MULTIPLE POSITIVE SOLUTIONS FOR BIHARMONIC EQUATION OF KIRCHHOFF TYPE INVOLVING CONCAVE-CONVEX NONLINEARITIES

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Abstract. In this article, we study the multiplicity of positive solutions for the biharmonic equation of Kirchhoff type involving concave-convex nonlinearities,

$$\Delta^2 u - \left( a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx \right) \Delta u + V(x)u = \lambda f_1(x)|u|^{q-2}u + f_2(x)|u|^{p-2}u.$$

Using the Nehari manifold, Ekeland variational principle, and the theory of Lagrange multipliers, we prove that there are at least two positive solutions, one of which is a positive ground state solution.

1. Introduction

In this article, we are concerned with the multiplicity of positive solutions for the biharmonic equations of Kirchhoff type

$$\Delta^2 u - \left( a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx \right) \Delta u + V(x)u = f(x, u) \quad \text{in } \mathbb{R}^N,$$  

(1.1)

with $u \in H^2(\mathbb{R}^N)$.

This problem is often referred to be nonlocal because of the presence of the term $\int_{\mathbb{R}^N} |\nabla u|^2 dx \Delta u$, which implies the problem is no longer a pointwise identity. This phenomenon provokes some mathematical difficulties, which make the study of such a class of problem particularly interesting. So there are many papers presented to study the nonlocal problems. We refer the reader to [2, 10, 13, 15, 16, 18, 20, 22, 30].

The motivation of this paper is from the studies on the dynamical system. Problem (1.1) is related to the stationary analog of the dissipative evolutionary equation

$$u_{tt} + h(u_t) + \Delta^2 u - \left( a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx \right) \Delta u + f(x, u) = 0.$$  

(1.2)

This equation arises as an evolutionary mathematical model in various systems for relevant physical applications, see [20] and the references therein.

The understanding of the asymptotic behavior of dynamical systems generated by dissipative evolutionary equation is an important problem of modern mathematical physics. The main method to treat this problem for a dissipative system is to consider the existence of a global attractor and analyze the structure of the global
attractor, which is an invariant compact set and attracts all bounded subsets in some space.

Recently, the authors in [19, 21, 33, 35] proved the existence of the multiple equilibrium points in the global attractors for the symmetric dynamical systems by estimating the lower bound of $Z_2$ index of two disjoint subsets of the global attractor for which one subset is located in the area where the Lyapunov function $F$ is positive and the other subset is located in the area where the Lyapunov function $F$ is negative. By the way, a fixed point, or a stationary point, or an equilibrium point for a semigroup of an evolutionary equation corresponds to the solution of the related stationary equation [24].

To have a better understanding of the asymptotic behavior of the dissipative system (1.2) for future studies, our aim in the present paper is to find the multiplicity of positive solutions for the corresponding stationary equation (1.1) based on the variational methods.

For $a = 1, b = 0$ in (1.1), we obtain the fourth-order elliptic equation

$$
\Delta^2 u - \Delta u + V(x)u = f(x, u) \quad \text{in } \mathbb{R}^N,
$$

with $u \in H^2(\mathbb{R}^N)$. The existence and multiplicity of positive, negative, sign-changing and high energy solutions of (1.3) have been the subject of extensive mathematical studies in recent years, see [1, 4, 31, 32] and the references therein.

Without the term $\Delta^2 u$, system (1.1) becomes the equation

$$
\left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right) \Delta u + V(x)u = f(x, u) \quad \text{in } \mathbb{R}^N.
$$

The solvability of the Kirchhoff-type Equation (1.4) has been widely studied by various authors. For example, Ma and Muñoz Rivera [18] investigated the existence of positive solutions of such problems by using variational methods. Perera and Zhang [22] obtained a nontrivial solution of (1.4) via Yang index and critical group. He and Zou [13] studied (1.4) the existence of infinitely many solutions by using the local minimum methods and the fountain theorems. Chen, Kuo and Wu [10] considered problem (1.4) with concave and convex nonlinearities by using Nehari manifold and fibering map methods, and the existence of multiple positive solutions were obtained. Li and Ye [16] considered (1.4) with super linear nonlinearities by using a monotonicity trick and a new version of global compactness lemma, and obtained the existence of positive ground state solutions. For other important results, see [2, 15, 30] and the references therein.

By using the mountain pass techniques and the truncation method, Wang, Avci and An [26] considered the existence of nontrivial solutions for problem (1.1) in bounded domain. By using variational methods and the truncation method, Wang, An and An [25] studied the positive solutions of (1.1) with $V(x) = 1$. Very recently, Khouiri and Chen [14] obtained the existence of ground state solutions and a least energy sign-changing solution of (1.1) by using the variational methods and the Nehari method with $f(x, u) = |u|^{p-2}u$. Wang, Ru and An [27] investigated the existence of nontrivial solutions of (1.1) via Galerkin method with $V(x) = 1$.

Concerning the concave-convex nonlinearity, there is a considerable literature that takes into account different type of problem, see for instance, the pioneering paper by Ambrosetti, Brezis and Cerami [3] for the elliptic problems in bounded domain, Wu [28] for the elliptic problem in unbounded domain, Liu and Wang [17] and Cao and Xu [9] for the Schrödinger equations, Zhang, Xu and Zhang [34] for...
the Choquard equation. However, very little work has been done for problem \((1.1)\) in \(\mathbb{R}^N\) inspired by the above-mentioned papers, we are focus on system \((1.1)\) in the case of \(f(x,u)\) involving a combination of convex and concave terms,

\[
\Delta^2 u - \left( a + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \Delta u + V(x)u = \lambda f_1(x)|u|^{q-2}u + f_2(x)|u|^{p-2}u, \tag{1.5}
\]

with \(u \in H^2(\mathbb{R}^N)\), where \(N \leq 7\), \(a, b\) are positive constants, \(\lambda > 0\) is a parameter \(1 < q < 2\), \(4 < p < 2^\ast\) \((2^\ast = \infty\) if \(N \leq 4\) and \(2^\ast = \frac{2N}{N-4}\) if \(N = 5, 6, 7\)).

We assume that \(V(x)\) and \(f(x)\) satisfy the following hypotheses:

(H1) \(V \in C(\mathbb{R}^N, \mathbb{R})\), \(\inf_{x \in \mathbb{R}^N} V(x) \geq a_0 > 0\), where \(a_0\) is a constant. Moreover, for every \(M > 0\), \(\text{meas}\{x \in \mathbb{R}^N : V(x) \leq M\} < \infty\), where \(\text{meas}\) denotes the Lebesgue measure in \(\mathbb{R}^N\);

(H2) \(f_1 \in L^{q^\ast}(\mathbb{R}^N) \cap C(\mathbb{R}^N) \setminus \{0\}\), where \(q^\ast = p/(p - q)\);

(H3) \(f_2 \in C(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)\) and \(f_2(x) > 0\) for almost every \(x \in \mathbb{R}^N\).

Our main result reads as follows.

**Theorem 1.1.** Suppose the conditions (H1)-(H3) hold. Let

\[
\lambda_0 = \frac{(p - 2)S_p^{q/2}}{(p - q)|f_1|_{q^\ast}} \left[ \frac{(2 - q)S_p^{q/2}}{(p - q)|f_2|_{\infty}} \right]^{\frac{2}{q-2}},
\]

with \(S_p\) defined in \((2.1)\) below. Then we have:

1. for each \(0 < \lambda < \lambda_0\), problem \((1.5)\) has at least two positive solutions, one of which has negative energy;

2. if \(0 < \lambda < \frac{q}{p-2} \lambda_0\), the solution corresponding to the negative energy is a positive ground state solution and the other one corresponds to positive energy.

When we restrict the space dimension to \(N \leq 7\), because \(4 < p < 2^\ast = \frac{2N}{N-4}\), then \(\frac{2N}{N-4} > 4\), hence \(N < 8\). In the present study, we mainly focus on the case \(p > 4\), whereas the different solution structure will appear if \(p < 4\) as in \([10]\), which we will investigate in the future study.

Since problem \((1.5)\) is defined in \(\mathbb{R}^N\) which is unbounded, the lack of compactness of the Sobolev embedding becomes more delicate by using variational techniques. To overcome the lack of compactness, the condition (V), which was first introduced by Bartsch and Wang in \([3]\), is always assumed to preserve the compactness of embedding of the working space. From \([11]\), we know that under the assumption (H1), the continuous embedding \(E \hookrightarrow L^s(\mathbb{R}^N)\) is compact for \(2 \leq s < 2^\ast\), where \(E\) is denoted in Section 2.1.

Since the functional of \((1.5)\) is concave-convex, it may have several critical points in the direction of nontrivial \(u\). Hence the standard method of Nehari manifold is invalid. Motivated by \([8, 10, 28, 29]\), we will make a partition of the Nehari manifold and figure out two non-degenerate submanifolds, and then consider minimization problems in the two submanifolds respectively to obtain one positive energy solution and one negative energy solution.

The remainder of this paper is organized as follows. After presenting some preliminary results in Section 2, we give the proof of our main result in Section 3.
2. Preliminaries

In the sequel, we shall use the following notation:

- \( H = H^2(\mathbb{R}^N) := \{ u \in L^2(\mathbb{R}^N) : |\nabla u|, \Delta u \in L^2(\mathbb{R}^N) \} \) is the usual Sobolev space endowed with the scalar product and norm
  \[
  \langle u, v \rangle_H = \int_{\mathbb{R}^N} (\Delta u \Delta v + \nabla u \nabla v + uv)dx, \quad \| u \|_H = \langle u, u \rangle_H^{1/2}.
  \]

In \( L^s(\mathbb{R}^N) \), we define the norm
  \[
  |u|_s = (\int_{\mathbb{R}^N} |u|^s dx)^{1/s} \quad \text{for} \quad 0 < s \leq \infty.
  \]

- \( E = \{ u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2dx < \infty \} \), with inner in \( E \) and norm
  \[
  \langle u, v \rangle = \int_{\mathbb{R}^N} (\Delta u \Delta v + a\nabla u \nabla v + V(x)uv)dx, \quad \| u \| = \langle u, u \rangle^{1/2}.
  \]

- Denote by \( S_p \) the best Sobolev constant for the embedding \( E \hookrightarrow L^p(\mathbb{R}^N) \) which is given by
  \[
  S_p = \inf_{E \setminus \{0\}} \frac{\| u \|^2}{\left( \int_{\mathbb{R}^N} |u|^p dx \right)^{2/p}} > 0. \tag{2.1}
  \]

Then we have
  \[
  |u|_p \leq S_p^{-1/2}\| u \|, \quad \forall u \in E. \tag{2.2}
  \]

- \( C, C_i \) denote different positive constants whose exact valued is inessential.

The energy functional corresponding to \( (1.5) \) is
  \[
  I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\Delta u|^2 + a|\nabla u|^2 + V(x)u^2)dx + \frac{b}{4} |\nabla u|^2 \]
  \[
  - \frac{\lambda}{q} \int_{\mathbb{R}^N} f_1(x)|u|^q dx - \frac{1}{p} \int_{\mathbb{R}^N} f_2(x)|u|^p dx, \quad u \in E.
  \]

By the assumptions (H1)–(H3), one has that \( I_\lambda(u) \in C^1(E, \mathbb{R}) \) and for any \( u, v \in E \),
  \[
  \langle I'_\lambda(u), v \rangle = \int_{\mathbb{R}^N} (\Delta u \Delta v + a\nabla u \nabla v + V(x)uv)dx + b \int_{\mathbb{R}^N} |\nabla u|^2 dx \int_{\mathbb{R}^N} \nabla u \nabla v dx
  \]
  \[
  - \lambda \int_{\mathbb{R}^N} f_1(x)|u|^{q-2}uv dx - \int_{\mathbb{R}^N} f_2(x)|u|^{p-2}uv dx.
  \]

It is well-known that \( u \) is a solution of system \( (1.5) \) if and only if \( u \in E \) is a critical point of \( I_\lambda \).

It is easy to verify that the energy functional \( I_\lambda \) is not bounded from below on \( E \). However, it is convenient to consider the functional restricted to a natural constraint, the Nehari manifold
  \[
  \mathcal{N}_\lambda = \{ u \in E \setminus \{0\} : \langle I'_\lambda(u), u \rangle = 0 \}.
  \]

Thus \( u \in \mathcal{N}_\lambda \), if and only if
  \[
  \| u \|^2 + b|\nabla u|^2 = \lambda \int_{\mathbb{R}^N} f_1(x)|u|^q dx + \int_{\mathbb{R}^N} f_2(x)|u|^p dx.
  \]

Obviously, \( \mathcal{N}_\lambda \) contains every nontrivial solution of \( (1.3) \). The following result is readily established.

**Lemma 2.1.** The energy functional \( I_\lambda \) is coercive and bounded from below on \( \mathcal{N}_\lambda \).
Proof. By Hölder’s inequality and (2.1), we have
\[ \int_{\mathbb{R}^N} f_1(x)|u|^q dx \leq \left( \int_{\mathbb{R}^N} |f_1(x)|^q dx \right)^{1/q} \left( \int_{\mathbb{R}^N} |u|^p dx \right)^{q/p} \]
\[ = |f_1|_{q^*} |u|^q \leq |f_1|_{q^*} \mathbb{S}^{-q/2}_p ||u||^q. \]  
(2.3)

For \( u \in \mathcal{N}_\lambda \), noting (2.3), we can conclude that
\[ I_\lambda(u) = I_\lambda(u) - \frac{1}{4} (I_\lambda'(u), u) \]
\[ = \frac{1}{4} ||u||^2 - \lambda \left( \frac{1}{q} - \frac{1}{4} \right) \int_{\mathbb{R}^N} f_1(x)|u|^q dx + \left( \frac{1}{4} - \frac{1}{p} \right) \int_{\mathbb{R}^N} f_2(x)|u|^p dx \]
\[ \geq \frac{1}{4} ||u||^2 - \lambda \left( \frac{1}{q} - \frac{1}{4} \right) |f_1|_{q^*} \mathbb{S}^{-q/2}_p ||u||^q. \]

Combining \( 1 < q < 2 < 4 < p < 2* \), we know that \( I_\lambda \) is coercive and bounded from below on \( \mathcal{N}_\lambda \). \( \square \)

The Nehari manifold \( \mathcal{N}_\lambda \) is closely linked to the behavior of fibering maps which is given by \( K_u(t) = I_\lambda(tu) \) for \( t > 0 \). The fibering map has been introduced by Drábek and Pohozaev in [12] and are also discussed in Brown and Zhang [8] and Brown and Wu [7]. If \( u \in E \), we have
\[ K_u(t) = \frac{1}{2} t^2 ||u||^2 + \frac{bt^4}{4} |\nabla u|^2 - \frac{\lambda t^q}{q} \int_{\mathbb{R}^N} f_1(x)|u|^q dx - \frac{t^p}{p} \int_{\mathbb{R}^N} f_2(x)|u|^p dx; \]
\[ K'_u(t) = t ||u||^2 + bt^3 |\nabla u|^2 - \lambda t^{q-1} \int_{\mathbb{R}^N} f_1(x)|u|^q dx - t^{p-1} \int_{\mathbb{R}^N} f_2(x)|u|^p dx; \]
\[ K''_u(t) = ||u||^2 + 3bt^2 |\nabla u|^2 - \lambda (q-1) t^{q-2} \int_{\mathbb{R}^N} f_1(x)|u|^q dx \]
\[ - (p-1) t^{p-2} \int_{\mathbb{R}^N} f_2(x)|u|^p dx. \]

It is easy to see that for \( u \in E \setminus \{0\} \) and \( t > 0 \), \( K''_u(t) = 0 \) if and only if \( tu \in \mathcal{N}_\lambda \), i.e., positive critical points of \( K_u \) correspond to points on the Nehari manifold. In particular, \( K''_u(1) = 0 \) if and only if \( u \in \mathcal{N}_\lambda \). Since \( K_u(t) \in C^2(\mathbb{R}^+, \mathbb{R}) \), we split \( \mathcal{N}_\lambda \) into three parts corresponding to local minima, points of inflection and local maxima.
\[ \mathcal{N}_\lambda^+ = \{ u \in \mathcal{N}_\lambda : K''_u(1) > 0 \}, \]
\[ \mathcal{N}_\lambda^0 = \{ u \in \mathcal{N}_\lambda : K''_u(1) = 0 \}, \]
\[ \mathcal{N}_\lambda^- = \{ u \in \mathcal{N}_\lambda : K''_u(1) < 0 \}. \]

For each \( u \in \mathcal{N}_\lambda \), we have
\[ K''_u(1) = ||u||^2 + 3bt |\nabla u|^2 - \lambda (q-1) \int_{\mathbb{R}^N} f_1(x)|u|^q dx - (p-1) \int_{\mathbb{R}^N} f_2(x)|u|^p dx \]
\[ = K''_u(1) - (q-1) \langle I'_\lambda(u), u \rangle \]
\[ = (2-q) ||u||^2 + b(4-q) |\nabla u|^2 - (p-q) \int_{\mathbb{R}^N} f_2(x)|u|^p dx \]
\[ = K''_u(1) - (p-1) \langle I'_\lambda(u), u \rangle \]  
(2.5)
which implies

\[ H\text{"} \text{ and Sobolev inequalities, we have} \]

\[ \therefore \text{ hence, we derive that} \]

\[ \text{which implies} \]

\[ \text{thus} \]

\[ \text{which implies} \]

\[ \text{Combining with (2.8) and (2.10), we deduce that} \]

\[ \therefore \lambda^* \geq \frac{(p-2)S_p^{q/2}}{(p-q)|f_1|_{q^*}} \left[ \frac{(2-q)S_p^{p/2}}{(p-q)|f_2|_{\infty}} \right]^{\frac{q-2}{q}} = \lambda_0 \]

\[ \text{which contradicts the assumptions. The proof is complete.} \]

\[ \text{to have a better understanding of the Nehari manifold } \mathcal{N}_\lambda \text{ and the fibering maps } K_u(t), \text{ we consider the function } h_b(t) : \mathbb{R}^+ \to \mathbb{R} \text{ defined by} \]

\[ h_b(t) = t^{2-q}\|u\|^2 + b(t^{4-q} - t^{p-q}) \int_{\mathbb{R}^N} f_2(x)|u|^p dx. \]

Then

\[ K_u(t) = t^{q-1} \left( h_b(t) - \lambda \int_{\mathbb{R}^N} f_1(x)|u|^q dx \right). \]

\[ \text{Clearly, } tu \in \mathcal{N}_\lambda \text{ if and only if } h_b(t) = \lambda \int_{\mathbb{R}^N} f_1(x)|u|^q dx, tu \in \mathcal{N}^+_\lambda \text{ (or } \mathcal{N}^-_\lambda \text{)} \text{ if and only if } h_b(t) > 0 \text{ (or } < 0). \]
For $u \in E \setminus \{0\}$ with $\int_{\mathbb{R}^N} f_2(x) |u|^p \, dx > 0$, it is obviously that $h_0(0) = 0$, $h_b(t) > 0$ for $t$ is small enough and $h_b(t) \to -\infty$ as $t \to \infty$. Note that $1 < q < 2$, $4 < p < 2$, and from
\[ h'_b(t) = t^{p-q-1} \left( (2-q)t^{2-q}||u||^2 + b(4-q)t^{4-p}|\nabla u|^4 - (p-q) \int_{\mathbb{R}^N} f_2(x) |u|^p \, dx \right) = 0, \]
we can infer that there is a unique $t_{b,\text{max}} > 0$ such that $h_b(t)$ achieves its maximum at $t_{b,\text{max}}$, increasing for $t \in [0, t_{b,\text{max}})$ and decreasing for $t \in (t_{b,\text{max}}, \infty)$ with
\[ \lim_{t \to \infty} h_b(t) = -\infty \] where $t_{b,\text{max}} \leq t_{b,\text{max}} \leq t_{b,\text{max}}$.

**Lemma 2.3.** Suppose that $0 < \lambda < \lambda_0$ and $u \in E \setminus \{0\}$. Then

(i) if $\int_{\mathbb{R}^N} f_1(x) |u|^q \, dx \leq 0$, then there at least exists a $t^+ > t_{b,\text{max}}$ such that $t^+u \in \mathcal{N}_b^+$, and
\[ I_\lambda(t^+u) = \sup_{t \geq 0} I_\lambda(tu); \]

(ii) if $\int_{\mathbb{R}^N} f_1(x) |u|^q \, dx > 0$, then there at least exists a $t^- > t_{b,\text{max}}$ and a $t^+ < t_{b,\text{max}}$, i.e. $0 < t^+ < t_{b,\text{max}} < t^-$ such that $t^+u \in \mathcal{N}_b^+$ and $t^-u \in \mathcal{N}_b^-$, satisfying
\[ I_\lambda(t^+u) = \inf_{t_{b,\text{max}} \leq t \geq 0} I_\lambda(tu), \quad I_\lambda(t^-u) = \sup_{t \geq t_{b,\text{max}}} I_\lambda(tu). \]

**Proof.** Denote $h_0(t) \doteq h_b(t)|_{b=0}$. Note that $b > 0$, we have
\[ h_b(t) > h_0(t) = t^{2-q}||u||^2 - t^{p-q} \int_{\mathbb{R}^N} f_2(x) |u|^p \, dx. \]
It is easy to see that $h_0(t)$ has a unique critical point
\[ t_{0,\text{max}} = \left( \frac{(2-q)||u||^2}{(p-q) \int_{\mathbb{R}^N} f_2(x) |u|^p \, dx} \right)^{\frac{1}{p-q}}, \]
and
\[ h_0(t_{0,\text{max}}) = \left( \frac{(2-q)||u||^2}{(p-q) \int_{\mathbb{R}^N} f_2(x) |u|^p \, dx} \right)^{\frac{p-q}{p-q}} ||u||^2 \]
\[ - \left( \frac{(2-q)||u||^2}{(p-q) \int_{\mathbb{R}^N} f_2(x) |u|^p \, dx} \right)^{\frac{p-q}{p-q}} \int_{\mathbb{R}^N} f_2(x) |u|^p \, dx \]
\[ = ||u||^q \left( \frac{||u||^p}{\int_{\mathbb{R}^N} f_2(x) |u|^p \, dx} \right)^{\frac{p-q}{p-q}} \left( \frac{2-q}{p-q} \right) \frac{p-2}{p-2} \]
\[ \geq ||u||^q \left( \frac{||u||^p}{\int_{f^2 ||S_p^{-1/2}||^2 ||u||^p}} \right)^{\frac{p-q}{p-q}} \left( \frac{2-q}{p-q} \right) \frac{p-2}{p-2} \]
\[ = ||u||^q \left( \frac{(2-q)S_p^{1/2}}{p-q} \right)^{\frac{p-2}{p-q}} \frac{p-2}{p-q} > 0. \]
Hence
\[ h_b(t_{b,\text{max}}) \geq h_b(t_{0,\text{max}}) > h_0(t_{0,\text{max}}) > 0. \]
From $0 < \lambda < \lambda_0$,
\[ \lambda \int_{\mathbb{R}^N} f_1(x) |u|^q \, dx \leq \lambda \left( \int_{\mathbb{R}^N} |f_1(x)|^q \, dx \right)^{1/q} \left( \int_{\mathbb{R}^N} |u|^p \, dx \right)^{q/p} \]
\[ \leq \lambda |f_1|_{q^*} |u|^q \leq \lambda |f_1|_{q^*} S_p^{-q/2} ||u||^q \]
Then for each \( u \) it follows that

\[
\alpha_{\lambda}^+ = \inf_{u \in \mathcal{N}_{\lambda}^+} I_{\lambda}(u), \quad \alpha_{\lambda}^- = \inf_{u \in \mathcal{N}_{\lambda}^-} I_{\lambda}(u).
\]

In the following part, we will drive some basic properties of \( \alpha_{\lambda}^+ \), \( \alpha_{\lambda}^- \).

**Lemma 2.4.** For given \( \lambda_0 \) in Theorem 1.1 we have

(i) \( \alpha_{\lambda}^+ < 0 \) for \( 0 < \lambda < \lambda_0 \);

(ii) \( \alpha_{\lambda}^- > 0 \), for \( 0 < \lambda < \frac{p}{p-2} \lambda_0 \).

**Proof.** (i) For each \( u \in \mathcal{N}_{\lambda}^+ \), \( K_{\lambda}(1) > 0 \). From (2.6), we have

\[
\lambda(p - q) \int_{\mathbb{R}^N} f_1(x)|u|^q dx > (p - 2)\|u\|^2 + b(p - 4)|\nabla u|^2.
\]

Then for each \( u \in \mathcal{N}_{\lambda}^+ \),

\[
I_{\lambda}(u) = I_{\lambda}(u) - \frac{1}{p}\langle I'_{\lambda}(u), u \rangle
\]

\[
= \frac{p - 2}{2p}\|u\|^2 + \frac{p - 4}{4p}b|\nabla u|^4 - \lambda \frac{p - q}{pq} \int_{\mathbb{R}^N} f_1(x)|u|^q dx
\]

\[
< \frac{p - 2}{2p}\|u\|^2 + \frac{p - 4}{4p}b|\nabla u|^4 - \frac{1}{pq}((p - 2)\|u\|^2 + (p - 4)b|\nabla u|^4)
\]

\[
= \frac{(p - 2)(q - 2)}{2pq}\|u\|^2 + \frac{(p - 4)(q - 4)}{4pq}b|\nabla u|^4 < 0,
\]

it follows that \( \alpha_{\lambda}^+ = \inf_{u \in \mathcal{N}_{\lambda}^+} I_{\lambda}(u) < 0 \).

(ii) Let \( u \in \mathcal{N}_{\lambda}^- \) by (2.3) and similar to the proof of (2.8), we have

\[
\|u\| \geq \left( \frac{(2 - q)S_p^{p/2}}{(p - q)|f_2|_\infty} \right)^{\frac{1}{p-2}}.
\]

Then

\[
I_{\lambda}(u) = I_{\lambda}(u) - \frac{1}{4} < I'_{\lambda}(u), u >
\]

\[
= \frac{1}{4}\|u\|^2 - \lambda \left( \frac{1}{q} - \frac{1}{4} \right) \int_{\mathbb{R}^N} f_1(x)|u|^q dx + \left( \frac{1}{4} - \frac{1}{p} \right) \int_{\mathbb{R}^N} f_2(x)|u|^p dx
\]

\[
\geq \frac{1}{4}\|u\|^2 - \lambda \left( \frac{1}{q} - \frac{1}{4} \right) |f_1|_q \cdot S_p^{-1/2}\|u\|^q
\]

\[
= \|u\|^\left( \frac{1}{4}\|u\|^{2-q} - \lambda \left( \frac{1}{q} - \frac{1}{4} \right) |f_1|_q \cdot S_p^{-1/2} \right)
\]

\[
\geq \left( \frac{(2 - q)S_p^{p/2}}{(p - q)|f_2|_\infty} \right)^{\frac{1}{p-2}} \left( \frac{1}{4} \left( \frac{(2 - q)S_p^{p/2}}{(p - q)|f_2|_\infty} \right)^{\frac{2}{p-2}} - \lambda \frac{p - q}{4q} |f_1|_q \cdot S_p^{-1/2} \right).
\]

Thus, if \( 0 < \lambda < \frac{p}{p-2} \lambda_0 \), we have \( I_{\lambda}(u) > c_0 \) for some \( c_0 > 0 \), which implies that \( \alpha_{\lambda}^- = \inf_{u \in \mathcal{N}_{\lambda}^-} I_{\lambda}(u) > 0 \). This completes the proof. \( \square \)
Lemma 2.5. If \(0 < \lambda < \lambda_0\), then \(\mathcal{N}_\lambda^-\) is closed in \(E\).

Proof. Let \(u_n \in \mathcal{N}_\lambda^-\) such that \(u_n \rightharpoonup u\) in \(E\). We need to prove \(u \in \mathcal{N}_\lambda^-\). Note that \(\langle I'_\lambda(u_n), u_n \rangle = 0\) and

\[
\langle I'_\lambda(u_n), u_n \rangle - \langle I'_\lambda(u), u \rangle
= \langle I'_\lambda(u_n), u_n - u \rangle - \langle I'_\lambda(u_n), u_n - u \rangle \to 0, \quad \text{as } n \to \infty,
\]

we have \(\langle I'_\lambda(u), u \rangle = 0\), which implies \(u \in \mathcal{N}_\lambda^\circ\).

For any \(u \in \mathcal{N}_\lambda^\circ\), by \(2.6\), we obtain that \(\mathcal{N}_\lambda^-\) is bound away from 0. By \(2.5\), it follows that \(K_u''(1) \to K_u''(1)\), noting that \(K_u''(1) < 0\), we have \(K_u''(1) \leq 0\). By Lemma 2.3, for \(\lambda < \lambda_0\), \(K_u''(1) < 0\). Therefore \(u \in \mathcal{N}_\lambda^-\).

The following lemma aims to find the critical point of \(I_\lambda\) on the whole space from the minimizer for \(I_\lambda\) on Nehari manifold.

Lemma 2.6. For \(\lambda \in (0, \lambda_0)\), if \(u_0\) is a local minimizer for \(I_\lambda\) on \(\mathcal{N}_\lambda\), then \(I'_\lambda(u_0) = 0\) in \(H^{-2}(\mathbb{R}^N)\), where \(H^{-2}(\mathbb{R}^N)\) is the dual space of \(H^2(\mathbb{R}^N)\).

Proof. From Lemma 2.3, we know that \(u_0 \notin \mathcal{N}_\lambda^0\). The rest of the proof is essentially the same as that in [8], see also in [6, 28] we omit it here.

By the above lemma, we know that the problem of finding solutions of (1.5) can be translated into that finding minimizers of \(I_\lambda\) on \(\mathcal{N}_\lambda\).

3. Proof of the main result

We first establish a lemma for locally compactness.

Lemma 3.1. Under assumptions (H1)–(H3), \(I_\lambda\) satisfies the (PS)_c condition with \(c \in \mathbb{R}\) on \(\mathcal{N}_\lambda^+\) (or \(\mathcal{N}_\lambda^-\), i.e. if \(u_n \in \mathcal{N}_\lambda^+\) (or \(\mathcal{N}_\lambda^-\)) such that \(I_\lambda(u_n) \to c\) and \(I'_\lambda(u_n) \to 0\), then there exists a convergent subsequence of \(u_n\).

Proof. Assume that \(u_n \in \mathcal{N}_\lambda^+\) (or \(\mathcal{N}_\lambda^-\)) such that \(I_\lambda(u_n) \to c\) and \(I'_\lambda(u_n) \to 0\). By Lemma 2.1, we infer that \(u_n\) is bounded in \(E\). Up to a subsequence, we may assume that \(u_n \rightharpoonup u\) in \(E\), \(u_n \to u\) in \(L^s(\mathbb{R}^n), \ s \in [2, 2^\ast)\).

It follows that

\[
b\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right) \int_{\mathbb{R}^N} \nabla u \nabla (u_n - u) dx \to 0 \quad \text{as } n \to \infty.
\]  

By using twice the Hölder inequality, the corresponding exponents are \((\frac{2}{p - q}, \frac{2}{q'})\) and \((q, \frac{q}{2})\) respectively, we obtain

\[
|\lambda \int_{\mathbb{R}^N} f_1(x)(|u_n|^{q - 2} u_n - |u|^{q - 2} u)(u_n - u) dx|
\leq \lambda \left(\int_{\mathbb{R}^N} |f_1(x)| q dx\right) 1/q' \left(\int_{\mathbb{R}^N} |u_n|^{q - 2} u_n - |u|^{q - 2} u |u_n - u|^{p/q} dx\right)^{q/p}
\leq \lambda C |f_1| q' \left(|u_n|^{q - 1} + |u|^{q - 1}\right)|u_n - u|_{p} \to 0, \quad \text{as } n \to \infty,
\]

where \(C\) is a positive constant. Similarly,

\[
|\int_{\mathbb{R}^N} f_2(x)(|u_n|^{p - 2} u_n - |u|^{p - 2} u)(u_n - u) dx| \to 0, \quad \text{as } n \to \infty.
\]
Lemma 3.2. Suppose that $u \in \mathcal{N}_\lambda^+$, there exist $\epsilon = \epsilon(u) > 0$ and a differentiable function $\psi^+: B_\epsilon(0) \to \mathbb{R}^+ := (0, +\infty)$ such that

1. $\psi^+(0) = 1$;
2. $(\psi^+(w))(u - w) \in \mathcal{N}_\lambda^+, \forall w \in B_\epsilon(0)$;
3. $\langle (\psi^+)(0), w \rangle = \frac{L(u,w)}{K_u(1)}(1)$, where

$$L(u, w) = 2(u,w) + 4b\int_{\mathbb{R}^N} |\nabla u|^2 \nabla u \cdot \nabla w \, dx - q\int_{\mathbb{R}^N} f_1(x)|u|^{q-2}uw \, dx - p\int_{\mathbb{R}^N} f_2(x)|u|^{p-2}uw \, dx.$$ 

Moreover, for any $C_1, C_2 > 0$, there exists $C > 0$ such that if $C_1 \leq \|u\| \leq C_2$, then $|\langle (\psi^+)'(0), w \rangle| \leq C\|w\|$. 

Proof. Define a $C^1$ mapping $J : \mathbb{R}^+ \times E \to \mathbb{R}$ by $J(t, w) = K_{u-w}(t)$, that is

$$J(t, w) = t\|u - w\|^2 + bt^3(\int_{\mathbb{R}^N} |\nabla u - w|^2 \, dx)^2 - \lambda t^{q-1}\int_{\mathbb{R}^N} f_1(x)|u - w|^q \, dx$$

$$- t^{p-1}\int_{\mathbb{R}^N} f_2(x)|u - w|^p \, dx.$$ 

Note that $u \in \mathcal{N}_\lambda^+$, then $J(1, 0) = 0$ and $J_t(1, 0) = K_u''(1) > 0$. Applying the implicit function theorem at point $(1, 0)$, we obtain that there exist $\epsilon = \epsilon(u) > 0$ and a differentiable function $\psi^+: B_\epsilon(0) \to \mathbb{R}^+ := (0, +\infty)$ such that

$$\psi^+(0) = 1, \quad J(\psi^+(w), w) = 0, \quad \forall w \in B_\epsilon(0).$$ 

Next, we prove that $\psi^+(u - w) \in \mathcal{N}_\lambda^+$ for all $w \in B_\epsilon(0)$. Indeed, by $u \in \mathcal{N}_\lambda^+$ and the set $\mathcal{N}_\lambda^- \cup \mathcal{N}_\lambda^0$ is closed, we can get $\text{dist}(u, \mathcal{N}_\lambda^- \cup \mathcal{N}_\lambda^0) > 0$. Note that $\psi^+(w)(u - w)$ is continuous with respect to $w$, choose $\epsilon = \epsilon(u) > 0$ small enough, such that

$$\|\psi^+(w)(u - w) - u\| < \frac{1}{2} \text{dist}(u, \mathcal{N}_\lambda^- \cup \mathcal{N}_\lambda^0), \quad \forall w \in B_\epsilon(0).$$
Hence
\[ \|\psi^+(w)(u - w) - N^+_\lambda \| \geq \text{dist}(u, N^-_\lambda \cup N^0_\lambda) - \text{dist}(\psi^+(w)(u - w), u) \]
\[ \geq \frac{1}{2} \text{dist}(u, N^-_\lambda \cup N^0_\lambda) > 0, \]
thus \( \psi^+(w)(u - w) \in N^+_\lambda \), for all \( w \in B_\epsilon(0) \).

By the differentiability of the implicit function theorem, we have
\[ \langle (\psi^+)'(0), w \rangle = -\frac{\langle J_w(1, 0), w \rangle}{J_I(1, 0)}. \]
Note that \( L(1, 0) = -\langle J_w(1, 0), w \rangle \), and \( K''_w(1) = J_I(1, 0) \), therefore, we have
\[ \langle (\psi^+)(0), w \rangle = \frac{L(u, w)}{K''_w(1)}. \]

Finally, we verify that there exists \( \delta > 0 \) such that \( K''_w(1) \geq \delta > 0 \) with \( C_1 \leq \|u\| \leq C_2 \), \( u \in N^+_\lambda \), where \( C_1, C_2 > 0 \). We will prove that by contradiction. Otherwise, if there exists a sequence \( \{u_n\} \in N^+_\lambda \) with \( C_1 \leq \|u_n\| \leq C_2 \), satisfying \( K''_{u_n}(1) \leq \delta_n \) for any \( \delta_n \) sufficiently small and \( \delta_n \rightarrow 0 \) as \( n \rightarrow \infty \). From (2.6), we have
\[ (2 - q)\|u_n\|^2 + b(4 - q) \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \right)^2 = (p - q) \int_{\mathbb{R}^N} f_2(x)|u_n|^p dx + o(\delta_n), \]
where \( o(\delta_n) \rightarrow 0 \) as \( n \rightarrow \infty \). Noting that \( 1 < q < 2, 4 < p < 2^*, C_1 \leq \|u_n\| \leq C_2 \) and (2.7), we have
\[ (2 - q)\|u_n\|^2 \leq (p - q)|f_2|_{\infty} S^{-p/2}_p \|u_n\|^p + o(\delta_n), \]
and hence
\[ \|u_n\| \geq \left( \frac{(2 - q)S^{-p/2}_p}{(p - q)|f_2|_{\infty}} \right)^{\frac{1}{p - q}} + o(\delta_n). \quad (3.2) \]
From (2.6), we have
\[ (p - 2)\|u_n\|^2 + b(p - 4) \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \right)^2 = \lambda(p - q) \int_{\mathbb{R}^N} f_1(x)|u_n|^q dx + o(\delta_n). \]
In view of (2.9), we have
\[ (p - 2)\|u_n\|^2 \leq \lambda(p - q)|f_1|_{q'} S^{-q/2}_p \|u_n\|^q + o(\delta_n), \]
which implies
\[ \|u_n\| \leq \left( \frac{\lambda_0(p - q)|f_1|_{q'}}{(p - 2)S^{-q/2}_p} \right)^{\frac{1}{p - q}} + o(\delta_n), \quad (3.3) \]
Combining this with (3.2) and (3.3), as \( n \rightarrow \infty \), we deduce a contradiction.

Therefore, if there exists \( C > 0 \) such that if \( C_1 \leq \|u\| \leq C_2 \), then \( \|(\psi^+)'(0), w)\| \leq C\|w\| \). This ends the proof of Lemma 3.2. \( \square \)

In the same way, to extract a \((PS)_{\alpha_L}\) sequence from the minimizing sequence of problem, we establish the following lemma.

**Lemma 3.3.** Suppose that \( u \in N^+_\lambda \), there exist \( \epsilon = \epsilon(u) > 0 \) and a differentiable function \( \psi^- : B_\epsilon(0) \rightarrow \mathbb{R}^+ := (0, +\infty) \) such that
(1) \( \psi^-(0) = 1; \)
(2) \( \psi^-(w)(u - w) \in N^+_\lambda \) for all \( w \in B_\epsilon(0) \);
(3) \( ((\psi^-)(0), w) = \frac{L(u, w)}{\mathcal{R}_{\mathcal{D}}(1)}, \) where
\[
L(u, w) = 2(u, w) + 4b \int_{\mathbb{R}^N} |\nabla u|^2 \nabla u \nabla w \, dx - q \int_{\mathbb{R}^N} f_1(x)|u|^{q-2}uw \, dx
- p \int_{\mathbb{R}^N} f_2(x)|u|^{p-2}uw \, dx.
\]
Moreover, for any \( C_1, C_2 > 0, \) there exists \( C > 0 \) such that if \( C_1 \leq \|u\| \leq C_2, \) then \( |((\psi^-)'(0), w)| \leq C\|w\|. \)

We are now ready to construct the \((PS)_{\alpha^+}\) (or \((PS)_{\alpha^-}\)) sequence from the minimizing sequence of the energy functional \( I_\lambda \) on the Nehari manifold \( \mathcal{N}_\lambda^+ \) (or \( \mathcal{N}_\lambda^- \)).

**Lemma 3.4.** Suppose (H1)–(H3) hold, and \( 0 < \lambda < \lambda_0 \), then there exists a sequence \( u_n \in \mathcal{N}_\lambda^+ \) such that \( I_\lambda(u_n) \to \alpha_\lambda^+ \) and \( I'_\lambda(u_n) \to 0 \) as \( n \to \infty \).

**Proof.** By Lemma 2.1 and the Ekeland Variational Principle on \( \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0 \), there exists a minimizing sequence \( \{u_n\} \subset \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0 \) such that
\[
\inf_{u \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0} I_\lambda(u) \leq I_\lambda(u_n) < \inf_{u \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0} I_\lambda(u) + \frac{1}{n},
\]
\[
I_\lambda(v) \geq I_\lambda(u_n) - \frac{1}{n}\|v - u_n\|, \quad \forall v \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0.
\]
Observe that \( \mathcal{N}_\lambda^0 = \emptyset \), then we have \( \inf_{u \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0} I_\lambda(u) = \inf_{u \in \mathcal{N}_\lambda^+} I_\lambda(u) = \alpha_\lambda^+ \).
Thus \( I_\lambda(u_n) \to \alpha_\lambda^+ \), and we may assume that \( u_n \in \mathcal{N}_\lambda^+ \). By Lemma 2.4, we know that \( \alpha_\lambda^+ < 0 \).

To finish the proof, we only need to verify that \( I'_\lambda(u_n) \to 0 \). Applying Lemma 3.2 with \( u_n \) to obtain the function \( \psi_n^+: \mathcal{B}_{\epsilon_n}(0) \to \mathbb{R}^+ \) such that
\[
\psi_n^+(w)(u_n - w) \in \mathcal{N}_\lambda^+, \quad \forall w \in \mathcal{B}_{\epsilon_n}(0).
\]
By the continuity of \( \psi_n^+(w) \) and \( \psi_n^+(0) = 1 \), without loss of generality, we can assume \( \epsilon_n \) is sufficiently small such that \( 1/2 \leq \psi_n^+(w) \leq 3/2 \) for \( \|w\| \leq \epsilon_n \). From \( \psi_n^+(w)(u_n - w) \in \mathcal{N}_\lambda^+ \) and (b), we have
\[
I_\lambda(\psi_n^+(w)(u_n - w)) - I_\lambda(u_n) \geq -\frac{1}{n}\|\psi_n^+(w)(u_n - w) - u_n\|,
\]
and by the mean value theorem, we have
\[
\langle I'_\lambda(u_n), \psi_n^+(w)(u_n - w) - u_n \rangle + o(\|\psi_n^+(w)(u_n - w) - u_n\|)
\geq -\frac{1}{n}\|\psi_n^+(w)(u_n - w) - u_n\|.
\]
Consequently,
\[
\psi_n^+(w)(I'_\lambda(u_n), w) + (1 - \psi_n^+(w))\langle I'_\lambda(u_n), u_n \rangle
\leq \frac{1}{n}\|((\psi_n^+)'(0), w)u_n\| + o(\|\psi_n^+(w)(u_n - w) - u_n\|).
\]
By the choice of \( \epsilon_n \) and \( \frac{1}{2} \leq \psi_n^+(w) \leq \frac{3}{2} \), we infer that there exists \( C_3 > 0 \) such that
\[
\langle I'_\lambda(u_n), w \rangle \leq \frac{1}{n}\|((\psi_n^+)'(0), w)u_n\| + \frac{C_3}{n}\|w\| + o(\|((\psi_n^+)'(0), w)(\|u_n\| + \|w\|)).
\]
For \( u_n \in \mathcal{N}_\lambda^+ \), we claim that \( \inf_{n \in \mathbb{N}} \| u_n \| \geq C_1 > 0 \), where \( C_1 \) is a constant. Otherwise, \( I_\lambda(u_n) \) would converge to zero, which contradict with \( I_\lambda(u_n) \to \alpha_\lambda^+ < 0 \). Moreover, by Lemma 2.1, we know that \( I_\lambda \) is coercive on \( \mathcal{N}_\lambda^+ \), \( \| u_n \| \) is bounded in \( E \). Thus, there exists \( C_2 > 0 \) such that \( 0 < C_1 \leq \| u_n \| \leq C_2 \). From Lemma 3.2, \( |\langle (\psi_n)^\prime(0), w \rangle| \leq C \| w \| \). Hence

\[
|\langle I_\lambda'(u_n), w \rangle| \leq \frac{C}{n} \| w \| + \frac{C}{n} \| w \| + o(\| w \|),
\]

\[
\| I_\lambda'(u_n) \| = \sup_{w \in E \setminus \{0\}} \frac{|\langle I_\lambda'(u_n), w \rangle|}{\| w \|} \leq \frac{C}{n} + o(1).
\]

Then \( \| I_\lambda(u_n) \| \to 0 \) as \( n \to \infty \). Thus, \( \{ u_n \} \subset \mathcal{N}_\lambda^+ \) is a \((PS)_{\alpha_\lambda^+}\) sequence for \( I_\lambda \) on \( E \). Similarly, we can construct the \((PS)_{\alpha_\lambda^-}\) sequence.

\[ \square \]

**Lemma 3.5.** Suppose (H1)–(H3) hold, and \( 0 < \lambda < \lambda_0 \), then there exists a sequence \( u_n \in \mathcal{N}_\lambda^- \) such that \( I_\lambda(u_n) \to \alpha_\lambda^- \) and \( I_\lambda'(u_n) \to 0 \) as \( n \to \infty \).

Now, we are in a position to give the proof of our main result.

**Proof of Theorem 1.1.** Firstly, we consider the minimization problem

\[
\alpha_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} I_\lambda(u).
\]

By Lemma 3.4, there exists \( u_n \in \mathcal{N}_\lambda^+ \) such that \( I_\lambda(u_n) \to \alpha_\lambda^+ \) and \( I_\lambda'(u_n) \to 0 \). From Lemma 3.1, there exists a strongly convergent subsequence of \( \{ u_n \} \), still denoted by \( \{ u_n \} \), satisfying \( u_n \to w_1 \) in \( E \). From the proof of Lemma 3.4, we know that there exist \( C_1, C_2 > 0 \) such that \( 0 < C_1 \leq \| u_n \| \leq C_2 \), then \( 0 < C_1 \leq \| w_1 \| \leq C_2 \). Thus \( w_1 \neq 0 \). Next we prove \( w_1 \in \mathcal{N}_\lambda^+ \). Indeed, by (2.5), it follows that \( K''_{w_1}(1) \to K''_{u_n}(1) \). From \( K''_{u_n}(1) > 0 \), we have \( K''_{w_1}(1) \geq 0 \). By Lemma 2.3, we know that \( K''_{w_1}(1) > 0 \). Hence

\[
w_1 \in \mathcal{N}_\lambda^+, \quad I(w_1) = \lim_{n \to \infty} I_\lambda(u_n) = \inf_{u \in \mathcal{N}_\lambda^+} I_\lambda(u).
\]

Thus \( w_1 \) is a nontrivial solution of 1.5 by Lemma 2.6. Since \( I_\lambda(w_1) = I_\lambda(\| w_1 \|) \) and \( \| w_1 \| \in \mathcal{N}_\lambda^+ \), we may assume that \( w_1 \) is a positive solution of 1.5. Therefore, we find a positive solution of 1.5.

Secondly, we consider the minimization problem \( \alpha_\lambda^- = \inf_{u \in \mathcal{N}_\lambda^-} I_\lambda(u) \). Similar to the above proof, we can also find a positive solution \( w_2 \in \mathcal{N}_\lambda^- \).

From the above proof, we know if \( 0 < \lambda < \lambda_0 \), then problem 1.5 has at least two positive solutions \( w_1 \in \mathcal{N}_\lambda^+ \) and \( w_2 \in \mathcal{N}_\lambda^- \). Combining with Lemma 2.4 (i), we have \( I_\lambda(w_1) < 0 \). Moreover, by Lemma 2.4 (ii), if \( 0 < \lambda < \frac{4}{p-2} \lambda_0 \), for any \( u \in \mathcal{N}_\lambda^- \), \( I_\lambda(u) > 0 \), then \( I_\lambda(w_2) > 0 \). Hence if \( 0 < \lambda < \frac{4}{p-2} \lambda_0 \), then \( I_\lambda(w_1) = \inf_{u \in \mathcal{N}_\lambda} I_\lambda(u) \), \( w_1 \) is a positive ground state solution of 1.5.

\[ \square \]

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