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# NON-PERTURBATIVE POSITIVITY AND WEAK HÖLDER CONTINUITY OF LYAPUNOV EXPONENT OF ANALYTIC QUASI-PERIODIC JACOBI COCYCLES DEFINED ON A HIGH DIMENSION TORUS

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Abstract. When analytic quasi-periodic cocycles are defined on a high dimension torus, their Lyapunov exponents have perturbative positivity and continuity. In this article, we study a class of analytic quasi-periodic Jacobi cocycles defined on a two dimension torus. We show that in the non-perturbative large coupling regimes, the Lyapunov exponent is positive for any frequency and weak Hölder continuous for the full-measured frequency.

## 1. Introduction

We consider the quasi-periodic Jacobi operator $H_{\underline{x},\underline{\omega},\lambda v,a}$ in $\ell^2(\mathbb{Z})$,

$$\begin{aligned}\big(H_{\underline{x},\underline{\omega},\lambda v,a}\phi\big)(n) &= -a(x_2+(n+1)\omega_2)\phi(n+1) - \bar{a}(x_2+n\omega_2)\phi(n-1)\\ &\quad + \lambda v(x_1+n\omega_1)\phi(n), \quad n\in\mathbb{Z},\end{aligned} \tag{1.1}$$

where $v:\mathbb{T}\to\mathbb{R}$ is a real analytic function called the potential, $a:\mathbb{T}\to\mathbb{C}$ is a complex analytic function and not identically zero, $\lambda$ is a real positive constant called the coupling number, $\underline{x}=(x_1,x_2)$ is the phase, and $\underline{\omega}=(\omega_1,\omega_2)$ is the frequency. Their characteristic equations $H_{\underline{x},\underline{\omega},\lambda v,a}\phi = E\phi$ can be expressed as

$$\begin{pmatrix}\phi(n+1)\\ \phi(n)\end{pmatrix} = M(\underline{x}+n\underline{\omega},E,\lambda v,a)\begin{pmatrix}\phi(n)\\ \phi(n-1)\end{pmatrix},$$

where

$$M(\underline{x}+n\underline{\omega},E,\lambda v,a) = \frac{1}{a(x_2+(n+1)\omega_2)}\begin{pmatrix}\lambda v(x_1+n\omega_1)-E & -\bar{a}(x_2+n\omega_2)\\ a(x_2+(n+1)\omega_2) & 0\end{pmatrix}.$$

In this article, we always fix the analytic functions $v$ and $a$, and suppress them from symbols. Then, we have the following analytic quasi-periodic Jacobi cocycles $(M_{\lambda,E},\underline{\omega})\in C^\omega(\mathbb{T}^2,M_2(\mathbb{C}))\times\mathbb{R}^2$ where $M_2(\mathbb{C})$ is the set of $2\times 2$ matrices with complex entries:

$$(M_{\lambda,E},\underline{\omega}):\mathbb{C}^2\times\mathbb{T}^2\to\mathbb{C}^2\times\mathbb{T}^2 \quad \text{with } (\underline{v},\underline{x})\to(M_{\lambda,E}(\underline{x})\underline{v},\underline{x}+\underline{\omega}),$$

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where

$$M_{\lambda,E}(\underline{x}) = \frac{1}{a(x_2+\omega_2)} \begin{pmatrix} \lambda v(x_1) - E & -\bar{a}(x_2) \\ a(x_2+\omega_2) & 0 \end{pmatrix}.$$

Because the complex function $a$ has only finite zero points in the complex plane, the matrix $M_{\lambda,E}$ and the Jacobi cocycles make sense almost everywhere.

Let $M(\underline{x}, E, \lambda) := M_{\lambda,E}(\underline{x})$ and define

$$\begin{aligned} M_n(\underline{x}, E, \underline{\omega}, \lambda) &= \prod_{j=n-1}^{0} M(\underline{x} + j\underline{\omega}, E, \lambda) \\ &= \prod_{j=n-1}^{0} \frac{1}{a(x_2+(j+1)\omega_2)} \begin{pmatrix} \lambda v(x_1+j\omega_1) - E & -\bar{a}(x_2+j\omega_2) \\ a(x_2+(j+1)\omega_2) & 0 \end{pmatrix}, \end{aligned}$$

which is called the transfer matrix of (1.1). Set

$$L_n(E, \underline{\omega}, \lambda) := \frac{1}{n} \int_{\mathbb{T}^2} \log \|M_n(\underline{x}, E, \underline{\omega}, \lambda)\| d\underline{x}.$$

From the Kingman's subadditive ergodic theorem, we have

$$L(E, \underline{\omega}, \lambda) := \lim_{n\to\infty} L_n(E, \underline{\omega}, \lambda) = \lim_{n\to\infty} \frac{1}{n} \log \|M_n(\underline{x}, E, \underline{\omega}, \lambda)\|$$

for almost every $\underline{x} \in \mathbb{T}^2$, which is called the Lyapunov exponent of (1.1).

Note that $L(E, \underline{\omega}, \lambda)$ is non-negative, as

$$\int_{\mathbb{T}^2} \log |\det M(\underline{x}, E, \lambda)| d\underline{x} \equiv 0.$$

In this article, we first show that the Lyapunov exponent is always positive when the coupling number is large.

**Theorem 1.1.** *For any $\kappa > 0$, there exists $\lambda_0 = \lambda_0(v, a, \kappa) > 0$ such that for any $\underline{\omega}$, if $|\lambda| > \lambda_0$ and $E$ is in the spectrum of* (1.1)*, then*

$$(1-\kappa)\log|\lambda| < L(E, \underline{\omega}, \lambda) < (1+\kappa)\log|\lambda|.$$

Because of the uniform hyperbolicity, the Lyapunov exponent is always positive when $E$ is in the resolvent set.

Secondly, we study the continuity of $L(E, \underline{\omega}, \lambda)$ in the energy $E$. It is well known that $L(E, \underline{\omega}, \lambda)$ is a $C^\infty$ function of $E$ on the resolvent set. So we only need to consider $E \in \mathscr{E}$, which contains the spectrum and will be defined in (2.1). What's more, we need to assume that $\omega_1$ and $\omega_2$ are both the Diophantine number (DN for short). Here when we say that a irrational number $\omega \in (0,1)$ is the DN, it means that $\omega$ satisfies the Diophantine condition

$$\|n\omega\| \geq \frac{C_\omega}{|n|^\alpha} \quad \text{for all } n \neq 0. \tag{1.2}$$

It is well known that for a fixed $\alpha > 1$, almost every $\omega \in \mathbb{T}$ satisfies (1.2). Thus, the set of $\underline{\omega}$ we assumed has full measure in $\mathbb{T}^2$. Then, we obtain the weak Hölder continuity of $L(E, \underline{\omega}, \lambda)$ in $E$.

**Theorem 1.2.** *Let $E \in \mathscr{E}$, both $\omega_1$ and $\omega_2$ be the DN, and $|\lambda| > \lambda_0$ where $\lambda_0$ comes from Theorem 1.1 with $\kappa = \frac{1}{100}$. Then $L(E, \underline{\omega}, \lambda)$ is a continuous function of $E$ with modulus of continuity*

$$h(t) = \exp\left(-c|\log t|^\tau\right),$$

*where $\tau = \tau(\alpha)$ and $c = c(\lambda v, a)$ are positive constants.*

**Remark 1.3.** Actually, the $d$-dimension Diophantine number (DN) is always defined by

$$\|\underline{n}\cdot\underline{\omega}\| := \|n_1\omega_1 + \dots + n_d\omega_d\| \geq \frac{c}{(|n_1| + \dots + |n_d|)^A}$$

for all $(n_1, \dots, n_d) \in \mathbb{Z}^d \backslash \{\underline{0}\}$ and $A > d$, which is also almost everywhere in $\mathbb{T}^d$. Obviously, this 2-dimension DN is a subset of our frequency.

The research on Lyapunov exponents has been a hot topic in several fields for a long time. In 2001 Goldstein and Schlag [8] developed two powerful techniques, the Large Deviation Theorem and the Avalanche Principle. These two techniques are widely applied in the literatures, to study the Schrödinger operator

$$\big(H^s_{\underline{x},\underline{\omega},\lambda}\phi\big)(n) = \phi(n+1) + \phi(n-1) + \lambda v(\underline{x} + n\underline{\omega})\phi(n), \quad n \in \mathbb{Z},$$

where the potential $v$ is a real analytic function on $\mathbb{T}^d$. Obviously, it is a special case of (1.1) with $a \equiv 1$, and $M^s_n(\underline{x}, E, \underline{\omega}, \lambda)$, $L^s(E, \underline{\omega}, \lambda)$ and $L^s_n(E, \underline{\omega}, \lambda)$ have the corresponding definitions. When $d = 1$ and $\underline{\omega} = \omega$ this is the Strong DN:

$$\|n\omega\| \geq \frac{C_\omega}{|n|(1 + \log|n|)^\alpha} \quad \text{for all } n \neq 0,$$

which is also almost everywhere in $\mathbb{T}$ for $\alpha > 1$. They obtained that $L^s(E, \omega, \lambda)$ is Hölder continuous in $E$ in the positive Lyapunov exponent regimes. When $d \geq 2$ and $\underline{\omega}$ is the d-dimension DN, they obtained the perturbative result that there exists a $\tilde{\lambda}^s_0 := \tilde{\lambda}^s_0(v, A, \underline{\omega})$ such that for any $|\lambda| > \tilde{\lambda}^s_0$, $L^s(E, \underline{\omega}, \lambda)$ is positive for all $E$ and weak Hölder continuous in $E$. Readers may have doubts when the Lyapunov exponent is positive for $d = 1$. Actually, Sorets-Spencer [14] proved in 1991 that for any nonconstant real analytic potential $v$, there exists $\lambda^s_0 = \lambda^s_0(v)$ such that for any $|\lambda| > \lambda^s_0$, the Lyapunov exponent is positive for any $\omega$. In 2002, Bourgain-Jitomirskaya [6] proved the joint continuity of $L^s(E, \omega, \lambda)$ in $(E, \omega)$ at every $(E, \omega_0)$ if $\omega_0$ is irrational and $L^s(E, \omega_0, \lambda)$ is positive. Then in 2005, Bourgain [3] extended this continuity and the result of the positive Lyapunov exponent in [14] from $\mathbb{T}$ to $\mathbb{T}^d$.

All above results depend on the fact that the determinants of the Schrödinger transfer matrices are always 1. For the analytic quasi-periodic $GL(2, \mathbb{C})$ cocycles

$$M(\underline{x}) = \begin{pmatrix} v_{11}(\underline{x}) & v_{12}(\underline{x}) \\ v_{21}(\underline{x}) & v_{22}(\underline{x}) \end{pmatrix},$$

where $v_{ij}$ $(i, j = 1, 2)$ are analytic function on $\mathbb{T}^d$, Jitomirskaya-Koslover-Schulteis [10] and Jitomirskaya-Marx [11] proved the weak Hölder continuity of the Lyapunov exponent in $v_{ij}$ over the analytic category for 1-dimension Diophantine frequency. Avila-Jitomirskaya-Sadel showed the continuity for any 1-dimension frequency in [1]. The author extended it to $d \geq 2$ for $d$-dimension Diophantine frequency in [16]. He also studied the following general analytic quasi-periodic Jacobi operators

$$\big(\tilde{H}_{\underline{x},\underline{\omega},\lambda v,a}\phi\big)(n) = -a(\underline{x} + (n+1)\underline{\omega})\phi(n+1) - \bar{a}(\underline{x} + n\underline{\omega})\phi(n-1) + \lambda v(\underline{x} + n\underline{\omega})\phi(n)$$

for $n \in \mathbb{Z}$, and proved in [17] that when $d = 1$ and $\underline{\omega} = \omega$ is the strong DN, the continuity of the Lyapunov exponent in $E$ can be Hölder.

In summary, the Lyapunov exponent of the $SL(2, \mathbb{C})$ cocycles is always positive for any $\underline{\omega}$ and any $d$ in the large coupling regimes. But when the cocycles become $GL(2, \mathbb{C})$, we have the same result only for $d = 1$. Therefore, the first highlight of our paper is that it is the first conclusion of the positive Lyapunov exponents of a class of $GL(2,\mathbb{C})$ cocycles defined on $\mathbb{T}^2$ for any frequency. Secondly, we prove the weak Hölder continuity for the more generic full-measured frequency (see Remark 1.3). Furthermore, both results are non-perturbative.

We organize this article as follows. In Section 2, we develop Bourgain-Goldstein's method, which was applied to the quasi-periodic Schrödinger equations in [4], to prove Theorem 1.1. With its help, we obtain the large deviation theorem and Theorem 1.2 in Section 3.

## 2. Positive Lyapunov exponent

It is well known that if $v$ is real analytic function on $\mathbb{T}$, then there exists some $\rho_v > 0$ such that

$$v(x) = \sum_{k\in\mathbb{Z}} \hat{v}(k) e^{2\pi i k x}, \quad \text{with } |\hat{v}(k)| \lesssim e^{-\rho_v |k|}.$$

So, it has a holomorphic extension

$$v(z) = \sum_{k\in\mathbb{Z}} \hat{v}(k) e^{2\pi i k z}$$

on the strip $|\Im z| < \frac{\rho_v}{10}$, satisfying

$$|v(z)| \le \sum_{k\in\mathbb{Z}} |\hat{v}(k)| e^{2\pi |k| |\Im z_v|} < \sum_{k\in\mathbb{Z}} e^{-\rho_v |k|} e^{\rho_v |k| \frac{\pi}{10}} < C_v.$$

Easy computations show that the spectrum of our operators must be in the interval

$$\mathscr{E} := [-2 \max_{x\in\mathbb{T}} |a| - |\lambda| C_v,\, 2 \max_{x\in\mathbb{T}} |a| + |\lambda| C_v]. \tag{2.1}$$

In the rest paper, we always fix the frequency $\underline{\omega}$ and suppress it for ease from now on. Define the analytic transfer matrix

$$M_n^a(\underline{x}, E, \lambda) := \prod_{j=n-1}^{0} M^a(\underline{x} + j\underline{\omega}, E, \lambda),$$

where

$$M^a(\underline{x}, E, \lambda) := a(x_2 + \omega_2) M(\underline{x}, E, \lambda) = \begin{pmatrix} \lambda v(x_1) - E & -\bar{a}(x_2) \\ a(x_2+\omega_2) & 0 \end{pmatrix}.$$

Then for fixed $\lambda, E$ and $x_2$, the function

$$u_n^a(\cdot, x_2, E, \lambda) = \frac{1}{n} \log \|M_n^a(\underline{x}, E, \lambda)\|$$

has a subharmonic extension $u_n^a(z, x_2, E, \lambda)$ ($u_n^a(z)$ for short) on $|\Im z| < \frac{\rho_v}{10}$, which is bounded by $\log(4\max_{x\in\mathbb{T}} |a| + 2|\lambda| C_v)$ for any $E \in \mathscr{E}$. If we choose

$$C_{\max} = 4 \max_{x\in\mathbb{T}} |a| + 2C_v,$$

then for any $x_1$, $x_2$, $\underline{\omega}$ and $E \in \mathscr{E}$, it holds, for any $|\lambda| \ge 1$,

$$u_n^a(x_1) \le \log C_{\max} |\lambda|. \tag{2.2}$$

Set

$$L_n^a(E,\lambda) := \frac{1}{n} \int_{\mathbb{T}^2} \log \|M_n^a(\underline{x}, E, \lambda)\| d\underline{x},$$
$$L^a(E,\lambda) := \lim_{n\to\infty} L_n^a(E,\lambda),$$

which also exists by the Kingman's subharmonic ergodic theorem. It is straightforward to check that

$$\log \|M_n^a(\underline{x}, E, \lambda)\| = \log \|M_n(\underline{x}, E, \lambda)\| + \sum_{j=1}^{n} \log |a(x_2 + (j+1)\omega_2)|,$$
$$L_n^a(E, \lambda) = L_n(E, \lambda) + D,$$
$$L^a(E, \lambda) = L(E, \lambda) + D,$$

where

$$D := \int_{\mathbb{T}} \log |a(x)|\, dx = \int_{\mathbb{T}} \log |\bar{a}(x)|\, dx, \tag{2.3}$$

which exists by the analyticity of $a$. Obviously, to obtain Theorem 1.1, we only need to prove that for any $|\lambda| > \lambda_0(v, a, \kappa)$,

$$\big(1 - \frac{\kappa}{2}\big) \log |\lambda| < L^a(\lambda, E) < \big(1 + \frac{\kappa}{2}\big) \log |\lambda|.$$

Actually, the second inequality is trivial by (2.2) with large $|\lambda|$.

Now, we start the proof of the first inequality. First, we recall the following lemmas from [4] and [18].

**Lemma 2.1** ([4, Lemma 14.5]). *For every $0 < \delta < \rho$, there is an $\epsilon$ such that*

$$\inf_{E_1} \sup_{\delta/2 < y < \delta} \inf_{x \in [0,1]} |v(x + iy) - E_1| > \epsilon.$$

**Lemma 2.2** ([18, Corollary 2]). *Let $u : \Omega \to [-\infty, +\infty)$ be an upper semicontinuous function. Then $u(z)$ is a subharmonic function on $\Omega$, if and only if for any Jordan subdomain $\Omega'$ satisfying $\overline{\Omega'} \subset \Omega$ and any $z \in \Omega'$, it satisfies*

$$u(z) \le \int_{\partial\Omega'} u(\zeta) d\mu_\zeta(z, \partial\Omega', \Omega'),$$

*where $\mu(z, \partial\Omega', \Omega')$ is the harmonic measure of $\partial\Omega'$ at $z \in \Omega'$.*

**Remark 2.3.** Here we emphasize that this harmonic measure depends only on the region $\Omega'$ and the point $z$, not on the subharmonic function $u(z)$. It is the key of our method applied in this section.

Without loss of generality, we assume $\lambda > 0$. Fix $0 < \delta \ll \rho$ and $\epsilon$ satisfying Lemma 2.1. Define

$$\tilde{\lambda}_0 = 200 C_{\max} \epsilon^{-100/\kappa} > 0$$

and let $\lambda > \tilde{\lambda}_0 > 0$. Then, for any fixed $E$, there is $\delta/2 < y_1 < \delta$ such that

$$\inf_{x_1 \in [0,1]} \big|v(x_1 + iy_1) - \frac{E}{\lambda}\big| > \epsilon.$$

Therefore,

$$\inf_{x_1 \in \mathbb{T}} |\lambda v(x_1 + iy_1) - E| > \lambda\epsilon > 200 C_{\max} \epsilon^{-\frac{100}{\kappa} + 1} > 200 C_{\max}. \tag{2.4}$$

For $n \ge 1$ we define

$$M_{n-1}^a(iy_1, x_2, E, \lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} w_1^{n-1} \\ w_2^{n-1} \end{pmatrix}. \tag{2.5}$$

Then

$$\begin{aligned}\begin{pmatrix} w_1^n \\ w_2^n \end{pmatrix} &= \begin{pmatrix} \lambda v(iy_1+n\omega_1)-E & -\bar{a}(x_2+n\omega_2) \\ a(x_2+(n+1)\omega_2) & 0 \end{pmatrix} \begin{pmatrix} w_1^{n-1} \\ w_2^{n-1} \end{pmatrix} \\ &= \begin{pmatrix} \big(\lambda v(iy_1+n\omega_1)-E\big)w_1^{n-1} - \bar{a}(x_2+n\omega_2)w_2^{n-1} \\ a\big(x_2+(n+1)\omega_2\big)w_1^{n-1} \end{pmatrix}.\end{aligned} \tag{2.6}$$

Now we use induction to show that for any $n \ge 1$,

$$|w_1^n| \ge |w_2^n|, \quad \text{and} \quad |w_1^n| \ge (\lambda\epsilon - 2C_{\max})|w_1^{n-1}| \ge (\lambda\epsilon - 2C_{\max})^n. \tag{2.7}$$

From definition (2.5), we have $w_1^0 = 1$ and $w_2^0 = 0$. Then

$$|w_1^1| = |\lambda v(iy_1+\omega_1) - E| > \lambda\epsilon > 200C_{\max}, \text{and } |w_2^1| < C_{\max} < |w_1^1|,$$

which satisfy (2.7) for $n = 1$. Let $n = t$ with

$$|w_1^t| \ge |w_2^t|, \quad \text{and} \quad |w_1^t| > (\lambda\epsilon - 2C_{\max})|w_1^{t-1}| > (\lambda\epsilon - 2C_{\max})^t. \tag{2.8}$$

From (2.6) and (2.8), we have

$$\begin{gathered} |w_1^{t+1}| \ge (\lambda\epsilon - 2C_{\max})w_1^t > (\lambda\epsilon - 2C_{\max})^{t+1}, \\ |w_2^{t+1}| \le 2C_{\max}|w_1^t| < 198C_{\max}|w_1^t| \le (\lambda\epsilon - 2C_{\max})|w_1^t| \le |w_1^{t+1}|, \end{gathered}$$

which also satisfy (2.7) for $n = t+1$. Thus, (2.7) holds for any $n \ge 1$. Then

$$\|M_n^a(iy_1, x_2, E, \lambda)\| > (\lambda\epsilon - 2C_{\max})^n \quad \text{and} \quad u_n^a(iy_1) > \log(\lambda\epsilon - 2C_{\max}).$$

We denote by $\mathbb{H} = \{z : \Im z > 0\}$ and $\mathbb{H}_\rho = \{z = x+iy : 0 < y < \frac{\rho}{2}\}$ strips of the complex plane. We denote by $\mu(z, \mathcal{E}, \mathbb{H})$ the harmonic measure of $\mathcal{E}$ at $z \in \mathbb{H}$ and $\mu_s(iy_1, \mathcal{E}_s, \mathbb{H}_\rho)$ the harmonic measure of $\mathcal{E}_s$ at $iy_1 \in \mathbb{H}_\rho$, where $\mathcal{E} \subset \partial\mathbb{H} = \mathbb{R}$ and $\mathcal{E}_s \subset \partial\mathbb{H}_\rho = \mathbb{R} \cup [y = \frac{\rho}{2}]$. Note that $\psi(z) = \exp\big(\frac{2\pi}{\rho}z\big)$ is a conformal map from $\mathbb{H}_\rho$ onto $\mathbb{H}$. From [7], we have

$$\begin{gathered} \mu_s(iy_1, \mathcal{E}_s, \mathbb{H}_\rho) \equiv \mu(\psi(iy_1), \psi(\mathcal{E}_s), \mathbb{H}), \\ \mu(z = x+iy, \mathcal{E}, \mathbb{H}) = \int_{\mathcal{E}} \frac{y}{(t-x)^2+y^2}\frac{dt}{\pi}. \end{gathered}$$

Easy computations show that

$$\mu_s[y = \frac{\rho}{10}] = \frac{10\pi y_1}{\pi\rho} < \frac{10\delta}{\rho} \quad \text{and} \quad \frac{d\mu_s(x)}{dx}\Big|_{y=0} < \frac{y_1}{x^2+y_1^2}.$$

So, the subharmonicity and Lemma 2.2 yield

$$\begin{aligned} \log(\lambda\epsilon - 2C_{\max}) < u_n^a(iy_1) &\le \int_{[y_1=0]\cup[y_1=\frac{\rho}{10}]} u_n^a(z_1)\mu_s(dz_1) \\ &= \int_{y_1=0} u_n^a(x_1)\mu_s(dx_1) + \int_{y_1=\frac{\rho}{10}} u_n^a(x_1+iy_1)\mu_s(dx_1) \\ &\le \int_{y_1=0} u_n^a(x_1)\mu_s(dx_1) + \frac{10\delta}{\rho}\big[\sup_{y_1=\frac{\rho}{10}} u_n^a(x_1+iy_1)\big] \\ &\le \int_{y_1=0} u_n^a(x_1)\mu_s(dx_1) + \frac{10(1+\kappa)\delta}{\rho}\log\lambda. \end{aligned}$$

Hence, by the definition of $\tilde\lambda_0$ and $\delta \ll \rho$, we have

$$\begin{aligned}\int_{\mathbb{R}} u_n^a(x_1)\mu_s(dx_1) &\geq \log(\lambda\epsilon - 2C_{\max}) - \frac{10(1+\kappa)\delta}{\rho}\log\lambda\\ &\geq \big(1-\frac{10(1+\kappa)\delta}{\rho}\big)\log\lambda + \log\epsilon \\ &> \big(1-\frac{\kappa}{2}\big)\log\lambda.\end{aligned}\tag{2.9}$$

Set

$$(u_n^a)^h(x_1) = u_n^a(x_1+h), \quad h\in\mathbb{T}.$$

Then, from Remark 2.3, and (2.4), it is easy to see that (2.9) also holds for $(u_n^a)^h(x_1)$. So, for any $h\in\mathbb{T}$, we have

$$\int_{\mathbb{R}} u_n^a(x_1+h)\mu_s(dx_1) > \big(1-\frac{\kappa}{2}\big)\log\lambda.\tag{2.10}$$

Define

$$L_n^a(x_2,E,\lambda) := \int_{\mathbb{T}} \frac{1}{n}\log\|M_n^a(x_1,x_2,E,\lambda)\|dx_1.$$

Using (2.10) and integrating for $h\in\mathbb{T}$, we obtain

$$\begin{aligned}L_n^a(x_2,E,\lambda) &= \int_0^1 u_n^a(x_1+h)dh\\ &\geq \Big(\int_{\mathbb{R}}\mu_s(dx_1)\Big)\Big(\int_0^1 u_n^a(x_1+h)dh\Big)\\ &= \int_0^1\int_{\mathbb{R}} u_n^a(x_1+h)\mu_s(dx_1)dh\\ &> \big(1-\frac{\kappa}{2}\big)\log\lambda, \quad \forall n\geq 0.\end{aligned}\tag{2.11}$$

Therefore,

$$L_n^a(E,\lambda) = \int_{\mathbb{T}} L_n^a(x_2,E,\lambda)dx_2 > \big(1-\frac{\kappa}{2}\big)\log\lambda, \quad \forall n\geq 0,\tag{2.12}$$

which completes the proof as $n\to+\infty$.

## 3. Large deviation theorems

As mentioned in the introduction, Goldstein and Schlag [8] introduced the large deviation theorem and the avalanche principle. These two methods are standard tools to study the continuity of the Lyapunov exponent. The avalanche principle read as follows.

**Proposition 3.1** ([8, Proposition 2.2]). *Let $A_1,\dots,A_n$ be a sequence of $2\times 2$-matrices whose determinants satisfy*

$$\max_{1\leq j\leq n}|\det A_j| \leq 1.\tag{3.1}$$

*Suppose that*

$$\min_{1\leq j\leq n}\|A_j\| \geq \gamma > n,\tag{3.2}$$

$$\max_{1\leq j<n}[\log\|A_{j+1}\| + \log\|A_j\| - \log\|A_{j+1}A_j\|] < \frac{1}{2}\log\gamma.\tag{3.3}$$

*Then*

$$\Big|\log\|A_n\cdots A_1\| + \sum_{j=2}^{n-1}\log\|A_j\| - \sum_{j=1}^{n-1}\log\|A_{j+1}A_j\|\Big| < C\frac{n}{\gamma} \tag{3.4}$$

*where* $C = \sum_{n=1}^{\infty} 4^n/n!$.

Because of assumption (3.1), the key in the references mentioned in Section 1 is to obtain a suitable large deviation theorem for some $SL(2,\mathbb{C})$ matrices related to the cocycles studied. In this paper, we also prove the corresponding statement, Lemma 3.2. Then, the other part of the proof of the weak Hölder continuity, including how to apply the large deviation theorem and the avalanche principle, can be found in [10, 16].

To state our large deviation theorem for the $SL(2,\mathbb{C})$ matrices, we define the unimodular matrices

$$M_n^u(\underline{x},E,\lambda) := \frac{M_n(\underline{x},E,\lambda)}{|\det M_n(\underline{x},E,\lambda)|^{1/2}} = \frac{M_n^a(\underline{x},E,\lambda)}{|\det M_n^a(\underline{x},E,\lambda)|^{1/2}}.$$

Because of the analyticity, this definition makes sense almost everywhere. It is straightforward to check that

$$\det M_n^a(\underline{x},E,\lambda) = \prod_{j=0}^{n-1}\bar{a}(x_2+j\omega_2)a(x_2+(j+1)\omega_2),$$

and

$$\begin{aligned}&\log\|M_n^u(\underline{x},E,\lambda)\|\\ &= \log\|M_n^a(\underline{x},E,\lambda)\| - \frac{1}{2}\sum_{j=0}^{n-1}\log|\bar{a}(x_2+j\omega_2)a(x_2+(j+1)\omega_2)|.\end{aligned} \tag{3.5}$$

Our desired large deviation theorem reads as follows.

**Lemma 3.2** (Large Deviation Theorem). *Let $E \in \mathscr{E}$, both $\omega_1$ and $\omega_2$ be the DN, and $|\lambda| > \lambda_0$ where $\lambda_0$ comes from Theorem 1.1 with $\kappa = \frac{1}{100}$. Then there exists an $n_0 = n_0(\lambda v, a, \underline{\omega})$ such that for any $n \geq n_0$,*

$$\begin{aligned}&\operatorname{meas}\big\{\underline{x}\in\mathbb{T}^2 : |\frac{1}{n}\log\|M_n^u(\underline{x},E,\lambda)\| - \langle\frac{1}{n}\log\|M_n^u(\cdot,E,\lambda)\|\rangle| > \frac{1}{10}\log\lambda\big\}\\ &\leq C\exp(-c\log\lambda n^{\frac{\sigma}{10}}).\end{aligned}$$

*where*

$$C = \sum_{n=1}^{\infty}\frac{4^n}{n!}, \quad c = 2^{-2}\log 2$$

*which are called the absolute constants, and $\sigma = \sigma(\alpha)$ is positive.*

To prove this lemma, we use the subharmonicity, which comes from the analyticity of $v$ and $a$, and is the most important hypothesis in the following four lemmas.

**Lemma 3.3** ([9, Lemma 2.1]). *Let $u : \Omega \to R$ be a subharmonic function on a domain $\Omega \subset C$. Suppose that $\partial\Omega$ consists of finitely many piece-wise $C^1$ curves. Then there exists a positive measure $\mu$ on $\Omega$ such that for any $\Omega_1 \Subset \Omega$ (i.e., $\Omega_1$ is a compactly contained subregion of $\Omega$)*

$$u(z) = \int_{\Omega_1}\log|z-\zeta|d\mu(\zeta) + h(z),$$

*where $h$ is harmonic on $\Omega_1$ and $\mu$ is unique with property. Moreover, $\mu$ and $h$ satisfy*

$$\mu(\Omega_1) \le C(\Omega, \Omega_1)(\sup_{\Omega} u - \sup_{\Omega_1} u),$$
$$\|h - \sup_{\Omega_1} u\|_{L^\infty(\Omega_2)} \le C(\Omega, \Omega_1, \Omega_2)(\sup_{\Omega} u - \sup_{\Omega_1} u),$$

*for any $\Omega_2 \Subset \Omega_1$.*

**Lemma 3.4** ([2, Corollary 4.7])**.** *Let $u$ be a subharmonic function defined in the annulus $\mathscr{A}_\rho = \{z : |\Im z| < \rho\}$. Suppose furthermore that $u(x) = \int \log|x-\zeta| d\mu(\zeta) + h(x)$ with $\|\mu\| + \|h\|_{L^\infty} \le \check{C}$. Then, the fourier coefficient of $u$ satisfies*

$$|\hat{u}(k)| \lesssim \frac{\check{C}}{|k|}.$$

**Lemma 3.5** ([5, Lemma 2.3])**.** *Suppose $u$ is subharmonic on $\mathscr{A}_\rho$ with $\sup_{\mathscr{A}_\rho} |u| \le n$. Furthermore, assume that $u = u_0 + u_1$, where*

$$\|u_0 - \langle u_0 \rangle\|_{L^\infty(\mathbb{T})} \le \epsilon_0 \quad \textit{and} \quad \|u_1\|_{L^1(\mathbb{T})} \le \epsilon_1.$$

*Then for some constant $C_\rho$ depending only on $\rho$,*

$$\|u\|_{BMO(\mathbb{T})} \le C_\rho \Big( \epsilon_0 \log(\frac{n}{\epsilon_1}) + \sqrt{n\epsilon_1} \Big).$$

**Remark 3.6.** BMO($\mathbb{T}$) is the space of functions of bounded mean oscillation on $\mathbb{T}$, see [13]. Identifying functions that differ only by an additive constant, the norm on BMO($\mathbb{T}$) is

$$\|f\|_{\rm BMO} := \sup_{I \subset \mathbb{T}} \frac{1}{|I|} \int_I |f - \langle f \rangle_I| dx,$$

where

$$\langle f \rangle_I = \frac{1}{|I|} \int_I f(x) dx.$$

**Lemma 3.7** ([17, Theorem 2.7])**.** *Let $u$ be a subharmonic function defined in the annulus $\mathscr{A}_\rho$. Suppose furthermore that $u(x) = \int \log|x-\zeta| d\mu(\zeta) + h(x)$ with $\|\mu\| + \|h\|_{L^\infty} \le \check{C}$. Then for any DN $\omega$, we have*

$$\text{meas}\big\{ x : |\sum_{j=1}^{n} u(x + j\omega) - n\langle u(\cdot) \rangle| > \delta n \big\} < \exp(-c\delta n), \tag{3.6}$$

*where $c = c(\check{C}, \omega)$.*

**Remark 3.8.** It is obvious that the subharmonic function $\log|\bar{a}(z) a(z+\omega_2)|$ has a upper bound $C_a$ on the annulus $\mathscr{A}_a = \{z : |\Im z| \le \rho_a\}$, and then Lemma 3.7 holds because

$$\text{meas}\big\{ x : |\sum_{j=1}^{n} \log|\bar{a}(x_2 + j\omega_2) a(x_2(j+1)\omega_2)| - 2nD| > \delta n \big\} < \exp(-c\delta n), \tag{3.7}$$

where $D$ is defined by (2.3). Combining this with (3.5), we obtain that the following large deviation theorem for $u_n^a$, which is the sufficient condition for Lemma 3.2:

$$\begin{aligned} &\text{meas}\big\{ \underline{x} \in \mathbb{T}^2 : |\frac{1}{n} \log\|M_n^a(\underline{x}, E, \lambda)\| - \langle \frac{1}{n} \log\|M_n^a(\cdot, E, \lambda)\| \rangle| > \frac{1}{20} \log\lambda \big\} \\ &\le C \exp(-c \log\lambda n^{\frac{\sigma}{10}}). \end{aligned} \tag{3.8}$$

Now, we start the proof of (3.8). Fixing $x_2$, $E \in \mathscr{E}$ and $\lambda > \lambda_0$ with $\kappa = \frac{1}{100}$, we expand $u_n^a$ into its Fourier series of $x_1$ and denote the Fourier coefficient as $\hat{u}_n^a(k, x_2, E, \lambda)$, i.e.,

$$u_n^a(\underline{x}, E, \lambda) = \sum_{k \in Z} \hat{u}_n^a(k, x_2, E, \lambda) e^{2\pi i k x_1},$$
$$\hat{u}_n^a(k, x_2, E, \lambda) = \int_{x_1 \in \mathbb{T}} u_n^a(x_1, x_2, E, \lambda) e^{-2\pi i k x_1} dx_1.$$

Combining Lemmas 3.3 and 3.4, we obtain that there exists a $C'_{\max}$ such that

$$\sup_{x_2 \in \mathbb{T}} |\hat{u}_n^a(k, x_2)| \le \frac{C'_{\max}}{|k|}, \quad \forall k \neq 0. \tag{3.9}$$

Here we suppress the fixed $\lambda > \lambda_0$ and $E \in \mathscr{E}$ from symbols for ease, if there is no doubt. Note that

$$u_n^a(x_1 + j\omega_1, x_2 + j\omega_2) = \langle u_n^a(\cdot, x_2 + j\omega_2)\rangle + \sum_{k \neq 0} \hat{u}_n^a\left(k, x_2 + j\omega_2\right) e^{2\pi i k (x_1 + j\omega_1)}.$$

Then

$$\begin{aligned}
&\frac{1}{N}\Big|\sum_{j=1}^{N}[u_n^a(x_1 + j\omega_1, x_2 + j\omega_2) - \langle u_n^a(\cdot, x_2 + j\omega_2)\rangle]\Big| \\
&= \frac{1}{N}\Big|\sum_{j=1}^{N}\sum_{k \in Z\setminus\{0\}} \hat{u}_n^a(k, x_2 + j\omega_2) e^{2\pi i k (x_1 + j\omega_1)}\Big| \\
&\le \frac{1}{N}\Big|\sum_{j=1}^{N}\sum_{0<|k|<K} \hat{u}_n^a(k, x_2 + j\omega_2) e^{2\pi i k (x_1 + j\omega_1)}\Big| \\
&\quad + \frac{1}{N}\Big|\sum_{j=1}^{N}\sum_{|k|>K} \hat{u}_n^a(k, x_2 + j\omega_2) e^{2\pi i k (x_1 + j\omega_1)}\Big| := (a) + (b)
\end{aligned}$$

From (3.9), we have

$$\|(b)\|_2^2 \le \sum_{|k|>K} \sup_j |\hat{u}(k, x_2 + j\omega)|^2 \le (C'_{\max})^2 K^{-1}.$$

On the other hand, by the Cauchy inequality,

$$\begin{aligned}
|(a)|^2 &\le N^{-2}\Big|\sum_{j=1}^{N}\sum_{0<|k|<K} \hat{u}_n^a(k, x_2 + j\omega_2) e^{2\pi i k (x_1 + j\omega_1)}\Big|^2 \\
&\le N^{-2}\Big(\sum_{j=1}^{N}\sum_{0<|k|<K} |\hat{u}_n^a(k, x_2 + j\omega_2)|^2\Big)\Big|\sum_{j=1}^{N}\sum_{0<|k|<K} e^{4\pi i k (x_1 + j\omega_1)}\Big| \\
&\le N^{-2}\Big(N\sup_j \sum_{0<|k|<K} |\hat{u}_n^a(k, x_2 + j\omega_2)|^2\Big)\Big|\sum_{0<|k|<K}\sum_{j=1}^{N} e^{4\pi i k (x_1 + j\omega_1)}\Big| \\
&\le N^{-1}(C'_{\max})^2\Big|\sum_{0<|k|<K}\sum_{j=1}^{N} e^{4\pi i k (x_1 + j\omega_1)}\Big|.
\end{aligned}$$

Easy computations show that

$$|\sum_{j=1}^{N} e^{2\pi ijk\omega}| = |\frac{\exp(2\pi ik\omega)\cdot(1-\exp(2\pi iNk\omega))}{1-\exp(2\pi ik\omega)}| \le \frac{1}{2\|k\omega\|}.$$

Combining this with (1.2), we have

$$|(a)|^2 \le N^{-1}\left(C'_{\max}\right)^2 \sum_{0<|k|<K} \frac{1}{2\|2k\omega\|} \le C\frac{K^{\alpha+1}}{N}.$$

Set $K = N^{\sigma}$ where $\sigma = 1/(2(\alpha+1))$. Then $|(a)|^2 \le CN^{-\frac{1}{2}}$ and $\|(b)\|_2^2 \le \left(C'_{\max}\right)^2 N^{-\sigma}$. For any fixed $x_2 \in \mathbb{T}$, we define

$$|u(x_1)| = \sum_{j=1}^{N}[u_n^a(x_1+j\omega_1, x_2+j\omega_2) - \langle u_n^a(\cdot, x_2+j\omega_2)\rangle].$$

Then

$$\operatorname{meas}\{x_1 \in \mathbb{T} : |u(x_1)| > N^{1-\frac{\sigma}{3}}\} < N^{-\frac{\sigma}{3}}. \tag{3.10}$$

We define $\mathscr{B}$ as the exceptional set for (3.10). Let $u(x_1) = u_0(x_1) + u_1(x_1)$ where $u_0(x_1) = 0$ on $\mathscr{B}$ and $u_1(x_1) = 0$ on $\mathbb{T}\backslash\mathscr{B}$. Thus

$$\|u_0(x_1)\|_{L^\infty(\mathbb{T})} \le N^{1-\frac{\sigma}{3}} \quad \text{and} \quad \|u_1(x_1)\|_{L^1(\mathbb{T})} \le N^{1-\frac{\sigma}{3}}.$$

From Lemma 3.5, we have

$$\|u\|_{\mathrm{BMO}(\mathbb{T})} \le C_\rho N^{1-\frac{\sigma}{7}}.$$

Recall the John-Nirenberg inequality ([13]),

$$\operatorname{meas}\{x \in \mathbb{T} : |u(x) - < u > | > \gamma\} \le C\exp\Big(-\frac{c\gamma}{\|u\|_{\mathrm{BMO}}}\Big),$$

with the absolute constants $C\sum_{n=1}^{\infty}\frac{4^n}{n!}$. $c = 2^{-2}\log 2$. Let $\gamma = \frac{1}{100}N\log\lambda$.

**Lemma 3.9.** *There exists an $N_0 := N_0(\lambda v, a)$ such that for any $N > N_0$, $E \in \mathscr{E}$, $x_2 \in \mathbb{T}$ and $DN\ \omega_1$, it holds*

$$\begin{aligned}&\operatorname{meas}\big\{x_1 \in \mathbb{T} : \frac{1}{N}|\sum_{j=1}^{N}[u_n^a(x_1+j\omega_1, x_2+j\omega_2) - \langle u_n^a(\cdot, x_2+j\omega_2)\rangle]| > \frac{1}{100}\log\lambda\big\}\\ &\le C\exp(-cN^{\frac{\sigma}{7}}\log\lambda).\end{aligned}$$

Obviously, comparing this with our desired (3.8), we need to obtain the following lemma, which studies the deviation between $u_n^a$ and its shifts.

**Lemma 3.10.** *There exists a constant $\tilde{C}_2 := \tilde{C}_2(\lambda v, a)$ such for any $C_2 \le \tilde{C}_2$, $\delta > 1$ and $N = C_2\delta n$, it holds*

$$\sup_{x_1\in\mathbb{T}} \operatorname{meas}\big\{x_2 \in \mathbb{T} : \frac{1}{N}|\sum_{j=1}^{N}[u_n^a(\underline{x}+j\underline{\omega}) - u_n^a(\underline{x})]| > \delta\big\} \le 2N\exp\big(-\frac{c\delta n}{4}\big). \tag{3.11}$$

*Proof.* Since $\det M^a(\underline{x}) = a(x_2+\omega_2)\bar{a}(x_2)$, it follows that

$$(M^a)^{-1}(\underline{x}) = \frac{1}{a(x_2+\omega_2)\bar{a}(x_2)}\begin{pmatrix} 0 & \bar{a}(x_2) \\ -a(x_2+\omega_2) & \lambda v(x_1+\omega_1) - E\end{pmatrix},$$

$$\sup_{x_1\in\mathbb{T}}\|(M^a)^{-1}(\underline{x})\| \le \frac{C_{\max}}{|a(x_2+\omega_2)\bar{a}(x_2)|}.$$

So,

$$\begin{aligned}\|M_n^a(\underline{x}+\underline{\omega})\| &\le \|M^a(\underline{x}+n\underline{\omega})\|\,\|M_n^a(\underline{x})\|\,\|(M^a)^{-1}(\underline{x})\|\\ &\le C_{\max}\|M_n^a(\underline{x})\|\frac{C_{\max}}{|\bar{a}(x_2)a(x_2+\omega_2)|},\end{aligned}$$

and

$$\|M_n^a(\underline{x})\| \le C_{\max}\|M_n^a(\underline{x}+\underline{\omega})\|\frac{C_{\max}}{|\bar{a}(x_2+(n-1)\omega_2)a(x_2+n\omega_2)|}.$$

Therefore,

$$\begin{aligned}-C_1+\log|\bar{a}(x_2)a(x_2+\omega_2)| &\le \log\|M_n^a(\underline{x})\|-\log\|M_n^a(\underline{x}+\underline{\omega})\|\\ &\le C_1-\log|\bar{a}(x_2+(n-1)\omega_2)a(x_2+n\omega_2)|,\end{aligned}$$

where $C_1=2\log C_{\max}$. Similarly,

$$\begin{aligned}&-\frac{kC_1}{n}+\sum_{j=0}^{k-1}\frac{1}{n}\log|a(x_2+(j+1)\omega_2)\bar{a}(x_2+j\omega_2)|\\ &\le u_n^a(\underline{x})-u_n^a(\underline{x}+k\underline{\omega})\\ &\le \frac{kC_1}{n}-\sum_{j=0}^{k-1}\frac{1}{n}\log|\bar{a}(x_2+(n+j-1)\omega_2)a(x_2+(n+j)\omega_2)|.\end{aligned}\tag{3.12}$$

Let

$$\mathscr{Y}_k^-=\big\{x_2\in\mathbb{T}:-\frac{kC_1}{n}+\sum_{j=0}^{k-1}\frac{1}{n}\log|a(x_2+(j+1)\omega_2)\bar{a}(x_2+j\omega_2)|<-\delta\big\}$$

and $N=C_2\delta n$ where $C_1C_2\le\frac12$. Then, for any $1\le k\le N$,

$$\mathscr{Y}_k^-\subset\mathscr{Y}_k^{-\prime}:=\big\{x_2\in\mathbb{T}:\sum_{j=0}^{k-1}\log|a(x_2+(j+1)\omega_2)\bar{a}(x_2+j\omega_2)|<-\frac{\delta n}{2}=-\frac{N}{2C_2}\big\}.$$

For $D>0$,

$$\mathscr{Y}_k^{-\prime}\subset\mathscr{Y}_k^{-\prime\prime}:=\big\{x_2\in\mathbb{T}:\sum_{j=0}^{k-1}\log|a(x_2+(j+1)\omega_2)\bar{a}(x_2+j\omega_2)|-2kD<-\frac{\delta n}{2}\big\}.$$

From (3.7), we have

$$\begin{aligned}&\operatorname{meas}\mathscr{Y}_k^-\le\operatorname{meas}\mathscr{Y}_k^{-\prime\prime}\\ &\le\operatorname{meas}\big\{x_2\in\mathbb{T}:|\sum_{j=0}^{k-1}\log|a(x_2+j\omega_2)a(x_2+(j+1)\omega_2)|-2kD|>\frac{\delta n}{2}\big\}\\ &\le\exp\big(-c\frac{\delta n}{2k}\cdot k\big)\\ &=\exp\big(-\frac{c\delta n}{2}\big).\end{aligned}$$

For $D<0$, let $8C_2|D|<1$, to make $\frac{1}{8C_2}+D>0$. This implies that $\frac{N}{4C_2}+2kD>0$ for $1\le k\le N$ and

$$\mathscr{Y}_k^{-\prime}\subset\mathscr{Y}_k^{-\prime\prime\prime}$$

$$:= \big\{x_2 \in \mathbb{T} : \sum_{j=0}^{k-1} \log |a(x_2 + (j+1)\omega_2)\bar{a}(x_2 + j\omega_2)| - 2kD < -\frac{N}{4C_2} = -\frac{\delta n}{4}\big\}.$$

From (3.7) again, it follows that $\operatorname{meas} \mathscr{Y}_k^- \le \exp(-\frac{c\delta n}{4})$ . Above all, there exists a constant $\tilde{C}_2 := \tilde{C}_2(\lambda v, a)$ such for any $C_2 < \tilde{C}_2$ and $1 \le k \le N = C_2\delta n$,

$$\operatorname{meas} \mathscr{Y}_k^- \le \exp\big(-\frac{c\delta n}{4}\big). \tag{3.13}$$

Similar calculations show that for the set

$$\mathscr{Y}_k^+ := \big\{x_2 \in \mathbb{T} : \frac{kC_1}{n} - \sum_{j=0}^{k-1} \frac{1}{n} \log |a(x_2 + (n+j-1)\omega_2)a(x_2 + (n+j)\omega_2)| > \delta\big\}$$

we have

$$\operatorname{meas} \mathscr{Y}_k^+ < \exp\big(-\frac{c\delta n}{4}\big).$$

Combining this with (3.12) and (3.13), we have that for any $1 \le k \le N$,

$$\operatorname{meas} \big\{x_2 \in \mathbb{T} : |u_n^a(\underline{x} + k\underline{\omega}) - u_n^a(\underline{x})| > \delta\big\} \le 2\exp\big(-\frac{c\delta n}{4}\big).$$

Then, this lemma is obtained by the drawer principle:

$$\begin{aligned}&\big\{x_2 \in \mathbb{T} : \frac{1}{N}\big|\sum_{j=1}^{N}[u_n^a(\underline{x} + j\underline{\omega}) - u_n^a(\underline{x})]\big| > \delta\big\}\\ &\subset \cup_{j=1}^{N}\{x_2 \in \mathbb{T} : |u_n^a(\underline{x} + j\underline{\omega}) - u_n^a(\underline{x})| > \delta\}.\end{aligned}$$

$\square$

**Remark 3.11.** Obviously, we also obtain the deviation between the integrations of $x_1$ for $u_n^a$ and its shifts: there exists a constant $\tilde{C}_2 := \tilde{C}_2(\lambda v, a)$ such for any $C_2 < \tilde{C}_2$, $\delta > 1$ and $N = C_2\delta n$,

$$\operatorname{meas} \big\{x_2 \in \mathbb{T} : \frac{1}{N}\big|\sum_{j=1}^{N}[\langle u_n^a(\cdot, x_2 + j\omega_2)\rangle - \langle u_n^a(\cdot, x_2)\rangle]\big| > \delta\big\} \le 2N\exp\big(-\frac{c\delta n}{4}\big).$$

Now, combining Lemmas 3.9 and 3.10, and Remark 3.11, there exists an $N_0 := N_0(\lambda v, a)$ such that for any $N = \tilde{C}_2\frac{\log\lambda}{100}n > N_0$, $E \in \mathscr{E}$ and DN $\omega_1$, we have

$$\begin{aligned}&\operatorname{meas} \big\{\underline{x} \in \mathbb{T}^2 : |u_n^a(\underline{x}) - \langle u_n^a(\cdot, x_2)\rangle| > \frac{1}{25}\log\lambda\big\}\\ &\le C\exp(-cN^{\frac{\sigma}{7}}\log\lambda) + 4N\exp\big(-\frac{c}{4}\frac{\log\lambda}{100}n\big)\\ &< C\exp(-cN^{\frac{\sigma}{10}}\log\lambda).\end{aligned} \tag{3.14}$$

At last, we need to exchange $\langle u_n^a(\cdot, x_2)\rangle$ by $\langle u_n^a(\cdot)\rangle$. It comes from Section Two. By (2.2), (2.11) and (2.12), for any $\lambda > \lambda_0\big(v, a, \frac{1}{100}\big)$, we have

$$\begin{gathered}\frac{199}{200}\log\lambda \le \langle u_n^a(\cdot, x_2)\rangle \le \frac{210}{200}\log\lambda,\\ \frac{199}{200}\log\lambda \le \langle u_n^a(\cdot)\rangle \le \frac{210}{200}\log\lambda.\end{gathered}$$

Therefore,

$$|\langle u_n^a(\cdot, x_2)\rangle - \langle u_n^a(\cdot)\rangle| \le \frac{1}{100}\log\lambda,$$

and then we obtain (3.8) by combining this with (3.14).

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