Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 45, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

POSITIVE SOLUTIONS FOR A NONLINEAR SYSTEM OF FOURTH-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this article, we consider the existence of positive solutions for a nonlinear system of fourth-order ordinary differential equations. By constructing a single cone P in the product space $C[0,1] \times C[0,1]$ and applying fixed point theorem in cones, we establish the existence of positive solutions for a system in which the nonlinear terms are both superlinear or sublinear. In addition, by the construction of the product cone $K_1 \times K_2 \subset C[0,1] \times C[0,1]$ along with the product formula of fixed point theory on a product cone, we investigate the existence of positive solutions involving nonlinear terms, one uniformly superlinear or sublinear, and the other locally uniformly sublinear or superlinear.

1. Introduction and main results

In this article, we consider the existence of positive solutions for the nonlinear system of fourth-order ordinary differential equations

$$\begin{gathered}
u^{(4)}(t) + \beta_1 u''(t) - \alpha_1 u(t) = f_1(t, u(t), v(t)), \quad t \in (0,1), \\
v^{(4)}(t) + \beta_2 v''(t) - \alpha_2 v(t) = f_2(t, u(t), v(t)), \quad t \in (0,1), \\
u(0) = u(1) = u''(0) = u''(1) = 0, \\
v(0) = v(1) = v''(0) = v''(1) = 0,
\end{gathered} \tag{1.1}$$

where $f_i \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $\mathbb{R}^+ = [0, +\infty)$ and $\beta_i, \alpha_i \in \mathbb{R}$ $(i = 1, 2)$ satisfy the following conditions:

$$\beta_i < 2\pi^2, \quad -\beta_i^2/4 \leqslant \alpha_i, \quad \alpha_i/\pi^4 + \beta_i/\pi^2 < 1.$$

These conditions involve a two-parameter nonresonance condition, see [5, 6].

In recent years, there have been extensive attention on the existence of positive solutions for second-order ordinary differential equations and systems, see [3, 4, 8] and the references therein. For example, in [8], by applying the product formula of fixed point theory on product cone and fixed point theory in cones, He-Yang discussed the existence and multiplicity of positive solutions for a system of nonlinear Sturm-Liouville equations. Clearly, when $\beta_1 = \beta_2 = \beta$, $\alpha_1 = \alpha_2 = \alpha$, $f_1(t, u, v) = f(t, u)$ and $f_2(t, u, v) = f(t, v)$, system (1.1) reduces to the following

2010 *Mathematics Subject Classification.* 34B18, 47H11, 47N20.

Key words and phrases. Positive solution; fixed point theory; ordinary differential equation.

Submitted December 9, 2019. Published May 19, 2020.

fourth-order ordinary differential equation

$$\begin{gathered} w^{(4)}(t)+\beta w''(t)-\alpha w(t)=f(t,w(t)), \quad t\in(0,1), \\ w(0)=w(1)=w''(0)=w''(1)=0. \end{gathered} \tag{1.2}$$

Equation (1.2) is used to describe the deformation of an elastic beam supported at the end points. Much has been studied for (1.2), see [9, 6, 1, 2] and references therein. For instance, in [9], by employing the fixed point theory in cones, Li presented the existence of positive solutions for (1.2) under the conditions that $f(t,w)$ is either superlinear or sublinear with respect to w at $w=0$ and $w=+\infty$.

Motivated by the works mentioned above, we shall deal with the existence of positive solutions for system (1.1). The purpose of this paper is to extend the result in [9] from two different aspects.

Firstly, we consider the existence of positive solutions for system (1.1) with superlinear or sublinear nonlinearities. For this problem, we can change the existence of positive solutions for system (1.1) into that of nontrivial fixed points for the corresponding compactly continuous mappings on a single cone P (see (2.2)) in product space $C[0,1]\times C[0,1]$ and then choose proper open sets $P_r\subset P$ in which the superlinearity or sublinearity can be applied directly. By using the fixed point theory in cones, we obtain the existence of positive solutions for system (1.1). Our main results are the following.

Theorem 1.1. *Assume that f_1 and f_2 satisfy the condition*

$$\limsup_{u+v\to0^+}\max_{t\in[0,1]}\frac{\sum_{i=1}^2 f_i(t,u,v)}{\lambda_1u+\lambda_2v}<1<\liminf_{u+v\to+\infty}\min_{t\in[0,1]}\frac{\sum_{i=1}^2 f_i(t,u,v)}{\lambda_1u+\lambda_2v}, \tag{1.3}$$

where $\lambda_i=\pi^4-\beta_i\pi^2-\alpha_i$ $(i=1,2)$. Then system (1.1) has at least one nonzero nonnegative solution. Moreover, if $f_1(t,0,v(t))\not\equiv0$ and $f_2(t,u(t),0)\not\equiv0$ for all $(u,v)\in P\backslash\{(0,0)\}$, then system (1.1) has at least one positive solution.

Theorem 1.2. *Suppose that f_1 and f_2 satisfy the following assumptions:*

$$\liminf_{u+v\to0^+}\min_{t\in[0,1]}\frac{\sum_{i=1}^2 f_i(t,u,v)}{\lambda_1u+\lambda_2v}>1>\limsup_{u+v\to+\infty}\max_{t\in[0,1]}\frac{\sum_{i=1}^2 f_i(t,u,v)}{\lambda_1u+\lambda_2v}, \tag{1.4}$$

where $\lambda_i=\pi^4-\beta_i\pi^2-\alpha_i$ $(i=1,2)$. Then system (1.1) has at least one nonzero nonnegative solution. Moreover, if $f_1(t,0,v(t))\not\equiv0$ and $f_2(t,u(t),0)\not\equiv0$ for all $(u,v)\in P\backslash\{(0,0)\}$, then system (1.1) has at least one positive solution.

Secondly, we investigate the existence of positive solutions for (1.1) involving nonlinear terms in which one is uniformly superlinear or sublinear and the other is locally uniformly sublinear or superlinear. In this case, it is very difficult to directly construct proper open sets in the single cone P on product space owing to the different features of nonlinear terms. In order to overcome the obstacle, we will construct a product cone $K_1\times K_2$ which is the Cartesian product of two cones $\{K_i\}_{i=1}^2\subset C[0,1]$ (see (2.2)) and then choose proper open sets $D=D_1\times D_2\subset K_1\times K_2$, such that the different features of nonlinearities can be exploited better. Applying the product formula for the fixed point index on product cone and the fixed point index theory in cones, we establish the existence of positive solutions for system (1.1). Our main results are the following.

Theorem 1.3. *Assume that f_1 and f_2 satisfy the following conditions:*

$$\limsup_{u\to 0^+}\max_{t\in[0,1]}\frac{f_1(t,u,v)}{u}<\lambda_1<\liminf_{u\to+\infty}\min_{t\in[0,1]}\frac{f_1(t,u,v)}{u} \tag{1.5}$$

uniformly with respect to $v\in\mathbb{R}^+$;

$$\liminf_{v\to 0^+}\min_{t\in[0,1]}\frac{f_2(t,u,v)}{v}>\lambda_2>\limsup_{v\to+\infty}\max_{t\in[0,1]}\frac{f_2(t,u,v)}{v} \tag{1.6}$$

locally uniformly with respect to $u\in[0,M]$ for all $M>0$, where $\lambda_i=\pi^4-\beta_i\pi^2-\alpha_i$ $(i=1,2)$. Then system (1.1) has at least one positive solution.

Theorem 1.4. *Suppose that f_1 and f_2 satisfy the following assumptions*

$$\liminf_{u\to 0^+}\min_{t\in[0,1]}\frac{f_1(t,u,v)}{u}>\lambda_1>\limsup_{u\to+\infty}\max_{t\in[0,1]}\frac{f_1(t,u,v)}{u} \tag{1.7}$$

uniformly with respect to $v\in\mathbb{R}^+$;

$$\limsup_{v\to 0^+}\max_{t\in[0,1]}\frac{f_2(t,u,v)}{v}<\lambda_2<\liminf_{v\to+\infty}\min_{t\in[0,1]}\frac{f_2(t,u,v)}{v} \tag{1.8}$$

locally uniformly with respect to $u\in[0,M]$ for all $M>0$, where $\lambda_i=\pi^4-\beta_i\pi^2-\alpha_i$ $(i=1,2)$. Moreover, if there exists a locally bounded function $g=g(u)$ such that

$$\limsup_{v\to+\infty}\max_{t\in[0,1]}f_1(t,u,v)=g(u) \tag{1.9}$$

locally uniformly with respect to $u\in[0,M]$ for all $M>0$, then system (1.1) has at least one positive solution.

Remark 1.5. If one of the following two conditions is valid

(i) f_1 and f_2 satisfy assumptions (1.5) and (1.8); or
(ii) f_2 and f_1 satisfy assumptions (1.6), (1.7) and (1.9),

then system (1.1) has at least one positive solution, which can be seen from the proofs of Theorems 1.3-1.4. In particular, when $\beta_1=\beta_2=\beta$, $\alpha_1=\alpha_2=\alpha$, $f_1(t,u,v)=f(t,u)$ and $f_2(t,u,v)=f(t,v)$, equation (1.2) has at least one positive solution if f_1 satisfies condition (1.5) or f_2 satisfies condition (1.6), which is just the result in [9].

The rest of this article is organized as follows: in Section 2, we present some preliminaries; in Section 3, we prove Theorems 1.1-1.4; in Section 4, we give some applications.

2. Preliminaries

In this section, we construct two classes of cones in which one is a single subcone $P\subset C[0,1]\times C[0,1]$, and the other is the Cartesian product $K_1\times K_2$ of subcones $\{K_i\}_{i=1}^2\subset C[0,1]$. And then we can change problem (1.1) into the fixed point problem in the constructed subcones. At the same time, we give some useful preliminaries for the proofs of our main results. Let

$$\mu_{i,1}=\frac{1}{2}\Big(-\beta_i+\sqrt{\beta_i^2+4\alpha_i}\Big),\quad \mu_{i,2}=\frac{1}{2}\Big(-\beta_i-\sqrt{\beta_i^2+4\alpha_i}\Big),\quad i=1,2,$$

and $G_{i,j}(t,s)$ $(j=1,2)$ be the Green's function of the linear boundary value problem

$$-u''(t)+\mu_{i,j}\,u(t)=0,\quad t\in(0,1),$$

$$u(0) = u(1) = 0,$$

then for $h_i \in C[0,1]$, the solution of linear boundary value problem

$$\begin{gathered} u^{(4)}(t) + \beta_i u''(t) - \alpha_i u(t) = h_i(t), \quad t \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{gathered} \tag{2.1}$$

can be expressed as

$$u(t) = \int_0^1 \int_0^1 G_{i,1}(t,\tau) G_{i,2}(\tau,s) h_i(s) \, \mathrm{d}s \, \mathrm{d}\tau.$$

Lemma 2.1 ([9]). *The function $G_{i,j}(t,s)$ $(i = 1,2\,; j = 1,2)$ has the following properties:*

- (i) *$G_{i,j}(t,s) > 0$ for all $t,s \in (0,1)$;*
- (ii) *$G_{i,j}(t,s) \leqslant C_{i,j} G_{i,j}(s,s)$ for all $t,s \in [0,1]$, where $C_{i,j} > 0$ is a constant;*
- (iii) *$G_{i,j}(t,s) \geqslant \delta_{i,j} G_{i,j}(t,t) G_{i,j}(s,s)$ for all $t,s \in [0,1]$, where $\delta_{i,j} > 0$ is a constant.*

It is well known that $C[0,1]$ is a Banach space with the maximum norm $\|u\| = \max_{t\in[0,1]} |u(t)|$ and $C^+[0,1] := \{u \in C[0,1] |\ u(t) \geqslant 0,\ \forall t \in [0,1]\}$ is a closed convex cone in $C[0,1]$. In addition, let

$$M_{i,j} = \max_{t\in[0,1]} G_{i,j}(t,t), \quad m_{i,j} = \min_{t\in[1/4,3/4]} G_{i,j}(t,t),$$

$$C_i = \int_0^1 G_{i,1}(\tau,\tau) G_{i,2}(\tau,\tau) \, \mathrm{d}\tau,$$

then $M_{i,j}$, $m_{i,j}$, $C_i > 0$ $(i = 1,2; j = 1,2)$.

Lemma 2.2 ([9]). *Let $h_i \in C^+[0,1]$ $(i = 1,2)$, then the solution of* (2.1) *satisfies*

$$u(t) \geqslant \frac{\delta_{i,1}\delta_{i,2} C_i}{C_{i,1} C_{i,2} M_{i,1}} G_{i,1}(t,t) \|u\|, \quad \forall t \in [0,1].$$

For $\lambda \in [0,1]$, $u,v \in C^+[0,1]$, we define the mappings $T_{\lambda,1}(\cdot,\cdot)$, $T_{\lambda,2}(\cdot,\cdot)$: $C^+[0,1] \times C^+[0,1] \to C^+[0,1]$ and $T_\lambda(\cdot,\cdot) : C^+[0,1] \times C^+[0,1] \to C^+[0,1] \times C^+[0,1]$ by

$$\begin{gathered} T_{\lambda,1}(u,v)(t) = \int_0^1 \int_0^1 G_{1,1}(t,\tau) G_{1,2}(\tau,s) h_{\lambda,1}(s,u(s),v(s)) \, \mathrm{d}s \, \mathrm{d}\tau, \\ T_{\lambda,2}(u,v)(t) = \int_0^1 \int_0^1 G_{2,1}(t,\tau) G_{2,2}(\tau,s) h_{\lambda,2}(s,u(s),v(s)) \, \mathrm{d}s \, \mathrm{d}\tau, \\ T_\lambda(u,v)(t) = (T_{\lambda,1}(u,v)(t), T_{\lambda,2}(u,v)(t)), \end{gathered}$$

where

$$\begin{aligned} h_{\lambda,1}(s,u(s),v(s)) &= \lambda f_1(s,u(s),v(s)) + (1-\lambda) f_1(s,u(s),0), \\ h_{\lambda,2}(s,u(s),v(s)) &= \lambda f_2(s,u(s),v(s)) + (1-\lambda) f_2(s,0,v(s)). \end{aligned}$$

It is obvious that the existence of positive solutions of system (1.1) is equivalent to the existence of nontrivial fixed points of T_1. We will find the nontrivial fixed point of T_1 by using the fixed point theory in cones. On this purpose, we introduce the following sub-cones:

$$\begin{gathered} K_i = \{u \in C^+[0,1] : u(t) \geqslant \sigma_i \|u\|,\ \forall t \in [1/4,3/4]\} \quad i = 1,2; \\ P = \{(u,v) \in C^+[0,1] \times C^+[0,1] : u(t) + v(t) \geqslant \sigma \|(u,v)\|,\ \forall t \in [1/4,3/4]\}, \end{gathered} \tag{2.2}$$

where $\sigma_i = m_{i,1}\delta_{i,1}\delta_{i,2}C_i/(C_{i,1}C_{i,2}M_{i,1})$, $\sigma = \min\{\sigma_1, \sigma_2\} > 0$ and $\|(u,v)\| = \|u\| + \|v\|$. For simplicity and convenience, we will use the following notation:

$$\begin{aligned} K_{r_i} &= \{u \in K_i : \|u\| < r_i\}, \quad P_r = \{(u,v) \in P : \|(u,v)\| < r\}, \\ \partial K_{r_i} &= \{u \in K_i : \|u\| = r_i\}, \quad \partial P_r = \{(u,v) \in P : \|(u,v)\| = r\}, \\ \overline{K_{r_i}} &= \{u \in K_i : \|u\| \leqslant r_i\}, \forall r_i > 0, \quad \overline{P_r} = \{(u,v) \in P : \|(u,v)\| \leqslant r\}, \forall r > 0. \end{aligned}$$

To calculate the fixed point index of T_1, we need the following results.

Lemma 2.3. (i) *$T_\lambda : K_1 \times K_2 \to K_1 \times K_2$ is completely continuous;*
(ii) *$T_1 : P \to P$ is completely continuous.*

Proof. (i) For $(u,v) \in K_1 \times K_2$, let $h_1(t) = h_{\lambda,1}(t, u(t), v(t))$, then $T_{\lambda,1}(u,v)(t)$ is the solution of equation (2.1) with $i = 1$. By Lemma 2.2, we have

$$\begin{aligned} T_{\lambda,1}(u,v)(t) &= \int_0^1 \int_0^1 G_{1,1}(t,\tau)G_{1,2}(\tau,s)h_1(s)\,\mathrm{d}s\,\mathrm{d}\tau \\ &\geqslant \frac{\delta_{1,1}\delta_{1,2}C_1}{C_{1,1}C_{1,2}M_{1,1}} G_{1,1}(t,t)\|T_{\lambda,1}(u,v)\|, \quad \forall t \in [0,1], \end{aligned}$$

which implies that $T_{\lambda,1}(u,v)(t) \geqslant \sigma_1 \|T_{\lambda,1}(u,v)\|$, $t \in [1/4, 3/4]$. Similarly,

$$T_{\lambda,\,2}(u,v)(t) \geqslant \sigma_2 \|T_{\lambda,\,2}(u,v)\|, \ t \in [1/4, 3/4].$$

Hence, $T_\lambda(K_1 \times K_2) \subset K_1 \times K_2$. By the Arzelà-Ascoli theorem, we know that $T_\lambda : K_1 \times K_2 \to K_1 \times K_2$ is completely continuous.

(ii) By (2.2) and the proof of Lemma 2.3 (i), we immediately obtain the desired conclusion. □

Remark 2.4. In fact, let $T(\lambda, u, v)(t) = T_\lambda(u,v)(t)$, then $\overline{T([0,1] \times K_{r_1} \times K_{r_2})}$ is a compact set by the Arzelà-Ascoli theorem.

Next, we will recall some concepts about the fixed point index, which will be used in the proofs of our theorems. Let E be a Banach space and $K \subset E$ is a closed convex cone in E. Assume Ω is a bounded open subset of E with boundary $\partial\Omega$ and $K \cap \partial\Omega \neq \emptyset$. Let $T : K \cap \overline{\Omega} \to K$ be a completely continuous operator. If $Tu \neq u$ for every $u \in K \cap \partial\Omega$, then the fixed point index $i(T, K \cap \Omega, K)$ is defined. One important fact is that if $i(T, K \cap \Omega, K) \neq 0$ then T has a fixed point in $K \cap \Omega$. The following results are useful in our proofs.

Lemma 2.5 ([7])**.** *Let E be a Banach space and $K \subset E$ is a closed convex cone in E. For $r > 0$, denote $K_r = \{u \in K : \|u\| < r\}$, $\partial K_r = \{u \in K : \|u\| = r\}$. Let $T : K \to K$ be a complete continuous mapping, then the following conclusions are valid:*

(i) *if $\mu Tu \neq u$, $\forall u \in \partial K_r$ and $\mu \in (0,1]$, then $i(T, K_r, K) = 1$;*
(ii) *if mapping T satisfies the following two conditions:*
 (a) $\inf_{u \in \partial K_r} \|Tu\| > 0$*;*
 (b) *$\mu Tu \neq u$, for all $u \in \partial K_r$ and $\mu \geqslant 1$,*

then $i(T, K_r, K) = 0$.

Lemma 2.6 ([4])**.** *Let X be a real Banach space, $P_i \subset X$ be a closed convex cone, W_i be a bounded open subset of X with the boundary ∂W_i $(i = 1,2)$, $P = P_1 \times P_2$ and $W = W_1 \times W_2$. Assume that $T : P \cap \overline{W} \to P$ is completely continuous and there exist compactly continuous mappings $A_i : P_i \cap \overline{W_i} \to P_i$ $(i = 1,2)$ and $H : (P \cap \overline{W}\,) \times [0,1] \to P$ such that*

(a) $H(\cdot,1)=T$ *and* $H(\cdot,0)=A$, *where* $A(u,v):=(A_1u,A_2v)$ *for all* $(u,v)\in P\cap\overline{W}$;
(b) $A_iu_i\neq u_i$ *for all* $u_i\in P_i\cap\partial W_i$; *and*
(c) $H(w,\tau)\neq w$ *for all* $(w,\tau)\in(P\cap\partial W)\times(0,1]$.

Then we have

$$i(T,P\cap W,P)=i(A_1,P_1\cap W_1,P_1)\cdot i(A_2,P_2\cap W_2,P_2).$$

3. Proof of main results

For Theorems 1.1 and 1.2, we choose a bounded open set $P_R\backslash\overline{P_r}$ in the single cone P on product space and calculate the fixed point index $i(T_1,P_R\backslash\overline{P_r},P)$.

Proof of Theorem 1.1. Firstly, by condition (1.3), there are $\varepsilon\in(0,1)$ and $r>0$ such that

$$\sum_{i=1}^{2}f_i(t,u,v)\leqslant(1-\varepsilon)(\lambda_1u+\lambda_2v), \tag{3.1}$$

for all $t\in[0,1]$, and all $(u,v)\in\{(u,v)\in(\mathbb{R}^+)^2|\ u+v\in[0,r]\}$. We claim that

$$\mu T_1(u,v)\neq(u,v),\quad \forall\mu\in(0,1],\ \forall(u,v)\in\partial P_r. \tag{3.2}$$

In fact, if this is not true, then there exist $\mu_0\in(0,1]$ and $(u_0,v_0)\in\partial P_r$, such that $\mu_0T_1(u_0,v_0)=(u_0,v_0)$; that is, (u_0,v_0) satisfies the system of differential equations

$$\begin{gathered}
u_0^{(4)}(t)+\beta_1u_0''(t)-\alpha_1u_0(t)=\mu_0f_1(t,u_0(t),v_0(t)),\quad\forall t\in(0,1),\\
v_0^{(4)}(t)+\beta_2v_0''(t)-\alpha_2v_0(t)=\mu_0f_2(t,u_0(t),v_0(t)),\quad\forall t\in(0,1),\\
u_0(0)=u_0(1)=u_0''(0)=u_0''(1)=0,\\
v_0(0)=v_0(1)=v_0''(0)=v_0''(1)=0.
\end{gathered}\tag{3.3}$$

It follows from (3.1) and (3.3) that

$$\begin{aligned}
&u_0^{(4)}(t)+\beta_1u_0''(t)-\alpha_1u_0(t)+v_0^{(4)}(t)+\beta_2v_0''(t)-\alpha_2v_0(t)\\
&\leqslant f_1(t,u_0(t),v_0(t))+f_2(t,u_0(t),v_0(t))\\
&\leqslant(1-\varepsilon)(\lambda_1u_0(t)+\lambda_2v_0(t)).
\end{aligned}$$

Furthermore,

$$\int_0^1(\lambda_1u_0(t)+\lambda_2v_0(t))\sin(\pi t)\,\mathrm{d}t\leqslant(1-\varepsilon)\int_0^1(\lambda_1u_0(t)+\lambda_2v_0(t))\sin(\pi t)\,\mathrm{d}t.$$

Noticing that $\int_0^1(\lambda_1u_0(t)+\lambda_2v_0(t))\sin(\pi t)\,\mathrm{d}t>0$, we get $1\leqslant1-\varepsilon$, which is a contradiction.

Secondly, from (1.3), there exist $\varepsilon>0$ and $m>0$ such that

$$\sum_{i=1}^{2}f_i(t,u,v)\geqslant(1+\varepsilon)(\lambda_1u+\lambda_2v), \tag{3.4}$$

for all $t\in[0,1]$ and all $(u,v)\in\{(u,v)\in(\mathbb{R}^+)^2|\ u+v\geqslant m\}$. Combining this with (3.4), it follows that

$$\sum_{i=1}^{2}f_i(t,u,v)\geqslant(1+\varepsilon)(\lambda_1u+\lambda_2v)-C,\quad\forall t\in[0,1],\ \forall(u,v)\in\mathbb{R}^+\times\mathbb{R}^+, \tag{3.5}$$

where $C = m(1+\varepsilon)(\lambda_1+\lambda_2)$. Now, we are in position to prove that there exists a $R > r$ such that

$$\inf_{(u,v)\in\partial P_R} \|T_1(u,v)\| > 0 \quad \text{and} \quad \mu T_1(u,v) \neq (u,v),\ \forall \mu \geqslant 1,\ \forall (u,v) \in \partial P_R. \tag{3.6}$$

In fact, if there are $\mu_0 \geqslant 1$ and $(u_0, v_0) \in P$ such that $\mu_0 T_1(u_0, v_0) = (u_0, v_0)$, then combining (3.3) with (3.5), we deduce that

$$\begin{aligned}
&u_0^{(4)}(t) + \beta_1 u_0''(t) - \alpha_1 u_0(t) + v_0^{(4)}(t) + \beta_2 v_0''(t) - \alpha_2 v_0(t) \\
&\geqslant f_1(t, u_0(t), v_0(t)) + f_2(t, u_0(t), v_0(t)) \\
&\geqslant (1+\varepsilon)(\lambda_1 u_0(t) + \lambda_2 v_0(t)) - C\,.
\end{aligned}$$

Then

$$\int_0^1 (\lambda_1 u_0(t) + \lambda_2 v_0(t)) \sin(\pi t)\,\mathrm{d}t \geqslant (1+\varepsilon)\int_0^1 (\lambda_1 u_0(t) + \lambda_2 v_0(t)) \sin(\pi t)\,\mathrm{d}t - \frac{2C}{\pi},$$

which along with the definition of cone P implies that

$$\|(u_0, v_0)\| \leqslant \frac{\sqrt{2}C}{\sigma\varepsilon \min\{\lambda_1, \lambda_2\}} =: \overline{R}. \tag{3.7}$$

Thus, for all $\mu \geqslant 1$ and $(u,v) \in \partial P_R$, $\mu T_1(u,v) \neq (u,v)$ as $R > \overline{R}$. In addition, if $R > m/\sigma$, then by Lemma 2.1 and (3.4) we have that for all $(u,v) \in \partial P_R$,

$$\begin{aligned}
\|T_1(u,v)\| &\geqslant (T_{1,1}(u,v) + T_{1,2}(u,v))(1/2) \\
&= \int_0^1 \int_0^1 \sum_{i=1}^2 G_{i,1}(1/2,\tau) G_{i,2}(\tau,s) f_i(s,u(s),v(s))\,\mathrm{d}s\mathrm{d}\tau \\
&\geqslant \sum_{i=1}^2 \delta_{i,1}\delta_{i,2} C_i G_{i,1}(1/2,1/2) \int_0^1 G_{i,2}(s,s) f_i(s,u(s),v(s))\,\mathrm{d}s \\
&\geqslant \min_{i=1,2}\{\delta_{i,1}\delta_{i,2} C_i m_{i,1} m_{i,2}\} \int_{1/4}^{3/4} \sum_{i=1}^2 f_i(s,u(s),v(s))\,\mathrm{d}s \\
&\geqslant \min_{i=1,2}\{\delta_{i,1}\delta_{i,2} C_i m_{i,1} m_{i,2}\}(1+\varepsilon) \int_{1/4}^{3/4} (\lambda_1 u(s) + \lambda_2 v(s))\,\mathrm{d}s \\
&\geqslant \min_{i=1,2}\{\delta_{i,1}\delta_{i,2} C_i m_{i,1} m_{i,2}\} \min\{\lambda_1,\lambda_2\}(1+\varepsilon)\sigma R/2,
\end{aligned}$$

which implies that $\inf_{(u,v)\in\partial P_R} \|T_1(u,v)\| > 0$. So, we choose $R > \max\{r, \overline{R}, m/\sigma\}$.

Finally, by (3.2), (3.6) and Lemmas 2.3-2.5, we have

$$i(T_1, P_R \backslash \overline{P_r}, P) = i(T_1, P_R, P) - i(T_1, P_r, P) = -1.$$

Hence, system (1.1) has at least one solution in $P\backslash\{(0,0)\}$. Moreover, if $f_1(t,0,v(t)) \not\equiv 0$ and $f_2(t,u(t),0) \not\equiv 0$, $\forall (u,v) \in P\backslash\{(0,0)\}$, then system (1.1) has at least one positive solution. □

Proof of Theorem 1.2. Firstly, from hypothesis (1.4), there exist $\varepsilon > 0$ and $r > 0$ such that

$$\sum_{i=1}^2 f_i(t,u,v) \geqslant (1+\varepsilon)(\lambda_1 u + \lambda_2 v), \tag{3.8}$$

for all $t \in [0,1]$ and all $(u,v) \in \{(u,v) \in (\mathbb{R}^+)^2 |\ u+v \in [0,r]\}$. Next, we can show that

$$\inf_{(u,v)\in\partial P_r} \|T_1(u,v)\| > 0 \quad \text{and} \quad \mu T_1(u,v) \neq (u,v),\ \forall \mu \geqslant 1,\ \forall (u,v) \in \partial P_r. \tag{3.9}$$

In fact, if there are $\mu_0 \geqslant 1$ and $(u_0,v_0) \in \partial P_r$ such that $(u_0,v_0) = \mu_0 T_1(u_0,v_0)$, by (3.3) and (3.8) we obtain

$$\begin{aligned} &u_0^{(4)}(t) + \beta_1 u_0''(t) - \alpha_1 u_0(t) + v_0^{(4)}(t) + \beta_2 v_0''(t) - \alpha_2 v_0(t) \\ &\geqslant f_1(t,u_0(t),v_0(t)) + f_2(t,u_0(t),v_0(t)) \\ &\geqslant (1+\varepsilon)(\lambda_1 u_0(t) + \lambda_2 v_0(t)) \,. \end{aligned}$$

Then

$$\int_0^1 (\lambda_1 u_0(t) + \lambda_2 v_0(t)) \sin(\pi t)\, dt \geqslant (1+\varepsilon) \int_0^1 (\lambda_1 u_0(t) + \lambda_2 v_0(t)) \sin(\pi t)\, dt.$$

Thus, $1 \geqslant 1+\varepsilon$, which is a contradiction. In addition, by Lemma 2.1 and (3.8) we obtain that for all $(u,v) \in \partial P_r$,

$$\begin{aligned} \|T_1(u,v)\| &\geqslant (T_{1,1}(u,v) + T_{1,2}(u,v))(1/2) \\ &\geqslant \min_{i=1,2}\{\delta_{i,1}\delta_{i,2} C_i m_{i,1} m_{i,2}\} \min\{\lambda_1,\lambda_2\}(1+\varepsilon)\sigma r/2, \end{aligned}$$

which implies that $\inf_{(u,v)\in\partial P_r} \|T_1(u,v)\| > 0$.

Secondly, in view of hypothesis (1.4), there are $\varepsilon \in (0,1)$ and $m > 0$ such that

$$\sum_{i=1}^{2} f_i(t,u,v) \leqslant (1-\varepsilon)(\lambda_1 u + \lambda_2 v), \tag{3.10}$$

for all $t \in [0,1]$ and all $(u,v) \in \{(u,v) \in (\mathbb{R}^+)^2 |\ u+v \geqslant m\}$. Furthermore, by the continuity of f_1 and f_2, there exists $C > 0$ such that

$$\sum_{i=1}^{2} f_i(t,u,v) \leqslant (1-\varepsilon)(\lambda_1 u + \lambda_2 v) + C, \quad \forall t \in [0,1],\ \forall (u,v) \in \mathbb{R}^+ \times \mathbb{R}^+. \tag{3.11}$$

Now, we can prove that there is a $R > r$ such that

$$\mu T_1(u,v) \neq (u,v), \quad \forall \mu \in (0,1],\ \forall (u,v) \in \partial P_R. \tag{3.12}$$

In fact, if there exist $\mu_0 \in (0,1]$ and $(u_0,v_0) \in P$ such that $\mu_0 T_1(u_0,v_0) = (u_0,v_0)$, then by (3.3) and (3.11) we can obtain that

$$\begin{aligned} &u_0^{(4)}(t) + \beta_1 u_0''(t) - \alpha_1 u_0(t) + v_0^{(4)}(t) + \beta_2 v_0''(t) - \alpha_2 v_0(t) \\ &\leqslant f_1(t,u_0(t),v_0(t)) + f_2(t,u_0(t),v_0(t)) \\ &\leqslant (1-\varepsilon)(\lambda_1 u_0(t) + \lambda_2 v_0(t)) + C \,. \end{aligned}$$

Then

$$\int_0^1 (\lambda_1 u_0(t) + \lambda_2 v_0(t)) \sin(\pi t)\, dt \leqslant (1-\varepsilon) \int_0^1 (\lambda_1 u_0(t) + \lambda_2 v_0(t)) \sin(\pi t)\, dt + \frac{2C}{\pi} \,.$$

Combining this with the definition of cone P, we obtain

$$\|(u_0,v_0)\| \leqslant \frac{\sqrt{2}C}{\sigma\varepsilon \min\{\lambda_1,\lambda_2\}} =: R^*. \tag{3.13}$$

Therefore, for all $\mu \in (0,1]$ and $(u,v) \in \partial P_R$, $\mu T_1(u,v) \neq (u,v)$ as $R > R^*$. Hence, we choose $R > \max\{r, R^*\}$.

Finally, from (3.9), (3.12) and Lemmas 2.3-2.5, we obtain

$$i(T_1, P_R\backslash\overline{P_r}, P) = i(T_1, P_R, P) - i(T_1, P_r, P) = 1.$$

Thus, (1.1) has at least one solution in $P\backslash\{(0,0)\}$. Moreover, if $f_1(t,0,v(t)) \not\equiv 0$ and $f_2(t,u(t),0) \not\equiv 0$, for all $(u,v) \in P\backslash\{(0,0)\}$, then system (1.1) has at least one positive solution. □

To prove Theorems 1.3 and 1.4, we choose a bounded open set $D = (K_{R_1}\backslash\overline{K_{r_1}}) \times (K_{R_2}\backslash\overline{K_{r_2}})$ in product cone $K_1 \times K_2$ and verify that a family of operators $\{T_\lambda\}_{\lambda\in[0,1]}$ satisfy the sufficient conditions for the homotopy invariance of fixed point index on ∂D.

Proof of Theorem 1.3. For convenience we present the proof four steps that determine r_1, R_1, r_2 and R_2 in turn.

Step 1. From assumption (1.5), there are $\varepsilon \in (0,\lambda_1)$ and $r_1 > 0$ such that

$$\lambda f_1(t,u,v)+(1-\lambda)f_1(t,u,0) \leqslant (\lambda_1-\varepsilon)u, \quad \forall t \in [0,1],\ \forall (u,v) \in [0,r_1]\times\mathbb{R}^+. \tag{3.14}$$

We claim that

$$\mu T_{\lambda,1}(u,v) \neq u, \quad \forall \mu \in (0,1],\ \forall (u,v) \in \partial K_{r_1} \times K. \tag{3.15}$$

In fact, if it is not valid, then there exist $\mu_0 \in (0,1]$ and $(u_0,v_0) \in \partial K_{r_1} \times K$, such that $\mu_0 T_{\lambda,1}(u_0,v_0) = u_0$, that is, (u_0,v_0) satisfies the differential equation

$$\begin{gathered} u_0^{(4)}(t) + \beta_1 u_0''(t) - \alpha_1 u_0(t) = \mu_0 h_{\lambda,1}(t,u_0(t),v_0(t)), \quad t \in (0,1),\\ u_0(0) = u_0(1) = u_0''(0) = u_0''(1) = 0, \end{gathered} \tag{3.16}$$

where $h_{\lambda,1}(t,u_0(t),v_0(t)) = \lambda f_1(t,u_0(t),v_0(t)) + (1-\lambda)f_1(t,u_0(t),0)$. In combination with (3.14), it follows that

$$u_0^{(4)}(t) + \beta_1 u_0''(t) - \alpha_1 u_0(t) \leqslant (\lambda_1-\varepsilon)u_0(t),$$

and then

$$\lambda_1 \int_0^1 u_0(t)\sin(\pi t)\,\mathrm{d}t \leqslant (\lambda_1-\varepsilon)\int_0^1 u_0(t)\sin(\pi t)\,\mathrm{d}t.$$

Noticing that $\int_0^1 u_0(t)\sin(\pi t)\,\mathrm{d}t > 0$, we have $\lambda_1 \leqslant \lambda_1 - \varepsilon$, which is a contradiction.

Step 2. By hypothesis (1.5), there exist $\varepsilon > 0$ and $m > 0$ such that

$$\lambda f_1(t,u,v) + (1-\lambda)f_1(t,u,0) \geqslant (\lambda_1+\varepsilon)u, \tag{3.17}$$

for all $t \in [0,1]$ and all $(u,v) \in [m,+\infty] \times \mathbb{R}^+$, which implies that

$$\lambda f_1(t,u,v) + (1-\lambda)f_1(t,u,0) \geqslant (\lambda_1+\varepsilon)u - C, \tag{3.18}$$

for all $t \in [0,1]$ and all $(u,v) \in \mathbb{R}^+ \times \mathbb{R}^+$, where $C = (\lambda_1+\varepsilon)m$. Now, we can prove that there exists a $R_1 > r_1$ such that

$$\mu T_{\lambda,1}(u,v) \neq u \quad \text{and} \quad \inf_{u\in\partial K_{R_1}} \|T_{\lambda,1}(u,v)\| > 0, \tag{3.19}$$

for all $\mu \geqslant 1$ and all $(u,v) \in \partial K_{R_1} \times K$.

First, if there are $(u_0,v_0) \in K \times K$ and $\mu_0 \geqslant 1$ such that $\mu_0 T_{\lambda,1}(u_0,v_0) = u_0$, then by (3.16) and (3.18), we obtain

$$u_0^{(4)}(t) + \beta_1 u_0''(t) - \alpha_1 u_0(t) \geqslant (\lambda_1+\varepsilon)u_0(t) - C.$$

It follows that

$$\lambda_1 \int_0^1 u_0(t)\sin(\pi t)\,\mathrm{d}t \geqslant (\lambda_1+\varepsilon)\int_0^1 u_0(t)\sin(\pi t)\,\mathrm{d}t - \frac{2C}{\pi};$$

that is,

$$\int_0^1 u_0(t)\sin(\pi t)\,\mathrm{d}t \leqslant \frac{2C}{\pi\varepsilon}.$$

Furthermore, in view of the definition of cone K_1, we know that

$$\|u_0\| \leqslant \frac{\sqrt{2}C}{\sigma_1\varepsilon} =: \widetilde{R}. \tag{3.20}$$

Therefore, as $R > \widetilde{R}$, $\mu T_{\lambda,1}(u,v) \neq u$ for all $\mu \geqslant 1$ and $(u,v) \in \partial K_R \times K$.

In addition, if $R > m/\sigma_1$, then by Lemma 2.1 and (3.17), for all $(u,v) \in \partial K_R \times K$,

$$\begin{aligned}
&\|T_{\lambda,1}(u,v)\| \geqslant T_{\lambda,1}(u,v)(1/2)\\
&= \int_0^1\int_0^1 G_{1,1}(1/2,\tau)G_{1,2}(\tau,s)[\lambda f_1(s,u(s),v(s)) + (1-\lambda)f_1(s,u(s),0)]\,\mathrm{d}s\mathrm{d}\tau\\
&\geqslant \delta_{1,1}\delta_{1,2}C_1G_{1,1}(\frac{1}{2},\frac{1}{2})\int_0^1 G_{1,2}(s,s)[\lambda f_1(s,u(s),v(s)) + (1-\lambda)f_1(s,u(s),0)]\,\mathrm{d}s\\
&\geqslant \delta_{1,1}\delta_{1,2}C_1 m_{1,1}m_{1,2}\int_{1/4}^{3/4}[\lambda f_1(s,u(s),v(s)) + (1-\lambda)f_1(s,u(s),0)]\,\mathrm{d}s\\
&\geqslant \delta_{1,1}\delta_{1,2}C_1 m_{1,1}m_{1,2}(\lambda_1+\varepsilon)\sigma_1 R/2,
\end{aligned}$$

which implies $\inf_{u\in\partial K_R}\|T_{\lambda,1}(u,v)\| > 0$. Hence, we choose $R_1 > \max\{r_1, \widetilde{R}, m/\sigma_1\}$.

Step 3. In view of condition (1.6), there exist $\varepsilon > 0$ and $r_2 > 0$ such that

$$\lambda f_2(t,u,v) + (1-\lambda)f_2(t,0,v) \geqslant (\lambda_2+\varepsilon)v, \tag{3.21}$$

for all $t \in [0,1]$ and all $(u,v) \in [0,R_1]\times[0,r_2]$. We claim that

$$\mu T_{\lambda,2}(u,v) \neq v \quad \text{and} \quad \inf_{v\in\partial K_{r_2}}\|T_{\lambda,2}(u,v)\| > 0, \tag{3.22}$$

for all $\mu \geqslant 1$ and all $(u,v) \in \overline{K_{R_1}} \times \partial K_{r_2}$. In fact, if there are $\mu_0 \geqslant 1$ and $(u_0,v_0) \in \overline{K_{R_1}} \times \partial K_{r_2}$ such that $\mu_0 T_{\lambda,2}(u_0,v_0) = v_0$, then (u_0,v_0) satisfies the differential equation

$$\begin{gathered}
v_0^{(4)}(t) + \beta_2 v_0''(t) - \alpha_2 v_0(t) = \mu_0 h_{\lambda,2}(t,u_0(t),v_0(t)), \quad t\in(0,1),\\
v(0) = v(1) = v''(0) = v''(1) = 0,
\end{gathered} \tag{3.23}$$

where $h_{\lambda,2}(t,u_0(t),v_0(t)) = \lambda f_2(t,u_0(t),v_0(t)) + (1-\lambda)f_2(t,0,v_0(t))$. In combination with (3.21), it follows that

$$v_0^{(4)}(t) + \beta_2 v_0''(t) - \alpha_2 v_0(t) \geqslant (\lambda_2+\varepsilon)v_0(t),$$

which implies that

$$\lambda_2\int_0^1 v_0(t)\sin(\pi t)\,\mathrm{d}t \geqslant (\lambda_2+\varepsilon)\int_0^1 v_0(t)\sin(\pi t)\,\mathrm{d}t.$$

By $\int_0^1 v_0(t)\sin(\pi t)\,\mathrm{d}t > 0$, we get $\lambda_2 \geqslant \lambda_2+\varepsilon$, which is a contradiction. In addition, by (3.21) and Lemma 2.1, we know that for all $(u,v)\in \overline{K_{R_1}}\times \partial K_{r_2}$,

$$\|T_{\lambda,2}(u,v)\| \geqslant T_{\lambda,2}(u,v)(1/2) \geqslant \delta_{2,1}\delta_{2,2}C_2 m_{2,1}m_{2,2}(\lambda_2+\varepsilon)\sigma_2 r_2/2.$$

So, $\inf_{v\in\partial K_{r_2}}\|T_{\lambda,2}(u,v)\|>0$.

Step 4. By hypothesis (1.6) and the continuity of f_2, there exist $\varepsilon\in(0,\lambda_2)$, $n>0$ and $C>0$ such that

$$\lambda f_2(t,u,v)+(1-\lambda)f_2(t,0,v) \leqslant (\lambda_2-\varepsilon)v, \tag{3.24}$$

for all $t\in[0,1]$ and all $(u,v)\in[0,R_1]\times[n,+\infty)$; and

$$\lambda f_2(t,u,v)+(1-\lambda)f_2(t,0,v) \leqslant (\lambda_2-\varepsilon)v+C, \tag{3.25}$$

for all $t\in[0,1]$ and all $(u,v)\in[0,R_1]\times\mathbb{R}^+$.

Now, we are in a position to prove that there is a $R_2>r_2$ such that

$$\mu T_{\lambda,2}(u,v)\neq v,\quad \forall\mu\in(0,1],\ \forall(u,v)\in\overline{K_{R_1}}\times\partial K_{R_2}. \tag{3.26}$$

In fact, if there are $\mu_0\in(0,1]$ and $(u_0,v_0)\in\overline{K_{R_1}}\times K$ such that $\mu_0 T_{\lambda,2}(u_0,v_0)=v_0$, then by (3.23) and (3.25), similar to the proof of (3.20), we obtain

$$\|v_0\| \leqslant \frac{\sqrt{2}C}{\sigma_2\varepsilon} =: R'. \tag{3.27}$$

Thus, as $R>R'$, $\mu T_{\lambda,2}(u,v)\neq v$ for all $\mu\in(0,1]$ and $(u,v)\in\overline{K_{R_1}}\times\partial K_R$. Hence, we choose $R_2>\max\{r_2,R'\}$.

Let $D=(K_{R_1}\backslash\overline{K_{r_1}})\times(K_{R_2}\backslash\overline{K_{r_2}})$, then from the expressions (3.15), (3.19), (3.22) and (3.26), it is easy to verify that $\{T_\lambda\}_{\lambda\in[0,1]}$ satisfy the sufficient conditions for the homotopy invariance of fixed point index on ∂D. Moreover, by Remark 2.4 and Lemmas 2.5-2.6, we have

$$i(T_1,D,K_1\times K_2)=i(T_{0,1},K_{R_1}\backslash\overline{K_{r_1}},K_1)\cdot i(T_{0,2},K_{R_2}\backslash\overline{K_{r_2}},K_2)=-1.$$

Hence, system (1.1) has at least one positive solution. □

Proof of Theorem 1.4. Similar to the proof of Theorem 1.3, we will determine r_1, R_1, r_2 and R_2 in turn.

Firstly, from assumption (1.7), there are $\varepsilon>0$ and $r_1>0$ such that

$$\lambda f_1(t,u,v)+(1-\lambda)f_1(t,u,0) \geqslant (\lambda_1+\varepsilon)u,\quad \forall t\in[0,1],\ \forall(u,v)\in[0,r_1]\times\mathbb{R}^+. \tag{3.28}$$

By (3.28) and the similar arguments in Step 3 of the proof for Theorem 1.3, we deduce that

$$\mu T_{\lambda,1}(u,v)\neq u,\quad \inf_{u\in\partial K_{r_1}}\|T_{\lambda,1}(u,v)\|>0,\quad \forall\mu\geqslant 1,\ \forall(u,v)\in\partial K_{r_1}\times K. \tag{3.29}$$

Secondly, by hypotheses (1.7) and (1.9), there exist $\varepsilon\in(0,\lambda_1)$, $m>0$ and $C>0$ such that

$$\lambda f_1(t,u,v)+(1-\lambda)f_1(t,u,0) \leqslant (\lambda_1-\varepsilon)u, \tag{3.30}$$

for all $t\in[0,1]$ and all $(u,v)\in[m,+\infty)\times\mathbb{R}^+$; and

$$\lambda f_1(t,u,v)+(1-\lambda)f_1(t,u,0) \leqslant (\lambda_1-\varepsilon)u+C, \tag{3.31}$$

for all $t\in[0,1]$ and all $(u,v)\in\mathbb{R}^+\times\mathbb{R}^+$.

In view of (3.30)-(3.31) and similar arguments to those in Step 4 of the proof for Theorem 1.3, we can prove that there is a $R_1>r_1$ such that

$$\mu T_{\lambda,1}(u,v)\neq u,\quad \forall\mu\in(0,1],\ \forall(u,v)\in\partial K_{R_1}\times K. \tag{3.32}$$

Thirdly, by assumption (1.8), there are $\varepsilon \in (0, \lambda_2)$ and $r_2 > 0$ such that

$$\lambda f_2(t,u,v) + (1-\lambda) f_2(t,0,v) \leqslant (\lambda_2 - \varepsilon) v, \tag{3.33}$$

for all $t \in [0,1]$ and all $(u,v) \in [0,R_1] \times [0,r_2]$. From (3.33) and similar arguments as those in Step 1 of the proof for Theorem 1.3, we can show that

$$\mu T_{\lambda,2}(u,v) \neq v, \quad \forall \mu \in (0,1],\ \forall (u,v) \in \overline{K_{R_1}} \times \partial K_{r_2}. \tag{3.34}$$

Finally, in view of condition (1.8), there exist $\varepsilon > 0$, $n > 0$ and $C > 0$ such that

$$\lambda f_2(t,u,v) + (1-\lambda) f_2(t,0,v) \geqslant (\lambda_2 + \varepsilon) v, \tag{3.35}$$

for all $t \in [0,1]$ and all $(u,v) \in [0,R_1] \times [n,+\infty)$; and

$$\lambda f_2(t,u,v) + (1-\lambda) f_2(t,0,v) \geqslant (\lambda_2 + \varepsilon) v - C, \tag{3.36}$$

for all $t \in [0,1]$ and all $(u,v) \in [0,R_1] \times \mathbb{R}^+$. By using (3.35)-(3.36) and the similar arguments as those in Step 2 of the proof for Theorem 1.3, we can deduce that there exists a $R_2 > r_2$ such that

$$\mu T_{\lambda,2}(u,v) \neq v \quad \text{and} \quad \inf_{v \in \partial K_{R_2}} \|T_{\lambda,2}(u,v)\| > 0, \tag{3.37}$$

for all $\mu \geqslant 1$ and all $(u,v) \in \overline{K_{R_1}} \times \partial K_{R_2}$.

Let $D = (K_{R_1} \backslash \overline{K_{r_1}}) \times (K_{R_2} \backslash \overline{K_{r_2}})$, then by (3.29), (3.32), (3.34) and (3.37), it is easy to verify that $\{T_\lambda\}_{\lambda \in [0,1]}$ satisfy the sufficient conditions for the homotopy invariance of fixed point index on ∂D. Furthermore, by Remark 2.4 and Lemmas 2.5-2.6, we have

$$i(T_1, D, K_1 \times K_2) = i(T_{0,1}, K_{R_1} \backslash \overline{K_{r_1}}, K_1) \cdot i(T_{0,2}, K_{R_2} \backslash \overline{K_{r_2}}, K_2) = -1.$$

Therefore, system (1.1) has at least one positive solution. $\square$

4. Applications

In this section, we present some corollaries to Theorems 1.1–1.4.

Corollary 4.1. *Assume $f_i \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ and $f_i(t,u,v) = g_i(t, \lambda_1 u + \lambda_2 v)$, where $g_i \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$ $(i = 1,2)$ satisfying the following conditions: there exists a $\theta \in (0,1)$ such that*

$$\limsup_{w \to 0^+} \max_{t \in [0,1]} \frac{g_1(t,w)}{w} < \theta \quad \textit{and} \quad \limsup_{w \to 0^+} \max_{t \in [0,1]} \frac{g_2(t,w)}{w} < 1 - \theta; \tag{4.1}$$

there exists a $\vartheta \in (0,1)$ such that

$$\liminf_{w \to +\infty} \min_{t \in [0,1]} \frac{g_1(t,w)}{w} > \vartheta \quad \textit{and} \quad \liminf_{w \to +\infty} \min_{t \in [0,1]} \frac{g_2(t,w)}{w} > 1 - \vartheta. \tag{4.2}$$

Then system (1.1) has at least one nonzero nonnegative solution. Moreover, if $g_1(t, \lambda_2 v(t)) \not\equiv 0$ and $g_2(t, \lambda_1 u(t)) \not\equiv 0$, $\forall (u,v) \in P \backslash \{(0,0)\}$, then system (1.1) has at least one positive solution.

Proof. It is easy to verify that f_1 and f_2 satisfy condition (1.3). Hence, by Theorem 1.1, we immediately get the desired conclusion. $\square$

Similarly, applying Theorem 1.2 we have the following result.

Corollary 4.2. *Suppose $f_i \in C([0,1]\times\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$ and $f_i(t,u,v)=g_i(t,\lambda_1 u+\lambda_2 v)$, where $g_i\in C([0,1]\times\mathbb{R}^+,\mathbb{R}^+)$ $(i=1,2)$ satisfying the following conditions: there is a $\theta\in(0,1)$ such that*

$$\liminf_{w\to 0^+}\min_{t\in[0,1]}\frac{g_1(t,w)}{w}>\theta \quad \textit{and} \quad \liminf_{w\to 0^+}\min_{t\in[0,1]}\frac{g_2(t,w)}{w}>1-\theta; \tag{4.3}$$

there is a $\vartheta\in(0,1)$ such that

$$\limsup_{w\to +\infty}\max_{t\in[0,1]}\frac{g_1(t,w)}{w}<\vartheta \quad \textit{and} \quad \limsup_{w\to +\infty}\max_{t\in[0,1]}\frac{g_2(t,w)}{w}<1-\vartheta. \tag{4.4}$$

Then system (1.1) has at least one nonzero nonnegative solution. Furthermore, if $g_1(t,\lambda_2 v(t))\not\equiv 0$ and $g_2(t,\lambda_1 u(t))\not\equiv 0$, for all $(u,v)\in P\backslash\{(0,0)\}$, then system (1.1) has at least one positive solution.

By Corollaries 4.1-4.2, it is easy to get the following result on the existence of positive solutions for equation (1.2) (see also [9, Theorem 1.1]).

Corollary 4.3. *Let $\lambda=\pi^4-\beta\pi^2-\alpha$. Assume that $f\in C([0,1]\times\mathbb{R}^+,\mathbb{R}^+)$ satisfies one of the following conditions:*

$$\limsup_{w\to 0^+}\max_{t\in[0,1]}\frac{f(t,w)}{w}<\lambda<\liminf_{w\to +\infty}\min_{t\in[0,1]}\frac{f(t,w)}{w}; \tag{4.5}$$

or

$$\liminf_{w\to 0^+}\min_{t\in[0,1]}\frac{f(t,w)}{w}>\lambda>\limsup_{w\to +\infty}\max_{t\in[0,1]}\frac{f(t,w)}{w}. \tag{4.6}$$

Then (1.2) has at least one positive solution.

Proof. Let $\beta_1=\beta_2=\beta$, $\alpha_1=\alpha_2=\alpha$, $\theta=\vartheta=1/2$ and

$$g_1(t,\lambda(u+v))=g_2(t,\lambda(u+v))=\frac{1}{2}f(t,u+v),$$

then it is not difficult to verify that g_1 and g_2 satisfy conditions (4.1) and (4.2) (resp. (4.3) and (4.4)) as f satisfies condition (4.5) (resp. (4.6)). Therefore, by Corollaries 4.1-4.2, system (1.1) has at least one nonzero nonnegative solution denoted by (u_0,v_0). Moreover, equation (1.2) has at least one positive solution $w=u_0+v_0$. □

Next, we give corollaries related to Theorems 1.3 and 1.4.

Corollary 4.4. *Suppose that $f_1(t,u,v)=g_1(t,u)+p_1(u,v)$, $f_2(t,u,v)=g_2(t,v)+p_2(u,v)$, where $g_i\in C([0,1]\times\mathbb{R}^+,\mathbb{R}^+)$ and $p_i\in C(\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$ $(i=1,2)$ satisfying the following conditions:*

$$\limsup_{u\to 0^+}\max_{t\in[0,1]}\frac{g_1(t,u)}{u}<\lambda_1<\liminf_{u\to +\infty}\min_{t\in[0,1]}\frac{g_1(t,u)}{u}; \tag{4.7}$$

$$\liminf_{v\to 0^+}\min_{t\in[0,1]}\frac{g_2(t,v)}{v}>\lambda_2>\limsup_{v\to +\infty}\max_{t\in[0,1]}\frac{g_2(t,v)}{v}; \tag{4.8}$$

$$\limsup_{u\to 0^+}\frac{p_1(u,v)}{u}=0 \quad \textit{uniformly w.r.t. } v\in\mathbb{R}^+; \tag{4.9}$$

$$\limsup_{v\to +\infty}\frac{p_2(u,v)}{v}=0 \quad \textit{locally uniformly w.r.t. } u\in[0,M] \quad (\forall M>0), \tag{4.10}$$

then system (1.1) has at least one positive solution.

Proof. It is clear that f_1 and f_2 satisfy all conditions in Theorem 1.3. Then, by Theorem 1.3, the desired conclusion yields. $\square$

Similarly, in view of Theorem 1.4 we obtain the following conclusion.

Corollary 4.5. *Assume that $f_1(t,u,v) = g_1(t,u) + p_1(u,v)$, $f_2(t,u,v) = g_2(t,v) + p_2(u,v)$, where $g_i \in C([0,1]\times\mathbb{R}^+,\mathbb{R}^+)$ and $p_i \in C(\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$ $(i = 1,2)$ satisfying the following conditions:*

$$\liminf_{u\to 0^+} \min_{t\in[0,1]} \frac{g_1(t,u)}{u} > \lambda_1 > \limsup_{u\to+\infty} \max_{t\in[0,1]} \frac{g_1(t,u)}{u}; \tag{4.11}$$

$$\limsup_{v\to 0^+} \max_{t\in[0,1]} \frac{g_2(t,v)}{v} < \lambda_2 < \liminf_{v\to+\infty} \min_{t\in[0,1]} \frac{g_2(t,v)}{v}; \tag{4.12}$$

$$\begin{gathered} \limsup_{u\to+\infty} \frac{p_1(u,v)}{u} = 0 \quad \text{uniformly w.r.t. } v \in \mathbb{R}^+, \\ \limsup_{v\to+\infty} p_1(u,v) = p(u) \quad \text{locally uniformly w.r.t. } u \in [0,M] \quad (\forall M > 0), \\ \text{where } p(u) \text{ is a locally bounded function}; \end{gathered} \tag{4.13}$$

$$\limsup_{v\to 0^+} \frac{p_2(u,v)}{v} = 0 \quad \text{locally uniformly w.r.t. } u \in [0,M] \quad (\forall M > 0), \tag{4.14}$$

then system (1.1) has at least one positive solution.

Remark 4.6. In particular, when $p_i(u,v) \equiv 0$ $(i = 1,2)$, Corollaries 4.4-4.5 reduce to the existence results of positive solutions to equation (1.2) (see Corollary 4.3).

Acknowledgement. The authors want to express their gratitude to anonymous referee for the valuable comments and suggestions. This work was partially supported by the Fundamental Research Funds for the Central Universities Grant (lzujbky-2018-112).

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