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GLOBAL WEAK SOLUTIONS TO DEGENERATE COUPLED TRANSPORT PROCESSES IN PARTIALLY SATURATED DEFORMABLE ELASTIC-INELASTIC POROUS MEDIA

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Abstract. In this work we prove the existence of global weak solutions to a degenerate and strongly coupled parabolic system arising from the transport processes through partially saturated deformable porous materials. The hygrothermal model is coupled with quasi-static evolution equations modeling elastic and inelastic mechanical deformations. Physically relevant Newton boundary conditions are considered for water pressure and temperature of the porous system. The traction boundary condition is imposed on the deformable solid skeleton of the porous material. Degeneration occurs in both elliptic and parabolic part of the balance equation for mass of water. The coupling between water pressure, temperature, stress tensor and internal variables occurs in transport coefficients, constitutive functions and the decomposition of the total strain tensor into elastic and plastic parts due to mechanical effect and strain tensor due to thermal expansion.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. Let $T \in (0,\infty)$ be fixed throughout the paper, $I := [0,T]$, $\Omega_T := \Omega \times I$ and $\partial\Omega_T := \partial\Omega \times I$. We consider the system

$$\partial_t \theta(p) + \nabla \cdot \mathbf{q}_p = 0 \quad \text{in } \Omega_T, \tag{1.1}$$

$$\partial_t \left[\theta(p)\vartheta + \rho\vartheta\right] + \nabla \cdot \mathbf{q}_\vartheta + \nabla \cdot [\vartheta \mathbf{q}_p] = \boldsymbol{\sigma}_d : \partial_t \boldsymbol{\varepsilon}^{p\ell} \quad \text{in } \Omega_T, \tag{1.2}$$

$$-\nabla \cdot [\boldsymbol{\sigma} - \mathbf{I}\chi(p)] = \mathbf{f} \quad \text{in } \Omega_T, \tag{1.3}$$

$$\partial_t \boldsymbol{\varepsilon}^{p\ell} = \mathbf{B}(\vartheta, \boldsymbol{\sigma}, \boldsymbol{\alpha}) \quad \text{in } \Omega_T, \tag{1.4}$$

$$\partial_t \boldsymbol{\alpha} = \mathbf{C}(\vartheta, \boldsymbol{\sigma}, \boldsymbol{\alpha}) \quad \text{in } \Omega_T \tag{1.5}$$

with the boundary conditions

$$\mathbf{q}_p \cdot \mathbf{n} = \gamma_p(\mathbf{x})(p - p_\infty) \quad \text{on } \partial\Omega_T, \tag{1.6}$$

$$\mathbf{q}_\vartheta \cdot \mathbf{n} = \gamma_\vartheta(\mathbf{x})(\vartheta - \vartheta_\infty) \quad \text{on } \partial\Omega_T, \tag{1.7}$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \check{\mathbf{t}} \quad \text{on } \partial\Omega_T \tag{1.8}$$

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and the initial conditions

$$p(\cdot,0)=p_0,\quad \vartheta(\cdot,0)=\vartheta_0,\quad \boldsymbol{\varepsilon}^{p\ell}(\cdot,0)=\boldsymbol{\varepsilon}_0^{p\ell},\quad \boldsymbol{\alpha}(\cdot,0)=\boldsymbol{\alpha}_0 \quad \text{in } \Omega. \tag{1.9}$$

System (1.1)–(1.9) arises from the coupled moisture movement and heat transport through partially saturated deformable porous media [6, 28, 31]. The model includes classical plasticity, elasto-plasticity, inelastic behaviour, hygro-thermal effects, creep as well as relaxation. In (1.1)–(1.9), $p:\Omega_T\to\mathbb{R}$, $\vartheta:\Omega_T\to\mathbb{R}$, $\boldsymbol{\sigma}:\Omega_T\to\mathbb{R}^4$, $\boldsymbol{\varepsilon}^{p\ell}:\Omega_T\to\mathbb{R}^4$ and $\boldsymbol{\alpha}:\Omega_T\to\mathbb{R}^d$, $d\in\mathbb{N}$, are the unknown functions. In particular, p corresponds to the water pressure, ϑ represents the temperature, $\boldsymbol{\sigma}$ denotes the symmetric stress tensor, $\boldsymbol{\varepsilon}^{p\ell}$ is the tensor of plastic deformation and $\boldsymbol{\alpha}$ stands for the vector of internal state variables (taking into account e.g. the work-hardening of the porous material). By $\mathbf{q}_p$ we denote the flow velocity of liquid water and $\mathbf{q}_\vartheta$ is the heat flux through the porous material. $p_0:\Omega\to\mathbb{R}$, $\vartheta_0:\Omega\to\mathbb{R}$, $\boldsymbol{\varepsilon}_0^{p\ell}:\Omega\to\mathbb{R}^4$ and $\boldsymbol{\alpha}_0:\Omega\to\mathbb{R}^d$ are given functions describing the initial state of the system. By $\mathbf{n}$ we denote the unit outward normal vector with respect to Ω along $\partial\Omega$. In (1.6) and (1.7), $\gamma_\vartheta:\partial\Omega\to\mathbb{R}$ represents the heat transfer coefficient function, $\gamma_p:\partial\Omega\to\mathbb{R}$ is a coefficient function associated with a measure of the permeability of the boundary to the moisture flow, ϑ_∞ is the temperature of the external environment and p_∞ is a fictitious water pressure related to the ambient conditions (the relative humidity, gas pressure and temperature). In (1.3), $\mathbf{f}:\Omega_T\to\mathbb{R}^2$ stands for given body forces and, in (1.8), $\check{\mathbf{t}}:\partial\Omega_T\to\mathbb{R}^2$ represents surface tractions. Further, $\theta:\mathbb{R}\to\mathbb{R}$, $\chi:\mathbb{R}\to\mathbb{R}$, $\mathbf{B}:\mathbb{R}\times\mathbb{R}^4\times\mathbb{R}^d\to\mathbb{R}^4$ and $\mathbf{C}:\mathbb{R}\times\mathbb{R}^4\times\mathbb{R}^d\to\mathbb{R}^d$ are given functions of primary unknowns. Namely, θ represents the water content, which is the volume of water per volume of porous medium [31, Chapter 3.7.6], $\mathbf{B}$ and $\mathbf{C}$ are given constitutive functions describing the elasto-plastic behavior of the solid material, see e.g. [22, 23]. ρ is a real positive constant associated with the density of the solid skeleton. For notational simplicity, we normalized the density of water, the heat capacity of water and the heat capacity of solid microstructure to 1. On the right-hand side of (1.2), $\boldsymbol{\sigma}_d$ represents the deviatoric part of the stress tensor $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}_d=\boldsymbol{\sigma}-\frac{1}{3}tr(\boldsymbol{\sigma})\mathbf{I}$, where $\mathbf{I}$ denotes the identity matrix. By $\boldsymbol{\varepsilon}$ we denote the symmetric small strain tensor composed of the plastic part $\boldsymbol{\varepsilon}^{p\ell}$, the elastic part $\boldsymbol{\varepsilon}^{e\ell}$ and the thermal dilatation strain $\boldsymbol{\varepsilon}^{\vartheta}$. The deformation of the domain Ω is described by small displacement theory and the strain tensor $\boldsymbol{\varepsilon}$ is defined by

$$\varepsilon_{ij}(\mathbf{u})=\frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\Big),\quad i,j=1,2, \tag{1.10}$$

where the vector $\mathbf{u}:\Omega_T\to\mathbb{R}^2$, $\mathbf{u}=(u_1,u_2)$, describes the displacement of the material.

In the model, we suppose the following constitutive equations

$$\mathbf{q}_p=-\left(\kappa(\boldsymbol{\sigma})k_R(p)/\nu(\vartheta)\right)(\nabla p-\mathbf{e}_g), \tag{1.11}$$

$$\mathbf{q}_\vartheta=-\lambda(p,\vartheta)\nabla\vartheta, \tag{1.12}$$

$$\varepsilon_{ij}=\varepsilon_{ij}^{e\ell}+\varepsilon_{ij}^{p\ell}+\varepsilon_{ij}^{\vartheta}=\frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\Big), \tag{1.13}$$

$$\varepsilon_{ij}^{e\ell}=\frac{\partial P}{\partial\sigma_{ij}}(\boldsymbol{\sigma},\boldsymbol{\alpha}), \tag{1.14}$$

$$\varepsilon_{ij}^{\vartheta}=\beta\delta_{ij}(\vartheta-\vartheta_{ref}). \tag{1.15}$$

Here, $\kappa : \mathbb{R}^4 \to \mathbb{R}$ is the intrinsic permeability, $k_R : \mathbb{R} \to \mathbb{R}$ represents the relative hydraulic conductivity, $\lambda : \mathbb{R}^2 \to \mathbb{R}$ is the thermal conductivity function, $\nu : \mathbb{R} \to \mathbb{R}$ is the temperature dependent kinematic viscosity of the fluid and $\mathbf{e}_g := (0, 1)$ stands for the normalized gravity vector. Note that the relation (1.14) is a general form of Hooke's law with a given function P. In (1.15), δ_{ij} is the Kronecker delta, β is the thermal expansion coefficient and ϑ_{ref} stands for the given reference temperature.

Under isothermal conditions, the quasi-static problem (1.3)–(1.5) and (1.14) with boundary and initial conditions (1.8) and $(1.9)_{3,4}$ has been first introduced and theoretically studied in [22], [23] and [27]. From the mathematical point of view, the existence and uniqueness theorems have been proven in [29] and [28, Chapter 9]. At the same time, the existence and uniqueness results for the displacement boundary-value problem were given in [19]. Existence and uniqueness results and the continuous dependence of the solution with respect to the input data for the problem with mixed boundary conditions have been proven in [32]. Uncoupled thermo-visco-plastic processes, where the absolute temperature has been introduced in the model as the internal variable, have been theoretically studied in [33, 34]. More recently, existence results for the coupled thermo-mechanical models can be found e.g. in [4, 5, 17, 18].

Further, assuming hygro-thermal processes and ignoring mechanical phenomena, the existence of the weak solution to the problem (1.1)–(1.2) with homogeneous Dirichlet boundary conditions or mixed homogeneous Dirichlet-Neumann boundary conditions is given in [8] and [10], respectively.

Our aim is to prove the existence of the solution to the fully coupled hygro-thermo-mechanical model (1.1)–(1.9). From theoretical point of view, the difficulty lies in the coupling of equations and nontrivial structure of the system with non-symmetrical parabolic part (see also [35]). Moreover, one easily verifies that (1.1) is a degenerate equation where the degeneracy occurs in both elliptic and parabolic terms ($\theta'(p) \to 0$ and $k_R(p) \to 0$ as $p \to -\infty$).

The rest of this paper is organized as follows. In Section 2, we introduce basic notation and suitable function spaces, specify our assumptions on the data in the problem and present auxiliary results which will be used throughout the paper. In Section 3, we formulate the problem in the variational sense and state the main result of the paper, the existence of the global weak solution to (1.1)–(1.9). The main result is proved in Section 4 by constructing approximates and limiting procedure.

2. Preliminaries

2.1. **Notation and function spaces.** Vectors, vector functions and matrices are denoted by boldface letters. Throughout the paper, we will always use positive constants c, c_1, c_2, ..., which are not specified and which may differ from line to line. We suppose $q, q' \in [1, \infty)$, q' denotes the conjugate exponent to q, $q > 1$, $1/q + 1/q' = 1$. $L^q(\Omega)$ denotes the usual Lebesgue space equipped with the norm $\|\cdot\|_{L^q(\Omega)}$ and $W^{k,q}(\Omega)$, $k \geq 0$ (k need not to be an integer, see [24]), denotes the usual Sobolev-Slobodecki space with the norm $\|\cdot\|_{W^{k,q}(\Omega)}$. By X' we denote the space of all continuous linear forms on Banach space X.

By $L^q(I; X)$ we denote the usual Bochner space (see e.g. [1]). Further, we define $C(I; X)$, the space of functions $u : I \to X$ continuous in I, equipped with the norm $\|u\|_{C(I;X)} = \max_{t\in I} \|u(t)\|_X$, where $\|\cdot\|_X$ denotes the norm in the space X.

Finally, let $\mathcal{S}$ be the Hilbert space of all symmetric tensor functions such that

$$\mathcal{S} = \{\boldsymbol{\tau} = (\tau_{ij}); \tau_{ij} \in L^2(\Omega),\ \tau_{ij} = \tau_{ji},\ i,j = 1,2\}$$

with the inner product

$$(\boldsymbol{\tau}, \boldsymbol{\sigma})_{\mathcal{S}} = \int_\Omega \tau_{ij}\sigma_{ij}\,\mathrm{d}x. \tag{2.1}$$

Here and subsequently, summation convention is used, i.e. summation is performed over repeated indices.

2.2. **Structure and data properties.** We now introduce our assumptions on functions and coefficients in (1.1)–(1.9).

(i) $\theta \in C^1(\mathbb{R})$ is a positive and strictly monotone function such that

$$0 < \theta(\xi) \le C_\theta < +\infty, \quad 0 < \theta'(\xi) \le C_L < +\infty \quad \forall \xi \in \mathbb{R}\ (C_\theta, C_L = \text{const}); \tag{2.2}$$

(ii) $\chi \in C^1(\mathbb{R})$ is Lipschitz continuous and κ, k_R, ν, λ are continuous functions satisfying

$$0 < \kappa_1 \le \kappa(\boldsymbol{\xi}) \le \kappa_2 < +\infty \quad (\kappa_1, \kappa_2 = \text{const}) \quad \forall \boldsymbol{\xi} \in \mathbb{R}^4, \tag{2.3}$$

$$0 < k_R(\xi) < k_2 < +\infty \quad (k_2 = \text{const}) \quad \forall \xi \in \mathbb{R}, \tag{2.4}$$

$$0 < \nu_1 \le \nu(\xi) \le \nu_2 < +\infty \quad (\nu_1, \nu_2 = \text{const}) \quad \forall \xi \in \mathbb{R}, \tag{2.5}$$

$$0 < \lambda_1 \le \lambda(\xi_1, \xi_2) \le \lambda_2 < +\infty \quad (\lambda_1, \lambda_2 = \text{const}) \quad \forall \xi_1, \xi_2 \in \mathbb{R}; \tag{2.6}$$

(iii) the constitutive functions P, B_{ij} and C_i are assumed to be *smooth enough*, such that $B_{ij} = B_{ji}$ and

$$\frac{\partial P}{\partial \tau_{ij}}(\mathbf{0}, \boldsymbol{\xi}) = 0, \tag{2.7}$$

$$\Big|\frac{\partial^2 P}{\partial \tau_{ij}\partial \xi_k}(\boldsymbol{\tau}, \boldsymbol{\xi})\Big| + \Big|\frac{\partial^2 P}{\partial \tau_{ij}\partial \tau_{kl}}(\boldsymbol{\tau}, \boldsymbol{\xi})\Big| + \Big|\frac{\partial^2 P}{\partial \xi_j\partial \xi_k}(\boldsymbol{\tau}, \boldsymbol{\xi})\Big| \le C_P \quad (C_P = \text{const}), \tag{2.8}$$

$$\frac{\partial^2 P}{\partial \tau_{ij}\partial \tau_{kl}}(\boldsymbol{\tau}, \boldsymbol{\xi})\zeta_{ij}\zeta_{kl} \ge c_1\zeta_{ij}\zeta_{ij}, \quad c_1 > 0, \tag{2.9}$$

$$\Big|\frac{\partial B_{ij}}{\partial \tau_{kl}}(\boldsymbol{\tau}, \boldsymbol{\xi})\Big| + \Big|\frac{\partial B_{ij}}{\partial \xi_k}(\boldsymbol{\tau}, \boldsymbol{\xi})\Big| + \Big|B_{ij}(\boldsymbol{\tau}, \boldsymbol{\xi})\Big| \le C_B \quad (C_B = \text{const}), \tag{2.10}$$

$$\Big|\frac{\partial C_i}{\partial \tau_{kl}}(\boldsymbol{\tau}, \boldsymbol{\xi})\Big| + \Big|\frac{\partial C_i}{\partial \xi_j}(\boldsymbol{\tau}, \boldsymbol{\xi})\Big| \le C_c \quad (C_c = \text{const}) \tag{2.11}$$

for all $\boldsymbol{\tau} \in \mathbb{R}^4$, $\boldsymbol{\xi} \in \mathbb{R}^d$ and $\boldsymbol{\zeta} \in \mathbb{R}^4$;

(iv) the given body forces $\mathbf{f}$ in $C(I; [L^2(\Omega)]^2)$ and surface tractions $\breve{\mathbf{t}}$ in $C(I; [L^2(\partial\Omega)]^2)$ satisfy the compatibility condition

$$\int_\Omega \mathbf{f}(t) \cdot \mathbf{v}\,\mathrm{d}x + \int_{\partial\Omega} \breve{\mathbf{t}}(t) \cdot \mathbf{v}\,\mathrm{d}S = 0 \tag{2.12}$$

for every t from I and $\mathbf{v} \in R$, where

$$R = \big\{\mathbf{v} \in [W^{1,2}(\Omega)]^2;\ \mathbf{v} = (a_1 - bx_2, a_2 + bx_1),\ a_1, a_2, b \in \mathbb{R}\big\}; \tag{2.13}$$

(v) the functions from (1.6), (1.7) and (1.9) have the following properties:

$$\gamma_p \in L^\infty(\partial\Omega),\ \gamma_\vartheta \in L^\infty(\partial\Omega), \tag{2.14}$$

such that

$$0 < \gamma_1 < \gamma_p(\cdot) < \gamma_2 \quad \text{on } \partial\Omega_T \quad (\gamma_1, \gamma_2 = \text{const}), \tag{2.15}$$

$$0 < \gamma_3 < \gamma_\vartheta(\cdot) < \gamma_4 \quad \text{on } \partial\Omega_T \quad (\gamma_3, \gamma_4 = \text{const}), \tag{2.16}$$

further

$$p_0 \in L^\infty(\Omega) \cap L^\infty(\partial\Omega), \quad \vartheta_0 \in L^2(\Omega), \quad \boldsymbol{\varepsilon}_0^{p\ell} \in L^2(\Omega)^4, \quad \boldsymbol{\alpha}_0 \in [L^2(\Omega)]^d, \tag{2.17}$$

$$p_\infty \in C(I; L^\infty(\partial\Omega)), \quad \vartheta_\infty \in C(I; L^\infty(\partial\Omega)), \tag{2.18}$$

such that

$$p_1 < p_\infty(\cdot) < p_2 \quad \text{on } \partial\Omega_T \quad (p_1, p_2 = \text{const}), \tag{2.19}$$

$$\vartheta_1 < \vartheta_\infty(\cdot) < \vartheta_2 \quad \text{on } \partial\Omega_T \quad (\vartheta_1, \vartheta_2 = \text{const}). \tag{2.20}$$

2.3. **Auxiliary results.** Here we present some auxiliary results which will be frequently used throughout the paper.

Remark 2.1 ([2, Section 1.1])**.** Let us note that (i) implies that there is a (strictly) convex C^1-function $\Phi : \mathbb{R} \to \mathbb{R}$, such that $b(z) - b(0) = \Phi'(z)$ $\forall z \in \mathbb{R}$. Introduce the Legendre transform

$$B(z) := (b(z) - b(0))z - \Phi(z) + \Phi(0) = \int_0^z (b(z) - b(s))\,\mathrm{d}s.$$

It is not difficult to verify that (see [2])

$$B(z) = \int_0^1 (b(z) - b(sz))z\,\mathrm{d}s \geq 0 \quad \forall z \in \mathbb{R}, \tag{2.21}$$

$$B(s) - B(r) \geq (b(s) - b(r))r \quad \forall r, s \in \mathbb{R}. \tag{2.22}$$

The following theorem is proven in [24, Theorem 6.4.2], see also [26, Section 2.4].

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^2$ and $q \geq 1$. Then there exists exactly one continuous linear mapping $\mathfrak{T} : W^{1,2}(\Omega) \to L^q(\partial\Omega)$, such that $\mathfrak{T}(u) = u$ on $\partial\Omega$ for all $u \in C^\infty(\overline{\Omega})$.*

As usual, we will denote the trace of $u \in W^{1,2}(\Omega)$ on $\partial\Omega$ again by u. We often use the inequality (see the proof of [26, Theorem 1.2] or [13, eq. (3.32)])

$$\int_{\partial\Omega} |u|^2\,\mathrm{d}S \leq \eta \int_\Omega |\nabla u|^2\,\mathrm{d}x + c(\eta) \int_\Omega |u|^2\,\mathrm{d}x \tag{2.23}$$

for all $u \in W^{1,2}(\Omega)$ and all sufficiently small $\eta > 0$.

The following useful assertion is proved in [13, Lemma 2 and 3]: let $0 < m < \infty$ and

$$\{w_k\}_{k=1}^\infty \subset L^2(I; W^{1,2}(\Omega)) \cap L^\infty(I; L^{m+1}(\Omega)),$$

$$\operatorname{ess\,sup}_{0 \leq t \leq T} \int_\Omega |w_k(t)|^{m+1}\,\mathrm{d}x + \int_0^T \|w_k(t)\|^2_{W^{1,2}(\Omega)}\,\mathrm{d}t < c, \quad k = 1, 2, \dots$$

Moreover, let $w_k \to w$ a.e. on Ω_T. Then

$$\begin{gathered} w_k \to w \quad \text{in } L^{q+1}(\Omega_T), \quad 0 \leq q < m+2, \\ w_k \to w \quad \text{in } L^{s+1}(\partial\Omega_T), \quad 0 < s < (2 + \min\{s, m\} + 1)/2. \end{gathered} \tag{2.24}$$

The next lemma (Korn's inequality) follows from [28, Chapter 10.2.2]. First, define

$$\mathcal{V}_R = \Big\{\mathbf{v} \in [W^{1,2}(\Omega)]^2 : \sum_{i=1}^3 q_i^2(\mathbf{v}) = 0\Big\}, \tag{2.25}$$

$$q_i(\mathbf{v}) = \int_\Gamma v_i \,\mathrm{d}S \quad (i = 1,2), q_3(\mathbf{v}) = \int_\Gamma (x_1 v_2 - x_2 v_1)\,\mathrm{d}S,$$

where Γ is an arbitrary part of the Lipschitz boundary $\widetilde{\Gamma}$ of a domain $\widetilde{\Omega} \subset \Omega$ such that Γ is open in $\widetilde{\Gamma}$ and its measure is positive ($\Gamma \subset \partial\Omega$ is also allowed).

Lemma 2.3 (Korn's inequality)**.** *The inequality*

$$\int_\Omega \varepsilon_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{v})\,\mathrm{d}x \geq c\|\mathbf{v}\|^2_{[W^{1,2}(\Omega)]^2} \tag{2.26}$$

holds for all $\mathbf{v} \in \mathcal{V}_R$. The constant c in the inequality (2.26) *depends only on the domain Ω.*

Lemma 2.4. *$\mathcal{V}_R$ is the orthogonal complement of R with respect to $[W^{1,2}(\Omega)]^2$ in the sense of the inner product*

$$(\mathbf{u},\mathbf{v}) = \int_\Omega \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})\,\mathrm{d}x + \sum_{i=1}^{3} q_i(\mathbf{u})q_i(\mathbf{v}).$$

The proof of the above lmma is analogous to the proof of [28, Chapter 7.3, Lemma 3.2]. Let $\omega : [W^{1,2}(\Omega)]^2 \to [L^2(\Omega)]^4$ be the mapping defined by

$$\omega_{ij}(\mathbf{u}) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) \tag{2.27}$$

for all $\mathbf{u} \in [W^{1,2}(\Omega)]^2$. Further, let $\mathcal{K} = \omega([W^{1,2}(\Omega)]^2)$. We have the following lemma, see [28, Chapter 9, Lemma 2.1].

Lemma 2.5. *$\mathcal{K}$ is a closed subspace in $\mathcal{S}$.*

Finally, let $\mathcal{H}$ be the orthogonal complement of the subspace $\mathcal{K}$ in $\mathcal{S}$, i.e. let

$$\mathcal{S} = \mathcal{K} \oplus \mathcal{H}. \tag{2.28}$$

3. Main result

The aim of this paper is to prove the existence of a weak solution to problem (1.1)–(1.9). Recall that the primary unknowns in the model are p, ϑ, $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}^{p\ell}$ and $\boldsymbol{\alpha}$. It is worth noting that the displacement field $\mathbf{u}$ is determined from (1.10), of course, except for a rigid-body translation and rotation.

We first formulate our problem in a variational sense.

Definition 3.1. By a weak solution of (1.1)–(1.9) we mean functions p, ϑ, $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}^{p\ell}$ and $\boldsymbol{\alpha}$ such that

$$p \in L^2(I; W^{1,2}(\Omega)), \quad \vartheta \in L^2(I; W^{1,2}(\Omega)), \quad \boldsymbol{\sigma} \in L^2(I; [L^2(\Omega)]^4),$$
$$\boldsymbol{\varepsilon}^{p\ell} \in C(I; [L^2(\Omega)]^4), \quad \boldsymbol{\alpha} \in C(I; [L^2(\Omega)]^d),$$

which satisfy the variational equations

$$\begin{aligned}
&- \int_{\Omega_T} \theta(p)\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} [(\kappa(\boldsymbol{\sigma})k_R(p)/\nu(\vartheta))\,(\nabla p - \mathbf{e}_g)]\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_{\partial\Omega_T} \gamma_p(\mathbf{x})p\varphi\,\mathrm{d}S\,\mathrm{d}t \\
&= \int_\Omega \theta(p_0)\varphi(\mathbf{x},0)\,\mathrm{d}x + \int_{\partial\Omega_T} \gamma_p(\mathbf{x})p_\infty\varphi\,\mathrm{d}S\,\mathrm{d}t
\end{aligned} \tag{3.1}$$

for any test function $\varphi \in C^\infty(\overline{\Omega}_T)$, $\varphi(x,T)=0$ and for all $x \in \Omega$;

$$
\begin{aligned}
&- \int_{\Omega_T} [\theta(p)\vartheta + \rho\vartheta]\partial_t \psi \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} \lambda(p,\vartheta)\nabla\vartheta\cdot\nabla\psi \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_{\partial\Omega_T} \gamma_\vartheta(\mathbf{x})\vartheta\psi \,\mathrm{d}S\,\mathrm{d}t \\
&+ \int_{\Omega_T} \vartheta[(\kappa(\boldsymbol{\sigma})k_R(p)/\nu(\vartheta))\,(\nabla p - \mathbf{e}_g)]\cdot\nabla\psi \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_{\partial\Omega_T} \vartheta\gamma_p(\mathbf{x})(p-p_\infty)\psi \,\mathrm{d}S\,\mathrm{d}t \\
&= \int_{\Omega_T} \boldsymbol{\sigma}_d : \mathbf{B}(\vartheta,\boldsymbol{\sigma},\boldsymbol{\alpha})\psi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_\Omega [\theta(p_0)\vartheta_0 + \rho\vartheta_0]\,\psi(\mathbf{x},0)\,\mathrm{d}x + \int_{\partial\Omega_T} \gamma_\vartheta(\mathbf{x})\vartheta_\infty\psi \,\mathrm{d}S\,\mathrm{d}t
\end{aligned}
\tag{3.2}
$$

for any test function $\psi \in C^\infty(\overline{\Omega}_T)$, $\psi(x,T)=0$ for all $x \in \Omega$;

$$
\int_{\Omega_T} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{v})\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} \nabla\chi(p)\cdot\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega_T} \breve{\mathbf{t}}\cdot\mathbf{v}\,\mathrm{d}S\,\mathrm{d}t \tag{3.3}
$$

for any $\mathbf{v} \in C^\infty(\overline{\Omega}_T)^2$ and

$$
\int_\Omega \big(\boldsymbol{\varepsilon}^{e\ell}(t) + \boldsymbol{\varepsilon}^{p\ell}(t) + \boldsymbol{\varepsilon}^{\vartheta}(t)\big) : \boldsymbol{\tau}\,\mathrm{d}x = 0 \tag{3.4}
$$

for all $\boldsymbol{\tau} \in \mathcal{H}$ and a.a. $t \in I$ and, finally,

$$
\boldsymbol{\varepsilon}^{p\ell}(t) = \boldsymbol{\varepsilon}_0^{p\ell} + \int_0^t \mathbf{B}\left(\vartheta(s),\boldsymbol{\sigma}(s),\boldsymbol{\alpha}(s)\right)\,\mathrm{d}s, \tag{3.5}
$$

$$
\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}_0 + \int_0^t \mathbf{C}(\vartheta(s),\boldsymbol{\sigma}(s),\boldsymbol{\alpha}(s))\,\mathrm{d}s. \tag{3.6}
$$

The main result of this paper reads as follows.

Theorem 3.2 (Main result)**.** *Let the assumptions* (i)–(v) *be satisfied. Then there exists at least one weak solution of the system* (1.1)–(1.9).

4. Proof of the main result

To prove the main result of the paper we use the method of semi-discretization in time by constructing temporal approximations and limiting procedure. We first approximate our problem by semi-discretizing the equations in time by the semi-implicit scheme and decompose the problem into hygral, thermal and mechanical parts. The decoupled steady problems are easier to solve combining the theory of pseudomonotone and potential operators and the Lax-Milgram lemma. We next construct piecewise constant time interpolants and derive suitable a priori estimates and employ the sequential weak-compactness arguments. Since we deal with the nonlinear problem, we need strong convergence. In this work, we apply the Aubin-Lions lemma [12] and use the technique introduced by Alt and Luckhaus in [2]. Finally, we pass to the limit in the discrete weak formulation to obtain the solution of the original problem (1.1)–(1.9).

4.1. **Discretized problem.** Let $h>0$ be a time step and suppose that T/h is an integer. We will work with a sequence $\{h_N\}_{N\in\mathbb{N}}$ such that $\lim_{N\to+\infty} h_N \to 0$. Let us fix $r\in\mathbb{N}$ and, for simplicity, assume that $h_N = T/(2^{N-1}r)$. In what follows we often omit the index N and write simply h instead of h_N. Further, let us define

$$
\begin{gathered}
(p_\infty)_N^n := \frac{1}{h}\int_{(n-1)h}^{nh} p_\infty(\cdot,s)\,\mathrm{d}s,\\
(\vartheta_\infty)_N^n := \frac{1}{h}\int_{(n-1)h}^{nh} \vartheta_\infty(\cdot,s)\,\mathrm{d}s,\\
\mathbf{f}_N^n := \frac{1}{h}\int_{(n-1)h}^{nh} \mathbf{f}(\cdot,s)\,\mathrm{d}s,\\
\breve{\mathbf{t}}_N^n := \frac{1}{h}\int_{(n-1)h}^{nh} \breve{\mathbf{t}}(\cdot,s)\,\mathrm{d}s,\\
p_N^0 := p_0, \quad \vartheta_N^0 := \vartheta_0,\\
(\boldsymbol{\varepsilon}^{p\ell})_N^0 := \boldsymbol{\varepsilon}_0^{p\ell}, \quad \boldsymbol{\alpha}_N^0 := \boldsymbol{\alpha}_0,
\end{gathered}
$$

$n=1,\dots,2^{N-1}r$.

We approximate our evolution problem by a semi-implicit discrete scheme. Then we define, in each time step $n=1,\dots,2^{N-1}r$, the set of functions p_N^n, ϑ_N^n, $\boldsymbol{\sigma}_N^n$, $(\boldsymbol{\varepsilon}^{p\ell})_N^n$ and $\boldsymbol{\alpha}_N^n$ as a solution of the following recurrence steady problem: for the given functions $p_N^{n-1}\in L^\infty(\Omega)$, $\vartheta_N^{n-1}\in L^2(\Omega)$, $(\boldsymbol{\varepsilon}^{p\ell})_N^{n-1}\in[L^2(\Omega)]^4$ and $\boldsymbol{\alpha}_N^{n-1}\in[L^2(\Omega)]^d$, $n=1,\dots,2^{N-1}r$, find $p_N^n\in W^{1,s}(\Omega)$ with some $s>2$, $\vartheta_N^n\in W^{1,2}(\Omega)$, $\boldsymbol{\sigma}_N^n\in[L^2(\Omega)]^4$, $(\boldsymbol{\varepsilon}^{p\ell})_N^n\in[L^2(\Omega)]^4$ and $\boldsymbol{\alpha}_N^n\in[L^2(\Omega)]^d$, such that

$$
\begin{aligned}
&\frac{1}{h}\int_\Omega \big(\theta(p_N^n)-\theta(p_N^{n-1})\big)\varphi\,\mathrm{d}x\\
&+\int_\Omega \big(\kappa(\boldsymbol{\sigma}_N^{n-1})k_R(p_N^{n-1})/\nu(\vartheta_N^{n-1})\big)(\nabla p_N^n-\mathbf{e}_g)\cdot\nabla\varphi\,\mathrm{d}x+\int_{\partial\Omega}\gamma_p(\mathbf{x})p_N^n\varphi\,\mathrm{d}S \\
&=\int_{\partial\Omega}\gamma_p(\mathbf{x})(p_\infty)_N^n\varphi\,\mathrm{d}S
\end{aligned}\tag{4.1}
$$

for any $\varphi\in W^{1,2}(\Omega)$;

$$
\begin{aligned}
&\frac{1}{h}\int_\Omega \big(\theta(p_N^n)\vartheta_N^n-\theta(p_N^{n-1})\vartheta_N^{n-1}\big)\psi\,\mathrm{d}x\\
&+\frac{\rho}{h}\int_\Omega\big(\vartheta_N^n-\vartheta_N^{n-1}\big)\psi\,\mathrm{d}x+\int_\Omega\lambda(p_N^{n-1},\vartheta_N^{n-1})\nabla\vartheta_N^n\cdot\nabla\psi\,\mathrm{d}x\\
&+\int_\Omega\vartheta_N^n\big(\kappa(\boldsymbol{\sigma}_N^{n-1})k_R(p_N^{n-1})/\nu(\vartheta_N^{n-1})\big)(\nabla p_N^n-\mathbf{e}_g)\cdot\nabla\psi\,\mathrm{d}x\\
&+\int_{\partial\Omega}\gamma_\vartheta(\mathbf{x})\vartheta_N^n\psi\,\mathrm{d}S+\int_{\partial\Omega}\vartheta_N^n\gamma_p(\mathbf{x})(p_N^n-(p_\infty)_N^n)\psi\,\mathrm{d}S\\
&=\int_\Omega(\boldsymbol{\sigma}_d)_N^n:\mathbf{B}\big(\vartheta_N^{n-1},\boldsymbol{\sigma}_N^n,\boldsymbol{\alpha}_N^n\big)\psi\,\mathrm{d}x+\int_{\partial\Omega}\gamma_\vartheta(\mathbf{x})(\vartheta_\infty)_N^n\psi\,\mathrm{d}S
\end{aligned}\tag{4.2}
$$

for any $\psi\in W^{1,2}(\Omega)$, where $(\boldsymbol{\sigma}_d)_N^n := \boldsymbol{\sigma}_N^n-\frac{1}{3}tr(\boldsymbol{\sigma}_N^n)\mathbf{I}$;

$$
\int_\Omega\boldsymbol{\sigma}_N^n:\boldsymbol{\varepsilon}(\mathbf{v})\,\mathrm{d}x+\int_\Omega\nabla\chi(p_N^n)\cdot\mathbf{v}\,\mathrm{d}x=\int_\Omega\mathbf{f}_N^n\cdot\mathbf{v}\,\mathrm{d}x+\int_{\partial\Omega}\breve{\mathbf{t}}_N^n\cdot\mathbf{v}\,\mathrm{d}S \tag{4.3}
$$

for any $\mathbf{v} \in [W^{1,2}(\Omega)]^2$ and

$$\int_\Omega \Big(\frac{\partial P}{\partial \sigma_{ij}}(\boldsymbol{\sigma}_N^n, \boldsymbol{\alpha}_N^n) + (\varepsilon_{ij}^{p\ell})_N^n + \beta\delta_{ij}(\vartheta_N^{n-1} - \vartheta_{ref})\Big)\tau_{ij}\,\mathrm{d}x = 0 \tag{4.4}$$

for all $\boldsymbol{\tau} \in \mathcal{H}$ and, finally,

$$(\boldsymbol{\varepsilon}^{p\ell})_N^n = (\boldsymbol{\varepsilon}^{p\ell})_N^{n-1} + h\mathbf{B}\Big(\vartheta_N^{n-1}, \boldsymbol{\sigma}_N^n, \boldsymbol{\alpha}_N^n\Big), \tag{4.5}$$

$$\boldsymbol{\alpha}_N^n = \boldsymbol{\alpha}_N^{n-1} + h\mathbf{C}\left(\vartheta_N^{n-1}, \boldsymbol{\sigma}_N^n, \boldsymbol{\alpha}_N^n\right). \tag{4.6}$$

The next result proves the xistence of the solution to (4.1)–(4.6).

Theorem 4.1. *Let $p_N^{n-1} \in L^\infty(\Omega)$, $\vartheta_N^{n-1} \in L^2(\Omega)$, $(\boldsymbol{\varepsilon}^{p\ell})_N^{n-1} \in [L^2(\Omega)]^4$ and $\boldsymbol{\alpha}_N^{n-1} \in [L^2(\Omega)]^d$ be given and the assumptions* (i)–(v) *be satisfied. Then for every $h \in (0, h_0)$ with h_0 small enough there exists a solution of* (4.1)–(4.6).

Proof. The existence of $p_N^n \in W^{1,2}(\Omega)$, the solution to the problem (4.1), follows from [36, Chapter 2.4]. Moreover, according to [14, Theorem 3, Chapter 4.1] we also have $p_N^n \in W^{1,s}(\Omega)$ with some $s > 2$. The $W^{1,s}$-regularity of p_N^n will be used later.

With $p_N^n \in W^{1,2}(\Omega)$ in hand, we now consider the problem to find the elements $\boldsymbol{\sigma}_N^n \in [L^2(\Omega)]^4$, $(\boldsymbol{\varepsilon}^{p\ell})_N^n \in [L^2(\Omega)]^4$ and $\boldsymbol{\alpha}_N^n \in [L^2(\Omega)]^d$ satisfying (4.3)–(4.6). Let us define $\mathcal{M}$, the class of statically admissible stress tensors. In particular, we require $\boldsymbol{\sigma} \in \mathcal{S}$ such that

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{v})\,\mathrm{d}x = \int_\Omega \mathbf{f}_N^n \cdot \mathbf{v}\,\mathrm{d}x + \int_{\partial\Omega} \breve{\mathbf{t}}_N^n \cdot \mathbf{v}\,\mathrm{d}S - \int_\Omega \nabla\chi(p_N^n)\cdot \mathbf{v}\,\mathrm{d}x \tag{4.7}$$

for all $\mathbf{v} \in [W^{1,2}(\Omega)]^2$. Let us fix arbitrary $\boldsymbol{\sigma} \in \mathcal{M}$. For all h, such that $hC_c < 1$, based on the Banach contraction principle, we have a unique solution $\boldsymbol{\alpha} \in [L^2(\Omega)]^d$ of the equation

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_N^{n-1} + h\mathbf{C}\left(\vartheta_N^{n-1}, \boldsymbol{\sigma}, \boldsymbol{\alpha}\right). \tag{4.8}$$

For arbitrary $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in \mathcal{M}$ we define $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in [L^2(\Omega)]^d$ satisfying

$$\boldsymbol{\alpha}_i = \boldsymbol{\alpha}_N^{n-1} + h\mathbf{C}\left(\vartheta_N^{n-1}, \boldsymbol{\sigma}_i, \boldsymbol{\alpha}_i\right), \quad i = 1, 2. \tag{4.9}$$

Using (2.11) we have

$$\begin{aligned}
\|\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2\|_{[L^2(\Omega)]^d} &\le h\|\mathbf{C}(\vartheta_N^{n-1}, \boldsymbol{\sigma}_1, \boldsymbol{\alpha}_1) - \mathbf{C}(\vartheta_N^{n-1}, \boldsymbol{\sigma}_2, \boldsymbol{\alpha}_2)\|_{[L^2(\Omega)]^d} \\
&\le hC_c\left(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_{\mathcal{M}} + \|\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2\|_{[L^2(\Omega)]^d}\right)
\end{aligned}$$

and hence

$$\|\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2\|_{[L^2(\Omega)]^d} \le \frac{hC_c}{1 - hC_c}\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_{\mathcal{M}}. \tag{4.10}$$

Define $\boldsymbol{\varepsilon}_i^{p\ell} \in [L^2(\Omega)]^4$, $i = 1, 2$, by

$$\boldsymbol{\varepsilon}_i^{p\ell} = (\boldsymbol{\varepsilon}^{p\ell})_N^{n-1} + h\mathbf{B}\left(\vartheta_N^{n-1}, \boldsymbol{\sigma}_i, \boldsymbol{\alpha}_i\right), \quad i = 1, 2. \tag{4.11}$$

Using (2.10) we can write

$$\begin{aligned}
\|\boldsymbol{\varepsilon}_1^{p\ell} - \boldsymbol{\varepsilon}_2^{p\ell}\|_{\mathcal{S}} &\le h\|\mathbf{B}(\vartheta_N^{n-1}, \boldsymbol{\sigma}_1, \boldsymbol{\alpha}_1) - \mathbf{B}(\vartheta_N^{n-1}, \boldsymbol{\sigma}_2, \boldsymbol{\alpha}_2)\|_{\mathcal{S}} \\
&\le hC_B\left(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_{\mathcal{M}} + \|\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2\|_{[L^2(\Omega)]^d}\right).
\end{aligned} \tag{4.12}$$

Choose $\boldsymbol{\sigma}_1 \in \mathcal{M}$ and define $\boldsymbol{\alpha}_1 \in [L^2(\Omega)]^d$ and $\boldsymbol{\varepsilon}_1^{p\ell} \in \mathcal{S}$ by

$$\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_N^{n-1} + h\mathbf{C}\left(\vartheta_N^{n-1}, \boldsymbol{\sigma}_1, \boldsymbol{\alpha}_1\right), \tag{4.13}$$

$$\boldsymbol{\varepsilon}_1^{p\ell} = (\boldsymbol{\varepsilon}^{p\ell})_N^{n-1} + h\mathbf{B}\left(\vartheta_N^{n-1}, \boldsymbol{\sigma}_1, \boldsymbol{\alpha}_1\right). \tag{4.14}$$

We now look for $\widetilde{\boldsymbol{\sigma}}_1 \in \mathcal{M}$ and define the mapping $\mathcal{Z}$ by $\widetilde{\boldsymbol{\sigma}}_1 = \mathcal{Z}(\boldsymbol{\sigma}_1)$ such that

$$\int_\Omega \big[\frac{\partial P}{\partial \sigma_{ij}}(\widetilde{\boldsymbol{\sigma}}_1, \boldsymbol{\alpha}_1) + (\varepsilon_{ij}^{p\ell})_1 + \beta\delta_{ij}(\vartheta_N^{n-1} - \vartheta_{ref})\big]\tau_{ij}\,\mathrm{d}x = 0 \tag{4.15}$$

for all $\boldsymbol{\tau} \in \mathcal{H}$. We find $\widetilde{\boldsymbol{\sigma}}_1 \in \mathcal{M}$ in the form

$$\widetilde{\boldsymbol{\sigma}}_1 = \boldsymbol{\omega} + \boldsymbol{\sigma}_0, \tag{4.16}$$

where $\boldsymbol{\omega} \in \mathcal{H}$ and $\boldsymbol{\sigma}_0$ is an arbitrary fixed element of $\mathcal{M}$. The left-hand side in (4.15) is a Gateaux differential of the functional

$$\Phi(\boldsymbol{\omega}) = \int_\Omega \big[P(\boldsymbol{\omega} + \boldsymbol{\sigma}_0, \boldsymbol{\alpha}_1) + (\varepsilon_{ij}^{p\ell})_1\omega_{ij} + \beta\delta_{ij}(\vartheta_N^{n-1} - \vartheta_{ref})\omega_{ij}\big]\,\mathrm{d}x \tag{4.17}$$

defined on $\mathcal{H}$. As we now see, the problem is to find all critical points of Φ. From (2.8) and (2.9) we have

$$|P(\boldsymbol{\omega} + \boldsymbol{\sigma}_0, \boldsymbol{\alpha})| \le c\,(1 + |\boldsymbol{\omega}| + |\boldsymbol{\alpha}|)^2\,, \tag{4.18}$$

further

$$P(\boldsymbol{\omega} + \boldsymbol{\sigma}_0, \boldsymbol{\alpha}) \ge c_1|\boldsymbol{\omega}|^2 - c_2|\boldsymbol{\alpha}|^2 - c_3, \tag{4.19}$$

$$\int_\Omega \Big(\frac{\partial P}{\partial \sigma_{ij}}(\boldsymbol{\omega} + \boldsymbol{\sigma}_0, \boldsymbol{\alpha}_1) - \frac{\partial P}{\partial \sigma_{ij}}(\widehat{\boldsymbol{\omega}} + \boldsymbol{\sigma}_0, \boldsymbol{\alpha}_1)\Big)\,(\omega_{ij} - \widehat{\omega}_{ij})\,\mathrm{d}x \ge c\|\boldsymbol{\omega} - \widehat{\boldsymbol{\omega}}\|_{\mathcal{S}}^2. \tag{4.20}$$

From (4.19) we further have

$$\Psi(\boldsymbol{\omega}) \to \infty \quad \text{as } \|\boldsymbol{\omega}\|_{\mathcal{S}} \to \infty. \tag{4.21}$$

We can now use the theory in [36, Chapter 4.1, Theorem 4.2], see also [28, Chapter 7.2, Theorem 2.1 and Theorem 2.2], to conclude that there exists $\boldsymbol{\omega} \in \mathcal{H}$, a point of minimum of Φ in $\mathcal{H}$. The uniqueness of such a point follows from (4.20). Note that $\widetilde{\boldsymbol{\sigma}}_1 \in \mathcal{M}$ in (4.16) is independent of the choice of $\boldsymbol{\sigma}_0 \in \mathcal{M}$.

For arbitrary $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in \mathcal{M}$ define $\widetilde{\boldsymbol{\sigma}}_1 = \mathcal{Z}(\boldsymbol{\sigma}_1)$ and $\widetilde{\boldsymbol{\sigma}}_2 = \mathcal{Z}(\boldsymbol{\sigma}_2)$, respectively. We have

$$\int_\Omega \big[\frac{\partial P}{\partial \sigma_{ij}}(\widetilde{\boldsymbol{\sigma}}_1, \boldsymbol{\alpha}_1) + (\varepsilon_{ij}^{p\ell})_1 + \beta\delta_{ij}(\vartheta_N^{n-1} - \vartheta_{ref})\big]\,((\widetilde{\sigma}_{ij})_1 - (\widetilde{\sigma}_{ij})_2)\,\mathrm{d}x = 0, \tag{4.22}$$

$$\int_\Omega \big[\frac{\partial P}{\partial \sigma_{ij}}(\widetilde{\boldsymbol{\sigma}}_2, \boldsymbol{\alpha}_2) + (\varepsilon_{ij}^{p\ell})_2 + \beta\delta_{ij}(\vartheta_N^{n-1} - \vartheta_{ref})\big]\,((\widetilde{\sigma}_{ij})_1 - (\widetilde{\sigma}_{ij})_2)\,\mathrm{d}x = 0. \tag{4.23}$$

Subtracting (4.23) from (4.22) and using (2.8) and (4.20) we obtain

$$\|\widetilde{\boldsymbol{\sigma}}_1 - \widetilde{\boldsymbol{\sigma}}_2\|_{\mathcal{M}} \le c\Big(\|\boldsymbol{\varepsilon}_1^{p\ell} - \boldsymbol{\varepsilon}_2^{p\ell}\|_{\mathcal{S}} + \|\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2\|_{[L^2(\Omega)]^d}\Big). \tag{4.24}$$

From this, (4.10) and (4.12), we deduce

$$\|\mathcal{Z}(\boldsymbol{\sigma}_1) - \mathcal{Z}(\boldsymbol{\sigma}_2)\|_{\mathcal{M}} := \|\widetilde{\boldsymbol{\sigma}}_1 - \widetilde{\boldsymbol{\sigma}}_2\|_{\mathcal{M}} \le ch\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_{\mathcal{M}}. \tag{4.25}$$

Hence, there exists $h_0 > 0$ *small enough*, such that for all h, $0 < h \le h_0$, $\mathcal{Z}$ realizes a contraction. Thus there exists a unique fixed point $\mathcal{Z}(\boldsymbol{\sigma}) = \boldsymbol{\sigma}$ in $\mathcal{M}$. We now set $\boldsymbol{\sigma}_N^n := \boldsymbol{\sigma}$ and compute $\boldsymbol{\alpha}_N^n := \boldsymbol{\alpha}_1$ and $(\boldsymbol{\varepsilon}^{p\ell})_N^n := \boldsymbol{\varepsilon}_1^{p\ell}$ from (4.13) and (4.14) (where we set $\boldsymbol{\sigma}_1 := \boldsymbol{\sigma}_N^n$), respectively.

Finally, we obtain the function $\vartheta_N^n \in W^{1,2}(\Omega)$ by solving (4.2) using the Lax-Milgram lemma. We also employ the $W^{1,s}$-regularity of p_N^n (with some $s > 2$) and the embedding $W^{1,s}(\Omega) \hookrightarrow C(\overline{\Omega})$ (recall that Ω is a bounded domain in $\mathbb{R}^2$ with

Lipschitz boundary). Let $p_N^n \in W^{1,s}(\Omega)$ with some $s > 2$ be the solution of (4.1). For $\phi, \psi \in W^{1,2}(\Omega)$ we define

$$\begin{aligned}
\mathfrak{a}(\phi,\psi) :=& \frac{1}{h}\int_\Omega [\theta(p_N^n)+\rho]\,\phi\psi\,\mathrm{d}x + \int_\Omega \lambda(p_N^{n-1},\vartheta_N^{n-1})\nabla\phi\cdot\nabla\psi\,\mathrm{d}x \\
&+ \int_\Omega \phi\left(\kappa(\boldsymbol{\sigma}_N^{n-1})k_R(p_N^{n-1})/\nu(\vartheta_N^{n-1})\right)(\nabla p_N^n - \mathbf{e}_g)\cdot\nabla\psi\,\mathrm{d}x \\
&+ \int_{\partial\Omega} [\gamma_\vartheta(\mathbf{x}) + \gamma_p(\mathbf{x})(p_N^n - (p_\infty)_N^n)]\,\phi\psi\,\mathrm{d}S.
\end{aligned}$$

The map $\mathfrak{a} : W^{1,2}(\Omega)\times W^{1,2}(\Omega)\to\mathbb{R}$ is clearly bilinear.

Using the Hölder inequality we have

$$\begin{aligned}
&\mathfrak{a}(\phi,\psi) \\
&\le \frac{1}{h}\|\theta(p_N^n)+\rho\|_{L^\infty(\Omega)}\|\phi\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)} \\
&\quad + c_1\|\phi\|_{W^{1,2}(\Omega)}\|\psi\|_{W^{1,2}(\Omega)} \\
&\quad + c_2\left\|\kappa(\boldsymbol{\sigma}_N^{n-1})k_R(p_N^{n-1})/\nu(\vartheta_N^{n-1})\right\|_{L^\infty(\Omega)}\|\nabla p_N^n\|_{[L^s(\Omega)]^2}\|\phi\|_{L^{2s/(s-2)}(\Omega)}\|\psi\|_{W^{1,2}(\Omega)} \\
&\quad + c_3\left\|\kappa(\boldsymbol{\sigma}_N^{n-1})k_R(p_N^{n-1})/\nu(\vartheta_N^{n-1})\right\|_{L^\infty(\Omega)}\|\phi\|_{L^2(\Omega)}\|\psi\|_{W^{1,2}(\Omega)} \\
&\quad + c_4\|\gamma_\vartheta(\mathbf{x})+\gamma_p(\mathbf{x})(p_N^n-(p_\infty)_N^n)\|_{L^\infty(\partial\Omega)}\|\phi\|_{L^2(\partial\Omega)}\|\psi\|_{L^2(\partial\Omega)} \\
&\le c\|\phi\|_{W^{1,2}(\Omega)}\|\psi\|_{W^{1,2}(\Omega)}
\end{aligned}$$

for all $\phi,\psi\in W^{1,2}(\Omega)$. Hence, $\mathfrak{a}$ is continuous.

Moreover, for *sufficiently small* h, it is also coercive, as it satisfies

$$\begin{aligned}
\mathfrak{a}(\phi,\phi) :=& \frac{1}{h}\int_\Omega \rho|\phi|^2\,\mathrm{d}x + \int_\Omega \lambda(p_N^{n-1},\vartheta_N^{n-1})|\nabla\phi|^2\,\mathrm{d}x + \int_{\partial\Omega}\gamma_\vartheta(\mathbf{x})|\phi|^2\,\mathrm{d}S \\
&+ \int_\Omega \phi\left(\kappa(\boldsymbol{\sigma}_N^{n-1})k_R(p_N^{n-1})/\nu(\vartheta_N^{n-1})\right)(\nabla p_N^n-\mathbf{e}_g)\cdot\nabla\phi\,\mathrm{d}x \\
&+ \frac{1}{h}\int_\Omega \theta(p_N^n)|\phi|^2\,\mathrm{d}x + \int_{\partial\Omega}\gamma_p(\mathbf{x})(p_N^n-(p_\infty)_N^n)|\phi|^2\,\mathrm{d}S \\
=& \frac{1}{h}\int_\Omega \rho|\phi|^2\,\mathrm{d}x + \int_\Omega \lambda(p_N^{n-1},\vartheta_N^{n-1})|\nabla\phi|^2\,\mathrm{d}x + \int_{\partial\Omega}\gamma_\vartheta(\mathbf{x})|\phi|^2\,\mathrm{d}S \\
&+ \frac{1}{2h}\int_\Omega \theta(p_N^n)|\phi|^2\,\mathrm{d}x + \frac{1}{2h}\int_\Omega \theta(p_N^{n-1})|\phi|^2\,\mathrm{d}x \\
&+ \frac{1}{2}\int_{\partial\Omega}\gamma_p(\mathbf{x})(p_N^n-(p_\infty)_N^n)|\phi|^2\,\mathrm{d}S \\
\ge& \frac{1}{h}\int_\Omega \rho|\phi|^2\,\mathrm{d}x + \int_\Omega \lambda(p_N^{n-1},\vartheta_N^{n-1})|\nabla\phi|^2\,\mathrm{d}x + \int_{\partial\Omega}\gamma_\vartheta(\mathbf{x})|\phi|^2\,\mathrm{d}S \\
&+ \frac{1}{2h}\int_\Omega \theta(p_N^n)|\phi|^2\,\mathrm{d}x + \frac{1}{2h}\int_\Omega \theta(p_N^{n-1})|\phi|^2\,\mathrm{d}x \\
&- \frac{1}{2}\|\gamma_p(\mathbf{x})(p_N^n-(p_\infty)_N^n)\|_{L^\infty(\partial\Omega)}\left(\delta\|\phi\|^2_{W^{1,2}(\Omega)} + C(\delta)\|\phi\|^2_{L^2(\Omega)}\right) \\
\ge& \Big(\frac{\rho}{h} - \frac{C(\delta)}{2}\|\gamma_p(\mathbf{x})(p_N^n-(p_\infty)_N^n)\|_{L^\infty(\partial\Omega)}\Big)\|\phi\|^2_{L^2(\Omega)} \\
&+ \Big(c_1 - \frac{\delta}{2}\|\gamma_p(\mathbf{x})(p_N^n-(p_\infty)_N^n)\|_{L^\infty(\partial\Omega)}\Big)\|\phi\|^2_{W^{1,2}(\Omega)},
\end{aligned} \tag{4.26}$$

where we have used (2.23) in the estimate

$$\begin{aligned}&\frac{1}{2}\int_{\partial\Omega}\gamma_p(\mathbf{x})(p_N^n-(p_\infty)_N^n)|\phi|^2\,\mathrm{d}S\\&\le\frac{1}{2}\left\|\gamma_p(\mathbf{x})(p_N^n-(p_\infty)_N^n)\right\|_{L^\infty(\partial\Omega)}\Big(\delta\|\phi\|^2_{W^{1,2}(\Omega)}+C(\delta)\|\phi\|^2_{L^2(\Omega)}\Big)\end{aligned}$$

and the identity (choosing $\varphi=\frac{1}{2}\phi^2$ in (4.1))

$$\begin{aligned}&\int_\Omega\phi\left(\kappa(\boldsymbol{\sigma}_N^{n-1})k_R(p_N^{n-1})/\nu(\vartheta_N^{n-1})\right)(\nabla p_N^n-\mathbf{e}_g)\cdot\nabla\phi\,\mathrm{d}x\\&=-\frac{1}{2h}\int_\Omega\left(\theta(p_N^n)-\theta(p_N^{n-1})\right)\phi^2\,\mathrm{d}x-\frac{1}{2}\int_{\partial\Omega}\gamma_p(\mathbf{x})\left((p_\infty)_N^n-p_N^n\right)\phi^2\,\mathrm{d}S.\end{aligned}\tag{4.27}$$

Now, it is easy to see in (4.26), that one can choose δ and then h small enough such that

$$\Big(\frac{\rho}{h}-\frac{C(\delta)}{2}\left\|\gamma_p(\mathbf{x})(p_N^n-(p_\infty)_N^n)\right\|_{L^\infty(\partial\Omega)}\Big)>0,\tag{4.28}$$

$$\Big(c_1-\frac{\delta}{2}\left\|\gamma_p(\mathbf{x})(p_N^n-(p_\infty)_N^n)\right\|_{L^\infty(\partial\Omega)}\Big)>0.\tag{4.29}$$

Hence, there exists $h_0>0$ (small enough) such that for all positive $h\le h_0$, the bilinear form $\mathfrak{a}$ is continuous and coercive. Applying the Lax-Milgram lemma we conclude that there exists $\vartheta_N^n\in W^{1,2}(\Omega)$ the solution to the problem (4.2). □

Remark 4.2. The solution of the first boundary value problem (4.3)–(4.6) also gives the displacement vector $\mathbf{u}_N^n\in\mathcal{V}_R$ which can be found from the compatible strain tensor

$$(\varepsilon_{ij})_N^n:=\frac{\partial P}{\partial\sigma_{ij}}(\boldsymbol{\sigma}_N^n,\boldsymbol{\alpha}_N^n)+(\varepsilon_{ij}^{p\ell})_N^n+\beta\delta_{ij}(\vartheta_N^{n-1}-\vartheta_{ref}).\tag{4.30}$$

The displacement field is determined except for a rigid-body displacement (translation and rotation) of the material (according to Lemma 2.4, every $\mathbf{v}\in[W^{1,2}(\Omega)]^2$ can be written as a sum $\mathbf{v}=\hat{\mathbf{v}}+\mathbf{v}_R$, $\hat{\mathbf{v}}\in\mathcal{V}_R$ and $\mathbf{v}_R\in R$).

4.2. **A priori estimates.** In this part we prove some uniform estimates (with respect to N). In the following estimates, many different constants will appear. Recall that, for simplicity of notation, we denote by c generic constants which may change their numerical value from one formula to another but do not depend on N and the functions under consideration.

In view of (1.11) and (2.4), degeneration occurs in the nonlinear transport coefficient k_R which is not assumed to be bounded below by some positive constant. This difficulty is avoided by deriving the uniform bound of p_N^n in Ω and on the boundary $\partial\Omega$. Let $\ell\in\mathbb{R}$ be an arbitrary real fixed number and set $(\xi-\ell)_-=\xi-\ell$ for $\xi<\ell$ and $(\xi-\ell)_-=0$ for $\xi\ge\ell$, $\xi\in\mathbb{R}$. Let us first observe that

$$\Phi_\ell(\xi_1)-\Phi_\ell(\xi_2)\le[\theta(\xi_1)-\theta(\xi_2)](\xi_1-\ell)_-\tag{4.31}$$

for all $\xi_1,\xi_2\in\mathbb{R}$, where $\Phi_\ell(\xi)=\int_\ell^\xi\theta'(s)(s-\ell)_-\,\mathrm{d}s$. It is easy to check that $\Phi_\ell(\xi)\ge0$ for any $\xi\in\mathbb{R}$. Now let k be sufficiently small, such that

$$k<p_0-x_2\quad\text{a.e. in }\Omega.\tag{4.32}$$

$$k<p_1-x_2\quad\text{a.e. on }\partial\Omega.\tag{4.33}$$

We choose $\varphi = (p_N^n - x_2 - k)_-$ as a test function in (4.1) to obtain

$$\begin{aligned} &\frac{1}{h}\int_\Omega \left(\theta(p_N^n) - \theta(p_N^{n-1})\right)(p_N^n - x_2 - k)_- \,\mathrm{d}x \\ &+ \int_\Omega \left(\kappa(\boldsymbol{\sigma}_N^{n-1}) k_R(p_N^{n-1})/\nu(\vartheta_N^{n-1})\right) \nabla(p_N^n - x_2 - k)\cdot\nabla(p_N^n - x_2 - k)_- \,\mathrm{d}x \\ &+ \int_{\partial\Omega} \gamma_p(\mathbf{x})\,(p_N^n - (p_\infty)_N^n)\,(p_N^n - x_2 - k)_- \,\mathrm{d}S = 0. \end{aligned} \tag{4.34}$$

In view of (4.32) and (4.33) we see that the second and third integrals are nonnegative. Hence, using (4.31), we obtain

$$\int_\Omega [\Phi_{k+x_2}(p_N^n) - \Phi_{k+x_2}(p_N^{n-1})]\,\mathrm{d}x \le 0. \tag{4.35}$$

Thus, $\Phi_{k+x_2}(p_N^{n-1}) = 0$ implies $\Phi_{k+x_2}(p_N^n) = 0$. By induction and taking into account (4.32), there exists $\underline{k} \in \mathbb{R}$ such that

$$\underline{k} \le k + x_2 \le p_N^n \tag{4.36}$$

a.e. in Ω, $n = 1, \dots, 2^{N-1}r$.

Note that since $p_N^n \in W^{1,s}_{\Gamma_D}(\Omega) \subset C(\overline{\Omega})$ (recall that $s > 2$ and Ω is the two-dimensional bounded domain with Lipschitz boundary) we also have $\underline{k} \le p_N^n(x)$ for all $x \in \overline{\Omega}$, $n = 1, \dots, 2^{N-1}r$. The upper bound can be shown analogously. Consequently,

$$\underline{k} \le p_N^n(x) \le \overline{k} \quad \text{for all } x \in \overline{\Omega}, n = 1, \dots, 2^{N-1}r \quad (\underline{k}, \overline{k}\text{const}). \tag{4.37}$$

Let us stress that the constants $\underline{k}$ and $\overline{k}$ are independent of N. Hence, by (2.3), (2.4) and (2.5), there exist constants K_1 and K_2 such that

$$0 < K_1 \le \left(\kappa(\boldsymbol{\sigma}_N^{n-1}) k_R(p_N^{n-1})/\nu(\vartheta_N^{n-1})\right) \le K_2 \quad \text{a.e. in } \Omega,\ n = 1, \dots, 2^{N-1}r. \tag{4.38}$$

We now test (4.1) with $\varphi = p_N^n$ and use (2.22) to obtain

$$\begin{aligned} &\frac{1}{h}\int_\Omega \left(B(p_N^n) - B(p_N^{n-1})\right)\,\mathrm{d}x \\ &+ \int_\Omega \left(\kappa(\boldsymbol{\sigma}_N^{n-1}) k_R(p_N^{n-1})/\nu(\vartheta_N^{n-1})\right) |\nabla p_N^n|^2 \,\mathrm{d}x + \int_{\partial\Omega} \gamma_p(\mathbf{x}) |p_N^n|^2 \,\mathrm{d}S \\ &\le \int_\Omega \left(\kappa(\boldsymbol{\sigma}_N^{n-1}) k_R(p_N^{n-1})/\nu(\vartheta_N^{n-1})\right) \mathbf{e}_g \cdot \nabla p_N^n \,\mathrm{d}x + \int_{\partial\Omega} \gamma_p(\mathbf{x})(p_\infty)_N^n p_N^n \,\mathrm{d}S. \end{aligned} \tag{4.39}$$

Applying Young's inequality to the right-hand side and summing for $n = 1, 2, \dots, k$ and using (4.38) we obtain

$$\begin{aligned} &\int_\Omega B(p_N^k)\,\mathrm{d}x + c_1 h \sum_{n=1}^k \int_\Omega |\nabla p_N^n|^2 \,\mathrm{d}x + c_2 h \sum_{n=1}^k \int_{\partial\Omega} |p_N^n|^2 \,\mathrm{d}S \\ &\le \int_\Omega B(p_N^0)\,\mathrm{d}x + c_3 k h + c_4 h \sum_{n=1}^k \int_{\partial\Omega} |(p_\infty)_N^n|^2 \,\mathrm{d}S, \quad k = 1, 2, \dots, 2^{N-1}r, \end{aligned}$$

and whence

$$\int_\Omega B(p_N^k)\,\mathrm{d}x + h \sum_{n=1}^k \|p_N^n\|^2_{W^{1,2}(\Omega)} + h \sum_{n=1}^k \|p_N^n\|^2_{L^2(\partial\Omega)} \le c. \tag{4.40}$$

Similarly, using $\psi = 2\vartheta_N^n$ as a test function in (4.2) and re-arranging the parabolic part we arrive at

$$\begin{aligned} &\frac{1}{h}\int_\Omega \left(\theta(p_N^n)[\vartheta_N^n]^2 - \theta(p_N^{n-1})[\vartheta_N^{n-1}]^2\right)\,\mathrm{d}x + \frac{\rho}{h}\int_\Omega \left([\vartheta_N^n]^2 - [\vartheta_N^{n-1}]^2\right)\,\mathrm{d}x \\ &+ \frac{1}{h}\int_\Omega \left(\theta(p_N^n) - \theta(p_N^{n-1})\right)[\vartheta_N^n]^2\,\mathrm{d}x + \int_\Omega \lambda(p_N^{n-1},\vartheta_N^{n-1})|\nabla\vartheta_N^n|^2\,\mathrm{d}x \\ &+ \int_\Omega 2\vartheta_N^n\left(\kappa(\boldsymbol{\sigma}_N^{n-1})k_R(p_N^{n-1})/\nu(\vartheta_N^{n-1})\right)(\nabla p_N^n - \mathbf{e}_g)\cdot\nabla\vartheta_N^n\,\mathrm{d}x \\ &+ 2\int_{\partial\Omega}\gamma_\vartheta(\mathbf{x})|\vartheta_N^n|^2\,\mathrm{d}S + 2\int_{\partial\Omega}|\vartheta_N^n|^2\gamma_p(\mathbf{x})(p_N^n - (p_\infty)_N^n)\,\mathrm{d}S \\ &\le 2\int_\Omega(\boldsymbol{\sigma}_d)_N^n : \mathbf{B}\left(\vartheta_N^{n-1},\boldsymbol{\sigma}_N^n,\boldsymbol{\alpha}_N^n\right)\vartheta_N^n\,\mathrm{d}x + 2\int_{\partial\Omega}\gamma_\vartheta(\mathbf{x})(\vartheta_\infty)_N^n\vartheta_N^n\,\mathrm{d}S. \end{aligned} \tag{4.41}$$

One is allowed to use $\varphi = [\vartheta_N^n]^2$ as a test function in (4.1) to obtain

$$\begin{aligned} &\frac{1}{h}\int_\Omega\left(\theta(p_N^n) - \theta(p_N^{n-1})\right)[\vartheta_N^n]^2\,\mathrm{d}x \\ &+ \int_\Omega 2\vartheta_N^n\left(\kappa(\boldsymbol{\sigma}_N^{n-1})k_R(p_N^{n-1})/\nu(\vartheta_N^{n-1})\right)(\nabla p_N^n - \mathbf{e}_g)\cdot\nabla\vartheta_N^n\,\mathrm{d}x \\ &+ \int_{\partial\Omega}|\vartheta_N^n|^2\gamma_p(\mathbf{x})(p_N^n - (p_\infty)_N^n)\,\mathrm{d}S \\ &= 0. \end{aligned} \tag{4.42}$$

Subtracting (4.42) from (4.41) and, again, re-arranging the parabolic part we deduce

$$\begin{aligned} &\int_\Omega(\theta(p_N^n)+\rho)[\vartheta_N^n]^2\,\mathrm{d}x - \int_\Omega\left(\theta(p_N^{n-1})+\rho\right)[\vartheta_N^{n-1}]^2\,\mathrm{d}x \\ &+ h\int_\Omega\lambda(p_N^{n-1},\vartheta_N^{n-1})|\nabla\vartheta_N^n|^2\,\mathrm{d}x + 2h\int_{\partial\Omega}\gamma_\vartheta(\mathbf{x})|\vartheta_N^n|^2\,\mathrm{d}S \\ &\le 2h\int_{\partial\Omega}\gamma_\vartheta(\mathbf{x})(\vartheta_\infty)_N^n\vartheta_N^n\,\mathrm{d}S - h\int_{\partial\Omega}|\vartheta_N^n|^2\gamma_p(\mathbf{x})(p_N^n - (p_\infty)_N^n)\,\mathrm{d}S \\ &\quad + 2h\int_\Omega(\boldsymbol{\sigma}_d)_N^n : \mathbf{B}\left(\vartheta_N^{n-1},\boldsymbol{\sigma}_N^n,\boldsymbol{\alpha}_N^n\right)\vartheta_N^n\,\mathrm{d}x. \end{aligned}$$

Using (2.15), (2.16), (2.18), (2.19), (2.20) and (4.37) and applying the discrete Gronwall lemma we obtain

$$\begin{aligned} &\max_{n=1,\dots,k}\int_\Omega|\vartheta_N^n|^2\,\mathrm{d}x + h\sum_{n=1}^k\int_\Omega|\nabla\vartheta_N^n|^2\,\mathrm{d}x + h\sum_{n=1}^k\int_{\partial\Omega}|\vartheta_N^n|^2\,\mathrm{d}S \\ &\le c_1 + c_2 h\sum_{n=1}^k\int_\Omega|(\boldsymbol{\sigma}_d)_N^n : \mathbf{B}\left(\vartheta_N^{n-1},\boldsymbol{\sigma}_N^n,\boldsymbol{\alpha}_N^n\right)|^2\,\mathrm{d}x, \quad k = 1,\dots,2^{N-1}r. \end{aligned} \tag{4.43}$$

We now proceed with the estimates for $\boldsymbol{\sigma}_N^n$, $(\boldsymbol{\varepsilon}^{p\ell})_N^n$ and $\boldsymbol{\alpha}_N^n$. First, substituting $\mathbf{v} = \mathbf{u}_N^n$, $\mathbf{u}_N^n \in \mathcal{V}_R$, into (4.3) we obtain

$$\int_\Omega\boldsymbol{\sigma}_N^n : \boldsymbol{\varepsilon}(\mathbf{u}_N^n)\,\mathrm{d}x = \int_\Omega\mathbf{f}_N^n\cdot\mathbf{u}_N^n\,\mathrm{d}x + \int_{\partial\Omega}\breve{\mathbf{t}}_N^n\cdot\mathbf{u}_N^n\,\mathrm{d}S - \int_\Omega\nabla\chi(p_N^n)\cdot\mathbf{u}_N^n\,\mathrm{d}x. \tag{4.44}$$

Employing the Green formula

$$\int_\Omega \nabla\chi(p_N^n)\cdot\mathbf{u}_N^n\,\mathrm{d}x = \int_{\partial\Omega}\chi(p_N^n)\mathbf{n}\cdot\mathbf{u}_N^n\,\mathrm{d}x - \int_\Omega \mathbf{I}\chi(p_N^n):\boldsymbol{\varepsilon}(\mathbf{u}_N^n)\,\mathrm{d}x \tag{4.45}$$

for the last term on the right-hand side in (4.44) and using (1.13)–(1.15), (2.7), (2.8) and (2.9) we arrive at

$$\begin{aligned} c\|\boldsymbol{\sigma}_N^n\|^2_{[L^2(\Omega)]^4} &\le \int_\Omega (\sigma_{ij})_N^n : \frac{\partial P}{\partial\sigma_{ij}}(\boldsymbol{\sigma}_N^n,\boldsymbol{\alpha}_N^n)\,\mathrm{d}x \\ &= \int_\Omega \mathbf{f}_N^n\cdot\mathbf{u}_N^n\,\mathrm{d}x + \int_{\partial\Omega}\breve{\mathbf{t}}_N^n\cdot\mathbf{u}_N^n\,\mathrm{d}S - \int_{\partial\Omega}\chi(p_N^n)\mathbf{n}\cdot\mathbf{u}_N^n\,\mathrm{d}S \\ &\quad + \int_\Omega \mathbf{I}\chi(p_N^n):\boldsymbol{\varepsilon}(\mathbf{u}_N^n)\,\mathrm{d}x - \int_\Omega \boldsymbol{\sigma}_N^n : \big[(\boldsymbol{\varepsilon}^{p\ell})_N^n + \beta\mathbf{I}(\vartheta_N^n - \vartheta_{ref})\big]\,\mathrm{d}x. \end{aligned}$$

Using the Young inequality with parameter δ we obtain

$$\begin{aligned} c\|\boldsymbol{\sigma}_N^n\|^2_{[L^2(\Omega)]^4} \le{}& C(\delta)\Big(\|\mathbf{f}_N^n\|^2_{[L^2(\Omega)]^2} + \|\breve{\mathbf{t}}_N^n\|^2_{[L^2(\partial\Omega)]^2}\Big) + \delta\|\mathbf{u}_N^n\|^2_{[L^2(\Omega)]^2} \\ &+ C(\delta)\|\chi(p_N^n)\|^2_{[L^2(\partial\Omega)]} + \delta\|\mathbf{u}_N^n\|^2_{[L^2(\partial\Omega)]^2} \\ &+ C(\delta)\|\mathbf{I}\chi(p_N^n)\|^2_{[L^2(\Omega)]^4} + \delta\|\boldsymbol{\varepsilon}(\mathbf{u}_N^n)\|^2_{[L^2(\Omega)]^4} + \delta\|\boldsymbol{\sigma}_N^n\|^2_{[L^2(\Omega)]^4} \\ &+ C(\delta)\Big(\|(\boldsymbol{\varepsilon}^{p\ell})_N^n\|^2_{[L^2(\Omega)]^4} + \|\beta\mathbf{I}(\vartheta_N^n - \vartheta_{ref})\|^2_{[L^2(\Omega)]^4}\Big). \end{aligned}$$

By (4.37) we can further simplify the latter inequality as

$$\begin{aligned} \|\boldsymbol{\sigma}_N^n\|_{[L^2(\Omega)]^4} \le{}& \delta\|\mathbf{u}_N^n\|_{[L^2(\Omega)]^2} + \delta\|\mathbf{u}_N^n\|_{[L^2(\partial\Omega)]^2} + \delta\|\boldsymbol{\varepsilon}(\mathbf{u}_N^n)\|_{[L^2(\Omega)]^4} \\ &+ C(\delta)\Big(1 + \|(\boldsymbol{\varepsilon}^{p\ell})_N^n\|_{[L^2(\Omega)]^4} + \|\vartheta_N^n\|_{L^2(\Omega)}\Big). \end{aligned} \tag{4.46}$$

Using Theorem 2.2 and Lemma 2.26 we can write

$$\|\mathbf{u}_N^n\|_{[L^2(\Omega)]^2} \le \|\mathbf{u}_N^n\|_{[W^{1,2}(\Omega)]^2} \le c\|\boldsymbol{\varepsilon}(\mathbf{u}_N^n)\|_{[L^2(\Omega)]^4}, \tag{4.47}$$

$$\|\mathbf{u}_N^n\|_{[L^2(\partial\Omega)]^2} \le \|\mathbf{u}_N^n\|_{[W^{1,2}(\Omega)]^2} \le c\|\boldsymbol{\varepsilon}(\mathbf{u}_N^n)\|_{[L^2(\Omega)]^4}. \tag{4.48}$$

Clearly, from (1.13) we have

$$\begin{aligned} &\|\boldsymbol{\varepsilon}(\mathbf{u}_N^n)\|_{[L^2(\Omega)]^4} \\ &\le \|(\boldsymbol{\varepsilon}^{e\ell})_N^n\|_{[L^2(\Omega)]^4} + \|(\boldsymbol{\varepsilon}^{p\ell})_N^n\|_{[L^2(\Omega)]^4} + \|\beta\mathbf{I}(\vartheta_N^n - \vartheta_{ref})\|_{[L^2(\Omega)]^4}. \end{aligned} \tag{4.49}$$

From (1.14) and using (2.8) we obtain

$$\|(\boldsymbol{\varepsilon}^{e\ell})_N^n\|_{[L^2(\Omega)]^4} \le c\left(1 + \|\boldsymbol{\sigma}_N^n\|_{[L^2(\Omega)]^4} + \|\boldsymbol{\alpha}_N^n\|_{[L^2(\Omega)]^d}\right). \tag{4.50}$$

From (4.5) and (4.6) one sees immediately that

$$\begin{aligned} (\boldsymbol{\varepsilon}^{p\ell})_N^n &= (\boldsymbol{\varepsilon}^{p\ell})_N^{n-1} + h\mathbf{B}(\vartheta_N^{n-1},\boldsymbol{\sigma}_N^n,\boldsymbol{\alpha}_N^n) \\ &= (\boldsymbol{\varepsilon}^{p\ell})_N^0 + h\sum_{j=1}^n \mathbf{B}(\vartheta_N^{j-1},\boldsymbol{\sigma}_N^j,\boldsymbol{\alpha}_N^j) \end{aligned} \tag{4.51}$$

and

$$\begin{aligned} \boldsymbol{\alpha}_N^n &= \boldsymbol{\alpha}_N^{n-1} + h\mathbf{C}\left(\vartheta_N^{n-1},\boldsymbol{\sigma}_N^n,\boldsymbol{\alpha}_N^n\right) \\ &= \boldsymbol{\alpha}_N^0 + h\sum_{j=1}^n \mathbf{C}(\vartheta_N^{j-1},\boldsymbol{\sigma}_N^j,\boldsymbol{\alpha}_N^j). \end{aligned} \tag{4.52}$$

Using (4.47)–(4.52) in (4.46) we deduce

$$
\begin{aligned}
&\|\boldsymbol{\sigma}_N^n\|_{[L^2(\Omega)]^4} \\
&\le (C(\delta)+\delta c) + \delta c\|\boldsymbol{\sigma}_N^n\|_{[L^2(\Omega)]^4} + (C(\delta)+\delta c)\,\|\vartheta_N^n\|_{L^2(\Omega)} \\
&\quad + \delta c\Big(\|\boldsymbol{\alpha}_N^0\|_{[L^2(\Omega)]^d} + h\sum_{j=1}^{n}\|\mathbf{C}(\vartheta_N^{j-1},\boldsymbol{\sigma}_N^j,\boldsymbol{\alpha}_N^j)\|_{[L^2(\Omega)]^d}\Big) \\
&\quad + (C(\delta)+\delta c)\Big(\|(\boldsymbol{\varepsilon}^{p\ell})_N^0\|_{[L^2(\Omega)]^4} + h\sum_{j=1}^{n}\|\mathbf{B}(\vartheta_N^{j-1},\boldsymbol{\sigma}_N^j,\boldsymbol{\alpha}_N^j)\|_{[L^2(\Omega)]^4}\Big)
\end{aligned}
\tag{4.53}
$$

and, choosing δ sufficiently small and using (2.10), we arrive at

$$
\|\boldsymbol{\sigma}_N^n\|_{[L^2(\Omega)]^4} \le c_1 + c_2\|\vartheta_N^n\|_{L^2(\Omega)} + c_3 h\delta\sum_{j=1}^{n}\|\mathbf{C}(\vartheta_N^{j-1},\boldsymbol{\sigma}_N^j,\boldsymbol{\alpha}_N^j)\|_{[L^2(\Omega)]^d}. \tag{4.54}
$$

Note that from (4.43), by (2.10), we have

$$
\|\vartheta_N^n\|_{L^2(\Omega)} \le c_1 + c_2 h\sum_{j=1}^{n}\|\boldsymbol{\sigma}_N^j\|_{[L^2(\Omega)]^4}. \tag{4.55}
$$

Using (4.55) and (2.11) in (4.54) we obtain

$$
\|\boldsymbol{\sigma}_N^n\|_{[L^2(\Omega)]^4} \le c_1 + c_2 h\sum_{j=1}^{n}\Big(\|\vartheta_N^{j-1}\|_{L^2(\Omega)} + \|\boldsymbol{\sigma}_N^j\|_{[L^2(\Omega)]^4} + \|\boldsymbol{\alpha}_N^j\|_{[L^2(\Omega)]^d}\Big). \tag{4.56}
$$

Similarly, from (4.52) we also have

$$
\|\boldsymbol{\alpha}_N^n\|_{[L^2(\Omega)]^d} \le c_1 + c_2 h\sum_{j=1}^{n}\Big(\|\vartheta_N^{j-1}\|_{L^2(\Omega)} + \|\boldsymbol{\sigma}_N^j\|_{[L^2(\Omega)]^4} + \|\boldsymbol{\alpha}_N^j\|_{[L^2(\Omega)]^d}\Big). \tag{4.57}
$$

Adding (4.55)–(4.57) and applying the discrete Gronwall inequality [36, Chapter 1.6] (provided h is sufficiently small) we arrive at

$$
\|\boldsymbol{\alpha}_N^n\|_{[L^2(\Omega)]^d} + \|\boldsymbol{\sigma}_N^n\|_{[L^2(\Omega)]^4} + \|\vartheta_N^n\|_{L^2(\Omega)} \le c, \quad n = 1,\dots,2^{N-1}r. \tag{4.58}
$$

Finally, from (4.51) and (2.10) we also have

$$
\|(\boldsymbol{\varepsilon}^{p\ell})_N^n\|_{[L^2(\Omega)]^4} \le c, \quad n = 1,\dots,2^{N-1}r. \tag{4.59}
$$

4.3. **Construction of temporal interpolants and passage to the limit.** Using the sequences $\{p_N^n\},\{\vartheta_N^n\},\{\boldsymbol{\sigma}_N^n\},\{(\boldsymbol{\varepsilon}^{p\ell})_N^n\}$ and $\{\boldsymbol{\alpha}_N^n\}$, we define the piecewise constant interpolants $\overline{\zeta}_N(t) = \zeta_N^n$ for $t \in ((n-1)h, nh]$ and, in addition if necessary, we extend $\overline{\zeta}_N$ for $t \le 0$ by $\overline{\zeta}_N(t) = \zeta_0$ for $t \in [-h, 0]$. For any function ζ we often use the simplified notation $\zeta := \zeta(t)$ and $\partial_t^{-h}\zeta(t) := \frac{\zeta(t)-\zeta(t-h)}{h}$. Then, by (4.1)–(4.3), $\overline{p}_N \in L^\infty(I; W^{1,s}(\Omega))$ with some $s > 2$, $\overline{\vartheta}_N \in L^\infty(I; W^{1,2}(\Omega))$, $\overline{\boldsymbol{\sigma}}_N \in L^\infty(I; [L^2(\Omega)]^4)$, $\overline{\boldsymbol{\varepsilon}^{p\ell}}_N \in L^\infty(I; [L^2(\Omega)]^4)$ and $\overline{\boldsymbol{\alpha}}_N \in L^\infty(I; [L^2(\Omega)]^d)$ satisfy the equations

$$\begin{aligned}
&\int_\Omega \partial_t^{-h}\theta(\overline{p}_N(t))\varphi\,\mathrm{d}x \\
&+\int_\Omega \left(\kappa(\overline{\boldsymbol{\sigma}}_N(t-h))k_R(\overline{p}_N(t-h))/\nu(\overline{\vartheta}_N(t-h))\right)(\nabla\overline{p}_N(t)-\mathbf{e}_g)\cdot\nabla\varphi\,\mathrm{d}x \\
&+\int_{\partial\Omega}\gamma_p(\mathbf{x})\overline{p}_N(t)\varphi\,\mathrm{d}S \\
&=\int_{\partial\Omega}\gamma_p(\mathbf{x})(\overline{p}_\infty)_N(t)\varphi\,\mathrm{d}S
\end{aligned}\tag{4.60}$$

for all $\varphi\in W^{1,2}(\Omega)$ and $t\in(0,T]$;

$$\begin{aligned}
&\int_\Omega \partial_t^{-h}[\theta(\overline{p}_N(t))\overline{\vartheta}_N(t)]\psi\,\mathrm{d}x+\rho\int_\Omega\partial_t^{-h}\overline{\vartheta}_N(t)\psi\,\mathrm{d}x \\
&+\int_\Omega\lambda(\overline{p}_N(t-h),\overline{\vartheta}_N(t-h))\nabla\overline{\vartheta}_N(t)\cdot\nabla\psi\,\mathrm{d}x \\
&+\int_\Omega\overline{\vartheta}_N(t)\left(\kappa(\overline{\boldsymbol{\sigma}}_N(t-h))k_R(\overline{p}_N(t-h))/\nu(\overline{\vartheta}_N(t-h))\right)(\nabla\overline{p}_N(t)-\mathbf{e}_g)\cdot\nabla\psi\,\mathrm{d}x \\
&+\int_{\partial\Omega}\gamma_\vartheta(\mathbf{x})\overline{\vartheta}_N(t)\psi\,\mathrm{d}S+\int_{\partial\Omega}\overline{\vartheta}_N(t)\gamma_p(\mathbf{x})(\overline{p}_N(t)-(\overline{p}_\infty)_N(t))\psi\,\mathrm{d}S \\
&=\int_{\partial\Omega}\gamma_\vartheta(\mathbf{x})(\overline{\vartheta}_\infty)_N(t)\psi\,\mathrm{d}S
\end{aligned}\tag{4.61}$$

for all $\psi\in W^{1,2}(\Omega)$ and $t\in(0,T]$;

$$\begin{aligned}
&\int_\Omega\overline{\boldsymbol{\sigma}}_N(t):\boldsymbol{\varepsilon}(\mathbf{v})\,\mathrm{d}x+\int_\Omega\nabla\chi(\overline{p}_N(t))\cdot\mathbf{v}\,\mathrm{d}x \\
&=\int_\Omega\overline{\mathbf{f}}_N(t)\cdot\mathbf{v}\,\mathrm{d}x+\int_{\partial\Omega}\overline{\breve{\mathbf{t}}}_N(t)\cdot\mathbf{v}\,\mathrm{d}S
\end{aligned}\tag{4.62}$$

for all $\mathbf{v}\in[W^{1,2}(\Omega)]^2$ and $t\in(0,T]$ and

$$\overline{\boldsymbol{\varepsilon}^{p\ell}}_N(t)=\overline{\boldsymbol{\varepsilon}^{p\ell}}_N(t-h)+h\mathbf{B}\left(\overline{\vartheta}_N(t-h),\overline{\boldsymbol{\sigma}}_N(t),\overline{\boldsymbol{\alpha}}_N(t)\right),\tag{4.63}$$

$$\overline{\boldsymbol{\alpha}}_N(t)=\overline{\boldsymbol{\alpha}}_N(t-h)+h\mathbf{C}\left(\overline{\vartheta}_N(t-h),\overline{\boldsymbol{\sigma}}_N(t),\overline{\boldsymbol{\alpha}}_N(t)\right).\tag{4.64}$$

From (4.37), (4.40), (4.43), (4.58) and (4.59) we see that

$$\sup_{0\le t\le T}\int_\Omega B(\bar{p}_N(t))\,\mathrm{d}x+\int_0^T\|\bar{p}(t)\|^2_{W^{1,2}(\Omega)}\,\mathrm{d}t+\int_0^T\|\bar{p}_N(t)\|^2_{L^2(\partial\Omega)}\,\mathrm{d}t\le c,\tag{4.65}$$

$$\|\bar{p}_N\|_{L^\infty(\Omega_T)}\le c,\tag{4.66}$$

$$\|\bar{p}_N\|_{L^\infty(\partial\Omega_T)}\le c,\tag{4.67}$$

$$\sup_{0\le t\le T}\int_\Omega|\bar{\vartheta}_N(t)|^2\,\mathrm{d}x+\int_0^T\|\bar{\vartheta}_N(t)\|^2_{W^{1,2}(\Omega)}\,\mathrm{d}t+\int_0^T\|\bar{\vartheta}_N(t)\|^2_{L^2(\partial\Omega)}\,\mathrm{d}t\le c,\tag{4.68}$$

$$\sup_{0\le t\le T}\|\boldsymbol{\sigma}_N(t)\|_{[L^2(\Omega)]^4}\le c,\tag{4.69}$$

$$\sup_{0\le t\le T}\|(\boldsymbol{\varepsilon}^{p\ell})_N(t)\|_{[L^2(\Omega)]^4}\le c,\tag{4.70}$$

$$\sup_{0\le t\le T}\|\overline{\boldsymbol{\alpha}}_N(t)\|_{[L^2(\Omega)]^d}\le c.\tag{4.71}$$

It follows from (4.65) and (4.68)–(4.71) that the sequences $\{\overline{p}_N\}$, $\{\overline{\vartheta}_N\}$, $\{\overline{\boldsymbol{\sigma}}_N\}$, $\{\overline{\boldsymbol{\varepsilon}^{p\ell}}_N\}$ and $\{\overline{\boldsymbol{\alpha}}_N\}$ are bounded in the spaces $L^2(I;W^{1,2}(\Omega))$, $L^2(I;W^{1,2}(\Omega))$, $L^\infty(I;[L^2(\Omega)]^4)$, $L^\infty(I;[L^2(\Omega)]^4)$ and $L^\infty(I;[L^2(\Omega)]^d)$, respectively, so that subsequences $\{\overline{p}_{N_k}\}$, $\{\overline{\vartheta}_{N_k}\}$, $\{\overline{\boldsymbol{\sigma}}_{N_k}\}$, $\{\overline{\boldsymbol{\varepsilon}^{p\ell}}_{N_k}\}$ and $\{\overline{\boldsymbol{\alpha}}_{N_k}\}$ can be chosen such that

$$
\begin{gathered}
\overline{p}_{N_k} \rightharpoonup p \quad \text{weakly in } L^2(I;W^{1,2}(\Omega)),\\
\overline{\vartheta}_{N_k} \rightharpoonup \vartheta \quad \text{weakly in } L^2(I;W^{1,2}(\Omega)),\\
\overline{\boldsymbol{\sigma}}_{N_k} \rightharpoonup \boldsymbol{\sigma} \quad \text{weakly star in } L^\infty(I;[L^\infty(\Omega)]^4),\\
\overline{\boldsymbol{\varepsilon}^{p\ell}}_{N_k} \rightharpoonup \boldsymbol{\varepsilon}^{p\ell} \quad \text{weakly star in } L^\infty(I;[L^2(\Omega)]^4),\\
\overline{\boldsymbol{\alpha}}_{N_k} \rightharpoonup \boldsymbol{\alpha} \quad \text{weakly star in } L^\infty(I;[L^2(\Omega)]^d).
\end{gathered}
$$

Since the problem is nonlinear, we need strong convergence to identify the limits with the weak solution of problem (1.1)–(1.9).

To simplify the notation, let us write, for a moment, $\overline{p}_N$, $\overline{\vartheta}_N$, $\overline{\boldsymbol{\sigma}}_N$, $\overline{\boldsymbol{\varepsilon}^{p\ell}}_N$, $\overline{\boldsymbol{\alpha}}_N$, instead of $\overline{p}_{N_k}$, $\overline{\vartheta}_{N_k}$, $\overline{\boldsymbol{\sigma}}_{N_k}$, $\overline{\boldsymbol{\varepsilon}^{p\ell}}_{N_k}$, $\overline{\boldsymbol{\alpha}}_{N_k}$, respectively. To show that $\overline{p}_N$ converges to p almost everywhere on Ω_T we follow [2]. Let $k \in \mathbb{N}$ and use

$$
\varphi(t) = \partial_t^{kh}\overline{p}_N(s)
$$

for $jh \le t \le (j+k)h$ with $(j-1)h \le s \le jh$ and $1 \le j \le \frac{T}{h} - k$, as a test function in (4.60). For the parabolic term, we can write

$$
\begin{aligned}
&\int_{jh}^{(j+k)h}\int_\Omega \partial_t^{-h}\theta(\overline{p}_N(t))\,\partial_t^{kh}\overline{p}_N(t)\,\mathrm{d}x\,\mathrm{d}t\\
&= \frac{1}{kh^2}\int_{(j-1)h}^{jh}\int_\Omega \left(\theta(\overline{p}_N(t+kh)) - \theta(\overline{p}_N(t))\right)\left(\overline{p}_N(t+kh) - \overline{p}_N(t)\right)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Hence, summing over $j = 1,\dots,m-k$ we obtain the estimate

$$
\begin{aligned}
&\sum_{j=1}^{m-k}\int_{jh}^{(j+k)h}\int_\Omega \partial_t^{-h}\theta(\overline{p}_N(t))\,\partial_t^{kh}\overline{p}_N(t)\,\mathrm{d}x\,\mathrm{d}t\\
&\ge \frac{1}{kh^2}\int_0^{T-kh}\int_\Omega \left(\theta(\overline{p}_N(t+kh)) - \theta(\overline{p}_N(t))\right)\left(\overline{p}_N(t+kh) - \overline{p}_N(t)\right)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{4.72}
$$

Similarly, for the elliptic term, using (4.65), we have

$$
\begin{aligned}
&\sum_{j=1}^{m-k}\int_{jh}^{(j+k)h}\int_\Omega \big(\kappa(\overline{\boldsymbol{\sigma}}_N(t-h))k_R(\overline{p}_N(t-h))/\nu(\overline{\vartheta}_N(t-h))\big)\\
&\quad\times \nabla\overline{p}_N\cdot\nabla\partial_t^{kh}\overline{p}_N\,\mathrm{d}x\,\mathrm{d}t\\
&= \sum_{\ell=1}^{k}\sum_{j=1}^{m-k}\int_{(j+\ell-1)h}^{(j+\ell)h}\int_\Omega \big(\kappa(\overline{\boldsymbol{\sigma}}_N(t-h))k_R(\overline{p}_N(t-h))/\nu(\overline{\vartheta}_N(t-h))\big)\\
&\quad\times \nabla\overline{p}_N\cdot\nabla\partial_t^{kh}\overline{p}_N\,\mathrm{d}x\,\mathrm{d}t\\
&= \sum_{\ell=1}^{k}\int_{\ell h}^{T-kh+\ell h}\int_\Omega \big(\kappa(\overline{\boldsymbol{\sigma}}_N(t-h))k_R(\overline{p}_N(t-h))/\nu(\overline{\vartheta}_N(t-h))\big)\\
&\quad\times \nabla\overline{p}_N(t)\cdot\nabla\partial_t^{kh}\overline{p}_N(t-\ell h)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{c_1}{h}\int_{\Omega_T} \big|\big(\kappa(\overline{\boldsymbol{\sigma}}_N(t-h))k_R(\overline{p}_N(t-h))/\nu(\overline{\vartheta}_N(t-h))\big) \\
&\quad \times \nabla\overline{p}_N\big|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{c_2}{h}\int_{\Omega_T} |\nabla\overline{p}_N|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\le \frac{C}{h}
\end{aligned}
$$

and by a similar computation and using (4.67) we arrive at

$$
\sum_{j=1}^{m-k}\int_{jh}^{(j+k)h}\int_{\partial\Omega} \gamma_p(\mathbf{x})\overline{p}_N\partial_t^{kh}\overline{p}_N \,\mathrm{d}S\,\mathrm{d}t \le \frac{C}{h}, \tag{4.73}
$$

$$
\sum_{j=1}^{m-k}\int_{jh}^{(j+k)h}\int_{\partial\Omega} \gamma_p(\mathbf{x})(\overline{p}_\infty)_N\partial_t^{kh}\overline{p}_N \,\mathrm{d}S\,\mathrm{d}t \le \frac{C}{h}. \tag{4.74}
$$

Combining (4.72)–(4.74) we obtain

$$
\int_0^{T-kh} \left(\theta(\overline{p}_N(s+kh)) - \theta(\overline{p}_N(s))\right)\left(\overline{p}_N(s+kh) - \overline{p}_N(s)\right)\mathrm{d}s \le Ckh. \tag{4.75}
$$

Using the compactness argument one can show in the same way as in [2, Lemma 1.9] and [13, Eqs. (2.10)–(2.12)] that

$$
\theta(\overline{p}_N) \to \theta(p) \text{ in } L^1(\Omega_T) \text{ and almost everywhere on } \Omega_T. \tag{4.76}
$$

Since θ is strictly monotone, it follows from (4.76) that (see [21, Proposition 3.35])

$$
\overline{p}_N \to p \quad \text{almost everywhere on } \Omega_T \tag{4.77}
$$

and taking into account the estimates (4.66) and (4.67) we conclude that

$$
\begin{gathered}
\overline{p}_N \to p \quad \text{in } L^{q+1}(\Omega_T), \quad 0 \le q < +\infty, \\
\overline{p}_N \to p \quad \text{in } L^{s+1}(\partial\Omega_T), \quad 0 \le s < +\infty.
\end{gathered} \tag{4.78}
$$

In what follows, we study the convergence of $\overline{\vartheta}_N$. From (4.61) we have

$$
\begin{aligned}
&\int_{\Omega_T} \partial_t^{-h}[(\theta(\overline{p}_N(t)) + \rho)\,\overline{\vartheta}_N(t)]\psi \,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_{\Omega_T} \lambda(\overline{p}_N(t-h), \overline{\vartheta}_N(t-h))\nabla\overline{\vartheta}_N(t)\cdot\nabla\psi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_{\Omega_T} \overline{\vartheta}_N(t)\left(\kappa(\overline{\boldsymbol{\sigma}}_N(t-h))k_R(\overline{p}_N(t-h))/\nu(\overline{\vartheta}_N(t-h))\right) \\
&\quad \times (\nabla\overline{p}_N(t) - \mathbf{e}_g)\cdot\nabla\psi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_{\partial\Omega_T} \gamma_\vartheta(\mathbf{x})\overline{\vartheta}_N(t)\psi \,\mathrm{d}S\,\mathrm{d}t \\
&\quad - \int_{\partial\Omega_T} \overline{\vartheta}_N(t)\gamma_p(\mathbf{x})(\overline{p}_N(t) - (\overline{p}_\infty)_N(t))\psi \,\mathrm{d}S\,\mathrm{d}t \\
&\quad + \int_{\partial\Omega_T} \gamma_\vartheta(\mathbf{x})(\overline{\vartheta}_\infty)_N(t)\psi \,\mathrm{d}S\,\mathrm{d}t
\end{aligned} \tag{4.79}
$$

for all $\psi \in W^{1,2}(\Omega)$. First, by means of an interpolation argument, see [1, Theorem 5.8], we deduce

$$
L^2(I; W^{1,2}(\Omega)) \cap L^\infty(I; L^2(\Omega)) \hookrightarrow L^4(\Omega_T). \tag{4.80}
$$

Now, let $\psi \in L^4(I;W^{1,4}(\Omega))$. For the convective term in (4.79), by Hölder's inequality and using (4.80), we have

$$\begin{aligned}
&\Big|\int_{\Omega_T} \overline{\vartheta}_N(t)\left(\kappa(\overline{\boldsymbol{\sigma}}_N(t-h))k_R(\overline{p}_N(t-h))/\nu(\overline{\vartheta}_N(t-h))\right)(\nabla\overline{p}_N(t)-\mathbf{e}_g)\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\le c\|\overline{\vartheta}_N(t)\left(\kappa(\overline{\boldsymbol{\sigma}}_N(t-h))k_R(\overline{p}_N(t-h))/\nu(\overline{\vartheta}_N(t-h))\right) \\
&\quad\times(\nabla\overline{p}_N(t)-\mathbf{e}_g)\|_{[L^{4/3}(\Omega_T)]^2}\|\psi\|_{L^4(I;W^{1,4}(\Omega))} \\
&\le c\|\overline{\vartheta}_N\|_{L^4(\Omega_T)}\left(\|\nabla\overline{p}_N\|_{[L^2(\Omega_T)]^2}+1\right)\|\psi\|_{L^4(I;W^{1,4}(\Omega))}.
\end{aligned} \tag{4.81}$$

The remaining terms on the right-hand side of (4.79) can be handled in a more straightforward and simpler way. Using (4.65)–(4.68) we conclude that

$$\Big|\int_{\Omega_T}\partial_t^{-h}[(\theta(\overline{p}_N)+\rho)\,\overline{\vartheta}_N]\psi\,\mathrm{d}x\,\mathrm{d}t\Big| \le c\|\psi\|_{L^4(I;W^{1,4}(\Omega))}$$

for all $\psi\in L^4(I;W^{1,4}(\Omega))$ and hence

$$\big\|\partial_t^{-h}[(\theta(\overline{p}_N)+\rho)\,\overline{\vartheta}_N]\big\|_{L^{4/3}(I;W^{1,4}(\Omega)')}\le c.$$

Further, from (4.65)–(4.68) we also have

$$\|\,(\theta(\overline{p}_N)+\rho)\,\overline{\vartheta}_N\|_{L^{4/3}(I;W^{1,4/3}(\Omega))}\le c.$$

Since

$$W^{1,4/3}(\Omega)\hookrightarrow\hookrightarrow W^{1-\beta,4/3}(\Omega)\hookrightarrow W^{1,4}(\Omega)',$$

where β is a small positive real number, the Aubin-Lions lemma [12] yields the existence of $g\in L^{4/3}(I;W^{1-\beta,4/3}(\Omega))$ such that (along a selected subsequence)

$$(\theta(\overline{p}_N)+\rho)\,\overline{\vartheta}_N\to g\quad\text{strongly in } L^{4/3}(I;W^{1-\beta,4/3}(\Omega)) \tag{4.82}$$

and almost everywhere on Ω_T. In view of (4.76) and (4.82) we can write

$$\overline{\vartheta}_N=\frac{(\theta(\overline{p}_N)+\rho)\,\overline{\vartheta}_N}{\theta(\overline{p}_N)+\rho}\to\frac{g}{\theta(p)+\rho}=\vartheta\quad\text{almost everywhere on }\Omega_T. \tag{4.83}$$

Now, using (4.68), (4.83) and (2.24) we also have

$$\begin{gathered}
\overline{\vartheta}_N\to\vartheta\quad\text{in } L^{q+1}(\Omega_T),\quad 0\le q<3,\\
\overline{\vartheta}_N\to\vartheta\quad\text{in } L^{s+1}(\partial\Omega_T),\quad 0<s<2.
\end{gathered} \tag{4.84}$$

We now prove that the sequence $\{\overline{\boldsymbol{\alpha}}_N(t)\}$ converges to the function $\boldsymbol{\alpha}(t)$ strongly in $[L^2(\Omega)]^d$, uniformly with respect to $t\in I$. To this aim, let us define

$$\mathbf{C}_N(t)=\begin{cases}\mathbf{C}(\vartheta_N^{n-1},\boldsymbol{\sigma}_N^n,\boldsymbol{\alpha}_N^n) & \text{for } t\in((n-1)h_N,nh_N],\ n=1,\dots,2^{N-1}r,\\ \mathbf{C}(\vartheta_N^0,\boldsymbol{\sigma}_N^1,\boldsymbol{\alpha}_N^1) & \text{for } t=0\end{cases} \tag{4.85}$$

and

$$\boldsymbol{\Psi}_N(t)=\int_0^t\mathbf{C}_N(s)\mathrm{d}s. \tag{4.86}$$

Indeed, for $t=0$, we have

$$\|\overline{\boldsymbol{\alpha}}_N(0)-\boldsymbol{\Psi}_N(0)\|_{[L^2(\Omega)]^d}\le ch_N. \tag{4.87}$$

Now, let $t\in(0,T]$ be arbitrary, say $t\in((n-1)h_N,nh_N]$. We have

$$\|\overline{\boldsymbol{\alpha}}_N(t)-\boldsymbol{\Psi}_N(t)\|_{[L^2(\Omega)]^d}\le\big\|\overline{\boldsymbol{\alpha}}_N(t)-\int_0^t\mathbf{C}_N(s)\mathrm{d}s\big\|_{[L^2(\Omega)]^d}\le ch_N. \tag{4.88}$$

Clearly, we can write

$$\begin{aligned} &\|\overline{\boldsymbol{\alpha}}_N(t)-\overline{\boldsymbol{\alpha}}_M(t)\|_{[L^2(\Omega)]^d} \\ &\le \|\overline{\boldsymbol{\alpha}}_N(t)-\boldsymbol{\Psi}_N(t)\|_{[L^2(\Omega)]^d} \\ &\quad + \|\overline{\boldsymbol{\alpha}}_M(t)-\boldsymbol{\Psi}_M(t)\|_{[L^2(\Omega)]^d} + \|\boldsymbol{\Psi}_M(t)-\boldsymbol{\Psi}_N(t)\|_{[L^2(\Omega)]^d} \end{aligned} \tag{4.89}$$

and, according to (4.87) and (4.88), we obtain

$$\begin{aligned} &\|\overline{\boldsymbol{\alpha}}_N(t)-\overline{\boldsymbol{\alpha}}_M(t)\|_{[L^2(\Omega)]^d} \\ &\le ch_N + ch_M + \Big\| \int_0^t \mathbf{C}_M(s)\mathrm{d}s - \int_0^t \mathbf{C}_N(s)\mathrm{d}s \Big\|_{[L^2(\Omega)]^d}. \end{aligned} \tag{4.90}$$

For the last term we have

$$\begin{aligned} &\Big\| \int_0^t \mathbf{C}_M(s)\mathrm{d}s - \int_0^t \mathbf{C}_N(s)\mathrm{d}s \Big\|_{[L^2(\Omega)]^d} \\ &\le \int_0^t \|\mathbf{C}_M(s)-\mathbf{C}_N(s)\|_{[L^2(\Omega)]^d}\,\mathrm{d}s \\ &\le c_1 \int_0^t \Big(\big\|\overline{\vartheta}_N(s-h_N)-\overline{\vartheta}_M(s-h_M)\big\|_{L^2(\Omega)} \\ &\quad + \|\overline{\boldsymbol{\sigma}}_N(s)-\overline{\boldsymbol{\sigma}}_M(s)\|_{[L^2(\Omega)]^4} \Big)\,\mathrm{d}s + c_2 \int_0^t \|\overline{\boldsymbol{\alpha}}_N(s)-\overline{\boldsymbol{\alpha}}_M(s)\|_{[L^2(\Omega)]^d}\,\mathrm{d}s \end{aligned} \tag{4.91}$$

and thus (4.90) becomes

$$\begin{aligned} &\|\overline{\boldsymbol{\alpha}}_N(t)-\overline{\boldsymbol{\alpha}}_M(t)\|_{[L^2(\Omega)]^d} \\ &\le c_1 \int_0^t \|\overline{\boldsymbol{\alpha}}_N(s)-\overline{\boldsymbol{\alpha}}_M(s)\|_{[L^2(\Omega)]^d}\,\mathrm{d}s + ch_N + ch_M \\ &\quad + c_1 \int_0^t \Big(\big\|\overline{\vartheta}_N(s-h_N)-\overline{\vartheta}_M(s-h_M)\big\|_{L^2(\Omega)} \\ &\quad + \|\overline{\boldsymbol{\sigma}}_N(s)-\overline{\boldsymbol{\sigma}}_M(s)\|_{[L^2(\Omega)]^4} \Big)\,\mathrm{d}s. \end{aligned} \tag{4.92}$$

A process similar to that used above implies that the same estimate can be obtained for $\overline{\boldsymbol{\varepsilon}^{p\ell}}_N$, i.e.

$$\begin{aligned} &\big\|\overline{\boldsymbol{\varepsilon}^{p\ell}}_N(t)-\overline{\boldsymbol{\varepsilon}^{p\ell}}_M(t)\big\|_{[L^2(\Omega)]^4} \\ &\le c_1 \int_0^t \|\overline{\boldsymbol{\alpha}}_N(s)-\overline{\boldsymbol{\alpha}}_M(s)\|_{[L^2(\Omega)]^d}\,\mathrm{d}s + ch_N + ch_M \\ &\quad + c_1 \int_0^t \Big(\big\|\overline{\vartheta}_N(s-h_N)-\overline{\vartheta}_M(s-h_M)\big\|_{L^2(\Omega)} \\ &\quad + \|\overline{\boldsymbol{\sigma}}_N(s)-\overline{\boldsymbol{\sigma}}_M(s)\|_{[L^2(\Omega)]^4} \Big)\,\mathrm{d}s. \end{aligned} \tag{4.93}$$

What remains is to estimate the last terms on the right-hand sides of (4.92) and (4.93). From (4.62) we have

$$\begin{aligned} &\int_\Omega [\overline{\boldsymbol{\sigma}}_N(t)-\overline{\boldsymbol{\sigma}}_M(t)] : \boldsymbol{\varepsilon}(\mathbf{v})\,\mathrm{d}x + \int_\Omega \nabla\cdot\mathbf{I}[\chi(\overline{p}_N(t))-\chi(\overline{p}_M(t))]\cdot\mathbf{v}\,\mathrm{d}x \\ &= \int_\Omega [\overline{\mathbf{f}}_N(t)-\overline{\mathbf{f}}_M(t)]\cdot\mathbf{v}\,\mathrm{d}x + \int_{\partial\Omega} [\bar{\breve{\mathbf{t}}}_N(t)-\bar{\breve{\mathbf{t}}}_M(t)]\cdot\mathbf{v}\,\mathrm{d}S \end{aligned} \tag{4.94}$$

for any $\mathbf{v} \in [W^{1,2}(\Omega)]^2$. Setting $\mathbf{v} = \overline{\mathbf{u}}_N(t) - \overline{\mathbf{u}}_M(t)$ and applying the Green formula to the second term we obtain

$$
\begin{aligned}
&\int_\Omega [\overline{\boldsymbol{\sigma}}_N(t) - \overline{\boldsymbol{\sigma}}_M(t)] : [\boldsymbol{\varepsilon}(\overline{\mathbf{u}}_N(t)) - \boldsymbol{\varepsilon}(\overline{\mathbf{u}}_M(t))] \,\mathrm{d}x \\
&+ \int_{\partial\Omega} [\chi(\overline{p}_N(t)) - \chi(\overline{p}_M(t))]\mathbf{n} \cdot [\overline{\mathbf{u}}_N(t) - \overline{\mathbf{u}}_M(t)] \,\mathrm{d}S \\
&- \int_\Omega \mathbf{I}[\chi(\overline{p}_N(t)) - \chi(\overline{p}_M(t))] : [\boldsymbol{\varepsilon}(\overline{\mathbf{u}}_N(t)) - \boldsymbol{\varepsilon}(\overline{\mathbf{u}}_M(t))] \,\mathrm{d}x \\
&= \int_\Omega [\overline{\mathbf{f}}_N(t) - \overline{\mathbf{f}}_M(t)] \cdot [\overline{\mathbf{u}}_N(t) - \overline{\mathbf{u}}_M(t)] \,\mathrm{d}x \\
&\quad + \int_{\partial\Omega} [\bar{\breve{\mathbf{t}}}_N(t) - \bar{\breve{\mathbf{t}}}_M(t)] \cdot [\overline{\mathbf{u}}_N(t) - \overline{\mathbf{u}}_M(t)] \,\mathrm{d}S.
\end{aligned} \tag{4.95}
$$

In view of (4.30) and (1.14), (4.95) becomes

$$
\begin{aligned}
&\int_\Omega [\overline{\boldsymbol{\sigma}}_N(t) - \overline{\boldsymbol{\sigma}}_M(t)] : \big[\frac{\partial P}{\partial \boldsymbol{\sigma}}(\overline{\boldsymbol{\sigma}}_N(t), \overline{\boldsymbol{\alpha}}_N(t)) - \frac{\partial P}{\partial \boldsymbol{\sigma}}(\overline{\boldsymbol{\sigma}}_M(t), \overline{\boldsymbol{\alpha}}_M(t))\big] \,\mathrm{d}x \\
&+ \int_\Omega [\overline{\boldsymbol{\sigma}}_N(t) - \overline{\boldsymbol{\sigma}}_M(t)] : [\overline{\boldsymbol{\varepsilon}^{p\ell}}_N(t) - \overline{\boldsymbol{\varepsilon}^{p\ell}}_M(t)] \,\mathrm{d}x \\
&+ \int_\Omega [\overline{\boldsymbol{\sigma}}_N(t) - \overline{\boldsymbol{\sigma}}_M(t)] : \beta \mathbf{I}(\overline{\vartheta}_N(t) - \overline{\vartheta}_M(t)) \,\mathrm{d}x \\
&+ \int_{\partial\Omega} [\chi(\overline{p}_N(t)) - \chi(\overline{p}_M(t))]\mathbf{n} \cdot [\overline{\mathbf{u}}_N(t) - \overline{\mathbf{u}}_M(t)] \,\mathrm{d}S \\
&- \int_\Omega \mathbf{I}[\chi(\overline{p}_N(t)) - \chi(\overline{p}_M(t))] : [\boldsymbol{\varepsilon}(\overline{\mathbf{u}}_N(t)) - \boldsymbol{\varepsilon}(\overline{\mathbf{u}}_M(t))] \,\mathrm{d}x \\
&= \int_\Omega [\overline{\mathbf{f}}_N(t) - \overline{\mathbf{f}}_M(t)] \cdot [\overline{\mathbf{u}}_N(t) - \overline{\mathbf{u}}_M(t)] \,\mathrm{d}x \\
&+ \int_{\partial\Omega} [\bar{\breve{\mathbf{t}}}_N(t) - \bar{\breve{\mathbf{t}}}_M(t)] \cdot [\overline{\mathbf{u}}_N(t) - \overline{\mathbf{u}}_M(t)] \,\mathrm{d}S.
\end{aligned} \tag{4.96}
$$

Using (4.20), (2.26) and Theorem 2.2 we deduce

$$
\begin{aligned}
&\|\overline{\boldsymbol{\sigma}}_N(t) - \overline{\boldsymbol{\sigma}}_M(t)\|^2_{[L^2(\Omega)]^4} \\
&\le c_1 \|\overline{\boldsymbol{\sigma}}_N(t) - \overline{\boldsymbol{\sigma}}_M(t)\|_{[L^2(\Omega)]^4} \|\overline{\boldsymbol{\alpha}}_N(t) - \overline{\boldsymbol{\alpha}}_M(t)\|_{[L^2(\Omega)]^d} \\
&+ c_2 \|\overline{\boldsymbol{\sigma}}_N(t) - \overline{\boldsymbol{\sigma}}_M(t)\|_{[L^2(\Omega)]^4} \left\|\overline{\boldsymbol{\varepsilon}^{p\ell}}_N(t) - \overline{\boldsymbol{\varepsilon}^{p\ell}}_M(t)\right\|_{[L^2(\Omega)]^4} \\
&+ c_3 \|\overline{\boldsymbol{\sigma}}_N(t) - \overline{\boldsymbol{\sigma}}_M(t)\|_{[L^2(\Omega)]^4} \left\|\overline{\vartheta}_N(t) - \overline{\vartheta}_M(t)\right\|_{L^2(\Omega)} \\
&+ c_4 \|\overline{p}_N(t) - \overline{p}_M(t)\|_{L^2(\partial\Omega)} \|\boldsymbol{\varepsilon}(\overline{\mathbf{u}}_N(t)) - \boldsymbol{\varepsilon}(\overline{\mathbf{u}}_M(t))\|_{[L^2(\Omega)]^4} \\
&+ c_5 \|\overline{p}_N(t) - \overline{p}_M(t)\|_{L^2(\Omega)} \|\boldsymbol{\varepsilon}(\overline{\mathbf{u}}_N(t)) - \boldsymbol{\varepsilon}(\overline{\mathbf{u}}_M(t))\|_{[L^2(\Omega)]^4} \\
&+ c_6 \left\|\overline{\mathbf{f}}_N(t) - \overline{\mathbf{f}}_M(t)\right\|_{[L^2(\Omega)]^2} \|\boldsymbol{\varepsilon}(\overline{\mathbf{u}}_N(t)) - \boldsymbol{\varepsilon}(\overline{\mathbf{u}}_M(t))\|_{[L^2(\Omega)]^4} \\
&+ c_7 \left\|\bar{\breve{\mathbf{t}}}_N(t) - \bar{\breve{\mathbf{t}}}_M(t)\right\|_{[L^2(\partial\Omega)]^2} \|\boldsymbol{\varepsilon}(\overline{\mathbf{u}}_N(t)) - \boldsymbol{\varepsilon}(\overline{\mathbf{u}}_M(t))\|_{[L^2(\Omega)]^4} .
\end{aligned} \tag{4.97}
$$

Indeed, from (1.13)–(1.15) we have

$$
\begin{aligned}
&\|\varepsilon(\overline{\mathbf{u}}_N(t)) - \varepsilon(\overline{\mathbf{u}}_M(t))\|_{[L^2(\Omega)]^4} \\
&\leq \Big\|\frac{\partial P}{\partial \boldsymbol{\sigma}}(\overline{\boldsymbol{\sigma}}_N(t), \overline{\boldsymbol{\alpha}}_N(t)) - \frac{\partial P}{\partial \boldsymbol{\sigma}}(\overline{\boldsymbol{\sigma}}_M(t), \overline{\boldsymbol{\alpha}}_M(t))\Big\|_{[L^2(\Omega)]^4} \\
&+ \Big\|\overline{\boldsymbol{\varepsilon}^{p\ell}}_N(t) - \overline{\boldsymbol{\varepsilon}^{p\ell}}_M(t)\Big\|_{[L^2(\Omega)]^4} + c\,\big\|\overline{\vartheta}_N(t) - \overline{\vartheta}_M(t)\big\|_{L^2(\Omega)} \\
&\leq c_1\,\|\overline{\boldsymbol{\sigma}}_N(t) - \overline{\boldsymbol{\sigma}}_M(t)\|_{[L^2(\Omega)]^4} + |\overline{\boldsymbol{\alpha}}_N(t) - \overline{\boldsymbol{\alpha}}_M(t)\|_{[L^2(\Omega)]^d} \\
&+ \Big\|\overline{\boldsymbol{\varepsilon}^{p\ell}}_N(t) - \overline{\boldsymbol{\varepsilon}^{p\ell}}_M(t)\Big\|_{[L^2(\Omega)]^4} + c_2\,\big\|\overline{\vartheta}_N(t) - \overline{\vartheta}_M(t)\big\|_{L^2(\Omega)}\,.
\end{aligned}
\tag{4.98}
$$

Finally, using (4.98) in (4.97) and applying the Young inequality with parameter $\delta > 0$ we arrive at

$$
\begin{aligned}
&\|\overline{\boldsymbol{\sigma}}_N(t) - \overline{\boldsymbol{\sigma}}_M(t)\|_{[L^2(\Omega)]^4} \\
&\leq c_1(\delta)\,\|\overline{\boldsymbol{\alpha}}_N(t) - \overline{\boldsymbol{\alpha}}_M(t)\|_{[L^2(\Omega)]^d} \\
&\quad + c_2(\delta)\,\Big\|\overline{\boldsymbol{\varepsilon}^{p\ell}}_N(t) - \overline{\boldsymbol{\varepsilon}^{p\ell}}_M(t)\Big\|_{[L^2(\Omega)]^4} + c_3(\delta)\,\big\|\overline{\vartheta}_N(t) - \overline{\vartheta}_M(t)\big\|_{L^2(\Omega)} \\
&\quad + c_4(\delta)\,\|\overline{p}_N(t) - \overline{p}_M(t)\|_{L^2(\partial\Omega)} + c_5(\delta)\,\|\overline{p}_N(t) - \overline{p}_M(t)\|_{L^2(\Omega)} \\
&\quad + c_6(\delta)\,\big\|\overline{\mathbf{f}}_N(t) - \overline{\mathbf{f}}_M(t)\big\|_{[L^2(\Omega)]^2} + c_7(\delta)\,\Big\|\overline{\breve{\mathbf{t}}}_N(t) - \overline{\breve{\mathbf{t}}}_M(t)\Big\|_{[L^2(\partial\Omega)]^2}\,.
\end{aligned}
\tag{4.99}
$$

Substituting (4.99) into (4.92) and (4.93) we obtain

$$
\begin{aligned}
&\|\overline{\boldsymbol{\alpha}}_N(t) - \overline{\boldsymbol{\alpha}}_M(t)\|_{[L^2(\Omega)]^d} \\
&\leq c_1 \int_0^t \|\overline{\boldsymbol{\alpha}}_N(s) - \overline{\boldsymbol{\alpha}}_M(s)\|_{[L^2(\Omega)]^d}\,\mathrm{d}s \\
&\quad + c_2 \int_0^t \Big\|\overline{\boldsymbol{\varepsilon}^{p\ell}}_N(s) - \overline{\boldsymbol{\varepsilon}^{p\ell}}_M(s)\Big\|_{[L^2(\Omega)]^4}\,\mathrm{d}s + c_3 h_N + c_3 h_M \\
&\quad + c_4 \int_0^t \big\|\overline{\vartheta}_N(s-h_N) - \overline{\vartheta}_M(s-h_M)\big\|_{L^2(\Omega)}\,\mathrm{d}s \\
&\quad + c_5 \int_0^t \big\|\overline{\vartheta}_N(s) - \overline{\vartheta}_M(s)\big\|_{L^2(\Omega)}\,\mathrm{d}s \\
&\quad + c_6 \int_0^t \|\overline{p}_N(s) - \overline{p}_M(s)\|_{L^2(\partial\Omega)}\,\mathrm{d}s + c_7 \int_0^t \|\overline{p}_N(s) - \overline{p}_M(s)\|_{L^2(\Omega)}\,\mathrm{d}s \\
&\quad + c_8 \int_0^t \big\|\overline{\mathbf{f}}_N(s) - \overline{\mathbf{f}}_M(s)\big\|_{[L^2(\Omega)]^2}\,\mathrm{d}s \\
&\quad + c_9 \int_0^t \Big\|\overline{\breve{\mathbf{t}}}_N(s) - \overline{\breve{\mathbf{t}}}_M(s)\Big\|_{[L^2(\partial\Omega)]^2}\,\mathrm{d}s
\end{aligned}
\tag{4.100}
$$

and

$$
\begin{aligned}
&\left\|\overline{\boldsymbol{\varepsilon}^{p\ell}}_N(t) - \overline{\boldsymbol{\varepsilon}^{p\ell}}_M(t)\right\|_{[L^2(\Omega)]^4} \\
&\le c_1 \int_0^t \|\overline{\boldsymbol{\alpha}}_N(s) - \overline{\boldsymbol{\alpha}}_M(s)\|_{[L^2(\Omega)]^d} \, \mathrm{d}s \\
&\quad + c_2 \int_0^t \left\|\overline{\boldsymbol{\varepsilon}^{p\ell}}_N(s) - \overline{\boldsymbol{\varepsilon}^{p\ell}}_M(s)\right\|_{[L^2(\Omega)]^4} \, \mathrm{d}s + c_3 h_N + c_3 h_M \\
&\quad + c_4 \int_0^t \left\|\overline{\vartheta}_N(s-h_N) - \overline{\vartheta}_M(s-h_M)\right\|_{L^2(\Omega)} \, \mathrm{d}s \\
&\quad + c_5 \int_0^t \left\|\overline{\vartheta}_N(s) - \overline{\vartheta}_M(s)\right\|_{L^2(\Omega)} \, \mathrm{d}s \\
&\quad + c_6 \int_0^t \|\overline{p}_N(s) - \overline{p}_M(s)\|_{L^2(\partial\Omega)} \, \mathrm{d}s + c_7 \int_0^t \|\overline{p}_N(s) - \overline{p}_M(s)\|_{L^2(\Omega)} \, \mathrm{d}s \\
&\quad + c_8 \int_0^t \left\|\overline{\mathbf{f}}_N(s) - \overline{\mathbf{f}}_M(s)\right\|_{[L^2(\Omega)]^2} \, \mathrm{d}s \\
&\quad + c_9 \int_0^t \left\|\overline{\breve{\mathbf{t}}}_N(s) - \overline{\breve{\mathbf{t}}}_M(s)\right\|_{[L^2(\partial\Omega)]^2} \, \mathrm{d}s,
\end{aligned}
\tag{4.101}
$$

respectively. Adding (4.100) with (4.101), applying the Gronwall lemma and using (4.78) and (4.84), we conclude that

$$
\begin{gathered}
\|\overline{\boldsymbol{\alpha}}_N(t) - \overline{\boldsymbol{\alpha}}_M(t)\|_{[L^2(\Omega)]^d} \to 0 \quad \text{as } M,\, N \to \infty, \\
\|\overline{\boldsymbol{\varepsilon}^{p\ell}}_N(t) - \overline{\boldsymbol{\varepsilon}^{p\ell}}_M(t)\|_{[L^2(\Omega)]^4} \to 0 \quad \text{as } M,\, N \to \infty
\end{gathered}
$$

independently of $t \in I$. This means that $\{\overline{\boldsymbol{\alpha}}_N(t)\}$ and $\left\{\overline{\boldsymbol{\varepsilon}^{p\ell}}_N(t)\right\}$ are Cauchy sequences and thus $\overline{\boldsymbol{\alpha}}_N(t) \to \boldsymbol{\alpha}(t)$ in $[L^2(\Omega)]^d$ and $\overline{\boldsymbol{\varepsilon}^{p\ell}}_N(t) \to \boldsymbol{\varepsilon}^{p\ell}(t)$ in $[L^2(\Omega)]^4$ uniformly with respect to $t \in I$.

By similar arguments, from (4.99) and in view of (4.78) and (4.84) we have

$$
\overline{\boldsymbol{\sigma}}_N \to \boldsymbol{\sigma} \quad \text{in } L^2(I;[L^2(\Omega)]^4)
$$

and almost everywhere on Ω_T.

The above established convergences are sufficient for taking the limit $N \to \infty$ in (4.60)–(4.64) (along a selected subsequence) to get the weak solution of the system (1.1)–(1.9) in the sense of Definition 3.1. This completes the proof of the main result stated in Theorem 3.2.

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