Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 43, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

μ PSEUDO ROTATING-PERIODIC SOLUTIONS FOR DIFFERENTIAL EQUATIONS

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Abstract. In this article, we combine rotating periodic functions with μ ergodic functions to obtain a new class of functions called μ pseudo rotating periodic functions. Then we study the existence and uniqueness of μ pseudo rotating periodic solutions for linear systems, semi-linear systems, and non-linear systems by exponential dichotomy.

1. Introduction

Researchers pay a lot of attention to the existence and uniqueness of all kinds of periodic solutions. Newton [16] was the first person who mentioned the existence of periodic solutions. Then, Bohl and Esclangon gave the concept of quasi-periodic functions. In 1931, Bohr [5] obtained the concept of almost periodic functions. Hale [12] studied their properties. In 1955, Bochner [2, 3] gave the concept of almost autonomous periodic functions. His student Veech [18] researched the group properties of this kind of functions. Shen and Yi [17], Liang and Xiao [10, 11, 13, 21], Campos and Tarallo [6], Wang and Li [19] also achieved a series of results in the field. Recently, many researchers have studied rotating periodic functions and obtained a series of results; see [14, 15, 20, 22, 23, 24, 25].

In the past, because of the development of ergodic perturbation theory, the study of systems with perturbations gradually became one of the focus in the field of dynamics. Soon after, Frechét gave the definition of asymptotically almost periodic function.

Definition 1.1. A function $f \in \mathcal{C}(\mathbb{R}^+)$ is called asymptotically almost periodic if

$$f = g + \phi, \tag{1.1}$$

where $g \in \mathcal{AP}(\mathbb{R}^+)$, and $\phi \in C_0(\mathbb{R}^+) = \{h \in C(\mathbb{R}^+) : \lim_{t\to+\infty} h(t) = 0\}$.

However, there were some shortcomings about asymptotically almost periodic functions. For example, the indefinite integral of asymptotically almost periodic function may not be asymptotically almost periodic function. Hence, Zhang [26] put forward the concept of pseudo almost periodic functions.

Definition 1.2. A function $f \in \mathcal{C}(\mathbb{R})(\mathcal{C}(\Omega \times \mathbb{R}))$ is called pseudo almost periodic if

$$f = g + \phi, \tag{1.2}$$

2010 *Mathematics Subject Classification.* 34D09, 34K14.
Key words and phrases. μ pseudo rotating periodic functions; exponential dichotomy.

Submitted November 24, 2019. Published May 16, 2020.

where $g \in \mathcal{AP}(\mathbb{R})(\mathcal{AP}(\Omega \times \mathbb{R}))$, and $\phi \in \mathcal{C}_1(\mathbb{R})(\mathcal{C}_1(\Omega \times \mathbb{R}))$. Here,

$$\begin{gathered}\mathcal{C}_1(\mathbb{R}) = \{h \in \mathcal{C}(\mathbb{R}),\ \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|h(t)\| dt = 0\}, \\ \mathcal{C}_1(\Omega \times \mathbb{R}) = \{h \in \mathcal{C}(\Omega \times \mathbb{R}) : \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|h(t,x)\| dt = 0\},\end{gathered} \tag{1.3}$$

where g and ϕ are called the almost periodic component and the ergodic perturbation, respectively, of the function of f.

Moreover, the completeness and translation invariance of pseudo almost periodic function were also proved in [26]. Then, Ait [1] gave the concept of generalized pseudo almost periodic functions by extending the ergodic perturbation part of the function f to an unbounded function. And he proved the existence and uniqueness of generalized pseudo almost periodic solutions for differential equations with exponential dichotomy.

In 2006, Diagana [9] introduced a new class of functions called weighted pseudo almost periodic functions, which generalize the classical pseudo almost periodic functions.

Definition 1.3. A function $f \in \mathcal{BC}(\mathbb{R}, \mathbb{X})$ is called weighted pseudo almost periodic if it can be expressed as $f = g + \phi$, where $g \in \mathcal{AP}(\mathbb{X})$ and $\phi \in \mathcal{C}_2(\mathbb{R}, \rho)$. Here,

$$\mathcal{C}_2(\mathbb{R}, \rho) = \{h \in \mathcal{BC}(\mathbb{R}, \mathbb{X}) : \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \|h(t)\| \rho(t) dt = 0\}. \tag{1.4}$$

Similarly, he obtained the completeness and translation invariance of weighted pseudo almost periodic function.

In 2012, Blot [4] gave the definition of μ pseudo almost periodic functions, which further expanded the weighted pseudo almost periodic functions.

Definition 1.4. A function $f \in \mathcal{BC}(\mathbb{R}, \mathbb{X})$ is called μ pseudo almost periodic if it can be expressed as $f = g + \phi$, where $g \in \mathcal{AP}(\mathbb{X})$, and $\phi \in \mathcal{C}_3(\mathbb{R}, \mu)$. Here,

$$\mathcal{C}_3(\mathbb{R}, \mu) = \{h \in \mathcal{BC}(\mathbb{R}, \mathbb{X}, \mu) : \lim_{r\to\infty} \frac{1}{\mu[-r,r]} \int_{-r}^{r} \|h(t)\| d\mu(t) = 0\}, \tag{1.5}$$

where μ represents a measure function defined on $\mathbb{R}$.

Blot et al, transformed the density function ρ of weighted pseudo almost periodic function into measure function by measure theory. They also proved the completeness and translation invariance of μ pseudo almost periodic function by introducing measure equivalence. In this article, we introduce a kind of pseudo rotating-periodic functions and prove the existence uniqueness of pseudo rotating periodic solutions.

This article is organized as follows: in section 2, we show the concept of μ pseudo rotating periodic function and exponential dichotomy. The basic properties of the space of μ pseudo rotating periodic function are shown in section 3. In section 4, we prove the existence and uniqueness of μ pseudo rotating periodic solution for linear systems, semi-linear systems, and non-linear systems. Finally, section 5 verifies the flexibility of our conditions by an example.

2. μ PSEUDO ROTATING PERIODIC FUNCTION

Firstly, we give some notation. $\mathcal{B}$ denotes the Lebesgue σ-field of $\mathbb{R}$. $\mathcal{M}$ is the set that consist of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < +\infty$, for all $a, b \in \mathbb{R}$ $(a \leq b)$. We denote the space of all n-order orthogonal matrices by $\mathbb{O}(n)$. $\mathcal{BC}(\mathbb{R}, \mathbb{R}^n)$ denotes the Banach space of all bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}^n$, equipped with the supremum norm $\|f\| = \sup_{t\in\mathbb{R}} \|f(t)\|$.

Definition 2.1. A function $g \in \mathcal{BC}(\mathbb{R}, \mathbb{R}^n)$ is said to be (T, Q) rotating periodic if there exist $T > 0$ and $Q \in \mathbb{O}(n)$ such that

$$g(t + T) = Qg(t) \quad \forall t \in \mathbb{R}.$$

Definition 2.2. A function $g \in \mathcal{BC}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is said to be (T, Q) rotating periodic if the exist $T > 0$ and $Q \in \mathbb{O}(n)$ such that

$$g(t + T, x) = Qg(t, Q^{-1}x) \quad \forall t \in \mathbb{R}.$$

We denote the space of rotating periodic functions by $\mathcal{RP}(\mathbb{R}, \mathbb{R}^n)$ ($\mathcal{RP}(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^n)$), equipped with the supremum norm.

Remark 2.3. If $Q = \mathrm{id}$ or $Q = -\mathrm{id}$, then a (T, Q) rotating periodic function is the usual T-periodic function or T-anti-periodic one, respectively. If for some positive integer k_0 such that $Q^{k_0} = id$, a (T, Q) rotating periodic function is a subharmonic one. If for all positive integer k_1 such that $Q^{k_1} \neq \mathrm{id}$, then a (T, Q) rotating periodic function is a quasi-periodic function.

Definition 2.4. Let $\mu \in \mathcal{M}$. A function $\phi \in \mathcal{BC}(\mathbb{R}, \mathbb{R}^n)$ is called μ-ergodic if

$$\lim_{r\to\infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \|\phi(t)\| d\mu(t) = 0.$$

Definition 2.5. Let $\mu \in \mathcal{M}$. A function $\phi \in \mathcal{BC}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is called μ-ergodic if

$$\lim_{r\to\infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \|\phi(t, x)\| d\mu(t) = 0. \tag{2.1}$$

We denote the space of all such functions by $\mathcal{E}(\mathbb{R}, \mathbb{R}^n, \mu)$, $\mathcal{E}(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^n, \mu)$, equipped with the supremum norm.

Remark 2.6. If μ is the Lebesgue measure, then the μ ergodic function ϕ is the same as the function given by Zhang [26]. If the Radon-Nikodym derivative of μ with respect to the Lebesgue measure on $\mathbb{R}$ is ρ, the μ-ergodic function is a Weighted ergodic one by Diagana [9].

Definition 2.7. Let $\mu \in \mathcal{M}$. A function $f \in \mathcal{BC}(\mathbb{R}, \mathbb{R}^n)$ is said to be μ pseudo rotating periodic if f is written in the form

$$f = g + \phi,$$

where $g \in \mathcal{RP}(\mathbb{R}, \mathbb{R}^n)$, and $\phi \in \mathcal{E}(\mathbb{R}, \mathbb{R}^n, \mu)$.

Definition 2.8. Let $\mu \in \mathcal{M}$. A function $f \in \mathcal{BC}(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^n)$ is said to be μ pseudo rotating periodic if f is written in the form

$$f = g + \phi,$$

where $g \in \mathcal{RP}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, and $\phi \in \mathcal{E}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n, \mu)$.

We denote the space of such functions by $\mathcal{PRP}(\mathbb{R}, \mathbb{R}^n, \mu)$ ($\mathcal{PRP}(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^n, \mu)$), equipped with the new norm $\|f\|_1 = \|g\| + \|\phi\|$.

Consider the system

$$\dot{x} = A(t)x, \tag{2.2}$$

where $A(t)$ is a continuous matrix on $\mathbb{R}$ and $X(t)$ is a fundamental matrix of system (2.2) satisfying $X(0) = \mathrm{id}$.

Definition 2.9 ([8]). *The system of differential equations* (2.2) *is said to possess exponential dichotomy on* $\mathbb{R}$, *if there exist a projection matrix* P *and constants* $K > 1, \alpha > 0$, *such that*

$$\|X(t)PX^{-1}(s)\| \le K\exp(-\alpha(t-s)), \quad t \ge s,$$
$$\|X(t)(I-P)X^{-1}(s)\| \le K\exp(-\alpha(s-t)), \quad t \le s.$$

We define the Green's function as follows

$$U(t,s) = \begin{cases} X(t)PX^{-1}(s) & t \ge s, \\ -X(t)(I-P)X^{-1}(s) & t \le s. \end{cases} \tag{2.3}$$

3. Basic properties

For $\mu \in \mathcal{M}$, and $\tau \in \mathbb{R}$, we define $\mu_\tau(A) = \mu(\{a+\tau : a \in A\})$ for $A \in \mathcal{B}$. Then we make the following hypothesis.

(H1) Let $\mu \in \mathcal{M}$, there exist $\beta > 0$ and a bounded interval I such that $\mu_\tau(A) \le \beta\mu(A)$, when $A \in \mathcal{B}$ satisfies $A \cap I = \emptyset$.

Lemma 3.1. *If* μ *satisfies* (H1), *then* $(\mathcal{PRP}(\mathbb{R},\mathbb{R}^n,\mu), \|\cdot\|_1)$ *is a Banach space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{PRP}(\mathbb{R},\mathbb{R}^n,\mu)$. For all $\varepsilon > 0$, there exist $N > 0$ such that

$$\|f_n(t) - f_m(t)\|_1 < \varepsilon \quad (m,n > N,\ \forall t \in \mathbb{R}),$$

and

$$f_n(t) = g_n(t) + \phi_n(t) \quad \forall t \in \mathbb{R},$$
$$f_m(t) = g_m(t) + \phi_m(t) \quad \forall t \in \mathbb{R}.$$

where $g_n, g_m \in \mathcal{RP}(\mathbb{R},\mathbb{R}^n)$, and $\phi_n, \phi_m \in \mathcal{E}(\mathbb{R},\mathbb{R}^n,\mu)$. So, we have

$$\begin{aligned} \|f_n(t) - f_m(t)\|_1 &= \|g_n(t) + \phi_n(t) - g_m(t) - \phi_m(t)\|_1 \\ &= \|g_n(t) - g_m(t)\| + \|\phi_n(t) - \phi_m(t)\| \\ &= \sup_{t\in\mathbb{R}} |g_n(t) - g_m(t)| + \sup_{t\in\mathbb{R}} |\phi_n(t) - \phi_m(t)| < \varepsilon. \end{aligned}$$

We deduce that

$$\sup_{t\in\mathbb{R}} |g_n(t) - g_m(t)| \le \varepsilon \quad \forall t \in \mathbb{R},$$
$$\sup_{t\in\mathbb{R}} |\phi_n(t) - \phi_m(t)| \le \varepsilon \quad \forall t \in \mathbb{R}.$$

According to the definition of Cauchy sequence, we can conclude that $\{g_n\}$ and $\{\phi_n\}$ are the Cauchy sequence of $\mathcal{RP}(\mathbb{R},\mathbb{R}^n)$ and $\mathcal{E}(\mathbb{R},\mathbb{R}^n,\mu)$, respectively. Then we use that $\mathcal{RP}(\mathbb{R},\mathbb{R}^n)$ and $\mathcal{E}(\mathbb{R},\mathbb{R}^n,\mu)$ are Banach space [7, 4]. Hence there exist $g_0(t) \in \mathcal{RP}(\mathbb{R},\mathbb{R}^n)$ and $\phi_0(t) \in \mathcal{E}(\mathbb{R},\mathbb{R}^n,\mu)$ such that

$$\lim_{t\to\infty} \|g_n(t) - g_0(t)\| = 0, \quad \lim_{t\to\infty} \|\phi_n(t) - \phi_0(t)\| = 0.$$

Let $f_0(t) = g_0(t) + \phi_0(t)$. We deduce that $f_0 \in \mathcal{PRP}(\mathbb{R},\mathbb{R}^n,\mu)$. So we can prove that $(\mathcal{PRP}(\mathbb{R},\mathbb{R}^n,\mu), \|\cdot\|_1)$ is a Banach space. $\square$

Lemma 3.2. *If μ satisfies* (H1)*, then $(\mathcal{PRP}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n,\mu),\|\cdot\|_1)$ is a Banach space.*

The proof of the above lemma is similar to that of Lemma 3.1.

Lemma 3.3. *Let μ satisfies* (H1)*. If $y\in\mathcal{PRP}(\mathbb{R},\mathbb{R}^n,\mu)$, $f\in\mathcal{PRP}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n,\mu)$, and f satisfies the Lipschitz condition in x, then the convolution product*

$$(f*U)(t)=\int_{-\infty}^{+\infty}U(t,s)f(s,y(s))ds \tag{3.1}$$

is also a μ pseudo rotating periodic function.

Proof. Since $y\in\mathcal{PRP}(\mathbb{R},\mathbb{R}^n,\mu)$ and $f\in\mathcal{PRP}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n,\mu)$, we have

$$\begin{aligned}f(t,y(t))&=f_1(t,y_1(t))+f(t,y(t))-f(t,y_1(t))+f_2(t,y_1(t))\\&=:f_1(t,y_1(t))+\widetilde{f}(t,y(t),y_1(t)),\end{aligned}$$

where $y_1\in\mathcal{RP}(\mathbb{R},\mathbb{R}^n)$, $y_2\in\mathcal{E}(\mathbb{R},\mathbb{R}^n,\mu)$, $f_1\in\mathcal{RP}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$, $f_2\in\mathcal{E}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n,\mu)$. Then

$$\begin{aligned}(f*U)(t)&=\int_{-\infty}^{+\infty}U(t,s)f_1(s,y_1(s))ds+\int_{-\infty}^{+\infty}U(t,s)\widetilde{f}(s,y(s),y_1(s))ds\\&=:(f*U)_1(t)+(f*U)_2(t).\end{aligned}$$

For $f_1\in\mathcal{RP}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$, it is easy to prove that $(f*U)_1(t)\in\mathcal{RP}(\mathbb{R},\mathbb{R}^n)$.

Next we prove that $(f*U)_2(t)$ is a μ ergodic function. i.e.

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|(f*U)_2(t)\|d\mu(t)=0.$$

Since f satisfies the Lipschitz condition, we have a constant N such that

$$\begin{aligned}\|f(s,y(s))-f(s,y_1(s))\|&\le\|f(s,y(s))-f(s,y_1(s))\|_1\\&\le N\|y(s)-y_1(s)\|_1\\&=N\|y_2(t)\|.\end{aligned}$$

Hence, we deduce that

$$\begin{aligned}&\int_{-r}^{r}\|\int_{-\infty}^{+\infty}U(t,s)\widetilde{f}(s,y(s),y_1(s))ds\|d\mu(t)\\&\le N\int_{-r}^{r}\int_{-\infty}^{+\infty}\|U(t,s)\|\|y_2(s)\|\,ds\,d\mu(t)\\&\quad+\int_{-r}^{r}\int_{-\infty}^{+\infty}\|U(t,s)\|\|f_2(s,y_1(s))\|\,ds\,d\mu(t)\\&=:Z_1(t)+Z_2(t).\end{aligned}$$

According to (2.3) and definition 2.5, we conclude that

$$\begin{aligned}Z_1(t)&\le\int_{-r}^{r}\int_{-\infty}^{t}KN\exp(-\alpha(t-s))\|y_2(s)\|\,ds\,d\mu(t)\\&\quad+\int_{-r}^{r}\int_{t}^{+\infty}KN\exp(\alpha(t-s))\|y_2(s)\|\,ds\,d\mu(t)\\&\le\int_{-\infty}^{0}\int_{-r}^{r}KN\|y_2(s+t)\|d\mu(t)ds\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{+\infty}\int_{-r}^{r} KN\|y_2(s+t)\|d\mu(t)ds \\
&= KN\int_{-\infty}^{+\infty}\int_{-r}^{r}\|y_2(s+t)\|d\mu(t)ds.
\end{aligned}
$$

For $y_2 \in \mathcal{E}(\mathbb{R},\mathbb{R}^n,\mu)$, Joël Blot [4] proved $y_2(t+s) \in \mathcal{E}(\mathbb{R},\mathbb{R}^n,\mu)$. Using Lebesgue's dominated convergence theorem, we deduce that

$$
\lim_{r\to\infty}\frac{Z_1(t)}{\mu([-r,r])} \le \int_{-\infty}^{+\infty}\lim_{r\to\infty}\frac{KN}{\mu([-r,r])}\int_{-r}^{r}\|y_2(s+t)\|d\mu(t)ds = 0.
$$

Similarly, $\lim_{r\to\infty}\frac{Z_2(t)}{\mu([-r,r])} = 0$. Therefore, we conclude that $(f*U)_2(t)$ is a μ ergodic function. Finally, we obtain that $(f*U)(t) \in \mathcal{PRP}(\mathbb{R},\mathbb{R}^n,\mu)$. □

4. Existence and uniqueness

Theorem 4.1. *Consider the linear system*

$$
\dot{x} = A(t)x + f(t), \tag{4.1}
$$

where $A(t) \in \mathcal{PR}(\mathbb{R},\mathbb{R}^n)$, $f(t) \in \mathcal{PRP}(\mathbb{R},\mathbb{R}^n,\mu)$. If the linear homogeneous system $\dot{x} = A(t)x$ satisfies exponential dichotomy, then (4.1) *has a unique μ pseudo rotating periodic solution.*

Proof. Since $f(t) \in \mathcal{PRP}(\mathbb{R},\mathbb{R}^n,\mu)$, we have a bounded solution (see [8])

$$
x(t) = \int_{-\infty}^{+\infty} U(t,s)g(s)ds + \int_{-\infty}^{+\infty} U(t,s)\phi(s)ds =: x_1(t) + x_2(t),
$$

where $g \in \mathcal{RP}(\mathbb{R},\mathbb{R}^n)$, $\phi \in \mathcal{E}(\mathbb{R},\mathbb{R}^n,\mu)$. It follows from $g(s) \in \mathcal{RP}(\mathbb{R},\mathbb{R}^n)$ that $x_1(t) \in \mathcal{RP}(\mathbb{R},\mathbb{R}^n)$.

Then we will prove that $x_2(t)$ is a μ ergodic function. By (2.3) and definition 2.5, we have

$$
\begin{aligned}
&\int_{-r}^{r}\|x_2(t)\|d\mu(t) \\
&\le \int_{-r}^{r}\int_{-\infty}^{t} K\exp(-\alpha(t-s))\|\phi(s)\|\,ds\,d\mu(t) \\
&\quad + \int_{-r}^{r}\int_{t}^{+\infty} K\exp(-\alpha(s-t))\|\phi(s)\|\,ds\,d\mu(t) \\
&\le K\int_{-r}^{r}\int_{-\infty}^{0}\|\phi(s+t)\|\,ds\,d\mu(t) + K\int_{-r}^{r}\int_{0}^{+\infty}\|\phi(s+t)\|\,ds\,d\mu(t) \\
&= K\int_{-\infty}^{0}\int_{-r}^{r}\|\phi(s+t)\|d\mu(t)ds + K\int_{0}^{+\infty}\int_{-r}^{r}\|\phi(s+t)\|d\mu(t)ds,
\end{aligned}
$$

by Fubini's theorem.

The translation invariance of $\mathcal{E}(\mathbb{R},\mathbb{R}^n,\mu)$ was proved in [4], i.e. $\phi(s+t) \in \mathcal{E}(\mathbb{R},\mathbb{R}^n,\mu)$. So we can obtain that $\lim_{r\to+\infty}\frac{x_2(t)}{\mu([-r,r])} = 0$ by Lebesgue's dominated convergence theorem. Then system (4.1) has a unique μ pseudo rotating periodic solution. □

Theorem 4.2. *Consider the semi-linear system*

$$\dot{x} = A(t)x + f(t, x), \tag{4.2}$$

where $A(t) \in \mathcal{RP}(\mathbb{R}, \mathbb{R}^n)$, $f(t, x) \in \mathcal{PRP}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n, \mu)$, and it satisfies the Lipschitz condition in x. If the linear homogeneous system $\dot{x} = A(t)x$ satisfies exponential dichotomy and there is a Lipschitz constant $N > 0$, such that $\frac{2KN}{\alpha} < 1$, then system (4.2) has a unique μ pseudo rotating periodic solution.

Proof. Firstly, for all $y(t) \in \mathcal{PRP}(\mathbb{R}, \mathbb{R}^n, \mu)$, we consider the system

$$\dot{x} = A(t)x + f(t, y(t)),$$

which has a bounded solution

$$x(t) = \int_{-\infty}^{+\infty} U(t, s) f(s, y(s)) ds$$

and $x(t) \in \mathcal{PRP}(\mathbb{R}, \mathbb{R}^n, \mu)$ by lemma 3.3. Then we define the mapping $H(y)(t)$: $\mathcal{PRP}(\mathbb{R}, \mathbb{R}^n, \mu) \to \mathcal{PRP}(\mathbb{R}, \mathbb{R}^n, \mu)$ by

$$H(y)(t) = \int_{-\infty}^{+\infty} U(t, s) f(s, y(s)) ds. \tag{4.3}$$

For all $y_1(t), y_2(t) \in \mathcal{PRP}(\mathbb{R}, \mathbb{R}^n, \mu)$, we deduce that

$$\begin{aligned}
\|H(y_1)(t) - H(y_2)(t)\|_1 &\le \int_{-\infty}^{+\infty} \|U(t, s)\|_1 N \|y_1(s) - y_2(s)\|_1 ds \\
&\le KN \int_{-\infty}^{t} \exp(-\alpha(t - s)) \|y_1(s) - y_2(s)\|_1 ds \\
&\quad + KN \int_{t}^{+\infty} \exp(\alpha(t - s)) \|y_2(s) - y_1(s)\|_1 ds \\
&\le \frac{2KN}{\alpha} \|y_1(s) - y_2(s)\|_1.
\end{aligned}$$

Since $\frac{2KN}{\alpha} < 1$, the mapping $H(y)(t)$ is a strict contraction. Using Banach fixed point theorem, there exists a unique $y(t) \in \mathcal{PRP}(\mathbb{R}, \mathbb{R}^n, \mu)$ such that $H(y)(t) = y(t)$. So system (4.2) has a unique μ pseudo rotating periodic solution. □

In the following, we replace the Lipschitz condition by the linear growth condition. Then the existence of μ pseudo rotating wave solutions for semi-linear systems is also going to be proved.

Corollary 4.3. *Consider system (4.2), where $A(t)$ is an rotating periodic matrix, $f(t, x) \in \mathcal{PRP}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n, \mu)$, and f is uniformly continuous function in x. Suppose that the homogeneous linear system $\dot{x} = A(t)x$ satisfies an exponential dichotomy, and the function f satisfies linear growth condition, i. e.*

$$\|f(t, x)\|_1 \le a\|x\|_1 + b \quad (t \in \mathbb{R}, x \in \mathbb{R}^n), \tag{4.4}$$

where $a > 0$, $\frac{2Ka}{\alpha} < 1$. Then system (4.2) has a μ pseudo rotating periodic solution.

Proof. Let $G := \{y \in \mathcal{PRP}(\mathbb{R}, \mathbb{R}^n, \mu), \|y\|_1 \le N_1\}$, where $N_1 = \frac{2Kb}{\alpha - 2Ka}$. We can deduce that G is a closed convex subset of the space of Banach. For all $y \in G$, we have

$$\|H(y)(t)\|_1 = \| \int_{-\infty}^{+\infty} U(t, s) f(s, y(s)) ds \|_1$$

$$\begin{aligned}
&\le \int_{-\infty}^{+\infty} \|U(t,s)\|_1 \|f(s,y(s))\|_1 ds \\
&\le (a\|y\|_1 + b)\Big(\int_{-\infty}^{t} K\exp(-\alpha(t-s))ds + \int_{t}^{+\infty} K\exp(\alpha(t-s))ds\Big) \\
&\le (aN_1 + b)\frac{2K}{\alpha} \\
&\le N_1.
\end{aligned}$$

By using the uniform continuity of f, for all $\varepsilon > 0$, there exist $\delta > 0$ such that $\|y_1 - y_2\|_1 < \delta$, and we have

$$\|f(t,y_1) - f(t,y_2)\|_1 < \varepsilon.$$

From the inequality

$$\begin{aligned}
&\|H(y_1)(t) - H(y_2)(t)\|_1 \\
&\le \Big\| \int_{-\infty}^{+\infty} U(t,s)f(s,y_1(s))ds - \int_{-\infty}^{+\infty} U(t,s)f(s,y_2(s))ds \Big\|_1 \\
&\le \int_{-\infty}^{t} \|X(t)PX^{-1}(s)\|_1 \|f(s,y_1(s)) - f(s,y_2(s))\|_1 ds \\
&\quad + \int_{t}^{+\infty} \|X(t)(I-P)X^{-1}(s)\|_1 \|f(s,y_2(s)) - f(s,y_1(s))\|_1 ds \\
&\le \int_{-\infty}^{t} K\exp(-\alpha(t-s))\varepsilon ds + \int_{t}^{+\infty} K\exp(\alpha(t-s))\varepsilon ds \\
&\le \frac{2K\varepsilon}{\alpha},
\end{aligned}$$

we deduce that $H(y)(t)$ is uniformly continuous in x and satisfies (4.2). Then we have

$$\|\dot{H}(y)(t)\|_1 = \|A(t)\|_1 \|H(y)(t)\|_1 + \|f(t,x)\|_1 \le (a + \sup_{t\in\mathbb{R}} |A(t)|)M_1 + b.$$

Therefore, the mapping $H(y)(t)$ is equicontinuous. By using Arzela-Ascoli theorem, there exist a subsequence of $\{f_k(t)\} \subset \{f_n(t)\} \in G$ such that

$$\lim_{k\to\infty} f_k(t) = y_0(t),$$

where $y_0(t) \in G$. We can prove that $H(y)(t)$ has fixed point in the space G by Schauder fixed point theorem. Then the system (4.2) has a μ pseudo rotating periodic solution. □

Theorem 4.4. *Consider the nonlinear system*

$$\dot{x} = f(t,x), \tag{4.5}$$

where $f \in \mathcal{PRP}(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^n, \mu)$, and f is a continuous differentiable function in x. Suppose

$$\dot{x} = \int_0^1 \frac{\partial f}{\partial x}(\theta y)d\theta \cdot x, \quad y \in \mathcal{PRP}(\mathbb{R}, \mathbb{R}^n, \mu) \tag{4.6}$$

satisfies the exponential dichotomy, i.e.

$$\begin{gathered}
\|\Phi(t,y)P_y\Phi^{-1}(s,y)\| \le K\exp(-\alpha(t-s)) \quad t \ge s, \\
\|\Phi(t,y)(I-P_y)\Phi^{-1}(s,y)\| \le K\exp(-\alpha(s-t)) \quad t \le s,
\end{gathered}$$

where $\Phi(t,y)$ *is a fundamental matrix of system* (4.6)*, and satisfies* $\Phi(0,x)=\mathrm{id}$*. Then* (4.5) *has a unique* μ *pseudo rotating periodic solution.*

Proof. Firstly, we prove uniqueness. Suppose x,y are μ pseudo rotating wave solutions of the system (4.5). Let $u=x-y$. Then we have

$$\begin{aligned}\dot u=\dot x-\dot y&=f(t,x)-f(t,y)\\&=\int_0^1\frac{\partial f}{\partial x}(\theta x+(1-\theta)y)d\theta\cdot(x-y)\\&=\int_0^1\frac{\partial f}{\partial x}(y+\theta u)d\theta\cdot u.\end{aligned}$$

Because (4.6) satisfies the exponential dichotomy, we get a unique solution $u=0$, i.e. $x=y$.

Next we prove existence. It is easy to obtain

$$\dot x=f(t,x)=\int_0^1\frac{\partial f}{\partial x}(\theta x)d\theta\cdot x+f(t,0).$$

For all $y\in\mathcal{PRP}(\mathbb{R},\mathbb{R}^n,\mu)$, we consider the equation

$$\dot x=\int_0^1\frac{\partial f}{\partial x}(\theta y)d\theta\cdot x+f(t,0),\tag{4.7}$$

which has a solution

$$\begin{aligned}x_y(t)=&\int_{-\infty}^t\Phi(t,y)P_y\Phi^{-1}(s,y)f(s,0)ds\\&-\int_t^{+\infty}\Phi(t,y)(I-P_y)\Phi^{-1}(s,y)f(s,0)ds.\end{aligned}$$

Because $f\in\mathcal{PRP}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n,\mu)$, we easily prove that $x_y\in\mathcal{PRP}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n,\mu)$. For all $s\in\mathbb{R}$, there exists a constant M_1 such that $\|f(s,0)\|_1\le M_1$. Then we consider the set

$$\mathcal{D}=\{h\in\mathcal{PRP}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n,\mu),\|h\|_1\le M\},$$

where $M=\frac{2KM_1}{\alpha}+1$. For all $y\in\mathcal{D}$, we define the mapping $T_{[y]}(t)=x_y(t)$. We have

$$\begin{aligned}\|T_{[y]}(t)\|_1\le&\int_{-\infty}^t\|\Phi(t,y)P_y\Phi^{-1}(s,y)\|_1\|f(s,0)\|_1ds\\&+\int_t^{\infty}\|\Phi(t,y)(I-P_y)\Phi^{-1}(s,y)\|_1\|f(s,0)\|_1ds\\\le&\int_{-\infty}^tK\exp(-\alpha(t-s))M_1ds+\int_t^{\infty}K\exp(\alpha(t-s))M_1ds\\\le&\frac{2KM_1}{\alpha}\\\le&M.\end{aligned}$$

Then $T_{[y]}:\mathcal{D}\to\mathcal{D}$ and uniformly bounded with respect to y. Because $T_{[y]}$ satisfies the system (4.7), we have

$$\|\dot T_{[y]}(t)\|_1\le\|\int_0^1\frac{\partial f}{\partial x}(\theta y)d\theta\|_1\|T_{[y]}(t)\|_1+\|f(t,0)\|_1$$

$$\leq \int_0^1 \|\frac{\partial f}{\partial x}(\theta y)\|_1 d\theta M + M_1.$$

It follows from f is a continuous differentiable function in x that $T_{[y]}$ is equicontinuous. Hence, (4.5) has μ pseudo rotating periodic solution by Schauder's fixed point theorem. □

5. Example

Consider the equation

$$\dot{x} = -3x + \exp(-t) + x^2 \exp(-t), \tag{5.1}$$

for all $t \in \mathbb{R}^+, x \in \Omega$, where $\Omega(\subseteq \mathbb{R})$ is a bounded set.

We will prove that (5.1) has a μ pseudo rotating periodic solution. Let

$$f(t,x) = (-3x + \exp(-t)) + x^2 \exp(-t) =: g(t,x) + \phi(t,x).$$

For a given $\tau \in \mathbb{R}$, let

$$\begin{aligned} g(t+\tau, x) &= -3x + \exp(-(t+\tau)) \\ &= \exp(-\tau)(-3\exp(\tau)x + \exp(-t)) \\ &= \exp(-\tau) g(t, \exp(\tau)x). \end{aligned}$$

Therefore, $g(t,x) \in \mathcal{PR}(\mathbb{R}^+ \times \Omega, \Omega)$.

If the Radon-Nikodym derivative ρ is defined by $\rho(t) = 1$, then

$$\begin{aligned} \lim_{r\to+\infty} \frac{\int_0^r x^2 \exp(-t) d\mu(t)}{\mu([0,r])} &= \lim_{r\to+\infty} \frac{\int_0^r x^2 \exp(-t) dt}{\int_0^r 1 dt} \\ &= \lim_{r\to+\infty} \frac{x^2(1-\exp(-r))}{r} = 0. \end{aligned}$$

Hence, $f \in \mathcal{PRP}(\mathbb{R}^+ \times \Omega, \Omega, \mu)$. By theorem 4.4, equation (5.1) has a unique μ pseudo rotating periodic solution.

References

[1] E. Ait Dads, K. Ezzinbi, O. Arino; *Pseudo almost periodic solutions for some differential equations in a Banach space*, Nonlinear Anal., 28 (1997), 1141–1155.

[2] S. Bochner; *Curvature and Betti numbers in real and complex vector bundles*, Univ. e Politec. Torino. Rend. Sem. Mat., 15 (1955-56), 225–253.

[3] S. Bochner; *Uniform convergence of monotone sequences of functions*, Proc. Nat. Acad. Sci. USA, 47 (1961), 582–585.

[4] J. Blot, P. Cieutat, K. Ezzinbi; *New approach for weighted pseudo-almost periodic functions under the light of measure theory, basic results and applications*, Appl. Anal. 92 (2013), 493–526.

[5] H. Bohr; *Zur theorie der fast periodischen funktionen I*, Acta Math., 45 (1925), 29–127.

[6] J. Campos, M. Tarallo; *Almost automorphic linear dynamics by Favard theory*, J. Differential Equations, 256 (2014), 1350–1367.

[7] C. Cheng, F. Huang, Y. Li; *Affine-periodic solutions and pseudo affine-periodic solutions for differential equations with exponential dichotomy and exponential trichotomy*, J. Appl. Anal. Comput. 6 (2016), 950–967.

[8] W. A. Coppel; *Dichotomy and reducibility*, J. Differential Equations, 3 (1967), 500–521.

[9] T. Diagana; *Weighted pseudo-almost periodic solutions to some differential equations*, Nonlinear Anal. 68 (2008), 2250–2260.

[10] H. Ding, T. Xiao, J. Liang; *Asymptotically almost automorphic solutions for some integrodifferential equations with nonlocal initial conditions*, J. Math. Anal. Appl., 338 (2008), 141–151.

[11] Z. Fan, J. Liang, T. Xiao; *Composition of Stepanov-like pseudo almost automorphic functions and applications to nonautonomous evolution equations*, Nonlinear Anal. Real World Appl., 13 (2012), 131–140.
[12] J. K. Hale; *Ordinary Differential Equations*, Wiley-interscience New York, 1969.
[13] J. Liang, J. Zhang, T. Xiao; *Composition of pseudo almost automorphic and asymptotically almost automorphic functions*, J. Math. Anal. Appl., 340 (2008) , 1493–1499.
[14] G. Liu, Y. Li, X. Yang; *Existence and multiplicity of rotating periodic solutions for resonant Hamiltonian systems*, J. Differential Equations, 265 (2018) , 1324–1352.
[15] G. Liu, Y. Li, X. Yang; *Rotating periodic solutions for asymptotically linear second-order Hamiltonian systems with resonance at infinity*, Math. Methods Appl. Sci., 40 (2017), 7139–7150.
[16] I. Newton; *Philosophiae naturalis principia mathematica*, Vol. I, Reprinting of the third edition with variant readings. Canbridge, Mass: Harvard University Press, 1972.
[17] W. Shen, Y. Yi; *Almost automorphic and almost periodic dynamics in skew-product semi-flows*, Mem. Amer. Math. Soc. 136 (1998) , x+93 pp.
[18] W. A. Veech; *Almost automorphic functions on groups*, Amer. J. Math., 87 (1965), 719–751.
[19] Y. Wang, Y. Li; *Alternative theorems for pseudo almost automorphic problems*, Taiwanese J. Math., 15 (2011) , 59–74.
[20] H. Wang, X. Yang, Y. Li; *Rotating-symmetric solutions for nonlinear systems with symmetry*, Acta Math. Appl., 31 (2015), 307–312.
[21] T. Xiao, J. Liang, J. Zhang; *Pseudo almost automorphic solutions to semilinear differential equations in Banach spaces*, Semigroup Forum. 76 (2008), 518–524.
[22] J. Xing, X. Yang, Y. Li; *Rotating periodic solutions for convex Hamiltonian systems*, Appl. Math. Lett., 89 (2019), 91–96.
[23] H. J. Yang, X. L. Han; *Existence and multiplicity of positive periodic solutions for fourth-order nonlinear differential equations*, Electron. J. Differential Equations, 2019 (2019) no. 119, 1–14.
[24] X. Yang, Y. Zhang, Y. Li; *Existence of rotating-periodic solutions for nonlinear systems via upper and lower solutions*, Rocky Mountain J. Math., 47 (2017), 2423–2438.
[25] Z. Yu, X.Yang, Y. Li; *Affine-periodic solutions for dissipative systems*, Abstr. Appl. Anal., (2013).
[26] C. Zhang; *Pseudo-almost-periodic solutions of some differential equations*, J. Math. Anal. Appl., 181 (1994), 62–76.

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