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POSITIVE SOLUTIONS OF SCHRÖDINGER-POISSON SYSTEMS WITH HARDY POTENTIAL AND INDEFINITE NONLINEARITY

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Abstract. In this article, we study the nonlinear Schrödinger-Poisson system

$$\begin{gathered}-\Delta u + u - \mu\frac{u}{|x|^2} + l(x)\phi u = k(x)|u|^{p-2}u \quad x \in \mathbb{R}^3,\\ -\Delta\phi = l(x)u^2 \quad x \in \mathbb{R}^3,\end{gathered}$$

where $k \in C(\mathbb{R}^3)$ and $4 < p < 6$, k changes sign in $\mathbb{R}^3$ and $\limsup_{|x|\to\infty} k(x) = k_\infty < 0$. We prove that Schrödinger-Poisson systems with Hardy potential and indefinite nonlinearity have at least one positive solution, using variational methods.

1. Introduction and statement of the main result

In recent years, the Schrödinger-Poisson system

$$\begin{gathered}-\Delta u + V(x)u + l(x)\phi u = f(x,u) \quad x \in \mathbb{R}^3,\\ -\Delta\phi = l(x)u^2 \quad x \in \mathbb{R}^3,\end{gathered} \tag{1.1}$$

have been widely investigated, because of its importance in quantum mechanics models and in semiconductor theory. For more details about its physical aspects, see [4, 7] and the references therein. There have been several results about nontrivial solutions, radial and nonradial solutions, ground states, multiplicity of solutions, and concentration of solutions, depending on assumptions of the potential V. Most of the literature focuses on the study of (1.1) with $V \equiv 1$, see e.g. [3, 5, 8, 11, 15, 22]. Azzollini and Pomponio [3] proved the existence of a ground state solution when $f(x,u) = |u|^{p-1}u$ with $p \in (3,5)$, $V \equiv 1$, and $l(x) \equiv 1$. Ruiz [15] proved the existence and nonexistence of nontrivial solutions by using a constrained minimization method. Cerami et al. [5] obtained the existence of ground states and bound states, under suitable assumptions. Huang et al. [11] considered the case when $f(x,u)$ is a combination of a superlinear and linear terms. More precisely, $f(x,u) = k(x)|u|^{p-1}u + \mu h(x)u$, where $4 < p < 6$ and $\mu > 0$, $k(x) \in C(\mathbb{R}^3)$, k changes sign in $\mathbb{R}^3$, and $\lim_{|x|\to\infty} k(x) = k_\infty < 0$. They proved the existence of at least two positive solutions. In [22] the authors obtained the existence of ground state and multiple solutions using critical growth by variational methods. Existence

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of multiple positive solutions of Schrödinger-Poisson type equations with indefinite nonlinearity was proved in [8] using the mountain pass theorem. For $V \not\equiv 1$ with $\inf_{x\in\mathbb{R}^3} V(x) > 0$, there are also many results (see [6, 10, 16, 17, 18, 21]). For other interesting results on the Schrödinger-Poisson system, we refer readers to [12, 13, 14, 19, 20] and references therein.

In this article, we consider the system

$$\begin{gathered} -\Delta u + u - \mu\frac{u}{|x|^2} + l(x)\phi u = k(x)|u|^{p-2}u \quad x \in \mathbb{R}^3, \\ -\Delta\phi = l(x)u^2 \quad x \in \mathbb{R}^3, \end{gathered} \tag{1.2}$$

where $0 < \mu < \overline{\mu} := \frac{(N-2)^2}{4} = \frac{1}{4}$, $4 < p < 6$, $k(x) \in C(\mathbb{R}^3)$, k changes sign in $\mathbb{R}^3$, and $\limsup_{|x|\to\infty} k(x) = k_\infty < 0$.

To the best of our knowledge, the literature does not have results on the existence of positive solutions to (1.2) with Hardy potential. The aim of this article is to show the existence of positive solutions of problem (1.2). Our approach combines variational techniques based on critical point theory and some analysis techniques.

Hereafter we use the following notation: For $1 \le s < +\infty$, $L^s(\mathbb{R}^3)$ is the Lebesgue space endowed with norm

$$\|u\|_s^s := \int_{\mathbb{R}^3} |u|^s \,\mathrm{d}x.$$

$H^1(\mathbb{R}^3)$ is the Sobolev space endowed with the scalar product and norm

$$(u, v) := \int_{\mathbb{R}^3} \Big(\nabla u \cdot \nabla v + uv - \mu\frac{uv}{|x|^2}\Big)\,\mathrm{d}x; \quad \|u\|^2 := \int_{\mathbb{R}^3} \Big(|\nabla u|^2 + u^2 - \mu\frac{u^2}{|x|^2}\Big)\,\mathrm{d}x.$$

By Hardy inequality [9], we easily derive that this norm is equivalent to the usual norm, in $H^1(\mathbb{R}^3)$,

$$\|u\|_0 = \Big(\int_{\mathbb{R}^3} |\nabla u|^2 + u^2 \,\mathrm{d}x\Big)^{1/2}.$$

$D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{D^{1,2}} := \Big(\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x\Big)^{1/2}.$$

$o_n(1)$ is a quantity that approaches zero as $n \to \infty$.

Recall that by the Lax-Milgram theorem, for each $u \in H^1(\mathbb{R}^3)$, there exists a unique solution $\phi_u \in D^{1,2}(\mathbb{R}^3)$ of

$$-\Delta\phi = l(x)u^2, \quad x \in \mathbb{R}^3. \tag{1.3}$$

Using this in (1.2) gives

$$-\Delta u + u - \mu\frac{u}{|x|^2} + l(x)\phi_u u = k(x)|u|^{p-2}u, \quad x \in \mathbb{R}^3. \tag{1.4}$$

Moreover one has

$$\begin{gathered} \phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{l(y)u^2(y)}{|x-y|}\,\mathrm{d}y, \\ \|\phi_u\|^2_{D^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\nabla\phi_u|^2 \,\mathrm{d}x = \int_{\mathbb{R}^3} l(x)\phi_u u^2 \,\mathrm{d}x. \end{gathered}$$

There is a one-to-one correspondence between the solution of (1.3) and the critical points of the functional defined in $H^1(\mathbb{R}^3)$ by

$$
\begin{aligned}
I(u) = \frac{1}{2}\int_{\mathbb{R}^3}\Big(|\nabla u|^2 + |u|^2 - \mu\frac{u^2}{|x|^2}\Big)\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} l(x)\phi_u(x)u^2(x)\,\mathrm{d}x \\
- \frac{1}{p}\int_{\mathbb{R}^3} k(x)|u|^p\,\mathrm{d}x\,.
\end{aligned}
\tag{1.5}
$$

Then $I'(u)$ is defined by

$$
\begin{aligned}
\langle I'(u), v\rangle = \int_{\mathbb{R}^3}\Big(\nabla u\cdot\nabla v + uv - \mu\frac{uv}{|x|^2}\Big)\,\mathrm{d}x + \int_{\mathbb{R}^3} l(x)\phi_u uv\,\mathrm{d}x \\
- \int_{\mathbb{R}^3} k(x)|u|^{p-2}uv\,\mathrm{d}x.
\end{aligned}
\tag{1.6}
$$

A pair of functions (u,ϕ) is called a positive solution of problem (1.2) if it satisfies (1.2) and $u>0$, $\phi>0$ for a.a. $x\in\mathbb{R}^3$.

Let us introduce some hypotheses on $k(x)$ and $l(x)$:

- (H1) $k\in C(\mathbb{R}^3)$ and k changes sign in $\mathbb{R}^3$;
- (H2) $\limsup_{|x|\to\infty} k(x) = k_\infty < 0$;
- (H3) $l\in L^2(\mathbb{R}^3)$, $l(x)\geq 0$ for all $x\in\mathbb{R}^3$ and $l\not\equiv 0$;
- (H4) $l=0$ a.e. in $\Omega^0 := \{x\in\mathbb{R}^3 : k(x)=0\}$ and Ω^0 coincides with the closure of its interior.

The main result of this article is the following theorem.

Theorem 1.1. *Assume that* (H1)–(H4) *hold, and* $4<p<6$. *Then problem* (1.2) *has at least one positive solution in* $H^1(\mathbb{R}^3)\times D^{1,2}(\mathbb{R}^3)$.

In the following discussions, c or c_i $(i=0,1,\dots)$ we denote positive constants.

2. Proof of Theorem 1.1

The proof is presented in three steps.

Step 1: The (PS) condition. Let $\{u_n\}$ be any sequence in $H^1(\mathbb{R}^3)$ such that $I(u_n)$ is bounded and $I'(u_n)$ converges to zero; that is,

$$
I(u_n) = \frac{1}{2}\|u_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} l(x)\phi_{u_n}(x)u_n^2(x)\,\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3} k(x)|u_n|^p\,\mathrm{d}x \to c, \tag{2.1}
$$

and

$$
\begin{aligned}
\langle I'(u_n),\varphi\rangle = \int_{\mathbb{R}^3}\Big(\nabla u_n\cdot\nabla\varphi + u_n\varphi - \mu\frac{u_n\varphi}{|x|^2}\Big)\,\mathrm{d}x + \int_{\mathbb{R}^3} l(x)\phi_{u_n}u_n\varphi\,\mathrm{d}x \\
- \int_{\mathbb{R}^3} k(x)|u_n|^{p-2}u_n\varphi\,\mathrm{d}x \to 0,
\end{aligned}
\tag{2.2}
$$

for any $\varphi\in H^1(\mathbb{R}^3)$ as $n\to\infty$.

We now prove that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. By contradiction, we assume $\|u_n\|\to\infty$. Let $v_n = u_n/\|u_n\|$, then $\|v_n\|=1$ for each $n\in\mathbb{N}$. Then there exists a

$v \in H^1(\mathbb{R}^3)$ such that for each bounded domain $\Omega \subset \mathbb{R}^3$,

$$\begin{gathered} v_n \rightharpoonup v \quad \text{in } H^1(\mathbb{R}^3), \\ v_n \to v \quad \text{a.e. in } \mathbb{R}^3, \\ v_n \to v \quad \text{in } L^s_{\text{loc}}(\mathbb{R}^3), \quad \text{where } 2 \le s < 2^* = 6, \\ \|v_n\| \le w_\Omega(x), \quad \text{for some } w_\Omega(x) \in L^s(\Omega). \end{gathered} \tag{2.3}$$

So, for any $\varphi \in H^1(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} \Big(\nabla v_n \nabla \varphi + v_n\varphi - \mu \frac{v_n\varphi}{|x|^2}\Big)\,\mathrm{d}x \to \int_{\mathbb{R}^3} \Big(\nabla v \nabla \varphi + v\varphi - \mu \frac{v\varphi}{|x|^2}\Big)\,\mathrm{d}x. \tag{2.4}$$

We claim that $v(x) = 0$ a.e. in $\mathbb{R}^3$. In fact, since $u_n = \|u_n\| v_n$, (2.2) becomes

$$\begin{aligned} &\int_{\mathbb{R}^3} \Big(\nabla v_n \nabla \varphi + v_n\varphi - \mu \frac{v_n\varphi}{|x|^2}\Big)\,\mathrm{d}x + \|u_n\|^2 \int_{\mathbb{R}^3} l(x)\phi_{v_n} v_n \varphi\,\mathrm{d}x \\ &- \|u_n\|^{p-2} \int_{\mathbb{R}^3} k(x)|v_n|^{p-2} v_n\varphi\,\mathrm{d}x \to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{2.5}$$

Next we prove the claim for x in Ω^+, Ω^- and Ω^0. From (H1), we see that $\Omega^+ \neq \emptyset$ and $\Omega^- \neq \emptyset$.

First, let $x \in \Omega^+$. Since $k \in C(\mathbb{R}^3)$, there exists $\delta > 0$ such that

$$k(y) > 0, \quad \forall y \in B_\delta(x). \tag{2.6}$$

We define a continuous function $\zeta_m \in C(\mathbb{R}^3)$ $(m > 2)$ such that $\zeta_m(y) \ge 0$ for any $y \in \mathbb{R}^3$ and

$$\zeta_m(y) = \begin{cases} 1, & \text{if } y \in B_{(\frac12 - \frac{1}{m^2})\delta}(x), \\ 0, & \text{if } y \in \mathbb{R}^3 \setminus B_{\delta/2}(x). \end{cases} \tag{2.7}$$

Taking $\varphi = v\zeta_m$ in (2.5), we know that $\operatorname{supp}\varphi \subset B_{\delta/2}(x)$ for any $m \in \mathbb{N}$ and $m > 2$. In view of (2.3), we have

$$k(y)|v_n(y)|^{p-2} v_n(y)\varphi(y) \to k(y)|v(y)|^{p-2} v(y)\varphi(y) \quad \text{for } y \in B_{\delta/2}(x),$$

and

$$|k(y)v_n(y)^{p-1}\varphi(y)| \le C|w_\Omega(y)|^{p-1}|\varphi(y)| \in L^1(B_{\delta/2}(x)).$$

Therefore, by the Lebesgue dominated convergent theorem, we have

$$\int_{B_{\delta/2}(x)} k(y)|v_n(y)|^{p-2} v_n(y)\varphi(y)\,\mathrm{d}y \to \int_{B_{\delta/2}(x)} k(y)|v(y)|^{p-2} v(y)\varphi(y)\,\mathrm{d}y. \tag{2.8}$$

Dividing (2.5) by $\|u_n\|^{p-2}$ and passing to the limit as $n \to \infty$, in view of the boundedness of v_n, we obtain

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int_{\mathbb{R}^3} k(y)|v_n(y)|^{p-2} v_n(y)\varphi(y)\,\mathrm{d}y \\ &= \int_{B_{\delta/2}(x)} k(y)|v(y)|^{p-2} v(y)\varphi(y)\,\mathrm{d}y \\ &= \int_{B_{(\frac12-\frac{1}{m^2})\delta}(x)} k(y)|v(y)|^p\,\mathrm{d}y + \int_{B_{\delta/2}(x)\setminus B_{(\frac12-\frac{1}{m^2})\delta}(x)} k(y)|v(y)|^p \zeta_m\,\mathrm{d}y, \end{aligned} \tag{2.9}$$

for any $m \in \mathbb{N}$ and $m > 2$. Passing to the limit in (2.9) as $m \to \infty$, we obtain

$$\int_{B_{\delta/2}(x)} k(y)|v(y)|^p\,\mathrm{d}y = 0.$$

It follows from (2.6) that $v = 0$ a.e. in $B_{\delta/2}(x)$. Since $x \in \Omega^+$ is arbitrarily, we can obtain that $v = 0$ a.e. in Ω^+.

A similar argument shows that $v = 0$ a.e. in Ω^-. Next, we prove that $v = 0$ a.e. in Ω^0. If $|\Omega^0| = 0$, the claim is true. If $|\Omega^0| \neq 0$, we take $\varphi \in C(\mathbb{R}^3)$ with $\operatorname{supp}\varphi \subset \Omega^0$ in (2.5). From the definition of Ω^0 and the assumption that $l = 0$ a.e. in Ω^0, it follows that

$$\int_{\mathbb{R}^3} k(y)|v_n(y)|^{p-2}v_n(y)\varphi(y)\,\mathrm{d}y = \int_{\operatorname{supp}\varphi} k(y)|v_n(y)|^{p-2}v_n(y)\varphi(y)\,\mathrm{d}y = 0, \tag{2.10}$$

$$\int_{\mathbb{R}^3} l(y)\phi_{v_n}v_n\varphi\,\mathrm{d}y = \int_{\operatorname{supp}\varphi} l(y)\phi_{v_n}v_n\varphi\,\mathrm{d}y = 0, \tag{2.11}$$

for any $n \in \mathbb{N}$. By (2.4), (2.10), (2.11), passing to the limit in (2.5) as $n \to \infty$ we obtain

$$\int_{\mathbb{R}^3} \Big(\nabla v\nabla\varphi + v\varphi - \mu\frac{v\varphi}{|x|^2}\Big)\,\mathrm{d}x = 0. \tag{2.12}$$

From (2.12) and $v = 0$ a.e. in $\Omega^+ \cup \Omega^-$, we obtain

$$\int_{\Omega^0} \Big(\nabla v\nabla\varphi + v\varphi - \mu\frac{v\varphi}{|x|^2}\Big)\,\mathrm{d}x = 0. \tag{2.13}$$

Therefore, $v = 0$ a.e. in Ω^0. Hence, $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$.

In the second place, choosing $\varphi = v_n$ in (2.2), dividing (2.1) by $\|u_n\|^2$ and dividing (2.2) by $\|u_n\|$, we obtain

$$\frac{1}{2} + \frac{1}{4}\|u_n\|^2 \int_{\mathbb{R}^3} l(x)\phi_{v_n}(x)v_n^2(x)\,\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3} k(x)|u_n|^{p-2}v_n^2\,\mathrm{d}x \to 0, \tag{2.14}$$

$$1 + \|u_n\|^2 \int_{\mathbb{R}^3} l(x)\phi_{v_n}(x)v_n^2(x)\,\mathrm{d}x - \int_{\mathbb{R}^3} k(x)|u_n|^{p-2}v_n^2\,\mathrm{d}x \to 0, \tag{2.15}$$

as $n \to \infty$.

From (2.14), (2.15) and the assumption of $4 < p < 6$, it follows that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} k(y)|u_n(y)|^{p-2}v_n^2(y)\,\mathrm{d}y = \frac{p}{4-p} < 0.$$

Moreover, in view of (2.14), we deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} k(y)|u_n(y)|^{p-2}v_n^2(y)\,\mathrm{d}y > 0.$$

which yields a contradiction. Hence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$.

Now we prove that $\{u_n\}$ has a convergent subsequence. Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Going if necessary to a subsequence (still denoted by $\{u_n\}$), we may assume that

$$\begin{gathered}
u_n \rightharpoonup u \quad \text{in } H^1(\mathbb{R}^3), \quad u_n \to u \quad \text{a.e. } \mathbb{R}^3,\\
\nabla u_n \rightharpoonup \nabla u \quad \text{in } L^2(\mathbb{R}^3), \quad u_n \rightharpoonup u \quad \text{in } L^2(\mathbb{R}^3),\\
u_n \to u \quad \text{in } L^s_{\rm loc}(\mathbb{R}^3), \quad \text{where } 2 \le s < 2^* = 6.
\end{gathered}$$

We define $w_n = k(x)|u_n|^{p-2}u_n$ and $w = k(x)|u|^{p-2}u$. Then $w_n \to w$ a.e. in $\mathbb{R}^3$. Since $\{u_n\}$ is bounded in $L^p(\mathbb{R}^3)$ for $4 < p < 6$ and k is bounded in $\mathbb{R}^3$, it follows that $\{w_n\}$ is bounded in $L^{\frac{p}{p-1}}(\mathbb{R}^3)$, so there exists $M > 0$ such that

$$\Big(\int_E |w_n(x) - w(x)|^{\frac{p}{p-1}}\,\mathrm{d}x\Big)^{\frac{p-1}{p}} \le M, \tag{2.16}$$

and there exists $\widetilde{w} \in L^{\frac{p}{p-1}}(\mathbb{R}^3)$ such that $w_n \rightharpoonup \widetilde{w}$ in $L^{\frac{p}{p-1}}(\mathbb{R}^3)$ with $4 < p < 6$. Moreover $w = \widetilde{w}$ a.e. in $\mathbb{R}^3$; indeed, let $f \in (L^{\frac{p}{p-1}}(\mathbb{R}^3))^* = L^p(\mathbb{R}^3)$, for any $\varepsilon > 0$ there exists $r > 0$ such that

$$\int_{\{x\in\mathbb{R}^3:|x|\geq r\}} |f(x)|^p \,\mathrm{d}x \leq \frac{\varepsilon^p}{2M^p}. \tag{2.17}$$

Moreover, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every $E \subseteq \mathbb{R}^3$ and $\operatorname{meas} E < \delta$, one has

$$\int_E |f(x)|^p \,\mathrm{d}x \leq \frac{\varepsilon^p}{2M^p}. \tag{2.18}$$

From Hölder's inequality and (2.16)–(2.18), for every $E \subseteq \mathbb{R}^3$ and $\operatorname{meas} E < \delta$, we have

$$\begin{aligned}
&\Big|\int_E f(x)(w_n(x) - w(x))\,\mathrm{d}x\Big| \\
&\leq \Big(\int_E |f(x)|^p \,\mathrm{d}x\Big)^{1/p}\Big(\int_E |w_n(x) - w(x)|^{\frac{p}{p-1}} \,\mathrm{d}x\Big)^{\frac{p-1}{p}} \\
&\leq M\Big(\int_{E\cap\{x\in\mathbb{R}^3:\,|x|\geq r\}} |f(x)|^p \,\mathrm{d}x + \int_{E\cap\{x\in\mathbb{R}^3:\,|x|\leq r\}} |f(x)|^p \,\mathrm{d}x\Big)^{1/p} \\
&\leq M\Big(\frac{\varepsilon^p}{2M^p} + \int_{E\cap\{x\in\mathbb{R}^3:\,|x|\leq r\}} |f(x)|^p \,\mathrm{d}x\Big)^{1/p} \\
&\leq M\Big(\frac{\varepsilon^p}{2M^p} + \frac{\varepsilon^p}{2M^p}\Big)^{1/p} \leq \varepsilon,
\end{aligned}$$

hence $\{\int_{\mathbb{R}^3} f(x)(w_n(x) - w(x))\,\mathrm{d}x,\ n \in N\}$ is equi-absolutely-continuous. It follows easily from Vitali Convergence Theorem that

$$\int_{\mathbb{R}^3} f(x)(w_n(x) - w(x))\,\mathrm{d}x \to \int_{\mathbb{R}^3} f(x)(w(x) - w(x))\,\mathrm{d}x = 0. \tag{2.19}$$

therefore $w = \widetilde{w}$ a.e. in $\mathbb{R}^3$.

Note that, for any $\psi \in H^1(\mathbb{R}^3)$, one has

$$\int_{\mathbb{R}^3} k(x)|u_n|^{p-2}u_n^{p-2}\psi \,\mathrm{d}x \to \int_{\mathbb{R}^3} k(x)|u|^{p-2}u^{p-2}\psi \,\mathrm{d}x, \tag{2.20}$$

and

$$\int_{\mathbb{R}^3} \Big(\nabla u_n \nabla\psi + u_n\psi - \mu\frac{u_n\psi}{|x|^2}\Big)\,\mathrm{d}x \to \int_{\mathbb{R}^3} \Big(\nabla u \nabla\psi + u\psi - \mu\frac{u\psi}{|x|^2}\Big)\,\mathrm{d}x, \tag{2.21}$$

as $n \to \infty$.

Now we prove that

$$\int_{\mathbb{R}^3} l(x)\phi_{u_n}(x)u_n^2 \,\mathrm{d}x \to \int_{\mathbb{R}^3} l(x)\phi_u(x)u^2 \,\mathrm{d}x, \tag{2.22}$$

as $n \to \infty$ and that for all $\psi \in H^1(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} l(x)\phi_{u_n}(x)u_n^2\psi \,\mathrm{d}x \to \int_{\mathbb{R}^3} l(x)\phi_u(x)u^2\psi \,\mathrm{d}x. \tag{2.23}$$

We borrow the strategy from [5]. In fact, by the continuity of embedding, we have $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } L^6(\mathbb{R}^3),\\ u_n^2 &\to u^2 \quad \text{in } L^3_{\rm loc}(\mathbb{R}^3),\\ \phi_{u_n} &\rightharpoonup \phi_u \quad \text{in } D^{1,2}(\mathbb{R}^3),\\ \phi_{u_n} &\rightharpoonup \phi_u \quad \text{in } L^3_{\rm loc}(\mathbb{R}^3). \end{aligned} \tag{2.24}$$

Thus, given $\varepsilon > 0$, we have

$$\big|\int_{\mathbb{R}^3} l(x)(\phi_{u_n}(x) - \phi_u(x))u_n^2 \,\mathrm{d}x\big| \le \varepsilon, \tag{2.25}$$

for n large enough. And for any fixed ψ,

$$\Big|\int_{\mathbb{R}^3} l(x)\phi_u(x)(u_n - u)(x) \,\mathrm{d}x\Big| \le \varepsilon, \tag{2.26}$$

for n large enough. Moreover, by $(2.24)_2$ and $(2.24)_4$, we can assert that for any choice of ε and $\rho > 0$, the relations

$$\Big(\int_{B_\rho(0)} |u_n^2 - u^2|^3 \,\mathrm{d}x\Big)^{1/3} \le \varepsilon, \tag{2.27}$$

$$\Big(\int_{B_\rho(0)} |\phi_{u_n} - \phi_u|^6 \,\mathrm{d}x\Big)^{1/6} \le \varepsilon, \tag{2.28}$$

hold for n large enough.

Noting that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. It is deduced from this and the continuity of the Sobolev embedding of $D^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$ that $\{\phi_{u_n}\}$ is bounded in $D^{1,2}(\mathbb{R}^3)$ and in $L^6(\mathbb{R}^3)$. Since $l \in L^2(\mathbb{R}^3)$, then lu_n^2 and lu^2 belong to $L^{6/5}(\mathbb{R}^3)$ and that for any $\varepsilon > 0$ there exists $\overline{\rho} = \overline{\rho}(\varepsilon)$ such that

$$\Big(\int_{\mathbb{R}^3\setminus B_\rho(0)} |l(x)|^2 \,\mathrm{d}x\Big)^{1/2} \le \varepsilon \quad \text{for } \rho \ge \overline{\rho}. \tag{2.29}$$

Thus, from (2.22), (2.25), (2.27) and (2.29), we obtain

$$\begin{aligned} &\Big|\int_{\mathbb{R}^3} l(x)\phi_{u_n}(x)u_n^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} l(x)\phi_u(x)u^2 \,\mathrm{d}x\Big|\\ &\le \Big|\int_{\mathbb{R}^3} l(x)\phi_{u_n}(x)(u_n^2 - u^2) \,\mathrm{d}x\Big| + \Big|\int_{\mathbb{R}^3} l(x)(\phi_{u_n}(x) - \phi_u(x))u^2 \,\mathrm{d}x\Big|\\ &\le \|\phi_{u_n}\|_6\Big(\int_{\mathbb{R}^3} |l(x)(u_n^2 - u^2)|^{6/5} \,\mathrm{d}x\Big)^{5/6} + \varepsilon\\ &\le C\Big(\int_{\mathbb{R}^3\setminus B_\rho(0)} |l(x)(u_n^2 - u^2)|^{6/5} \,\mathrm{d}x + \int_{B_\rho(0)} |l(x)(u_n^2 - u^2)|^{6/5} \,\mathrm{d}x\Big)^{5/6} + \varepsilon\\ &\le C\Big[\Big(\int_{\mathbb{R}^3\setminus B_\rho(0)} |l(x)|^2 \,\mathrm{d}x\Big)^{3/5} |u_n^2 - u^2|_3^{6/5}\\ &\quad + |l|_2^{6/5}\Big(\int_{B_\rho(0)} |u_n^2 - u^2|^3 \,\mathrm{d}x\Big)^{2/5}\Big]^{5/6} + \varepsilon \le C\varepsilon, \end{aligned}$$

for n large enough.

By a similar argument, we conclude from (2.23) (2.26), (2.28) and (2.29) that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^3} l(x)\phi_{u_n}(x)u_n\psi(x)\,\mathrm{d}x - \int_{\mathbb{R}^3} l(x)\phi_u(x)u\psi(x)\,\mathrm{d}x\Big| \\
&\le \Big|\int_{\mathbb{R}^3} l(x)\phi_u(x)(u_n-u)\psi(x)\,\mathrm{d}x\Big| + \Big|\int_{\mathbb{R}^3} l(x)(\phi_{u_n}(x)-\phi_u(x))u_n\psi(x)\,\mathrm{d}x\Big| \\
&\le \|u_n\|_6\|\psi\|_6\Big(\int_{\mathbb{R}^3} |l(x)(\phi_{u_n}(x)-\phi_u(x))|^{\frac{3}{2}}\,\mathrm{d}x\Big)^{2/3} + \varepsilon \\
&\le C\varepsilon,
\end{aligned}$$

for n large enough.

From (2.20) and (2.21) with (2.23), one has

$$\begin{aligned}
&\langle I'(u_n),\psi\rangle \\
&= \int_{\mathbb{R}^3}\Big(\nabla u_n\nabla\psi + u_n\psi - \mu\frac{u_n\psi}{|x|^2}\Big)\,\mathrm{d}x + \int_{\mathbb{R}^3} l(x)\phi_{u_n}(x)u_n\psi(x)\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3} k(x)|u_n|^{p-2}u_n\psi\,\mathrm{d}x \\
&\to \int_{\mathbb{R}^3}\Big(\nabla u\nabla\psi + u\psi - \mu\frac{u\psi}{|x|^2}\Big)\,\mathrm{d}x + \int_{\mathbb{R}^3} l(x)\phi_u(x)u\psi(x)\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3} k(x)|u|^{p-2}u\psi\,\mathrm{d}x \\
&= \langle I'_\mu(u),\psi\rangle.
\end{aligned}$$

Since $I'(u_n)\to 0$ in $H^{-1}(\mathbb{R}^3)$, we have $\langle I'(u_n),\psi\rangle \to 0$ for any $\psi\in H^1(\mathbb{R}^3)$. So $\langle I'(u),\psi\rangle = 0$ for any $\psi\in H^1(\mathbb{R}^3)$, and

$$\langle I'(u),u\rangle = 0. \tag{2.30}$$

We denote $v_n = u_n - u$. Then $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$. By using this and (2.22), we deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} l(x)\phi_{v_n}(x)v_n^2(x)\,\mathrm{d}x = 0. \tag{2.31}$$

From the Brézis-Lieb lemma, we derive

$$\begin{gathered}
\|u_n\|^2 = \|v_n\|^2 + \|u\|^2 + o(1), \\
\int_{\mathbb{R}^3} k(x)|u_n|^p\,\mathrm{d}x = \int_{\mathbb{R}^3} k(x)|v_n|^p\,\mathrm{d}x + \int_{\mathbb{R}^3} k(x)|u|^p\,\mathrm{d}x + o(1),
\end{gathered}$$

for n large enough.

It follows from (2.22) that

$$\langle I'(u_n),u_n\rangle = \langle I'(u),u\rangle + \|v_n\|^2 + \int_{\mathbb{R}^3} l(x)\phi_{v_n}(x)v_n^2(x)\,\mathrm{d}x - \int_{\mathbb{R}^3} k(x)|v_n|^p\,\mathrm{d}x + o(1).$$

By using this, (2.30), and (2.31), we deduce that

$$\lim_{n\to\infty}\Big(\|u_n-u\|^2 - \int_{\mathbb{R}^3} k(x)|u_n-u|^p\,\mathrm{d}x\Big) = 0. \tag{2.32}$$

Next, without loss of generality we can assume that $k_\infty < -1$. By (H2), there is $R_0 > 0$ such that

$$k(x) < -1 \quad \text{if } |x| > R_0. \tag{2.33}$$

Moreover, since $k \in C(\mathbb{R}^3)$ and $4 < p < 6$, we obtain

$$\int_{|x|\le R_0} k(x)|u_n - u|^p \,\mathrm{d}x \to 0, \tag{2.34}$$

as $n \to \infty$. It follows from (2.32)–(2.34) that

$$\begin{aligned} 0 &\le \lim_{n\to\infty} \|u_n - u\|^2 \\ &= \liminf_{n\to\infty} \int_{\mathbb{R}^3} k(x)|u_n - u|^p \,\mathrm{d}x \\ &\le \lim_{n\to\infty} \int_{|x|\le R_0} k(x)|u_n - u|^p \,\mathrm{d}x = 0. \end{aligned} \tag{2.35}$$

Thus we have $u_n \to u$ in $H^1(\mathbb{R}^3)$, which means that I satisfies (PS) condition.

Step 2: Mountain-pass geometric structure. It follows from (H3) that

$$\int_{\mathbb{R}^3} l(x)\phi_u(x)u^2(x)\,\mathrm{d}x \ge 0.$$

From (H1) and (H2), we have k is bounded in $\mathbb{R}^3$. It follows from the continuity of the Sobolev embedding of $H^1(\mathbb{R}^3)$ in $L^p(\mathbb{R}^3)$ that

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} l(x)\phi_u(x)u^2(x)\,\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3} k(x)|u|^p\,\mathrm{d}x \\ &\ge \frac{1}{2}\|u\|^2 - C\|u\|^p. \end{aligned}$$

Choosing $\rho = \|u\|$ small enough such that $\frac{1}{2}\|u\|^2 - C\|u\|^p > 0$, we obtain $I(u) > 0$.

Choose $\varphi \in H^1(\mathbb{R}^3)$ with $\operatorname{supp}\varphi \subseteq \Omega^+$ such that $\varphi(x) \ge 0$ with strict inequality holding on a subset of positive measure, for all $x \in \Omega^+$. Then we have

$$I(s\varphi) = \frac{s^2}{2}\|\varphi\|^2 + \frac{s^4}{4}\int_{\mathbb{R}^3} l(x)\phi_\varphi(x)\varphi^2(x)\,\mathrm{d}x - \frac{s^p}{p}\int_{\mathbb{R}^3} k(x)|\varphi|^p\,\mathrm{d}x \to -\infty,$$

as $s \to +\infty$. Thus there is $u_1 := s\varphi \in H^1(\mathbb{R}^3)$ with $\|u_1\| > \rho$ such that $I(u_1) < 0$. Therefore I has a mountain pass geometry.

Step 3: Critical value of I. For u_1 in step 2, we define

$$\Gamma := \{\gamma : C[0,1] \to H^1(\mathbb{R}^3) | \gamma(0) = 0,\, \gamma(1) = u_1\},$$
$$c := \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} I(\gamma(t)).$$

It turns out that the Mountain Pass Theorem holds. Then $c > 0$ is critical value of I.

Since $I(u) = I(|u|)$ in $H^1(\mathbb{R}^3)$, we conclude that $u \ge 0$ a.e. in $\mathbb{R}^3$ with $I(u) > 0$ and it is a solution of (1.3). The strong Maximum Principle implies that $u > 0$ in $\mathbb{R}^3$.

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