problem here with a new indexing: find in \(Q_T\) a solution of the hyperbolic equation
\[
u_{tt} - (a(x, t)u_x)_x + c(x, t)u = f(x, t)
\] (2.1)
satisfying the initial conditions
\[
u(x, 0) = 0, \quad \nu_t(x, 0) = 0, \quad x \in [0, l]
\] (2.2)
and nonlocal conditions \((t \in [0, T])\),
\[
a_1u_x(0, t) + b_1u_x(l, t) + a_0u(0, t) + b_0u(l, t) - \int_0^l H_1(x, t)u(x, t)dx = g_1(t), \\
c_1u_x(0, t) + d_1u_x(l, t) + c_0u(0, t) + d_0u(l, t) - \int_0^l H_2(x, t)u(x, t)dx = g_2(t).
\] (2.3)

Note that there is no loss of generality in supposing that initial conditions are homogeneous.
2.2. Criterium I. For all $t \in [0, T]$,
\[ \Delta_1 = a_1d_1 - b_1c_1 \neq 0. \] (2.4)
Solving (2.3) as a system with respect to $u_x(0, t), u_x(l, t)$, we obtain
\[
\begin{align*}
    a(0, t)u_x(0, t) + \alpha_{11}(t)u(0, t) + \alpha_{12}(t)u(l, t) + \int_0^l H_{11}(x, t)u(x, t)dx &= g_{11}(t), \\
    a(l, t)u_x(l, t) + \alpha_{21}(t)u(0, t) + \alpha_{22}(t)u(l, t) + \int_0^l H_{12}(x, t)u(x, t)dx &= g_{12}(t),
\end{align*}
\] (2.5)
\[
\begin{align*}
    \alpha_{11}(t) &= \frac{a_0d_1 - c_0b_1}{\Delta_1} a(0, t), & \alpha_{12}(t) &= \frac{b_0d_1 - d_0b_1}{\Delta_1} a(0, t), \\
    \alpha_{21}(t) &= \frac{a_0c_1 - c_0a_1}{\Delta_1} a(l, t), & \alpha_{22}(t) &= \frac{b_0c_1 - d_0a_1}{\Delta_1} a(l, t), \\
    H_{11}(x, t) &= \frac{d_1H_1(x, t) - b_1H_2(x, t)}{\Delta_1} a(0, t), \\
    H_{12}(x, t) &= \frac{c_1H_1(x, t) - a_1H_2(x, t)}{\Delta_1} a(l, t), \\
    g_{11}(t) &= \frac{(d_1g_1(t) - b_1g_2(t))a(0, t)}{\Delta_1}, & g_{12}(t) &= \frac{(c_1g_1(t) - a_1g_2(t))a(l, t)}{\Delta_1}.
\end{align*}
\]

This form of integral conditions enables to apply, with only little modifications, a well-known method for boundary-value problem [21], based on a priori estimates. In our view, this approach is effective for studying nonlocal problems with conditions of the form (2.5). It was used for some particular cases [6, 27] so we will not demonstrate it here in detail.

Problem 1. Find a solution $u(x, t)$ to (2.1) satisfying (2.2) and (2.5).

We consider the Sobolev space $W_2^1(Q_T)$ and denote
\[ \tilde{W}_2^1(Q_T) = \{ v(x, t) : v \in W_2^1(Q_T), \, v(x, T) = 0 \}. \]
Let $u(x, t)$ be a solution to the Problem I and $v \in \tilde{W}_2^1(Q_T)$. Using a standard method [21, p. 92] and taking into account (2.5) and $u_t(x, 0) = 0$ we derive the equality
\[
\begin{align*}
    &\int_0^T \int_0^l (-u_t v_t + au_x v_x + cuv) \, dx \, dt \\
    &- \int_0^T v(0, t)[\alpha_{11}u(0, t) + \alpha_{12}u(l, t)] \, dt + \int_0^T v(l, t)[\alpha_{21}u(0, t) + \alpha_{22}u(l, t)] \, dt \\
    &- \int_0^T v(0, t) \int_0^l H_{11}(x, t)u(x, t) \, dx \, dt + \int_0^T v(l, t) \int_0^l H_{12}(x, t)u(x, t) \, dx \, dt \\
    &= \int_0^T \int_0^l f v \, dx \, dt + \int_0^T v(0, t)g_{11}(t) \, dt - \int_0^T v(l, t)g_{12}(t) \, dt.
\end{align*}
\] (2.6)
A function $u \in W_2^1(Q_T)$ is said to be a weak solution to the Problem I if $u(x, 0) = 0$ and for every $v \in \tilde{W}(Q_T)$ the identity (2.6) holds.

Theorem 2.1. Assume that
(i) $a \in C(\bar{Q}_T) \cap C^1(\bar{Q}_T), \, c \in C(\bar{Q}_T), \, a(x, t) > 0$ for all $(x, t) \in \bar{Q}_T$;
(ii) \( H_{1i}, H_{1tt} \in C(Q_T), f \in L_2(Q_T), g_{1i} \in W^1_2(0,T), i = 1,2; \)

(iii) \( \alpha_{12} + \alpha_{21} = 0, \alpha_{11}\xi^2 + 2\alpha_{12}\xi\eta - \alpha_{22}\eta^2 \leq 0. \)

Then there exists a unique weak solution to Problem I.

The proof of this theorem for \( a(x,t) = 1 \) one can find in [27]. It is not too difficult to show this theorem for \( a(x,t) \) not constant.

2.3. Criterium II. Now let \( \Delta \equiv a_1d_1 - b_1c_1 = 0 \) and \( |a_1| + |b_1| > 0 \). We will not lose too much generality if suppose that the coefficients in (2.1) do not depend on \( t \). This assumption simplifies the computational work. Then (2.3) can be written as

\[
a_1 u_x(0,t) + b_1 u_x(l,t) + a_0 u(0,t) + b_0(t) u(l,t) - \int_0^l H_1(x) u(x,t) dx = g_1(t),
\]

\[
c_0 u(0,t) + d_0 u(l,t) - \int_0^l H_2(x) u(x,t) dx = g_2(t).
\]

Problem 2. Find a solution \( u(x,t) \) to (2.1) satisfying initial conditions

\[
u(x,0) = 0, \quad u_t(x,0) = 0
\]

and nonlocal conditions (2.7).

We can not give at once a definition of a weak solution to this problem as for Problem 1. In response to this, the following can be done.

Let \( u(x,t) \) be a solution to the Problem 2. Differentiating the second relation of (2.7) with respect to \( t \) twice we obtain:

\[
c_0 u_{tt}(0,t) + d_0 u_{tt}(l,t) + \int_0^l H_2(x) u_{tt} dx = g_2''(t).
\]

As \( u(x,t) \) satisfies (2.1), we have

\[
\int_0^l H_2(x) u_{tt}(x,t) dx = \int_0^l H_2(x)[(au_x)_x - cu + f] dx.
\]

After some manipulation,

\[
\int_0^l H_2(x)(au_x)_x dx = H_2(l)a(l)u_x(l,t) - H_2(0)a(0)u_x(0,t) - H_2'(0)a(0)u(0,t)
\]

\[
+ H_2'(0)a(0)u(0,t) + \int_0^l (H_2'(x)a(x))_x u(x,t) dx.
\]

Then the second relation in (2.7) becomes

\[
c_0 u_{tt}(0,t) + d_0 u_{tt}(l,t) - H_2(l)a(l)u_x(l,t) + H_2(0)a(0)u_x(0,t) + H_2'(0)a(l)u(l,t)
\]

\[
- H_2'(0)a(0)u(0,t) - \int_0^l [(H_2'(x)a(x))_x - H_2(x)c(x)] u(x,t) dx = g_{22}(t)
\]
Integrating this equality with respect to \( t \)
where \( g \). Equation (2.7) can be written as

\[
\begin{align*}
& a_1 u_x(0, t) + b_1 u_x(l, t) + a_0 u(0, t) + b_0(t)u(l, t) - \int_0^t H_1 u \, dx = g_1(t), \\
& - H_2(0)a(0)u_x(0, t) + H_2(l)a(l)u_x(l, t) + H'_2(0)a(0)u(0, t) - H'_2(l)a(l)u(l, t) \\
& + c_0 u_{tt}(0, t) + d_0 u_{tt}(l, t) - \int_0^t [(H'_2(x)a(x))_x - H_2(x)c(x)] u(x, t) \, dx \\
& = g_{22}(t).
\end{align*}
\]

If \( \Delta = a_1 H_2(l)a(l) + b_1 H_2(0)a(0) \neq 0 \), then we can solve system (2.8) with respect to \( u_x(0, t) \) and \( u_x(l, t) \):

\[
\begin{align*}
& a(0)u_x(0, t) = \alpha_{11} u(0, t) + \alpha_{12} u(l, t) + \beta_{11} u_{tt}(0, t) \\
& \quad + \beta_{12} u_{tt} + \int_0^t P_1 u \, dx + G_1(t), \\
& a(l)u_x(l, t) = \alpha_{21} u(0, t) + \alpha_{22} u(l, t) + \beta_{21} u_{tt}(0, t) \\
& \quad + \beta_{22} u_{tt} + \int_0^t P_2 u \, dx + G_2(t), \tag{2.9}
\end{align*}
\]

where \( \alpha_{ij}, \beta_{ij}, P_i(x), G_i(t) \) \( i, j = 1, 2 \) can be find easily, for example,

\[
\begin{align*}
& \alpha_{11} = \frac{H'_2(0)b_1a(0) - H_2(l)a_0a(l)}{\Delta_2} a(0), \quad \beta_{11} = \frac{c_0 b_1}{\Delta_2} a(0), \\
& P_1(x) = \frac{H_1(x)H_2(l)a(l) + (H'_2(x)a(x))_x b_1 - H_2(x)c(x)b_1}{\Delta_2} a(0).
\end{align*}
\]

(We do not cite all formulas because of their length). Conditions (2.9) are known as dynamical conditions \[11, 20, 33\].

Thus if (2.7) holds then (2.9) also holds. The converse is also true if \( g_2(0) = g'_2(0) = 0 \). Indeed, let \( u(x, t) \) be a solution of (2.1) and let (2.9) hold. Then (2.8) holds. Integrating \( \int_0^t (H'_2(x)a(x))_x u(x, t) \, dx \) by parts and taking into account that \( u(x, t) \) is a solution to (2.1) we easily arrive to

\[
\frac{d^2}{dt^2} [c_0 u(0, t) + d_0 u(l, t) + \int_0^t H_2(x)u(x, t) \, dx - g_2(t)] = 0.
\]

Integrating this equality with respect to \( t \) twice, taking into account homogeneous initial data \( c_0 u(0, 0) + d_0 u(l, 0) + \int_0^t H_2(x)u(x, 0) \, dx - g_2(0) = 0 \), \( c_0 u(0, 0) + d_0 u(l, 0) + \int_0^t H_2(x)u(x, 0) \, dx - g'_2(0) = 0 \) we obtain (2.7).

Thus the nonlocal conditions (2.7) and (2.9) are equivalent, so we will consider the Problem 2 as follows: find a solution \( u(x, t) \) to (2.1) satisfying (2.2) and (2.9).
This form of nonlocal conditions enables us to introduce a notation of a weak solution. Following [21, p. 92], we obtain
\[
\int_0^T \int_0^t (-u_t v_t + au_t v + cuv) \, dx \, dt + \int_0^T v(0, t)[\alpha_{11} u(0, t) + \alpha_{12} u(t)] \, dt \\
- \int_0^T v_t(0, t)[\beta_{11} u_t(0, t) + \beta_{12} u_t(t)] \, dt + \int_0^T v(0, t) \int_0^t P_1(x) u(x, t) \, dx \, dt \\
- \int_0^T v(l, t)[\alpha_{21} u(0, t) + \alpha_{22} u(t)] \, dt \\
+ \int_0^T v(l, t)[\beta_{21} u_t(0, t) + \beta_{22} u_t(t)] \, dt - \int_0^T v(l, t) \int_0^t P_2(x) u(x, t) \, dx \, dt
\] (2.10)

Let us denote
\[ \Gamma_0 = \{(x, t) : x = 0, t \in [0, T]\}, \quad \Gamma_1 = \{(x, t) : x = l, t \in [0, T]\}, \quad \Gamma = \Gamma_0 \cup \Gamma_1, \]
\[ W(Q_T) = \{u : u \in W_2^1(Q_T), \quad u_t \in L_2(\Gamma)\}, \]
\[ \bar{W}(Q_T) = \{v(x, t) : v(x, t) \in W(Q_T), \quad v(x, T) = 0\}. \]

A function \( u \in W(Q_T) \) is said to be a weak solution to the Problem 2 if (2.10) holds.

**Theorem 2.2.** Assume the following conditions
(i) \( a \in C(\bar{Q}_T), \quad a(x, t) > 0 \) for all \((x, t) \in \bar{Q}_T, \quad c \in C(\bar{Q}_T);\)
(ii) \( P_i \in C(Q_T), \quad f_i \in L_2(Q_T), \quad G_i \in C[0, T] \cap C^1(0, T);\)
(iii) \( \beta_{11} \xi^2 + 2\beta_{21} \xi \eta - \beta_{22} \eta^2 \geq 0;\)
(iv) \( \alpha_{12} + \alpha_{21} = 0, \quad \beta_{12} + \beta_{21} = 0;\)
\[ p(v) \beta_{11} > 0, \quad \beta_{22} < 0, \quad \beta_{11} - |\beta_{21}| > 0, \quad -\beta_{22} - |\beta_{21}| > 0. \]

Then there exists a unique weak solution to Problem 2.

**Proof. Uniqueness.** Suppose that \( u_1 \) and \( u_2 \) are two solutions to Problem 2. Then \( u = u_1 - u_2 \) satisfies initial condition \( u(x, 0) = 0 \), and the equation
\[
\int_0^T \int_0^t (-u_t v_t + au_t v + cuv) \, dx \, dt + \int_0^T v(0, t)[\alpha_{11} u(0, t) + \alpha_{12} u(t)] \, dt \\
- \int_0^T v_t(0, t)[\beta_{11} u_t(0, t) + \beta_{12} u_t(t)] \, dt + \int_0^T v(0, t) \int_0^t P_1(x) u(x, t) \, dx \, dt \\
- \int_0^T v(l, t)[\alpha_{21} u(0, t) + \alpha_{22} u(t)] \, dt \\
+ \int_0^T v(l, t)[\beta_{21} u_t(0, t) + \beta_{22} u_t(t)] \, dt - \int_0^T v(l, t) \int_0^t P_2(x) u(x, t) \, dx \, dt = 0.
\]

Setting
\[ v(x, t) = \begin{cases} \int_0^t u(x, \eta) \, d\eta, & 0 \leq t \leq \tau, \\ 0, & \tau \leq t \leq T, \end{cases} \]

...
where $\tau \in [0,T]$ is arbitrary, and after some manipulation we obtain

$$
\int_0^l [u^2(x, \tau) + a(x)v_x^2(x, 0)] dx
$$

$$
= 2 \int_0^l \int_0^\tau c u v dx dt - \beta_{11}u^2(0, \tau) + 2\beta_{21}u(0, \tau)u(l, \tau) + \beta_{22}u^2(l, \tau)
+ \alpha_{22}v_x^2(l, 0) + 2\alpha_{21}v(0, 0)v(l, 0) - \alpha_{11}v^2(0, 0)
+ 2 \int_0^l (\alpha_{12} + \alpha_{21})v(0, t)u_t(l, t) dt - 2 \int_0^\tau (\beta_{12} + \beta_{21})u(0, t)u_t(l, t) dt
+ \int_0^\tau v(0, t) \int_0^l P_1(x)u(x, t) dx dt - \int_0^\tau v(l, t) \int_0^l P_2(x)u(x, t) dx dt.
$$

(2.11)

Taking into account condition (iii), namely $\beta_{11}\xi^2 + 2\beta_{21}\xi\eta - \beta_{22}\eta^2 \geq 0$, and (iii) of Theorem 2.2 we obtain

$$
\int_0^l [u^2(x, \tau) + a(x)v_x^2(x, 0)] dx
\leq 2 \int_0^l \int_0^\tau c(x)u(x, t)v(x, t) dx dt + \alpha_{22}v_x^2(l, 0) + 2\alpha_{21}v(0, 0)v(l, 0)
- \alpha_{11}v^2(0, 0) + \int_0^\tau v(0, t) \int_0^l P_1(x)u(x, t) dx dt
- \int_0^\tau v(l, t) \int_0^l P_2(x)u(x, t) dx dt.
$$

(2.12)

Note that under conditions of Theorem 2.2 there exist positive numbers $a_0, c_0, p$ such that

$$
\max_{[0,l]} |c(x)| \leq c_0, \quad a(x) \geq a_0, \quad \max_i \int_0^l P_{i2}^2(x) dx \leq p.
$$

Let us denote $A = \max_{ij} |\alpha_{ij}|$. Now we estimate right-hand side of (2.12). Firstly, we use Cauchy and Cauchy-Bunyakovskii-Schwartz inequalities to obtain

$$
\int_0^l [u^2(x, \tau) + a(x)v_x^2(x, 0)] dx
\leq \int_0^l [u^2(x, \tau) + a(x)v_x^2(x, 0)] dx
\leq c_0 \int_0^\tau \int_0^l [u^2(x, t) + v^2(x, t)] dx dt + A[v^2(0, 0) + v^2(l, 0)]
+ \int_0^\tau [v^2(0, \tau) + v^2(l, \tau)] dt + 2p \int_0^\tau \int_0^l u^2(x, t) dx dt.
$$

Using trace inequalities,

$$
v^2(z_1, t) \leq 2l \int_0^l v_x^2(x, t) dx + \frac{2}{l} \int_0^l v^2(x, t) dx, \quad z_1 = 0, z_2 = l,
$$

(both are derived from $v(z_1, t) = \int_{z_1}^{z_2} v_\xi(\xi, t)d\xi + v(z_1, t)$), we obtain

$$
\int_0^\tau [v^2(0, \tau) + v^2(l, \tau)] dt \leq 4l \int_0^\tau \int_0^l v_x^2(x, t) dx dt + \frac{4}{l} \int_0^\tau \int_0^l v^2(x, t) dx dt.
$$
To estimate $A[v^2(0, 0) + v^2(l, 0)]$ we use the following inequalities (a partial case for $n = 1$ in [21 p.77]):

$$v^2(z_i, t) \leq \varepsilon \int_0^l v_x^2(x, t)dx + c(\varepsilon) \int_0^l v^2(x, t)dx,$$

where $z_1 = 0$, $z_2 = l$ and $t \in [0, \tau]$. Then we obtain

$$v^2(0, 0) \leq \varepsilon \int_0^l v_x^2(x, 0)dx + c(\varepsilon) \int_0^l v^2(x, 0)dx,$$

$$v^2(l, 0) \leq \varepsilon \int_0^l v_x^2(x, 0)dx + c(\varepsilon) \int_0^l v^2(x, 0)dx,$$

where $c(\varepsilon) = (l + \varepsilon)/l \varepsilon$. We note also that from representation of $v(x, t)$ it follows that

$$v^2(x, t) \leq \tau \int_0^\tau u^2(x, t) dt.$$

Hence,

$$\int_0^\tau \int_0^l v^2(x, t) dt dx dt \leq \tau^2 \int_0^\tau \int_0^l u^2(x, t) dx dt,$$

$$\int_0^l v^2(x, 0)dx \leq \tau \int_0^\tau \int_0^l u^2(x, t) dx dt.$$

Choosing $\varepsilon$ with due care ($\varepsilon = a_0/4$, then $a_0 - 2\varepsilon > 0$) we obtain

$$\int_0^l [u^2(x, \tau) + \frac{a_0}{2} v_x^2(x, 0)]dx \leq M \int_0^\tau \int_0^l (u^2(x, t) + v_x^2(x, t)) dx dt, \quad (2.13)$$

where $M = \max\{c_0 + 2p, (c_0 + \frac{2}{3})^2 A, 4l\}$.

Let $w(x, t) = \int_0^t u_x(x, \eta) d\eta$. Then

$$v_x(x, t) = w(x, t) - w(x, \tau), \quad v_x(x, 0) = -w(x, \tau).$$

With the aid of these equalities we obtain

$$\int_0^l [u^2(x, \tau) + \frac{a_0}{2} w^2(x, \tau)]dx \leq 2M \int_0^\tau \int_0^l [u^2 + w^2] dx dt + 2M \tau \int_0^l w^2(x, \tau) dx dt.$$

As $\tau$ is arbitrary we choose it so that $a_0 - 4M\tau > 0$. To be specific, let $a_0 - 4M\tau \geq \frac{a_0}{3M}$. Then for all $\tau \in [0, \frac{a_0}{3M}]$

$$m_1 \int_0^l [u^2(x, \tau) + w^2(x, \tau)]dx \leq 2M \int_0^\tau \int_0^l (u^2 + w^2) dx dt,$$

where $m_1 = \min\{1, a_0/4\}$.

From Gronwall’s lemma it follows that $\int_0^l [u^2(x, \tau) + w^2(x, \tau)]dx = 0$. Hence $u(x, \tau) = 0$ for all $\tau \in [0, \frac{a_0}{3M}]$. Using the same reasoning as in [21 p.212], we obtain $u(x, t) = 0$ in $Q_T$. It means that there cannot be more than one weak solution to the Problem II.

**Existence.** First, we construct approximations of the weak solution by the Faedo-Galerkin method. Let $w_k(x) \in C^2[0, l]$ be a basis in $W^1_2(0, l)$. We define the approximations as follows

$$u^m(x, t) = \sum_{k=1}^m c_k(t) w_k(x) \quad (2.14)$$
and shall seek \( c_k(t) \) from the equations
\[
\int_0^t \left( u_i^m w_j + a u_i^m w_j' + cu^m w_j \right) dx + w_k(0) \left[ \alpha_{11} u_m^m(0, t) + \alpha_{12} u^m(l, t) \right] \\
+ \beta_{11} u_i^m(0, t) + \beta_{12} u^m(l, t) + \int_0^t P_1(x) u(x, t) dx \\
- w_k(l) \left[ \alpha_{21} u^m(0, t) + \alpha_{22} u^m(l, t) + \beta_{21} u_i^m(0, t) + \beta_{22} u^m(l, t) \right] \\
+ \int_0^t P_2(x) u(x, t) dx \\
= \int_0^t f(x, t) w_j(x) dx - w_j(0) G_1(t) + w_j(t) G_2(t).
\]
(2.15)

For every \( m \), (2.15) represents a system of second-order ODE’s with respect to \( c_k(t) \),
\[
\sum_{k=1}^m A_{kj} c_k''(t) + \sum_{k=1}^m B_{kj} c_k(t) = f_j(t),
\]
(2.16)
where
\[
A_{kj} = \int_0^t w_k(x)w_j(x) dx + \beta_{11} w_k(0)w_j(0) + \beta_{12} w_k(l)w_j(0) \\
- \beta_{21} w_k(0)w_j(l) - \beta_{22} w_k(l) w_j(l),
\]
\[
B_{kj} = \int_0^t \left( a(x)w_j'(x)w_j'(x) + c(x)w_k(x)w_j(x) \right) dx + w_k(0) \int_0^t P_1(x) w_j(x) dx \\
- w_k(l) \int_0^t P_2(x) w_j(x) dx + \alpha_{11} w_k(0)w_j(0) + \alpha_{12} w_k(l)w_j(0) \\
- \alpha_{21} w_k(0) w_j(l) - \alpha_{22} w_k(l) w_j(0),
\]
\[
f_j(t) = \int_0^t f(x, t) w_j(x) dx - w_j(0) G_1(t) + w_j(t) G_2(t).
\]

Firstly we prove that this system is solvable with respect to \( c_k''(t) \). To this end consider the matrix \( \{ A_{ij} \} \) and show that it is positive definite.

Consider the quadratic form \( q = \sum_{i,j=1}^m A_{kj} \xi_k \xi_j \) and denote \( z(x) = \sum_{k=1}^m \xi_k w_k(x) \). After substituting \( A_{kj} \) in \( q \) we obtain
\[
q = \int_0^t |z(x)|^2 dx + \beta_{11} |z(0)|^2 + 2 \beta_{12} |z(0)||z(l)| - \beta_{22} |z(l)|^2.
\]

From (iii) of Theorem 2.2 \( q \geq 0 \). As \( q = 0 \) if and only if \( z = 0 \) and \( \{ w_k \} \) is linearly independent then \( \xi_k = 0 \) for \( k = 1, \ldots, m \), therefore \( q \) is positive definite. Hence (2.16) is solvable with respect to \( c_k''(t) \). Thus we can state that under conditions of Theorem 2.2 Cauchy problem for (2.16) with initial conditions \( c_k(0) = 0 \), \( c_k'(0) = 0 \) has a solution for every \( m \) and \( \{ u^m \} \) is constructed.

To derive the estimate we multiply (2.15) by \( c_j'(t) \), sum over \( j = 1, \ldots, m \) and integrate over \( (0, \tau) \), where \( \tau \in [0, T] \) is arbitrary:
\[
\int_0^\tau \int_0^t \left( u_i^m u_t^m + a u_i^m u_{xt}^m + cu^m u_{tt}^m \right) dx dt
\]
Integration by parts and condition (iii) lead to

\[ \int_0^1 [(u_i^m(x, \tau))^2 + a(x)(u_i^m(x, \tau))^2] dx + \beta_{11}(u_i^m(0, \tau))^2 - \beta_{22}(u_i^m(l, \tau))^2 \]

\[ = 2\beta_{21}u_i^m(0, \tau)u_i^m(l, \tau) - \alpha_{11}(u_i^m(0, \tau))^2 - 2\alpha_{21}u_i^m(0, \tau)u_i^m(l, \tau) \]

\[ - \alpha_{22}(u_i^m(l, \tau))^2 - 2 \int_0^\tau \int_0^1 cu_i^m u_i^m dx dt \]

\[ + 2 \int_0^\tau u_i^m(0, t) \int_0^1 P_1(x)u_i^m dx dt - 2u_i^m(0, \tau) \int_0^1 P_1(x)u_i^m(x, \tau)dx \]

(2.17)

\[ - 2 \int_0^\tau u_i^m(l, t) \int_0^1 P_2(x)u_i^m dx dt + 2u_i^m(l, \tau) \int_0^1 P_2(x)u_i^m(x, \tau)dx \]

\[ + 2 \int_0^\tau \int_0^1 f u_i^m dx dt + 2 \int_0^\tau u_i^m(0, t)G_{1t}(t) dt - 2 \int_0^\tau u_i^m(l, t)G_{2t}(t) dt \]

\[ + 2u_i^m(0, \tau)G_{1}(\tau) - 2u_i^m(l, \tau)G_{2}(\tau). \]

As \( \beta_{11} > 0 \) and \( \beta_{22} < 0 \), By (iii) of Theorem 2.2, the left-hand side of (2.17) is positive. To estimate right-hand side of (2.17) we use the same technique as in the subsection for uniqueness. Therefore we demonstrate this procedure briefly. Using Cauchy and Cauchy-Bunyakovskii-Schwartz inequalities we obtain

\[ \int_0^l [(u_i^m(x, \tau))^2 + a_0(u_i^m(x, \tau))^2] dx + \beta_{11}(u_i^m(0, \tau))^2 - \beta_{22}(u_i^m(l, \tau))^2 \]

\[ \leq C_1 \int_0^\tau \int_0^1 (u_i^m(x, t))^2 dx dt + 2p \int_0^\tau (u_i^m(x, \tau))^2 dx dt \]

\[ + 2 \int_0^\tau [(u_i^m(0, t))^2 + (u_i^m(l, t))^2] dt \]

(2.18)

\[ + (2 + \sqrt{|a_{21}|})(u_i^m(0, \tau))^2 + (u_i^m(l, \tau))^2 \]

\[ + \sqrt{|b_{21}|}(u_i^m(0, \tau))^2 + (u_i^m(l, \tau))^2 + \int_0^\tau \int_0^1 f^2(x, t) dx dt \]

\[ + \int_0^\tau [(G_{1t}(t))^2 + (G_{2t}(t))^2] dt + G_1^2(\tau) + G_2^2(\tau). \]
where $C_1$ depends on $a_0, c_0, p, l, T$ and do not depend on $m$. Using the inequality
\[ v^2(z_1, t) \leq \epsilon \int_0^l v_x^2(x, t)dx + c(\epsilon) \int_0^l v^2(x, t)dx, \quad z_1 = 0, \ z_2 = l \]
we obtain
\[
(u^m(0, \tau))^2 + (u^m(l, \tau))^2 \leq 2\epsilon \int_0^l (u^m_x(x, \tau))^2 dx + 2c(\epsilon) \int_0^l (u^m(x, \tau))^2 dx,
\]
\[
\int_0^\tau [(u^m(0, \tau))^2 + (u^m(l, \tau))^2] dt \leq 2\epsilon \int_0^l \int_0^l (u^m_x(x, \tau))^2 dx dt + 2c(\epsilon) \int_0^l \int_0^l (u^m(x, \tau))^2 dx dt.
\]
Taking into account (iv) in Theorem 2.2.2, $(u^m(x, \tau))^2 \leq \tau \int_0^\tau (u^m_t(x, t))^2 dt$, and adding the inequality $\int_0^l (u^m(x, \tau))^2 dx \leq \tau \int_0^\tau \int_0^l (u^m_t(x, t))^2 dx dt$ to the both sides of (2.18), we obtain
\[
\int_0^l [(u^m(x, \tau))^2 + (u^m_t(x, \tau))^2 + a_0(u^m_x(x, \tau))^2] dx
+ (\beta_{11} - \sqrt{|b_{21}|})(u^m_t(0, \tau))^2 + (-\beta_{22} - \sqrt{|b_{21}|})(u^m_t(l, \tau))^2
\leq C_2 \int_0^\tau \int_0^l [(u^m(x, t))^2 + (u^m_t(x, t))^2 + (u^m_x(x, t))^2] dx dt
+ 2\sqrt{|a_{21}|}\epsilon \int_0^l (u^m_x(x, \tau))^2 dx + \int_0^\tau \int_0^l f^2(x, t) dx dt
+ \int_0^\tau [(G_{1t}(\tau))^2 + (G_{2t}(\tau))^2] dt + G_2^2(\tau) + G_3^2(\tau).
\]
Choosing $\epsilon$ such that $\nu = a_0 - 2\sqrt{|a_{21}|}\epsilon > 0$, we can carry $2\sqrt{|a_{21}|}\epsilon \int_0^l (u^m_x(x, \tau))^2 dx$ to the left-hand side of (2.19). Consequently,
\[
\int_0^l [(u^m(x, \tau))^2 + (u^m_t(x, \tau))^2 + (u^m_x(x, \tau))^2] dx + [(u^m_t(0, \tau))^2 + (u^m_t(l, \tau))^2]
\leq C_3 \int_0^\tau \int_0^l [(u^m(x, t))^2 + (u^m_t(x, t))^2 + (u^m_x(x, t))^2] dx dt
+ C_4 \left( \int_0^\tau \int_0^l f^2(x, t) dx dt + \int_0^\tau [(G_{1t}(\tau))^2 + (G_{2t}(\tau))^2] dt
+ G_2^2(\tau) + G_3^2(\tau) \right).
\]
In particular,
\[
\int_0^l [(u^m(x, \tau))^2 + (u^m_t(x, \tau))^2] dx
\leq C_3 \int_0^\tau \int_0^l [(u^m(x, t))^2 + (u^m_t(x, t))^2] dx dt
+ C_4 \left( \int_0^\tau \int_0^l f^2(x, t) dx dt + \int_0^\tau [(G_{1t}(\tau))^2 + (G_{2t}(\tau))^2] dt
+ G_2^2(\tau) + G_3^2(\tau) \right).
with respect to $v$ function $v$. The main idea of this method is the following: we introduce a new unknown $\text{Namely, we differentiate both equations in (2.22) with respect to $t$ not depend on $S$ at dynamic nonlocal conditions. This method works for for a partial case: Find a solution $u(T)$.

Applying Gronwall's lemma to the above inequality, after integrating over $(0, T)$, we obtain

$$
\|u^m\|_{W^2_t(Q_T)} \leq r_1,
$$

where $r_1 = C_4 T e^{C_5 T} (\|f\|_{L^2(Q_T)}^2 + \|G_1\|_{L^2_t(0, T)}^2 + \|G_2\|_{L^2_t(0, T)}^2)$.

Moreover, it follows also from (2.20) that

$$(u_t^m(0, \tau))^2 + (u_t^m(l, \tau))^2 \leq C_3 \|u\|_{W^2_t(Q_T)}^2 + C_4 \int_0^\tau \int_0^l f^2(x, t) \, dx \, dt + C_4 \left( \int_0^\tau [(G_{11}(t))^2 + (G_{21}(t))^2] \, dt + G_1^2(\tau) + G_2^2(\tau) \right).$$

Then

$$
\|u^m\|_{L^2(\Gamma)} \leq r_2,
$$

where $r_2 = TC_3 r_1 + TC_4 (\|f\|_{L^2_t(Q_T)}^2 + \|G_1\|_{L^2_t(0, T)}^2 + \|G_2\|_{L^2_t(0, T)}^2)$.

Thus we have a priori estimate,

$$
\|u^m\|_{W^2(\Omega_T)} \leq R, \quad R = \max_i \{r_i\}, \quad i = 1, 2.
$$

Because of (2.21), we can extract a subsequence $\{u^\mu\}$ from $\{u^m\}$ such that as $\mu \to \infty$ $\{u^\mu\}$ converges weakly to $u \in W(Q_T)$. This enables us to use standard technique [21, pp. 214-215] and show that the limit of $\{u^\mu\}$ is the required weak solution to Problem 2.

2.4. **Criterium III.** Let $a_1 = b_1 = c_1 = d_1 = 0$, $\Delta_3 = a_0 d_0 - b_0 c_0 \neq 0$. Then (2.3) can be write as

$$
\begin{align*}
&u(0, t) + \int_0^l S_1(x, t) u(x, t) \, dx = g_{31}(t), \\
&u(l, t) + \int_0^l S_2(x, t) u(x, t) \, dx = g_{32}(t),
\end{align*}
$$

where

$$
\begin{align*}
S_1(x, t) &= \frac{b_0 H_2(x, t) - d_0 H_1(x, t)}{\Delta_3}, \\
S_2(x, t) &= \frac{c_0 H_1(x, t) - a_0 H_2(x, t)}{\Delta_3}, \\
g_{31}(t) &= \frac{d_0 g_1(t) - b_0 g_2(t)}{\Delta_3}, \\
g_{32}(t) &= \frac{a_0 g_2(t) - c_0 g_1(t)}{\Delta_3}.
\end{align*}
$$

**Problem 3.** Find a solution $u(x, t)$ to equation (2.1) satisfying (2.2) and (2.22).

We can use at least two methods to show the solvability of the Problem 3. One of them may be considered as a particular case of the method used in earlier section. Namely, we differentiate both equations in (2.22) with respect to $t$ twice and arrive at dynamic nonlocal conditions. This method works for for a partial case: $S_i$ does not depend on $t$ [28]. However, this method is not effective when $S_i$ depend on $t$ also.

The second method we can apply is a particular case of the technique initiated in [17]. The main idea of this method is the following: we introduce a new unknown function $v(x, t)$ and arrive at the boundary-value problem for a loaded equation with respect to $v(x, t)$ and can use the technique represented in [17].
Here we propose a third way. Using the idea in [17] to form a new unknown function. We suggest a different method.

A function $u(x, t)$ is said to be the solution to the Problem 3 if it satisfies equation (2.1) for almost all $(x, t) \in Q_T$, the initial condition (2.2), and conditions (2.22) in the $L_2(0, T)$ sense.

**Theorem 2.3.** Assume that: $a, a_t, a_x, a_{tt}, c, c_t \in C(Q_T)$, $a_0, b_0, c_0, d_0 \in C^2(0, T)$,

$$S_i, S_{tt} \in C^2(\bar{Q}_T), \quad S_i(0, t) = S_i(t, 0) = 0, \quad \frac{2}{3} \int_0^t (S_1 + S_2)^2 d\xi < 1$$

for all $t \in [0, T]$, $g_{3i} \in C^3[0, T]$, $g_{3i}(x, 0) = g'_{3i}(x, 0) = 0$, and $f, f_t \in L_2(Q_T)$. Then there exists a unique solution $u(x, t)$ to the problem 3.

The proof is rather long, so we break it up into 3 steps.

**Step 1.** Reduction to a problem for a loaded equation. Let $u(x, t)$ be a solution to the Problem 3. We introduce a new function

$$v(x, t) = u(x, t) + \int_0^l \tilde{S}(x, \xi, t)u(\xi, t)d\xi - \tilde{g}(x, t)$$

where

$$\tilde{S}(x, \xi, t) = \frac{l - x}{l}S_1(\xi, t) + \frac{x}{l}S_2(\xi, t), \quad \tilde{g}(x, t) = \frac{l - x}{l}g_{31}(t) + \frac{x}{l}g_{32}(t).$$

Then $v(x, t)$ satisfies the equation

$$v_{tt} - (av_x)_x + cv - \int_0^l (\tilde{S}u)_{tt}d\xi + (a(x, t)\int_0^l \tilde{S}_x(x, \xi, t)u(\xi, t))_x d\xi$$

$$- c(x, t) \int_0^l \tilde{S}(x, \xi, t)u(\xi, t)d\xi$$

$$= f + \tilde{g}_{tt} + c\tilde{g} + a_x \tilde{g}_x.$$  

As $u(x, t)$ satisfies (2.1), then

$$\int_0^l (\tilde{S}u)_{tt}d\xi = \int_0^l \tilde{S}_{tt}d\xi + 2\int_0^l \tilde{S}_t u_d\xi + \int_0^l \tilde{S}[(au_x)_x - c(\xi, t)u + f]d\xi.$$  

After little manipulations and taking into account the assumptions of Theorem 2.3 we obtain

$$v_{tt} - (av_x)_x + cv$$

$$= \int_0^l M(x, \xi, t)u(\xi, t)d\xi + 2\int_0^l \tilde{S}_t u_d\xi$$

$$+ \tilde{S}(x, 0, t)a(0, t)u(0, t) - \tilde{S}(x, l, t)a(l, t)u(l, t) + G(x, t),$$  

where

$$M(x, \xi, t) = \tilde{S}_{tt}(x, \xi, t) + (a(\xi, t)\tilde{S}_x(x, \xi, t))_x - (a(x, t)\tilde{S}_x(x, \xi, t))_x$$

$$+ |c(x, t) - c(\xi, t)|\tilde{S}(x, \xi, t),$$

$$G(x, t) = f(x, t) + \tilde{g}_{tt}(x, t) + c(x, t)\tilde{g}(x, t) + a_x(x, t)\tilde{g}_x(x, t)$$

$$+ \int_0^l \tilde{S}(x, \xi, t)f(\xi, t)d\xi.$$
It is easy to see that
\[ v(x,0) = v_t(x,0) = 0, \quad v(0,t) = v(l,t) = 0 \] (2.24)
and we arrive to the next problem.

**Problem 4.** Find a solution to equation (2.23) satisfying (2.24) Note that we are required to find not only \( v(x,t) \), but also \( u(x,t) \). Let us denote
\[ W^1_{2,0}(Q_T) = \{ v(x,t) : v \in W^1_2(Q_T), v(0,t) = v(l,t) = 0 \} \]
\[ \tilde{W}^1_{2,0}(Q_T) = \{ \eta(x,t) : \eta \in W^1_{2,0}(Q_T), \eta(x,T) = 0 \}. \]

A pair \((u,v)\) is said to be a weak solution to Problem 4 if \( u \in W^1_2(Q_T), v \in W^1_{2,0}(Q_T), v(x,0) = 0 \), for every \( \eta \in \tilde{W}^1_{2,0}(Q_T) \):
\[
\int_0^T \int_0^l (-v_t \eta_t + av_x \eta_x + cv \eta) \, dx \, dt \\
= \int_0^T \int_0^l \eta(x,t) \int_0^l M(x,\xi,t) u(\xi,t) \, d\xi \, dx \, dt \\
+ \int_0^T \int_0^l \eta(x,t) [\tilde{S}_\xi(x,0,t)a(0,t)u(0,t) - \tilde{S}_\xi(x,l,t)a(l,t)u(l,t)] \, dx \, dt \\
+ 2 \int_0^T \int_0^l \eta(x,t) \int_0^l \tilde{S}_i u_i \, d\xi \, dx \, dt + \int_0^T \int_0^l G(x,t) \eta(x,t) \, dx \, dt
\] (2.25)
and \( u, v \) are related by
\[
v(x,t) = u(x,t) + \int_0^l \tilde{S}(x,\xi,t) u(\xi,t) \, d\xi - \tilde{g}(x,t). \] (2.26)

**Step 2.** Solvability of Problem 4.

**Theorem 2.4.** Under the assumptions of Theorem 2.3 there exists a unique weak solution \((u,v)\) to Problem 4.

**Proof.** We approximate our weak solution as follows. Let \( u^0 = 0 \) and define \((u^n, v^n)\) by
\[
\int_0^T \int_0^l (-v^n_t \eta_t + av^n_x \eta_x + cv^n \eta) \, dx \, dt \\
= \int_0^T \int_0^l \eta(x,t) \int_0^l M(x,\xi,t) u^{n-1}(\xi,t) \, d\xi \, dx \, dt \\
+ \int_0^T \int_0^l \eta(x,t) [\tilde{S}_\xi(x,0,t)a(0,t)u^{n-1}(0,t) - \tilde{S}_\xi(x,l,t)a(l,t)u^{n-1}(l,t)] \, dx \, dt \\
+ 2 \int_0^T \int_0^l \eta(x,t) \int_0^l \tilde{S}_i u^{n-1}_i \, d\xi \, dx \, dt + \int_0^T \int_0^l G(x,t) \eta(x,t) \, dx \, dt,
\] (2.27)
\[
v^n(x,t) = u^n(x,t) + \int_0^l \tilde{S}(x,\xi,t) u^n(\xi,t) \, d\xi - \tilde{g}(x,t). \] (2.28)
As \( u_0 = 0 \), then for \( v^1 \) we have

\[
\int_0^T \int_0^l (-v^1_{tt} \eta_t + a v^1_x \eta_x + c v^1) \, dx \, dt = \int_0^T \int_0^l G \eta \, dx \, dt.
\]

This means that \( v^1(x, t) \) is a weak solution of the first initial boundary problem for

\[
v_{tt} - (av_x)_x + cv = G(x, t).
\]  

(2.29)

It is known 21 pp. 213-215 that this solution is unique and \( \|v^1\|_{W^1_2} (Q_T) \leq C \|G\|_{L_2(Q_T)} \). Moreover, as \( G_t \in L_2(Q_T) \) and \( a, a_t, a_{tt}, c_t \) are bounded then \( v^1 \in W^2_2(Q_T) \)21 pp. 216-219.

Now we can find \( u^1(x, t) \) from (2.28) as under assumptions of Theorem 2.4 provide that for every \( n \) \( G_n, G_{nt} \in L_2(Q_T) \). So \( u^n, v^n \in W_2^3(Q_T) \) and the sequence of pairs \( (u^n, v^n) \) is well defined.

Now we show that this sequence converges as \( n \rightarrow \infty \) in \( W^1_2 \) and the limit pair \( (u, v) \) is the weak solution of the Problem 4.

Let \( z^n = v^{n+1} - v^n, r^n = u^{n+1} - u^n \). From (2.27) and (2.28) we have

\[
\int_0^T \int_0^l (-z^n_{tt} \eta_t + a z^n_x \eta_x + c z^n) \, dx \, dt
= \int_0^T \int_0^l \eta(x, t) \int_0^l M(x, \xi, t) r^{n-1}(\xi, t) \, d\xi \, dx \, dt
+ \int_0^T \int_0^l \eta(x, t) [\tilde{S}_\xi(x, 0, t) a(0, t) r^{n-1}(0, t) - \tilde{S}_\xi(x, l, t) a(l, t) r^{n-1}(l, t)] \, dx \, dt
+ 2 \int_0^T \int_0^l \eta(x, t) \int_0^l \tilde{S}_{\xi} r^{n-1}_t \, d\xi \, dx \, dt,
\]

(2.30)

\[
z^n(x, t) = r^n(x, t) + \int_0^t \tilde{S}(x, \xi, t) r^n(\xi, t) \, d\xi.
\]  

(2.31)

The assumptions of Theorem 2.4 provide the existence of positive number \( s_0 \) such that

\[
\max_{Q_T} \{ \int_0^l \tilde{S}^2 \, d\xi, \int_0^l \tilde{S}_x^2 \, d\xi, \int_0^l \tilde{S}_x^2 \, d\xi \} \leq s_0.
\]

Then for \( 1 - 2s_0l > 0 \) from (2.31), we obtain

\[
\|r^n\|_{L_2(Q_T)} \leq \frac{2}{1 - 2s_0l}\|z^n\|_{L_2(Q_T)}.
\]  

(2.32)
From \( z^n = r^n + \int_0^t \dot{S}_t r^n d\xi + \int_0^t \ddot{S}_t r^n d\xi \) and \( z^n_\tau = r^n + \int_0^\tau \ddot{S}_\tau r^n d\xi \), for \( 1 - 3s_0l > 0 \), we obtain

\[
\|r^n_\tau\|_{L^2(Q_T)}^2 \leq \frac{3}{1 - s_0l} \|z^n_\tau\|_{L^2(Q_T)}^2 + \frac{6s_0l}{(1 - 2s_0l)(1 - 3s_0l)} \|z^n\|_{L^2(Q_T)}^2,
\]

(2.33)

\[
\|r^n_\tau\|_{L^2(Q_T)}^2 \leq \|z^n_\tau\|_{L^2(Q_T)}^2 + \frac{4s_0l}{(1 - 2s_0l)^2} \|z^n\|_{L^2(Q_T)}^2.
\]

(2.34)

Then from (2.32)–(2.34),

\[
\|r^n\|_{W^2_0(Q_T)}^2 \leq A \|z^n\|_{W^2_0(Q_T)}^2
\]

(2.35)

where \( A \) depends on \( a_0, a_1, c_0, s_0, l, T \). To proceed further we prove the following statement.

If \( v \in W^2_0(Q_T) \) is a solution to the first initial-boundary problem in (2.29), then for almost all \( \tau \in [0, T] \),

\[
\int_0^l \left( (v^2(x, \tau))^2 + a(x, \tau)v^2_\tau(x, \tau) \right) dx = \int_0^\tau \int_0^l a(t)v^2 dx dt - 2 \int_0^\tau \int_0^l c(t)v_\tau dx dt + 2 \int_0^\tau \int_0^l G v_\tau dx dt.
\]

(2.36)

We obtain (2.36) after integrating the equality \( (v_t - (av_x)_x + cv)v_t = G(t)v_t \) over \( Q_\tau = (0, l) \times (0, \tau) \).

Note that we seek \( z^n \) as a solution of (2.29) with

\[
G(x, t) = \int_0^l M(x, \xi, t) r^n\xi^{-1}(\xi, t) d\xi + \left[ \ddot{S}_{\xi}(x, 0, t) a(0, t) r^n\xi^{-1}(0, t) - \ddot{S}_{\xi}(x, 0, t) a(0, t) r^n\xi^{-1}(0, t) \right] + 2 \int_0^l \dot{S}_{\xi}(x, \xi, t) r^n\xi^{-1}(\xi, t) d\xi.
\]

Under the assumptions of Theorem 2.4 there exist positive numbers \( a_0, a_1, c_0, s_1 \) such that \( a(x, t) \geq a_0, |c(t, x)| \leq c_0, |a(x, t), a_2(x, t), a_1(x, t)| \leq a_1, s_1 = \max \left\{ \int_0^l \left( \ddot{S}_{\xi}(t) \right)^2 d\xi, \int_0^l \int_0^l \left( \ddot{S}_{\xi\xi}(t) \right)^2 d\xi \right\} \).

From Theorem 2.1 and some manipulations (integrating and using Cauchy and Cauchy with \( \varepsilon \) inequalities) we obtain

\[
\int_0^l \left[ (z^n(x, \tau))^2 + a(x, \tau)(z^n_x(x, \tau))^2 \right] dx 
\]

\[
\leq A_1 \int_0^\tau \int_0^l \left[ (z^n)^2 + (z^n_x)^2 \right] dx dt 
\]

\[
+ c(\varepsilon) \int_0^\tau \int_0^l (z^n_x)^2 dx dt + \varepsilon \int_0^\tau \int_0^l G^2(x, t) dx dt.
\]

Estimating the last term with the help of Cauchy-Bunyakovskii-Schwartz inequality, and the trace inequalities

\[
(r^n\xi^{-1}(z_t, t))^2 \leq 2l \int_0^l (r^n\xi^{-1}(x, t))^2 dx + \frac{2}{l} \int_0^l (r^n\xi^{-1}(x, t))^2 dx, \quad z_1 = 0, \quad z_2 = l,
\]
we obtain
\[
\int_0^l [(z^n(x, \tau))^2 + a(x, \tau)(z^n_x(x, \tau))^2] \, dx \\
\leq A_2 \int_0^T \int_0^l [(z^n)^2 + (z^n_x)^2] \, dx \, dt \\
+ \varepsilon A_3 \int_0^T \int_0^l [(r^n-1)^2 + (r^n_x-1)^2] \, dx \, dt.
\]
From Gronwall’s lemma and integrating over \((0, T)\) it follows that
\[
\|z^n\|^2_{W^1_2(Q_T)} \leq \varepsilon B\|r^n-1\|^2_{W^1_2(Q_T)},
\]
(2.37)
where \(B = T A_3 e^{A_3 T}, A_i, i = 1, 2, 3\), depend only on \(a_0, a_1, c_0, s_1, l, T\). From (2.35) and (2.37) it follows that
\[
\|z^n\|^2_{W^1_2(Q_T)} \leq AB\varepsilon\|z^n-1\|^2_{W^1_2(Q_T)}, \quad \|r^n\|^2_{W^1_2(Q_T)} \leq AB\varepsilon\|r^n-1\|^2_{W^1_2(Q_T)}.
\]
(2.38)
We select a small \(\varepsilon\) such that \(0 < \varepsilon AB < 1\). Hence, \(\{u^n, v^n\}\) is a Cauchy sequence in \(W^1_2(Q_T)\). Thus, there exists a unique pair \((u, v)\) such that \(u^n \to u, v^n \to v\). Let \(n \to \infty\) in (2.27), (2.28). From the converges of \(\{u^n, v^n\}\) we see that \((u, v)\) is the required solution of the Problem 4.

**Step 3.** Solvability of the Problem 3. As we noted above, under assumptions of Theorem 2.4 the weak solution \((u, v)\) belongs to \(W^1_2(Q_T)\). That is why we can rewrite (2.25) as
\[
\int_0^T \int_0^l \left( v_{tt} - (av_x)_x + cv \right) \eta \, dx \, dt \\
= \int_0^T \int_0^l \eta(x, t) \int_0^l M(x, \xi, t) u(\xi, t) d\xi \, dx \, dt \\
+ \int_0^T \int_0^l \eta(x, t) \left[ \tilde{S}_\xi(x, 0, t) a(0, t) u(0, t) - \tilde{S}_\xi(x, l, t) a(l, t) u(l, t) \right] \, dx \, dt \\
+ 2 \int_0^T \int_0^l \tilde{S}_t u_x \eta \, dx \, dt + \int_0^T \int_0^l G(x, t) \eta(x, t) \, dx \, dt.
\]
(2.39)
Substituting \(v(x, t)\) represented by (2.26) into (2.39), after some manipulations, we obtain
\[
\int_0^T \int_0^l \left( u_{tt} - (au_x)_x + cu \right) \eta(x, t) \, dx \, dt = \int_0^T \int_0^l f(x, t) \eta(x, t) \, dx \, dt
\]
for all \(\eta \in W^1_2(Q_T)\). So, \(u(x, t)\) is the solution to the (2.1). Obviously, the conditions (2.22) are fulfilled. This completes the proof of Theorem 2.3.

**Acknowledgments.** I want to thank the anonymous referees for his/her comments that improved this article, and Professor Julio G. Dix for managing this submission to the Electron. J. Differential Equ.
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