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EXISTENCE AND MULTIPLICITY FOR A SUPERLINEAR ELLIPTIC PROBLEM UNDER A NON-QUADRADICITY CONDITION AT INFINITY

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Abstract. In this article, we study the existence and multiplicity of solutions of the boundary-value problem

$$\begin{gathered} -\Delta u = f(x,u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered}$$

where Δ denotes the N-dimensional Laplacian, Ω is a bounded domain with smooth boundary, $\partial\Omega$, in $\mathbb{R}^N$ ($N \geqslant 3$), and f is a continuous function having subcritical growth in the second variable.

Using infinite-dimensional Morse theory, we extended the results of Furtado and Silva [9] by proving the existence of a second nontrivial solution under a non-quadradicity condition at infinity on the non-linearity. Assuming more regularity on the non-linearity f, we are able to prove the existence of at least three nontrivial solutions.

1. Introduction

Furtado and Silva [9] studied the existence and multiplicity of solutions for the boundary-value problem (BVP)

$$\begin{gathered} -\Delta u = f(x,u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Δ denotes the N-dimensional Laplacian, Ω is a bounded domain with smooth boundary, $\partial\Omega$, in $\mathbb{R}^N$ ($N \geqslant 3$), and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies the following conditions:

(H1) There exist constants $a_1 > 0$ and $p \in (2, 2^*)$ such that

$$|f(x,s)| \le a_1(1 + |s|^{p-1}), \quad \text{for } (x,s) \in \Omega \times \mathbb{R}, \tag{1.2}$$

where $2^* = 2N/(N-2)$ is the critical Sobolev exponent.

(H2) For $F(x,s) = \int_0^s f(x,\xi)\,d\xi$, for all $s \in \mathbb{R}$,

$$\lim_{|s|\to\infty} (f(x,s)s - 2F(x,s)) = +\infty \quad \text{uniformly for } x \in \Omega. \tag{1.3}$$

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(H3) The following limit holds

$$\lim_{|s|\to+\infty} \frac{2F(x,s)}{s^2} = +\infty \quad \text{uniformly for } x \in \Omega\,. \tag{1.4}$$

Furtado and Silva [9] proved the following existence and multiplicity result for problem (1.1).

Theorem 1.1 ([9, Theorem 1.2])**.** *Suppose f satisfies* (H1)–(H4)*. Then problem* (1.1) *has at least one nontrivial solution provided that*

$$\limsup_{s\to 0} \frac{F(x,s)}{s^2} = 0, \quad \text{uniformly for } x \in \Omega. \tag{1.5}$$

If $f(x,s)$ is odd in s, condition (1.5) *can be dropped and and problem* (1.1) *has infinitely many weak solutions.*

Remark 1.2. We remark first that condition (1.5) implies that

$$f(x,0) = 0, \quad \text{for all } x \in \Omega. \tag{1.6}$$

Thus, the first part of Theorem 1.1 asserts that problem (1.1) has at least two solutions, one of them being the trivial solution.

Remark 1.3. Condition (H2) is denoted (NQ) in [9]; this is the non-quadraticity condition introduced by Costa and Magalhães in [7]. Condition (H3) is denoted (SL) in [9]; it imposes superlinear growth in the second variable of the nonlinearity f of problem (1.1).

To prove Theorem 1.1, the authors of [9] first showed that the energy functional associated with problem (1.1) satisfies the Cerami condition. They then showed that the energy functional satisfies the conditions of the mountain pass theorem of Ambrosetti and Rabinowitz (see [2, 19]). For the second part of the theorem, assuming that f is odd, they showed that the conditions of the symmetric mountain pass theorem of Rabinowitz [19] were satisfied. Furtado and Silva also presented many examples in the literature that could be included in their framework, such as a double resonance problems. The authors of [9] also showed how to obtain a positive and negative solution of problem (1.1) by using a cutoff-technique presented in [2]. For more details, see [9] and references therein.

Condition (H2) was first introduced by Costa and Magalhães in [7]. It allowed the authors to treat resonant and double resonant problems without a restriction on the quotient $f(x,s)/s$. By considering some additional assumptions on the function f and its primitive, Costa and Magalhães proved that the associated energy functional satisfies the geometric conditions of the mountain-pass theorem and the saddle-point theorem of Rabinowitz (see [19]), and, consequently, proved the existence of a nontrivial solution for problem (1.1). Furtado and Silva [9] assumed the non-quadradicity condition (H2) for the superlinear problem proposed in this work. One of the main motivations for (H2) and (1.5) is that there are many non-linearities that do not satisfy the Ambrosetti-Rabinowitz condition:

(H4) There exist constants $\mu > 2$ and $R > 0$ such that

$$0 < \mu F(x,s) < sf(x,s), \quad \text{for } |s| > R, \text{ and } x \in \Omega,$$

(see [2, 19]). Such problems were also studied by Miyagaki and Souto [16], Liu [13], and Li and Wang [11].

In this article, we use infinite-dimensional Morse theory to extend the results of Furtado and Silva for the case in which f is not assumed to be odd. We prove that, under the same hypotheses in Theorem 1.1, problem (1.1) has at least two nontrivial solutions; this is the content of Theorem 5.1 in Section 5. To prove Theorem 5.1, we compute the critical groups at the origin and at infinity of the energy functional associated with problem (1.1). To compute the critical groups at the origin, we show that the trivial solution of problem (1.1) is a local minimum of the associated energy functional. To compute the critical groups at infinity, we use an argument similar to that presented in [21, Section 3] by using a standard argument involving a long exact sequence of reduced homology groups. Using the same techniques, we prove the existence of three nontrivial solutions of problem (1.1) for the case in which the nonlinearity f is differentiable with continuous derivative, and the condition on its primitive, F, at the origin in (1.5) is stated in terms of the limit as s approaches 0, and not the limit superior.

This article is organized as follows: In Section 2, we present the variational framework that will be used throughout this work. In Section 3, we present the computation of the critical groups at the origin. In Section 4, we compute the critical groups at infinity. In Section 5, we use a standard argument involving the Morse relation to show the existence of a second nontrivial solution of problem (1.1) under the assumptions of Theorem 1.1. Finally, in Section 6, we prove the existence of three nontrivial solutions for problem (1.1) by assuming that the nonlinearity f is C^1 and a strengthening of condition (1.5).

2. Variational framework

Let X denote the Sobolev space $H_0^1(\Omega)$ obtained through completion of $C_c^\infty(\Omega)$ with respect to the metric induced by the norm

$$\|u\| = \Big(\int_\Omega |\nabla u|^2 dx\Big)^{1/2}, \quad \text{for all } u \in X.$$

Weak solutions of (1.1) are critical points of the functional $J\colon X \to \mathbb{R}$ given by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \int_\Omega F(x,u)\, dx, \quad \text{for } u \in X. \tag{2.1}$$

The functional J belongs to $C^1(X,\mathbb{R})$ and its Fréchet derivative at $u \in X$ is given by

$$\langle J'(u), v\rangle = \int_\Omega \nabla u \cdot \nabla v\, dx - \int_\Omega f(x,u)v\, dx, \quad \text{for any } v \in X. \tag{2.2}$$

We say that a functional $J \in C^1(X,\mathbb{R})$ satisfies the Cerami condition at level c (denoted $(C)_c$), if every sequence (u_j) in X such that

$$J(u_j) \to c, \quad (1 + \|u_j\|)J'(u_j) \to 0, \quad \text{as } j \to \infty,$$

called a $(C)_c$ sequence, has a convergent subsequence. We say that J satisfies (C) if it satisfies $(C)_c$ for every c. This condition was introduced by Cerami [4]. It is a weaker condition compared to the Palais-Smale condition, but the main deformation lemmas used in critical point theory are still valid assuming the Cerami condition (see [18, Chapter 1]). For a full exposition of the various compactness conditions used in critical point theory, we refer the reader to Mawhin and Willem [15] and references therein.

Based on conditions (H2) and (H3), the authors in [9] proved that the energy functional given in (2.1) associated with problem (1.1) satisfies a Cerami condition at any level $c \in \mathbb{R}$ (see [9, Theorem 1]). This condition is needed in the use of infinite-dimensional Morse theory, which is an important tool in the arguments presented in this paper.

Let A, B be two topological spaces with $B \subset A$. Denote by $H_q(A, B)$ the q-singular relative homology group of the pair (A, B) with coefficients in a field $\mathbb{F}$. Let $c = J(u_0)$, where u_0 is an isolated critical point of J, and set

$$J^c = \{u \in X : J(u) \leqslant c\}.$$

The q-critical groups of J at u_0, with coefficients in $\mathbb{F}$, are given by

$$C_q(J, u_0) = H_q(J^c \cap U, (J^c \cap U)\backslash\{u_0\}), \quad q \in \mathbb{Z}, \tag{2.3}$$

(see [5, Definition 4.1, p. 32]), where U is an open neighborhood of u_0 such that u_0 is the unique critical point of J in U. The critical groups of isolated critical points are well–defined and they do not depend on the choice of the neighborhood U. This follows from the excision property of homology theory.

Assume that J satisfies the Cerami condition and let $\mathcal{K} = \{u \in X : J'(u) = 0\}$ be the set of critical points of J and $-a < \inf_{u\in\mathcal{K}} J(\mathcal{K})$. The critical groups at infinity were first introduced by Barstch and Li [3] and are

$$C_q(J, \infty) = H_q(X, J^{-a}), \quad \text{for all } q \in \mathbb{Z}.$$

By the second deformation theorem (see [5, Theorem 3.2, Chapter I]), these critical groups are well-defined. We will also denote by $\widetilde{H}_q(A)$ the reduced homology groups of the topological space $A \subset X$ with coefficients in a field $\mathbb{F}$ defined by

$$\widetilde{H}_q(A) = H_q(A) \quad \text{for } q > 0,$$
$$H_0(A) = \widetilde{H}_0(A) \oplus \mathbb{F}.$$

The reduced homology groups for pair (A, B) are defined in a similar manner. For more details on these definitions, we refer the reader to Hatcher [10, Chapter 2, page 110].

3. Critical groups at the origin

Note that in view of (1.6) and the definition of the Fréchet derivative of J in (2.2), the origin of X is a critical point of J. In this section, we compute the critical groups at the origin of the functional J defined in (2.1). Before we prove the main results of this section, we need some estimates on the function F.

By condition (1.5), given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|s| < \delta \Rightarrow F(x, s) < \frac{\varepsilon}{2} s^2, \quad \text{for } x \in \Omega. \tag{3.1}$$

Next, condition (H1) implies that there exists a constant $A = A(\delta)$ such that

$$|F(x, s)| \leqslant A|s|^p, \quad \text{for all } |s| \geqslant \delta \text{ and a.e. } x \in \Omega. \tag{3.2}$$

In fact, assume $s \geqslant \delta$, and use (H1) to obtain the estimate

$$|F(x, s)| \leqslant \int_0^s |f(x, \xi)|\, d\xi \leqslant a_1 s + \frac{a_1}{p} s^p,$$

for $x \in \Omega$; so that,

$$|F(x,s)| \leqslant a_1\big[\delta\left(\frac{s}{\delta}\right) + \frac{\delta^p}{p}\left(\frac{s}{\delta}\right)^p\big], \quad \text{for } x \in \Omega. \tag{3.3}$$

Consequently, since we are assuming that $s \geqslant \delta$; so that $\frac{s}{\delta} \geqslant 1$, it follows from (3.3) that

$$|F(x,s)| \leqslant a_1\big[\delta\left(\frac{s}{\delta}\right)^p + \frac{\delta^p}{p}\left(\frac{s}{\delta}\right)^p\big], \quad \text{for } x \in \Omega \text{ and } s \geqslant \delta,$$

from which we obtain that

$$|F(x,s)| \leqslant \frac{a_1}{\delta^p}\left(\delta + \delta^p\right) s^p, \quad \text{for } x \in \Omega \text{ and } s \geqslant \delta, \tag{3.4}$$

where we have used the fact that $p > 1$, in view of that assumption $p \in (2, 2^*)$ in (H1). Setting $A = A(\delta) = \frac{a_1}{\delta^p}\left(\delta + \delta^p\right)$, we see that (3.2) follows from (3.4) for the case $s \geqslant \delta$. The case for $s \leqslant -\delta$ is analogous. Therefore, the estimate (3.2) is valid for all $|s| \geqslant \delta$.

Combining the estimates (3.1) and (3.2), we obtain that

$$F(x,s) \leqslant \frac{\varepsilon}{2}s^2 + A|s|^p, \quad \text{for } x \in \Omega \text{ and } s \in \mathbb{R}. \tag{3.5}$$

Next, use this estimate in (3.5) to obtain

$$\int_\Omega F(x,u)\,dx \leqslant \frac{\varepsilon}{2}\int_\Omega |u|^2\,dx + A\int_\Omega |u|^p;$$

so that, using the Poincaré and Sobolev inequalities,

$$\int_\Omega F(x,u)\,dx \leqslant C\left(\frac{\varepsilon}{2} + A\|u\|^{p-2}\right)\|u\|^2, \tag{3.6}$$

for some positive constant C.

Setting $\rho = (\frac{\varepsilon}{2A})^{1/(p-2)}$, we see from (3.6) that

$$\|u\| < \rho \implies \int_\Omega F(x,u)\,dx \leqslant C\varepsilon\|u\|^2. \tag{3.7}$$

Lemma 3.1. *Assume that f satisfies* (H1) *and* (1.5). *Then, the critical groups of J at the origin are*

$$C_q(J,0) = H_q(J^0 \cap B_\rho(0), (J^0 \cap B_\rho(0))\backslash\{0\}) \cong \delta_{q,0}\mathbb{F} \quad \textit{for } q \in \mathbb{Z}.$$

Proof. It follows from (3.7) and the definition of J in (2.1) that

$$J(u) \geqslant \left(\frac{1}{2} - C\varepsilon\right)\|u\|^2,$$

so that, since ε is arbitrary, we can choose $\varepsilon = 1/(4C)$ to obtain

$$J(u) \geqslant \frac{1}{4}\|u\|^2 > J(0), \quad \text{for } 0 < \|u\| < \rho, \tag{3.8}$$

where $\rho > 0$ is sufficiently small. Consequently, $u = 0$ is a local minimum of J in $B_\rho(0)$. Then, by (2.3) with $U = B_\rho(0)$, it follows from [5, Example 1, page 33] that

$$C_q(J,0) \cong \delta_{q,0}\mathbb{F}, \quad \text{for } q \in \mathbb{Z}. \tag{3.9}$$

□

4. Critical groups at infinity

In this section, we compute the critical groups at infinity of the functional J given in (2.1). We assume that the functions f and F satisfy the conditions in (H1) and (H2).

Let $\mathcal{K} = \{u \in X : J'(u) = 0\}$ be the critical set of J. We first show that the functional J is bounded from below in $\mathcal{K}$. It follows from (2.2) that

$$\|u_0\|^2 = \int_\Omega f(x, u_0)u_0 \, dx, \quad \text{for } u_0 \in \mathcal{K}. \tag{4.1}$$

Substituting (4.1) into the definition of J in (2.1) yields

$$J(u_0) = \frac{1}{2}\int_\Omega (f(x, u_0)u_0 - 2F(x, u_0)) \, dx, \quad \text{for } u_0 \in \mathcal{K}. \tag{4.2}$$

Now, by condition (H2), there exists $R_1 > 0$ such that

$$|s| > R_1 \implies f(x, s)s - 2F(x, s) > 1, \quad \text{for } x \in \overline{\Omega}. \tag{4.3}$$

Next, denote $f(x, s)s - 2F(x, s)$ by $H(x, s)$, for $(x, s) \in \Omega \times \mathbb{R}$, to rewrite (4.2) as follows

$$J(u_0) = \frac{1}{2}\int_{|u_0|\leqslant R_1} H(x, u_0) \, dx + \frac{1}{2}\int_{|u_0|>R_1} H(x, u_0) \, dx, \quad \text{for } u_0 \in \mathcal{K};$$

so that, in view of (4.3),

$$J(u_0) \geqslant \frac{1}{2}\int_{|u_0|\leqslant R_1} H(x, u_0) \, dx, \quad \text{for } u_0 \in \mathcal{K}. \tag{4.4}$$

Thus, letting

$$C_0 = \max_{x\in\overline{\Omega},\, |s|\leqslant R_1} |H(x, s)|, \tag{4.5}$$

from (4.4) we obtain

$$J(u_0) \geqslant -\frac{C_0}{2}|\Omega|, \quad \text{for } u_0 \in \mathcal{K}. \tag{4.6}$$

It follows from (4.6) that the set of critical values of J is bounded below. Thus, the critical groups of J at infinity are well-defined.

In what follows, let $a_0 > 0$ be such that $-a_0 < \inf_{u\in\mathcal{K}} J(u)$.

Lemma 4.1. *Let J be the $C^1(X, \mathbb{R})$ functional defined in* (2.1)*, and assume that* (H1) *and* (H2) *are satisfied. There exists a constant $M \geqslant a_0$ such that any compact subset of the sub–level set J^{-M} is contractible in J^{-M}.*

Proof. We show that we can deform the level set J^{-M} to the level set J^{-2M} for an $M \geqslant a_0$ that will be chosen shortly. The rest of the proof of Lemma 4.1 will follow the same steps shown in the proof of [12, Proposition 2.1].

Let $u \in J^{-M}$ and A denote a compact subset of J^{-M}. Using the definition of the functional J given in (2.1) we have that

$$J(tu) = \frac{t^2}{2}\|u\|^2 - \int_\Omega F(x, tu) \, dx, \quad \text{for } t \in \mathbb{R}. \tag{4.7}$$

Consequently,

$$\frac{d}{dt}[J(tu)] = t\|u\|^2 - \int_\Omega f(x, tu)u \, dx, \quad \text{for } t \in \mathbb{R},$$

which we can rewrite as

$$\frac{d}{dt}[J(tu)] = \frac{2}{t}\Big[\frac{t^2}{2}\|u\|^2 - \frac{1}{2}\int_\Omega f(x,tu)tu\,dx\Big], \quad \text{for } t \neq 0;$$

so that, by (4.7),

$$\frac{d}{dt}[J(tu)] = \frac{2}{t}\Big[J(tu) - \frac{1}{2}\int_\Omega (f(x,tu)tu\,dx - 2F(x,u))\Big], \quad \text{for } t \neq 0. \tag{4.8}$$

Next, define

$$\Omega_1^t = \{x \in \Omega : |tu(x)| \leqslant R_1\} \quad \text{and} \quad \Omega_2^t = \Omega\backslash\Omega_1^t, \tag{4.9}$$

where $R_1 > 0$ is the constant from (4.3). Then, denoting $f(x,s)s - 2F(x,s)$ by $H(x,s)$, for $(x,s) \in \Omega \times \mathbb{R}$, we can write (4.8) as

$$\frac{d}{dt}[J(tu)] = \frac{2}{t}\Big[J(tu) - \frac{1}{2}\int_{\Omega_1^t} H(x,tu)\,dx - \frac{1}{2}\int_{\Omega_2^t} H(x,tu)\,dx\Big], \tag{4.10}$$

for $t \neq 0$.

Now, in view of (4.3) and the definition of Ω_2^t in (4.9) we have that

$$\int_{\Omega_2^t} H(x,tu)\,dx \geqslant 0, \quad \text{for all } t. \tag{4.11}$$

Combining (4.10) and (4.11) yields

$$\frac{d}{dt}[J(tu)] \leqslant \frac{2}{t}\Big[J(tu) - \frac{1}{2}\int_{\Omega_1^t} H(x,tu)\,dx\Big], \quad \text{for } t \neq 0. \tag{4.12}$$

On the other hand,

$$\Big|\int_{\Omega_1^t} H(x,tu)\,dx\Big| \leq C_0|\Omega|, \quad \text{for } x \in \overline{\Omega}, \tag{4.13}$$

where C_0 is the constant given in (4.5). It then follows from (4.12) and (4.13) that

$$\frac{d}{dt}[J(tu)] \leqslant \frac{2}{t}\Big[J(tu) + \frac{C_0}{2}|\Omega|\Big], \quad \text{for } t \neq 0, \tag{4.14}$$

which we can rewrite as

$$\frac{d}{dt}[J(tu)] - \frac{2}{t}J(tu) \leqslant \frac{C_0|\Omega|}{t}, \quad \text{for } t \neq 0. \tag{4.15}$$

Multiplying (4.15) by the integrating factor $1/t^2$, and integrating from 1 to $t > 1$, we obtain

$$\int_1^t \frac{d}{d\xi}\big[\frac{1}{\xi^2}J(\xi u)\big]\,d\xi \leqslant C_0|\Omega|\int_1^t \frac{1}{\xi^3}\,d\xi,$$

from which we obtain

$$J(tu) \leqslant t^2 J(u) - \frac{C_0|\Omega|}{2}(t^2-1), \quad \text{for } t \geqslant 1. \tag{4.16}$$

Since we are assuming that $u \in J^{-M}$,from (4.16) we obtain

$$J(tu) \leqslant -t^2 M, \quad \text{for all } t \geqslant 1. \tag{4.17}$$

It then follows from (4.17) that

$$J(tu) \leqslant -M, \quad \text{for all } t \geqslant 1 \text{ and all } u \in J^{-M}. \tag{4.18}$$

It also follows from (4.17) that

$$J(tu) \to -\infty \quad \text{as } t \to \infty, \text{ for all } u \in J^{-M}. \tag{4.19}$$

Next, observe that, in view of (4.14) and (4.18),

$$\frac{d}{dt}[J(tu)] \leqslant \frac{2}{t}\Big[-M + \frac{C_0}{2}|\Omega|\Big], \quad \text{for all } t \geqslant 1 \text{ and } u \in J^{-M},$$

which we can rewrite as

$$\frac{d}{dt}[J(tu)] \leqslant -\frac{2}{t}\Big[M - \frac{C_0}{2}|\Omega|\Big], \quad \text{for all } t \geqslant 1 \text{ and } u \in J^{-M}. \tag{4.20}$$

Setting $a_1 = C_0|\Omega|/2$, we see that, if $M > \max\{a_0, a_1\}$, then from (4.20) it follows that

$$\frac{d}{dt}[J(tu)] < 0, \quad \text{for all } t \geqslant 1 \text{ and } u \in J^{-M}. \tag{4.21}$$

This determines our choice of M in the statement of Lemma 4.1.

Now, it follows from (4.19), the intermediate value theorem, and the estimate in (4.21) that there exists $t^* \geqslant 1$ such that $J(t^*u) \leqslant -2M$. As a consequence of (4.21) and the implicit function theorem, we also get that t^* is a continuous function of u, for $u \in J^{-M}$. Thus, for any compact subset, A, of J^{-M}, we can define a continuous map $\eta_1 : [0,1] \times A \to X$ by

$$\eta_1(t,u) = [(1-t) + tt^*(u)]\, u, \quad \text{for } (t,u) \in [0,1] \times A. \tag{4.22}$$

Hence, in view of (4.18), η_1 defines a continuous map from $[0,1] \times A$ to J^{-M}.

Set $A_1 = \eta_1(1, A)$. Then, A_1 is also a compact set and $A_1 \subset J^{-2M}$. Thus, any compact subset, A, of J^{-M} can be deformed in J^{-M} to a compact subset of J^{-2M}.

The rest of the proof follows the same steps outlined in the proof of [12, Proposition 2.1], or the proof of [20, Proposition 7.1]. □

As a consequence of Lemma 4.1, we conclude that, for $M > \max\{a_0, a_1\}$,

$$\widetilde{H}_q(J^{-M}) \cong 0, \quad \text{for } q \in \mathbb{Z}. \tag{4.23}$$

The computation of the critical groups of J at infinity follows by using a standard argument with the following long exact sequence of reduced homology groups

$$\dots \to \widetilde{H}_q(J^{-M}) \xrightarrow{i_*} \widetilde{H}_q(X) \xrightarrow{j_*} \widetilde{H}_q(X, J^{-M}) \xrightarrow{\partial_*} \widetilde{H}_{q-1}(J^{-M}) \xrightarrow{i_*} \dots \tag{4.24}$$

where i_* and j_* are the induced homomorphisms of the inclusion maps

$$i\colon J^{-M} \to X, \quad j\colon (X, \emptyset) \to (X, J^{-M}),$$

respectively, and $\partial_* : \widetilde{H}_q(X, J^{-M}) \to \widetilde{H}_{q-1}(J^{-M})$ is a homomorphism.

Using the fact that X is contractible and the assertion in (4.23), we deduce from the long exact sequence in (4.24) and the definition of reduced homology groups that

$$C_q(J, \infty) = H_q(X, J^{-M}) \cong \delta_{q,0}\mathbb{F}, \quad \text{for } q \in \mathbb{Z}. \tag{4.25}$$

For more details on this calculation, we refer the reader to [21, Section 3].

5. Existence of a second nontrivial solution

In this section, we prove the existence of a second nontrivial solution of problem (1.1) under the assumptions of Theorem 1.1. To do that, we will use an argument by contradiction involving the Morse relation.

First, let u_1 denote the nontrivial solution of problem (1.1) found in [9, Theorem 1.2] by means of the mountain-pass theorem. Assume, by way of contradiction, that 0 and u_1 are the only critical points of J. Then, the critical groups $C_q(J, u_1)$ are given by

$$C_q(J, u_1) \cong \delta_{q,1}\mathbb{F}, \quad \text{for } q \in \mathbb{Z}, \tag{5.1}$$

(see [17, Proposition 6.101]).

Before presenting the final argument, we briefly discuss the Morse relation. Let $J \in C^1(X, \mathbb{R})$ be a functional that satisfies the Cerami condition. If J has a finite number of critical points, we define the Morse–type numbers of the pair (X, J^{-M}) by

$$M_q := M_q(X, J^{-M}) = \sum_{u\in\mathcal{K}} \dim C_q(J, u), \quad q = 0, 1, 2, \dots, \tag{5.2}$$

where $-M < \inf_{u\in\mathcal{K}} J(u)$. Applying the infinite-dimensional Morse-theory developed in [5], [14], or [17], we can derive the Morse relation

$$\sum_{q=0}^{\infty} M_q t^q = \sum_{q=0}^{\infty} \beta_q t^q + (1 + t) \sum_{q=0}^{\infty} a_q t^q, \tag{5.3}$$

where $\beta_q = \dim C_q(J, \infty)$ and a_q are non-negative numbers. The integers β_q, for $q \in \mathbb{Z}$, are called the Betti numbers of the pair (X, J^{-M}).

Let M be the constant from Lemma 4.1. We first note that J satisfies the Cerami condition as a consequence of [9, Theorem 1, 1]. Hence, by (3.9) and (5.1), we obtain the Morse type numbers of the pair (X, J^{-M}) as

$$M_0 = \dim C_0(J, 0) = 1, \quad M_1 = \dim C_1(J, u_1) = 1, \quad M_q = 0, \quad \text{for } q > 1. \tag{5.4}$$

On the other hand, it follows from (4.25) that the Betti numbers of the pair (X, J^{-M}) are given by

$$\beta_0 = 1 \quad \text{and} \quad \beta_q = 0, \quad \text{for } q > 0. \tag{5.5}$$

Therefore, by (5.4) and (5.5), it follows from the Morse relation (5.3) with $t = -1$ that

$$M_0(-1)^0 + M_1(-1)^1 = \beta_0(-1)^0,$$

that is, $0 = 1$, which is a contradiction. Thus, J must have another critical point.

Hence, assuming the same hypotheses in Theorem 1.1, we have proved the following result.

Theorem 5.1. *Suppose f satisfies* (H1), (H2), (H4)*, Then problem* (1.1) *has at least two nontrivial solutions, provided that*

$$\limsup_{s\to 0} \frac{F(x, s)}{s^2} = 0, \quad \textit{uniformly for } x \in \Omega. \tag{5.6}$$

6. Existence of three nontrivial solutions

In this section we show that, under an additional regularity assumption on the nonlinearity f, we can obtain three nontrivial solutions of problem (1.1). This result is motivated by the final remark in [9]. Furtado and Silva obtained two nontrivial solutions of mountain pass type using a cutoff technique. We will use the arguments of the previous section to prove that problem (1.1) has a third nontrivial solution, provided that f is assumed to be C^1 and that the assumption (5.6) is replaced by

$$\lim_{s\to 0} \frac{F(x, s)}{s^2} = 0, \quad \text{uniformly for } x \in \Omega. \tag{6.1}$$

This is the content of the following theorem

Theorem 6.1. *Suppose f satisfies* (H1)–(H3)*. Assume also that $f \in C^1(\Omega\times\mathbb{R},\mathbb{R})$ and that* (6.1) *holds. Then, problem* (1.1) *has at least three nontrivial solutions.*

Remark 6.2. We remark that the assumption that $f \in C^1(\Omega\times\mathbb{R},\mathbb{R})$ and condition (6.1) imply that

$$\frac{\partial f}{\partial s}(x,0) = 0, \qquad \text{uniformly for } x \in \Omega, \tag{6.2}$$

as a consequence of L'Hospital's rule.

Proof of Theorem 6.1. We start the proof by showing the existence of two nontrivial solutions of the mountain-pass type as described in [9]. We present the details here for the reader's convenience. We then proceed with the argument using the Morse relation to obtain a third nontrivial solution of (1.1).

First, we obtain a positive solution u_1 of problem (1.1). A negative solution, u_2, can be obtained in an analogous way. The assumption $f \in C^1$ will imply that weak solutions of (1.1) are also classical solutions (see Agmon [1]). This will allow us to use the maximum principle and obtain a positive solution and a negative solution. We will use the arguments presented in [19, Corollary 2.23].

Consider the truncated version of the function f,

$$\overline{f}(x,s) = \begin{cases} f(x,s), & \text{for } s \geqslant 0; \\ 0, & \text{for } s < 0, \end{cases} \tag{6.3}$$

and its primitive

$$\overline{F}(x,s) = \int_0^s \overline{f}(x,\xi)\, d\xi, \quad \text{for all } (x,s) \in \Omega\times\mathbb{R}. \tag{6.4}$$

Define the associated functional $J^+ : X \to \mathbb{R}$ by

$$J^+(u) = \frac{1}{2}\|u\|^2 - \int_\Omega \overline{F}(x,u)\, dx, \quad \text{for } u \in X. \tag{6.5}$$

We note that J^+ is Fréchet differentiable with derivative given by

$$\langle {J^+}'(u), v\rangle = \int_\Omega \nabla u\cdot\nabla v\, dx - \int_\Omega \overline{f}(x,u)v\, dx, \quad \text{for all } u, v \in X,$$

which, in view of the definition of $\overline{f}$ in (6.3) is equivalent to

$$\langle {J^+}'(u), v\rangle = \int_\Omega \nabla u\cdot\nabla v\, dx - \int_\Omega \chi_{\{u\geqslant 0\}} f(x,u)v\, dx, \quad \text{for all } u, v \in X, \tag{6.6}$$

where χ_A denotes the indicator function of $A \subseteq \Omega$, and $\{u \geqslant 0\}$ denotes the set $\{x \in \Omega\colon u(x) \geqslant 0\}$.

We will verify that the the functional J^+ given in (6.5) satisfies the conditions of the mountain-pass theorem:

(1) $J^+(0) = 0$;
(2) there exist constants $\alpha > 0$ and $\rho > 0$ such that
$$J^+(v) \geqslant \alpha, \quad \text{for all } v \in X \text{ with } \|v\| = \rho;$$
(3) there exists $v_1 \in X$ such that $\|v_1\| > \rho$ and $J^+(v_1) \leqslant 0$;
(4) J^+ satisfies the Cerami condition.

First, observe that (1) follows from the definition of J^+ in (6.5) and the definition of $\overline{F}$ in (6.4).

Next, note that by conditions (H2) and (H3) in Theorem 5.1, we can show that

$$\lim_{s\to+\infty}(s\overline{f}(x,s)-2\overline{F}(x,s))=+\infty, \quad \text{uniformly for } x\in\Omega, \tag{6.7}$$

$$\lim_{s\to+\infty}\frac{2\overline{F}(x,s)}{s^2}=+\infty \quad \text{uniformly for } x\in\Omega. \tag{6.8}$$

Consequently, $\overline{f}$ and $\overline{F}$ satisfy the non-quadraticity condition in (6.7) and the superlinearity condition in (6.8) at infinity, respectively. Therefore, the Cerami condition can be verified for J^+ using the arguments in the proof of [9, Therorem 1.1]. Hence, condition (4) is verified.

Next, observe that the assumption (5.6) in Theorem 5.1, together with the definition of $\overline{F}$ in (6.4), can be used to show that there exists $\rho>0$ such that

$$\|u\|<2\rho \implies J^+(u)\geqslant\frac{1}{4}\|u\|^2, \tag{6.9}$$

using the calculations leading to (3.8) in Section 3. Thus, setting $\alpha=\rho^2/4$, we obtain from (6.9) that that $J^+(u)\geqslant\alpha$ for $u\in\partial B_\rho(0)$, which shows that (2) is verified.

To verify condition (3) of the mountain-pass theorem, let φ_1 be an eigenfunction of the Laplacian over Ω, with Dirichlet boundary conditions, associated with the first eigenvalue, λ_1, of the Laplacian, and satisfying $\varphi_1>0$ and $\|\varphi_1\|=1$. Then, using the definition of J^+ in (6.5),

$$J^+(t\varphi_1)=\frac{t^2}{2}-\int_\Omega F(x,t\varphi_1)\,dx, \quad \text{for } t>0. \tag{6.10}$$

Now, by conditions (6.8), given any $M>0$ (to be chosen shortly), there exists $R_1>0$ such that

$$s>R_1 \implies \frac{2F(x,s)}{s^2}>M, \quad \text{for all } x\in\Omega. \tag{6.11}$$

With R_1 dictated by our choice of M (to be given shortly), define the sets

$$\Omega_1^t=\{x\in\Omega : t\varphi(x)>R_1\} \quad \text{and} \quad \Omega_2^t=\Omega\backslash\Omega_1^t. \tag{6.12}$$

We can then rewrite (6.10) as

$$J^+(t\varphi_1)=\frac{t^2}{2}-\int_{\Omega_1^t}F(x,t\varphi_1)\,dx-\int_{\Omega_2^t}F(x,t\varphi_1)\,dx, \quad \text{for } t>0,$$

or

$$J^+(t\varphi_1)=\frac{t^2}{2}\Big(1-\int_{\Omega_1^t}\frac{2F(x,t\varphi_1)}{t^2\varphi^2}\varphi_1^2\,dx\Big)-\int_{\Omega_2^t}F(x,t\varphi_1)\,dx, \tag{6.13}$$

for $t>0$.

By the definition of Ω_2^t in (6.12),

$$\Omega_2^t=\big\{x\in\Omega\colon \varphi_1(x)\leqslant\frac{R_1}{t}\big\} \quad \text{for } t>0.$$

Consequently,

$$\lim_{t\to\infty}|\Omega_2^t|=0, \tag{6.14}$$

where $|A|$ denotes the Lebesgue measure of a measurable subset, A, of $\mathbb{R}^N$.

Now, it follows from condition (H1) that

$$|F(x,s)| \leqslant C\left(|s| + |s|^p\right), \quad \text{for } (x,s) \in \Omega \times \mathbb{R}, \tag{6.15}$$

for some positive constant C. Thus, by the Sobolev embedding theorem and the assumption that $2 < p < 2^*$, it follows from (6.15) that $F(\cdot, u(\cdot)) \in L^1(\Omega)$ for all $u \in X$. Thus, in view of (6.14),

$$\lim_{t\to\infty} \int_{\Omega_2^t} F(x, t\varphi_1)\, dx = 0. \tag{6.16}$$

On the other hand, using (6.11) and the definition of Ω_1^t in (6.12),

$$\int_{\Omega_1^t} \frac{2F(x,t\varphi_1)}{t^2\varphi^2} \varphi_1^2\, dx > M \int_{\Omega_1^t} \varphi_1^2\; dx, \quad \text{for } t > 0, \tag{6.17}$$

where

$$\Omega_1^t = \Big\{x \in \Omega\colon \varphi_1(x) > \frac{R_1}{t}\Big\} \quad \text{for } t > 0.$$

Consequently,

$$\lim_{t\to\infty} \int_{\Omega_1^t} \varphi_1^2\; dx = \int_{\Omega} \varphi_1^2\; dx = \frac{1}{\lambda_1}, \tag{6.18}$$

since we are assuming that $\|\varphi_1\| = 1$. It follows from (6.18) that there exists $R_2 > 0$ such that

$$\int_{\Omega_1^t} \varphi_1^2\; dx > \frac{2}{3\lambda_1}, \quad \text{for } t \geqslant R_2. \tag{6.19}$$

Combining (6.17) and (6.19) we get

$$\int_{\Omega_1^t} \frac{2F(x,t\varphi_1)}{t^2\varphi^2} \varphi_1^2\, dx > \frac{2M}{3\lambda_1}, \quad \text{for } t \geqslant R_2. \tag{6.20}$$

Thus, choosing

$$M = \frac{9\lambda_1}{2},$$

from (6.20) there exists $R_2 > 0$ such that

$$\int_{\Omega_1^t} \frac{2F(x,t\varphi_1)}{t^2\varphi^2} \varphi_1^2\, dx > 3, \quad \text{for } t \geqslant R_2. \tag{6.21}$$

Using estimate (6.21) in (6.13) yields

$$J^+(t\varphi_1) < -t^2 - \int_{\Omega_2^t} F(x, t\varphi_1)\, dx, \quad \text{for } t \geqslant R_2. \tag{6.22}$$

The estimate in (6.22), together with limit fact in (6.16), yields that

$$J^+(t\varphi_1) \to -\infty \quad \text{as } t \to \infty. \tag{6.23}$$

To complete the verification of (3), use (6.23) to find $R_3 > \rho$ such that $J^+(R_3\varphi_1) \leqslant 0$ and set $v_1 = R_3\varphi_1$.

Therefore, the conditions for the mountain-pass theorem have been verified for J^+. Hence, J^+ has a nontrivial critical point, u_1, which corresponds to a weak solution of the elliptic boundary-value problem

$$\begin{gathered} -\Delta u = \overline{f}(x,u), \quad \text{in } \Omega; \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{6.24}$$

Since we are assuming that f is a C^1 function, we can apply elliptic regularity theory (see Agmon [1]) to conclude that u_1 is also a classical solution of (6.24).

Next, we proceed to show that $u_1 > 0$ in Ω. First, we show that $u_1 \geqslant 0$ in Ω. To see this, let $\Omega^- = \{x \in \Omega : u_1(x) < 0\}$. Then, by the definition of $\overline{f}$ in (6.3), u_1 is a solution of the BVP

$$\begin{aligned} -\Delta v &= 0, \quad \text{in } \Omega^-; \\ v &= 0, \quad \text{on } \partial\Omega^-, \end{aligned} \tag{6.25}$$

which has only the trivial solution $v \equiv 0$ in Ω^-; this assertion can be proved, for instance, by applying the maximum principle. Consequently, $\Omega^- = \emptyset$, which proves that $u_1 \geqslant 0$ in Ω.

Thus, u_1 is a non-negative solution of the BVP (6.24). Hence, by the definition of $\overline{f}$ in (6.3), u_1 is also a solution of the BVP

$$\begin{aligned} -\Delta u &= f(x,u), \quad \text{in } \Omega; \\ u &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{6.26}$$

Define

$$g(x) = \begin{cases} \frac{f(x,u_1(x))}{u_1(x)}, & \text{if } u_1(x) > 0; \\ 0, & \text{if } u_1(x) = 0; \end{cases} \tag{6.27}$$

so that, in view of (6.2), $g\colon \Omega \to \mathbb{R}$ is a continuous function. Thus, since u_1 is a non-negative solution of the BVP in (6.26), u_1 is also a solution of the linear BVP

$$\begin{aligned} -\Delta v &= g(x)v, \quad \text{in } \Omega; \\ v &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{6.28}$$

where g is the function defined in (6.27).

Write $g(x) = g^+(x) - g^-(x)$, for $x \in \Omega$, where $g^+(x) = \max\{g(x), 0\}$ is the positive part of the function g defined in (6.27), and $g^-(x) = \max\{-g(x), 0\}$ is the negative part. Then, the BVP in (6.28) can be written as

$$\begin{aligned} -\Delta v + g^-(x)v &= g^+(x)v, \quad \text{in } \Omega; \\ v &= 0, \quad \text{on } \partial\Omega; \end{aligned} \tag{6.29}$$

so that, since u_1 is a non-negative solution of (6.29), u_1 satisfies

$$\begin{aligned} -\Delta v + g^-(x)v &\geqslant 0, \quad \text{in } \Omega; \\ v &= 0, \quad \text{on } \partial\Omega; \end{aligned} \tag{6.30}$$

Therefore, we can apply Hopf's maximum principle (see, for instance, [8, Theorem 4 on page 333]), to conclude that $u_1(x) > 0$, for all $x \in \Omega$, because u_1 is nontrivial. Since, we are assuming that Ω has smooth boundary, it is also the case that $\frac{\partial u_1}{\partial \nu} < 0$, on $\partial\Omega$, where ν denotes the outward unit normal vector to $\partial\Omega$ (see Hopf's Lemma on page 330 in [8]).

We have therefore shown that J^+ has a critical point, u_1, that is given by the mountain-pass theorem and is positive in Ω. We show presently that u_1 is also a critical point of J. Indeed, since $u_1 > 0$ in Ω, it follows from the definition of the Fréchet derivative of J^+ in (6.6) that

$$\begin{aligned} \langle J'(u_1), v\rangle &= \int_\Omega \nabla u_1 \cdot \nabla v\, dx - \int_\Omega f(x,u_1)v\, dx \\ &= \int_\Omega \nabla u_1 \cdot \nabla v\, dx - \int_\Omega \chi_{\{u_1 \geqslant 0\}} f(x,u_1)v\, dx \end{aligned}$$

$$= \langle J^{+\prime}(u_1), v\rangle = 0,$$

for any $v \in X$

The existence of another non-trivial critical point, u_2, of J satisfying $u_2 < 0$ in Ω can be proved by similar arguments to those presented above. This negative solution, u_2, is also obtained as an application of the mountain-pass theorem.

Using arguments similar to those found in [6, Theorem A], it can be shown that

$$C_q(J, u_1) \cong C_q(J^+, u_1) \cong \delta_{q,1}\mathbb{F}, \quad \text{for } q \in \mathbb{Z}. \tag{6.31}$$

A similar result can also be obtained for the negative solution u_2.

Next, we show the existence of a third nontrivial critical point of J. Assume that J has only three critical points: $0, u_1$, and u_2. We will show that this leads to a contradiction. Since u_1 and u_2 are of mountain-pass type, it follows from (6.31) that the critical groups of J at u_1 and u_2 are given by

$$C_q(J, u_1) \cong C_q(J, u_2) \cong \delta_{q,1}\mathbb{F}, \quad \text{for } q \in \mathbb{Z}. \tag{6.32}$$

Hence, by (3.9), (6.32), and (4.25), it follows from the Morse relation (5.3), with $t = -1$, that

$$M_0(-1)^0 + M_1(-1)^1 + M_2(-1)^1 = \beta_0(-1)^0, \tag{6.33}$$

where the Morse type numbers are given by $M_0 = \dim C_0(J, 0) = 1$, $M_1 = \dim C_1(J, u_1) = 1$, $M_2 = C_1(J, u_2) = 1$, and, by (5.5), the Betti number β_0 is $\beta_0 = 1$.

Then, it follows from (6.33) that $-1 = 1$ which is a contradiction. Hence, J must have a fourth critical point. This concludes the proof of the theorem. □

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