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CONVERGENCE OF DELAY EQUATIONS DRIVEN BY A HÖLDER CONTINUOUS FUNCTION OF ORDER $1/3 < \beta < 1/2$

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Abstract. In this article we show that, when the delay approaches zero, the solution of multidimensional delay differential equations driven by a Hölder continuous function of order $1/3 < \beta < 1/2$ converges with the supremum norm to the solution for the equation without delay. Finally we discuss the applications to stochastic differential equations

1. Introduction

Hu and Nualart [9] used fractional calculus to establish the existence and uniqueness of a solution for the dynamical system

$$dx_t = f(x_t)\, dy_t,$$

where y is a Hölder continuous function of order $1/3 < \beta < 1/2$. They give an explicit expression for the integral $\int_0^t f(x_s)dy_s$ that depends on the functions x, y and a quadratic multiplicative functional $x \otimes y$. As an example of a path-wise approach to classical stochastic calculus, they apply these results to solve stochastic differential equations driven by a multidimensional Brownian motion. Using the same approach, Besalú and Nualart [2] obtained estimates for the supremum norm of the solution and Besalú et al. [1] studied delay equations with non-negativity constraints.

The work by Hu and Nualart [9] is an extension of the previous paper of Nualart and Răşcanu [16], where they study the dynamical systems $dx_t = f(x_t)dy_t$ and the control function y is Hölder continuous of order $\beta > 1/2$. In this case the Riemann-Stieltjes integral $\int_0^t f(x_s)dy_s$ can be expressed as a Lebesgue integral using fractional derivatives following the ideas by Zähle [19].

All these papers have to be seen in the framework of the theory of rough path analysis and the path-wise approach to classical stochastic calculus. This theory has been developed from the initial paper by Lyons [12] and has generated many publications (see, for instance, Lyons and Qian [13], Friz and Victoir [5], Lejay [11] or Gubinelli [8]). We refer to Coutin and Lejay [3], Friz and Victoir [6], Friz [7] and Ledoux et al. [14] for some applications of rough path analysis to stochastic calculus.

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Delay differential equations rise from the need to study models that behave more like real processes. They find their applications in dynamical systems with aftereffects or when the dynamics are subjected to propagation delay. Some examples are epidemiological models with incubation periods that postpone the transmission of disease, or neuronal models where the spatial distribution of neurons can cause a delay in the transmission of the impulse. Sometimes the delay avoids some usual problems, but in general, it adds difficulties and cumbersome notations.

The purpose of our paper is to consider the differential equation with delay

$$\begin{gathered} x_t^r = \eta_0 + \int_0^t b(u, x^r)\,du + \int_0^t \sigma(x_{u-r}^r)\,dy_u, \quad t \in (0, T], \\ x_t^r = \eta_t, \quad t \in [-r, 0], \end{gathered}$$

where r denotes a strictly positive time delay, $\eta : [-r, 0] \to \mathbb{R}^d$ is a smooth function, y is a Hölder continuous function of order $\beta \in (\frac{1}{3}, \frac{1}{2})$ and the hereditary term $b(u, x)$ depends on the path $\{x_s,\, 0 \le s \le u\}$. From Hu and Nualart [9] and Besalú et al. [1] it is easy to check that there exists a unique solution to this equation. Our aim is to prove that it converges almost surely in the supremum norm to the solution of the differential equation without delay

$$x_t = \eta_0 + \int_0^t b(u, x_u)\,du + \int_0^t \sigma(x_u)\,dy_u, \quad t \in [0, T],$$

when the delay tends to zero. Our approach is based on the techniques of the classical fractional calculus and it is inspired by [9]. Finally, we apply these results to stochastic differential equations driven by Brownian motion.

The case when $\beta > 1/2$ has been studied by Ferrante and Rovira in [4]. They proved that the solution to the delay equation converges, almost surely and in L^p, to the solution to the equation without delay and then apply the result pathwise to fractional Brownian motion with Hurst parameter $H > 1/2$.

With a different approach based on a slight variation of the Young integration theory, called algebraic integration, León and Tindel [10] prove the existence of a unique solution for a general class of delay differential equations driven by a Hölder continuous function with parameter greater than 1/2. They obtain some estimates of the solution which allow to show that the solution to a delay differential equation driven by a fractional Brownian motion with Hurst parameter $H > 1/2$ has a $\mathcal{C}^\infty$-density.

When $\beta < 1/2$ more difficulties appear. In the literature we find results only up to the value $\beta > 1/3$, eventually extended to $\beta > 1/4$. In [15], Neuenkirch, Nourdin and Tindel consider delay differential equation driven by a β-Hölder continuous function with $\beta > 1/3$. The authors show the existence of a unique solution for these equations under suitable hypothesis. Then, they apply these results to a delay differential equation driven by a fractional Brownian motion with Hurst parameter $H > 1/3$. These results are extended by Tindel and Torrecilla in [18] to the deterministic case with $\beta > 1/4$ and the corresponding stochastic case with Hurst parameter $H > 1/4$.

This article is organized as follows. The following section is devoted to introduce some notation. In section 3 we define the equations and the solutions we work with and we describe our main result. Section 4 contains technical estimates for the study of the integrals. In section 5 we give some estimates for the solutions of our equations. In section 6 we give the proof of the main theorem. In the last section we give an example of applications of the main theorem, studying stochastic differential equations driven by Brownian motion.

2. Preliminaries

First, we recall some definitions and results presented in Hu and Nualart [9]. Fix a time interval $[0,T]$ and $0<\beta\le 1$. For any function $x:[0,T]\to\mathbb{R}^d$, the β-Hölder norm of x on the interval $[s,t]\subset[0,T]$ will be denoted by

$$\|x\|_{\beta(s,t)}=\sup_{s\le u<v\le t}\frac{|x_v-x_u|}{(v-u)^\beta}.$$

If $\Delta_T:=\{(s,t):0\le s<t\le T\}$, for any $(s,t)\in\Delta_T$ and for any $g:\Delta_T\to\mathbb{R}^d$ we set

$$\|g\|_{\beta(s,t)}=\sup_{s\le u<v\le t}\frac{|g_{u,v}|}{(v-u)^\beta}.$$

We will also set $\|x\|_\beta=\|x\|_{\beta(0,T)}$ and $\|x\|_{\beta(r)}=\|x\|_{\beta(-r,T)}$. Moreover, $\|\cdot\|_{\infty(s,t)}$ will denote the supremum norm in the interval $[s,t]$, and for simplicity $\|x\|_\infty=\|x\|_{\infty(0,T)}$ and $\|x\|_{\infty(r)}=\|x\|_{\infty(-r,T)}$.

Hu and Nualart [9] proved an explicit formula for the integral $\int_a^b f(x_u)\,dy_u$ in terms of x, y and $x\otimes y$, and transformed the dynamical system $dx_t=f(x_t)\,dy_t$ into a closed system of equations involving only x, $x\otimes y$ and $x\otimes(y\otimes y)$. From Lyons [12] we introduce the definition of $x\otimes y$.

Definition 2.1. We say that $(x,y,x\otimes y)$ is a (d,m)-dimensional β-Hölder continuous multiplicative functional if:

(1) $x:[0,T]\to\mathbb{R}^d$ and $y:[0,T]\to\mathbb{R}^m$ are β-Hölder continuous functions,
(2) $x\otimes y:\Delta_T\to\mathbb{R}^d\otimes\mathbb{R}^m$ is a continuous function satisfying the following properties:
   (a) (Multiplicative property) For all $s\le u\le t$ we have
   $$(x\otimes y)_{s,u}+(x\otimes y)_{u,t}+(x_u-x_s)\otimes(y_t-y_u)=(x\otimes y)_{s,t}.$$
   (b) For all $(s,t)\in\Delta_T$, $|(x\otimes y)_{s,t}|\le c|t-s|^{2\beta}$.

We denote by $M^\beta_{d,m}(0,T)$ the space of (d,m)-dimensional β-Hölder continuous multiplicative functionals. Furthermore, we will denote by $M^\beta_{d,m}(a,b)$ the obvious extension of the definition $M^\beta_{d,m}(0,T)$ to a general interval $[a,b]$. We introduce the following functional defined on $M^\beta_{d,m}(0,T)$ for $(a,b)\in\Delta_T$:

$$\Phi_{\beta(a,b)}(x,y)=\|x\otimes y\|_{2\beta(a,b)}+\|x\|_{\beta(a,b)}\|y\|_{\beta(a,b)}. \tag{2.1}$$

Moreover, if $(x,y,x\otimes y)$ and $(y,z,y\otimes z)$ belongs to $M^\beta_{d,m}(0,T)$ we define

$$\begin{aligned}\Phi_{\beta(a,b)}(x,y,z)&=\|x\|_{\beta(a,b)}\|y\|_{\beta(a,b)}\|z\|_{\beta(a,b)}+\|z\|_{\beta(a,b)}\|x\otimes y\|_{2\beta(a,b)}\\&\quad+\|x\|_{\beta(a,b)}\|y\otimes z\|_{2\beta(a,b)}.\end{aligned} \tag{2.2}$$

From these definitions it follows that

$$\|(x\otimes y)_{\cdot,b}\|_{\beta(a,b)}\le\Phi_{\beta(a,b)}(x,y)(b-a)^\beta, \tag{2.3}$$

$$\big\|x\otimes(y\otimes z)_{\cdot,b}\big\|_{2\beta(a,b)}\le K\Phi_{\beta(a,b)}(x,y,z)(b-a)^\beta \tag{2.4}$$

which are equations (3.29) and (3.30) in [9]. We refer to [9] and [12] for a more detailed presentation on β-Hölder continuous multiplicative functionals.

To define the integral $\int_a^b f(x_u)\,dy_u$ we use the construction of the integral given by Hu and Nualart in [9], who were inspired by the work of Zälhe [19] and use fractional derivatives. We refer to Hu and Nualart in [9] for the details.

In the sequel, K denotes a generic constant that may depend on the parameters β, α, λ and T and can vary from line to line.

3. Main result

Consider the differential equation with delay on $\mathbb{R}^d$,

$$
\begin{gathered}
x_t^r = \eta_0 + \int_0^t b(u, x_u^r)\,du + \int_0^t \sigma(x_{u-r}^r)\,dy_u, \quad t \in [0,T], \\
x_t^r = \eta_t, \quad t \in [-r, 0),
\end{gathered} \tag{3.1}
$$

where x and y are Hölder continuous functions of order $\beta \in (\frac{1}{3}, \frac{1}{2})$, η is a continuous function and r denotes a strictly positive time delay. We set the following hypotheses:

- (H1) $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^m$ is a bounded and continuously twice differentiable function such that σ' and σ'' are bounded and λ-Hölder continuous for $\lambda > \frac{1}{\beta} - 2$.
- (H2) $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a measurable function such that there exists $b_0 \in L^\rho(0,T;\mathbb{R}^d)$ with $\rho \geq 2$ and $\forall N \geq 0$ there exists $L_N > 0$ such that:
  - (1) $|b(t,x_t) - b(t,y_t)| \leq L_N |x_t - y_t|$, $\forall x, y$ such that $|x_t| \leq N$, $|y_t| \leq N$ $\forall t \in [0,T]$,
  - (2) $|b(t,x_t)| \leq L_0 |x_t| + b_0(t)$, $\quad \forall t \in [0,T]$.
- (H3) σ and b are bounded functions.

Conditions (H1) and (H2) are a particular case of the hypotheses for the proof of existence and uniqueness of solution to the delay equation (3.1), while condition (H3) is necessary to prove that the solution is bounded.

We denote by $(x, y, x \otimes y) \in M_{d,m}^\beta(0,T)$ the solution to the stochastic differential equation on $\mathbb{R}^d$ without delay

$$
x_t = \eta_0 + \int_0^t b(u, x_u)\,du + \int_0^t \sigma(x_u)\,dy_u, \quad t \in [0,T]. \tag{3.2}
$$

Assuming that $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^m$ is a continuously differentiable function such that σ' is λ-Hölder continuous with $\lambda > \frac{1}{\beta} - 2$, σ and σ' are bounded and $(y, y, y \otimes y) \in M_{m,m}^\beta(0,T)$, Hu and Nualart [9] prove the existence of a bounded solution in $M_{d,m}^\beta(0,T)$ for the differential equation (3.2) with $b \equiv 0$. Moreover, if σ is twice continuously differentiable with bounded derivatives and σ'' is λ-Hölder continuous, with $\lambda > \frac{1}{\beta} - 2$, the solution is unique. Here the authors deal with the equation without the hereditary term, but the results can be easily extended to the case where the hereditary term does not vanish. If $(y, y, y \otimes y) \in M_{m,m}^\beta(0,T)$, then we can consider

$$
(x \otimes y)_{s,t} = \int_s^t (y_t - y_u) b(u, x_u)\,du + \int_s^t \sigma(x_u)\,d_u (y \otimes y)_{\cdot,t}\,. \tag{3.3}
$$

And $(x, y, x \otimes y) \in M_{d,m}^\beta(0,T)$ will be a solution to (3.2) for x and $(x \otimes y)$ such that (3.2) and (3.3) hold, respectively.

On the other hand, following the ideas contained in [1], it is easy to show that there exists a unique solution to the delay equation (3.1) in $M^{\beta}_{d,m}(-r,T)$. It can be easily proved assuming that σ and b satisfy the hypotheses (H1) and (H2), respectively, with $\rho \geq \frac{1}{1-\beta}$ and that $(\eta_{\cdot-r}, y, \eta_{\cdot-r}\otimes y) \in M^{\beta}_{d,m}(0,r)$ and $(y_{\cdot-r}, y, y_{\cdot-r}\otimes y) \in M^{\beta}_{m,m}(r,T)$. If we also assume that hypothesis (H3) is satisfied, we can obtain that the solution is bounded. So, $(x^r, y, x^r\otimes y) \in M^{\beta}_{d,m}(-r,T)$ is the unique solution to (3.1) for x^r such that (3.1) holds and $(x^r\otimes y)_{s,t}$ is defined as follows:

- for $s<t\in[-r,0)$,
$$(x^r\otimes y)_{s,t} = (\eta\otimes y)_{s,t} = \int_s^t (y_t - y_u)\, d\eta_u,$$
- for $s\in[-r,0)$ and $t\in[0,T]$,
$$\begin{aligned}(x^r\otimes y)_{s,t} &= (\eta\otimes y)_{s,0} + \int_0^t (y_t-y_u) b(u,x^r_u)\, du \\ &\quad + \int_0^t \sigma(x^r_{u-r})\, d_u(y\otimes y)_{\cdot,t} + (\eta_0-\eta_s)\otimes(y_t-y_0),\end{aligned}$$
- for $0\leq s<t\leq T$,
$$(x^r\otimes y)_{s,t} = \int_s^t (y_t-y_u) b(u,x^r_u)\, du + \int_s^t \sigma(x^r_{u-r})\, d_u(y\otimes y)_{\cdot,t}.$$

Let $\beta\in(\frac13,\frac12)$ and set $\beta'=\beta-\varepsilon$, where $\varepsilon>0$ is such that $\beta-2\varepsilon>0$ and $\lambda>\frac{1}{\beta-\varepsilon}-2$. Set $r_0\in(0,T)$. The main result of this article is the following theorem.

Theorem 3.1. *Suppose that $(y,y,y\otimes y)$ belongs to $M^{\beta}_{m,m}(0,T)$ and $(y_{\cdot-r}, y, y_{\cdot-r}\otimes y)$ belongs to $M^{\beta}_{m,m}(r,T)$ for all $0<r\leq r_0$. Assume that σ and b satisfy* (H1) *and* (H2)*, respectively, and both satisfy* (H3)*. Assume also that $(\eta_{\cdot-r_0}, y, \eta_{\cdot-r_0}\otimes y)\in M^{\beta}_{d,m}(0,r_0)$, $\|\eta\|_{\beta(-r_0,0)}<\infty$ and $\sup_{r\leq r_0}\Phi_{\beta(0,r)}(\eta_{\cdot-r},y)<\infty$. Suppose that $\|(y-y_{\cdot-r})\otimes y\|_{2\beta'(r,T)}\to 0$ and $\|y_{\cdot-r}\otimes(y-y_{\cdot-r})\|_{2\beta'(r,T)}\to 0$ when r tends to zero. Then, $(x,y,x\otimes y)\in M^{\beta}_{d,m}(0,T)$ the solution to the stochastic differential equation without delay* (3.2) *and $(x^r,y,x^r\otimes y)\in M^{\beta}_{d,m}(-r,T)$ the solutions to the stochastic differential equations with delay* (3.1) *satisfy that*
$$\lim_{r\to0}\|x-x^r\|_\infty = 0 \quad \text{and} \quad \lim_{r\to0}\|(x\otimes y)-(x^r\otimes y)\|_\infty = 0.$$

4. Estimates of integrals

In this section we will give some estimates for the integrals appearing in our equations. We begin recalling versions of [9, Propositions 3.4 and 3.9].

Proposition 4.1. *Let $(x,y,x\otimes y)$ be in $M^{\beta}_{d,m}(0,T)$. Assume that $f:\mathbb{R}^d\to\mathbb{R}^m$ is a continuous differentiable function such that f' is bounded and λ-Hölder continuous, where $\lambda>\frac{1}{\beta}-2$. Then, for any $0\leq a<b\leq T$, we have*
$$\begin{aligned}\Big|\int_a^b f(x_u)\, dy_u\Big| &\leq K|f(x_a)|\,\|y\|_{\beta(a,b)}(b-a)^\beta + K\,\Phi_{\beta(a,b)}(x,y) \\ &\quad\times\Big(\|f'\|_\infty + \|f'\|_\lambda \|x\|^{\lambda}_{\beta(a,b)}(b-a)^{\lambda\beta}\Big)(b-a)^{2\beta},\end{aligned}$$

where $\Phi_{\beta(a,b)}(x,y)$ is defined by (2.1)*.*

Proposition 4.2. *Suppose that $(x,y,x\otimes y)$ and $(y,z,y\otimes z)$ belong to $M^{\beta}_{d,m}(0,T)$. Let $f:\mathbb{R}^d\to\mathbb{R}^m$ be a continuously differentiable function such that f' is λ-Hölder continuous and bounded, where $\lambda>\frac{1}{\beta}-2$. Then, the following estimate holds*

$$
\begin{aligned}
&\Big|\int_a^b f(x_u)\,d_u(y\otimes z)_{\cdot,b}\Big| \\
&\le K|f(x_a)|\,\Phi_{\beta(a,b)}(y,z)(b-a)^{2\beta} \\
&\quad + K\Big(\|f'\|_\infty+\|f'\|_\lambda\|x\|^{\lambda}_{\beta(a,b)}(b-a)^{\lambda\beta}\Big)\,\Phi_{\beta(a,b)}(x,y,z)(b-a)^{3\beta},
\end{aligned}
$$

where $\Phi_{\beta(a,b)}(x,y,z)$ is defined in (2.2)*.*

The following propositions give some useful estimates for proving Theorem 3.1. First, we give an estimate for a function b that fulfills conditions (H2).

Proposition 4.3. *Assume that b satisfies* (H2)*. Let $x,\widetilde{x}\in C(0,T;\mathbb{R}^d)$ such that $\|x\|_\infty\le N$ and $\|\widetilde{x}\|_\infty\le N$. Then, for $0\le a<b\le T$,*

$$
\Big|\int_a^b \big[b(u,x_u)-b(u,\widetilde{x}_u)\big]\,du\Big| \le L_N(b-a)\|x-\widetilde{x}\|_{\infty(a,b)}.
$$

The proof of the above proposition follows easily using the Lipschitz property of hypothesis (H2). To give some results for a function f under conditions (H1) we need to introduce some notation. Let

$$
\begin{aligned}
G^1_{\beta(a,b)}(f,x,\widetilde{x},y) &= K\Big[\|y\|_\beta\,\|f'\|_\infty+\big(\|f''\|_\infty+\|f''\|_\lambda(\|x\|^{\lambda}_{\beta(a,b)} \\
&\quad +\|\widetilde{x}\|^{\lambda}_{\beta(a,b)})(b-a)^{\lambda\beta}\big)\big(\Phi_{\beta(a,b)}(x,y)+\|y\|_\beta\,\|\widetilde{x}\|_{\beta(a,b)}\big)\Big], \\
G^2_{\beta(a,b)}(f,x,\widetilde{x},y) &= K\Big[\|y\|_\beta\,\|f'\|_\infty+\|f''\|_\infty\big(\Phi_{\beta(a,b)}(x,y) \\
&\quad +\|y\|_\beta\,\|\widetilde{x}\|_{\beta(a,b)}\big)(b-a)^{\beta}\Big], \\
G^3_{\beta(a,b)}(f,\widetilde{x}) &= K\Big[\|f'\|_\infty+\|f''\|_\infty\|\widetilde{x}\|_{\beta(a,b)}(b-a)^{\beta}\Big].
\end{aligned}
$$

The first result corresponds to Hu and Nualart [9, Proposition 6.4].

Proposition 4.4. *Suppose that $(x,y,x\otimes y)$ and $(\widetilde{x},y,\widetilde{x}\otimes y)$ belong to $M^{\beta}_{d,m}(0,T)$. Assume that f satisfies* (H1)*. Then, for $0\le a<b\le T$,*

$$
\begin{aligned}
\Big|\int_a^b [f(x_u)-f(\widetilde{x}_u)]\,dy_u\Big| &\le G^1_{\beta(a,b)}(f,x,\widetilde{x},y)(b-a)^{2\beta}\|x-\widetilde{x}\|_{\infty(a,b)} \\
&\quad + G^2_{\beta(a,b)}(f,x,\widetilde{x},y)(b-a)^{2\beta}\|x-\widetilde{x}\|_{\beta(a,b)} \\
&\quad + G^3_{\beta(a,b)}(f,\widetilde{x})(b-a)^{2\beta}\|(x-\widetilde{x})\otimes y\|_{2\beta(a,b)}.
\end{aligned}
$$

From this we can deduce the following estimate.

Proposition 4.5. *Assume $(x,y,x\otimes y)$ and $(x_{\cdot-r},y,x_{\cdot-r}\otimes y)$ belong to $M^{\beta}_{d,m}(0,T)$, and f satisfies* (H1)*. Then, for $0\le a<b\le T$,*

$$
\Big|\int_a^b [f(x_u)-f(x_{u-r})]\,dy_u\Big| \le G^1_{\beta(a,b)}(f,x,x_{\cdot-r},y)(b-a)^{2\beta}\|x-x_{\cdot-r}\|_{\infty(a,b)}
$$

$$
\begin{aligned}
&+ G^2_{\beta(a,b)}(f, x, x_{\cdot-r}, y)(b-a)^{2\beta}\|x - x_{\cdot-r}\|_{\beta(a,b)} \\
&+ G^3_{\beta(a,b)}(f, x_{\cdot-r})(b-a)^{2\beta}\|(x - x_{\cdot-r}) \otimes y\|_{2\beta(a,b)}.
\end{aligned}
$$

The above proposition is a particular case of Proposition 4.4 with $\widetilde{x} \equiv x_{\cdot-r}$. Let us introduce more useful notation:

$$
\begin{aligned}
G^4_{\beta(a,b)}(f, x, \widetilde{x}, y, z) &= K\Big[\|f'\|_\infty \Phi_{\beta(a,b)}(y, z) \\
&\quad + \big(\|f''\|_\infty + \|f''\|_\lambda(\|x\|^\lambda_{\beta(a,b)} + \|\widetilde{x}\|^\lambda_{\beta(a,b)})(b-a)^{\lambda\beta}\big) \\
&\quad \times \big(\Phi_{\beta(a,b)}(x, y, z) + \|\widetilde{x}\|_{\beta(a,b)}\Phi_{\beta(a,b)}(y, z)\big)\Big], \\
G^5_{\beta(a,b)}(f, x, \widetilde{x}, y, z) &= K\Big[\big(\|f'\|_\infty + \|f''\|_\infty\|\widetilde{x}\|_{\beta(a,b)}(b-a)^\beta\big)\Phi_{\beta(a,b)}(y, z) \\
&\quad + \|f''\|_\infty\Phi_{\beta(a,b)}(x, y, z)(b-a)^\beta\Big], \\
G^6_{\beta(a,b)}(f, \widetilde{x}, z) &= KG^3_{\beta(a,b)}(f, \widetilde{x})\|z\|_{\beta(a,b)}.
\end{aligned}
$$

From the previous results it is possible to prove the following two propositions.

Proposition 4.6. *Suppose that $(x, y, x \otimes y)$, $(\widetilde{x}, y, \widetilde{x} \otimes y)$ and $(y, z, y \otimes z)$ belong to $M^\beta_{d,m}(0, T)$. Assume that f satisfies* (H1)*. Then, for $0 \le a < b \le T$,*

$$
\begin{aligned}
\Big|\int_a^b [f(x_u) - f(\widetilde{x}_u)]\, d_u(y \otimes z)_{\cdot,b}\Big| &\le G^4_{\beta(a,b)}(f, x, \widetilde{x}, y, z)(b-a)^{3\beta}\|x - \widetilde{x}\|_{\infty(a,b)} \\
&\quad + G^5_{\beta(a,b)}(f, x, \widetilde{x}, y, z)(b-a)^{3\beta}\|x - \widetilde{x}\|_{\beta(a,b)} \\
&\quad + G^6_{\beta(a,b)}(f, \widetilde{x}, z)(b-a)^{3\beta}\|(x - \widetilde{x}) \otimes y\|_{2\beta(a,b)}.
\end{aligned}
$$

Proof. To simplify the proof we will assume $d = m = 1$. Observe that from inequalities (2.3) and (2.4) we obtain

$$
\Phi_{\beta(a,b)}(x, y \otimes z) \le K\Phi_{\beta(a,b)}(x, y, z)(b-a)^\beta, \tag{4.1}
$$

$$
\begin{aligned}
\big\|(x - \widetilde{x}) \otimes (y \otimes z)_{\cdot,b}\big\|_{2\beta(a,b)} &\le K\Phi_{\beta(a,b)}(y, z)(b-a)^\beta\|x - \widetilde{x}\|_{\beta(a,b)} \\
&\quad + K\|z\|_{\beta(a,b)}(b-a)^\beta\|(x - \widetilde{x}) \otimes y\|_{2\beta(a,b)}.
\end{aligned} \tag{4.2}
$$

The proof of the proposition is obtained by applying Proposition 4.4 and using inequalities (2.3), (2.4), (4.1) and (4.2). □

Proposition 4.7. *Suppose that $(x, y, x \otimes y)$, $(x_{\cdot-r}, y, x_{\cdot-r} \otimes y)$ and $(y, z, y \otimes z)$ belong to $M^\beta_{d,m}(0, T)$. Assume that f satisfies* (H1)*. Then, for $0 \le a < b \le T$,*

$$
\begin{aligned}
&\Big|\int_a^b [f(x_u) - f(x_{u-r})]\, d_u(y \otimes z)_{\cdot,b}\Big| \\
&\le G^4_{\beta(a,b)}(f, x, x_{\cdot-r}, y, z)(b-a)^{3\beta}\|x - x_{\cdot-r}\|_{\infty(a,b)} \\
&\quad + G^5_{\beta(a,b)}(f, x, x_{\cdot-r}, y, z)(b-a)^{3\beta}\|x - x_{\cdot-r}\|_{\beta(a,b)} \\
&\quad + G^6_{\beta(a,b)}(f, x_{\cdot-r}, z)(b-a)^{3\beta}\|(x - x_{\cdot-r}) \otimes y\|_{2\beta(a,b)}.
\end{aligned} \tag{4.3}
$$

The above proposition is a particular case of Proposition 4.6 with $\widetilde{x} \equiv x_{\cdot-r}$. We conclude this section with a general result on β-Hölder functions.

Lemma 4.8. *Let $y : [0,T] \to \mathbb{R}^m$ be a β-Hölder continuous function and $\beta' = \beta - \varepsilon > 0$ with $\varepsilon > 0$, then*

$$\|y - y_{\cdot - r}\|_{\infty(r,T)} \le \|y\|_\beta r^\beta, \tag{4.4}$$

$$\|y - y_{\cdot - r}\|_{\beta'(r,T)} \le 2\|y\|_\beta r^\varepsilon. \tag{4.5}$$

Proof. On the one hand,

$$\|y - y_{\cdot - r}\|_{\infty(r,T)} = \sup_{t\in[r,T]} \frac{|y_t - y_{t-r}|}{r^\beta} \cdot r^\beta \le \|y\|_\beta r^\beta.$$

On the other hand,

$$\begin{aligned} &\sup_{\substack{s<t\in[r,T]\\ t-s\le r}} \frac{|(y - y_{\cdot-r})_t - (y - y_{\cdot-r})_s|}{(t-s)^{\beta'}} \\ &\le \sup_{\substack{s<t\in[r,T]\\ t-s\le r}} \frac{|y_t - y_s|}{(t-s)^\beta} \cdot \frac{(t-s)^\beta}{(t-s)^{\beta'}} + \sup_{\substack{s<t\in[r,T]\\ t-s\le r}} \frac{|y_{t-r} - y_{s-r}|}{(t-s)^\beta} \cdot \frac{(t-s)^\beta}{(t-s)^{\beta'}} \\ &\le 2\|y\|_\beta r^\varepsilon \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} &\sup_{\substack{s<t\in[r,T]\\ t-s\ge r}} \frac{\big|(y - y_{\cdot-r})_t - (y - y_{\cdot-r})_s\big|}{(t-s)^{\beta'}} \\ &\le \sup_{\substack{s<t\in[r,T]\\ t-s\ge r}} \frac{|y_t - y_{t-r}|}{r^\beta} \cdot \frac{r^\beta}{(t-s)^{\beta'}} + \sup_{\substack{s<t\in[r,T]\\ t-s\ge r}} \frac{|y_s - y_{s-r}|}{r^\beta} \cdot \frac{r^\beta}{(t-s)^{\beta'}} \\ &\le 2\|y\|_\beta r^\varepsilon. \end{aligned} \tag{4.7}$$

The proof finishes by putting together (4.6) and (4.7). $\square$

5. Estimates of the solutions

In this section we obtain some estimates on the solutions to our equations. Let us recall that $\varepsilon > 0$ with $\beta - 2\varepsilon > 0$, $\lambda > \frac{1}{\beta-\varepsilon} - 2$ and $\beta' = \beta - \varepsilon$. First of all, let us introduce $\widehat{x}^r_t = x^r_{t-r}$ where x^r is the solution to (3.1). Then $(\widehat{x}^r \otimes y)_{s,t}$ can be expressed as follows:

- for $s < t \in [0,r)$,

$$(\widehat{x}^r \otimes y)_{s,t} = (\eta_{\cdot-r} \otimes y)_{s,t} = \int_s^t (y_t - y_u)\, d\eta_{u-r}, \tag{5.1}$$

- for $s \in [0,r)$ and $t \in [r,T]$,

$$\begin{aligned} (\widehat{x}^r \otimes y)_{s,t} &= (\eta_{\cdot-r} \otimes y)_{s,r} + \int_r^t (y_t - y_u) b(u-r, \widehat{x}^r_u)\, du \\ &\quad + \int_r^t \sigma(\widehat{x}^r_{u-r})\, d_u(y_{\cdot-r} \otimes y)_{\cdot,t} + (\eta_0 - \eta_{s-r}) \otimes (y_t - y_r), \end{aligned} \tag{5.2}$$

- for $s < t \in [r,T]$,

$$(\widehat{x}^r \otimes y)_{s,t} = \int_s^t (y_t - y_u) b(u-r, \widehat{x}^r_u)\, du + \int_s^t \sigma(\widehat{x}^r_{u-r})\, d_u(y_{\cdot-r} \otimes y)_{\cdot,t}.$$

We will prove that the norms $\|\widehat{x}^r\|_{\beta'}$ and $\|\widehat{x}^r\otimes y\|_{2\beta'}$ are bounded and their upper bound does not depend on r. To this aim, the following lemma will be useful.

Lemma 5.1. *Let* $(\eta_{\cdot-r}, y, \eta_{\cdot-r}\otimes y) \in M^{\beta}_{d,m}(0,r)$ *and* $(y_{\cdot-r}, y, y_{\cdot-r}\otimes y) \in M^{\beta}_{d,d}(r,T)$. *Let* $(x^r, y, x^r\otimes y) \in M^{\beta}_{d,m}(0,T)$ *be the solution to* (3.1). *Then*

$$\|\widehat{x}^r\|_{\beta'} \le \|\widehat{x}^r\|_{\beta'(0,r)} + \|\widehat{x}^r\|_{\beta'(r,T)}, \tag{5.3}$$

$$\|\widehat{x}^r\otimes y\|_{2\beta'} \le \|\widehat{x}^r\otimes y\|_{2\beta'(0,r)} + \|\widehat{x}^r\otimes y\|_{2\beta'(r,T)} + \|\eta\|_{\beta'(-r,0)}\|y\|_{\beta'}. \tag{5.4}$$

Proof. On the one hand, observe that

$$\|\widehat{x}^r\|_{\beta'} \le \max\Big(\sup_{0\le s<t<r}\frac{|\widehat{x}^r_t-\widehat{x}^r_s|}{(t-s)^{\beta'}}, \sup_{0\le s<r\le t\le T}\frac{|\widehat{x}^r_t-\widehat{x}^r_s|}{(t-s)^{\beta'}}, \sup_{r\le s<t\le T}\frac{|\widehat{x}^r_t-\widehat{x}^r_s|}{(t-s)^{\beta'}}\Big),$$

and

$$\sup_{0\le s<r\le t\le T}\frac{|\widehat{x}^r_t-\widehat{x}^r_s|}{(t-s)^{\beta'}} \le \sup_{r\le s<t\le T}\frac{|\widehat{x}^r_t-\widehat{x}^r_s|}{(t-s)^{\beta'}} + \sup_{0\le s<t<r}\frac{|\widehat{x}^r_t-\widehat{x}^r_s|}{(t-s)^{\beta'}}.$$

So, we easily get (5.3).

On the other hand, observe that from the multiplicative property we obtain

$$\begin{aligned}
&\sup_{0\le s<r\le t\le T}\frac{|(\widehat{x}^r\otimes y)_{s,t}|}{(t-s)^{2\beta'}}\\
&\le \sup_{0\le s<r\le t\le T}\Big[\frac{|(\widehat{x}^r\otimes y)_{s,r}|}{(t-s)^{2\beta'}} + \frac{|(\widehat{x}^r\otimes y)_{r,t}|}{(t-s)^{2\beta'}} + \frac{|(\widehat{x}^r_r-\widehat{x}^r_s)\otimes(y_t-y_r)|}{(t-s)^{2\beta'}}\Big],
\end{aligned}$$

and using the same argument as before (5.4) follows easily. ☐

Proposition 5.2. *Let* $(\eta_{\cdot-r}, y, \eta_{\cdot-r}\otimes y) \in M^{\beta}_{d,m}(0,r)$ *and* $(y_{\cdot-r}, y, y_{\cdot-r}\otimes y) \in M^{\beta}_{m,m}(r,T)$ *for all* $r\le r_0$. *Assume that* σ *and* b *satisfy* (H1) *and* (H2) *respectively, and both satisfy* (H3). *Let* $(x^r, y, x^r\otimes y) \in M^{\beta}_{d,m}(0,T)$ *be the solution to equation* (3.1). *Assume also that* $\|\eta\|_{\beta(-r_0,0)}<\infty$ *and* $\sup_{r\le r_0}\|\eta_{\cdot-r}\otimes y\|_{2\beta(0,r)}<\infty$. *Then, for* $r\le r_0$, *we have the following estimates:*

$$\|\widehat{x}^r\|_{\infty(0,T+r)} \le M_{\eta,y}, \tag{5.5}$$

$$\|\widehat{x}^r\|_{\beta'(0,T+r)} \le K\rho_{\eta,b,\sigma}\Lambda_y(1+2M_{\eta,y}), \tag{5.6}$$

$$\|\widehat{x}^r\otimes y\|_{2\beta'(0,T+r)} \le K\rho_{\eta,b,\sigma}\Lambda_y\big(2+(T+r_0)(K\rho_{\eta,b,\sigma}\Lambda_y)^{1/\beta}\big), \tag{5.7}$$

where $K\ge 1$ *and*

$$\rho_{\eta,b,\sigma} := 2\|\eta\|_{\beta(-r_0,0)} + \|b\|_\infty T^{1-\beta} + \|\sigma\|_\infty + \|\sigma'\|_\infty + \|\sigma'\|_\lambda, \tag{5.8}$$

$$\Lambda_y := \|y\|_\beta + \max(1, \|y\|^2_\beta + \|y\otimes y\|_{2\beta}), \tag{5.9}$$

$$M_{\eta,y} := |\eta_0| + (T+r_0)(K\rho_{\eta,b,\sigma}\Lambda_y)^{\frac{1}{\beta}} + 1. \tag{5.10}$$

Proof. For simplicity, we assume that $d=m=1$. Firstly, note that, if $\|\eta\|_{\beta(-r_0,0)} < C$ and $\sup_{r\le r_0}\|\eta_{\cdot-r}\otimes y\|_{2\beta(0,r)} < C'$, with C and C' two positive constants, then $\|\eta\|_{\beta'(-r_0,0)} < Cr_0^{\varepsilon}$ and $\sup_{r\le r_0}\|\eta_{\cdot-r}\otimes y\|_{2\beta'(0,r)} < C' r_0^{2\varepsilon}$.

Secondly, notice that by (5.1)

$$\begin{aligned}
\|\eta_{\cdot-r}\otimes y\|_{2\beta(0,r)} &= \sup_{s,t\in[0,r)}\frac{\big|\int_s^t (y_t-y_u)\,d\eta_{u-r}\big|}{(t-s)^{2\beta}}\\
&\le \|\eta_{\cdot-r}\|_{\beta(0,r)}\|y\|_\beta\\
&\le \|\eta\|_{\beta(-r_0,0)}\|y\|_\beta.
\end{aligned}$$

To prove the result we will follow the ideas in [2, Theorem 4.1]. Consider the mapping $J : M^{\beta}_{1,1}(0, T + r) \to M^{\beta}_{1,1}(0, T + r)$ given by $J(\widehat{x}^r, y, \widehat{x}^r \otimes y) = (J_1, y, J_2)$, where J_1 and J_2 are the right-hand sides of the definition of $\widehat{x}^r$ and $(\widehat{x}^r \otimes y)$, respectively:

$$J_1(\widehat{x}^r, y, \widehat{x}^r \otimes y)(t) = \begin{cases} \eta_{t-r} & \text{for } 0 \le t < r \\ \eta_0 + \int_0^{t-r} b(u, \widehat{x}^r_{u+r})\,du + \int_0^{t-r} \sigma(\widehat{x}^r_u)\,dy_u & \text{for } r \le t \le T, \end{cases}$$

$$\begin{aligned} &J_2(\widehat{x}^r, y, \widehat{x}^r \otimes y)(s,t) \\ &= \begin{cases} \int_s^t (y_t - y_u)\,d\eta_{u-r} & \text{for } 0 \le s \le t < r, \\ (\eta_0 - \eta_{s-r}) \otimes (y_t - y_r) + \int_s^r (y_r - y_u)\,d\eta_{u-r} & \\ + \int_r^t (y_t - y_u) b(u - r, \widehat{x}^r_u)\,du & \\ + \int_r^t \sigma(\widehat{x}^r_{u-r})\,d_u(y_{\cdot-r} \otimes y)_{\cdot,t} & \text{for } 0 \le s < r \le t \le T, \\ \int_s^t (y_t - y_u) b(u - r, \widehat{x}^r_u)\,du & \\ + \int_s^t \sigma(\widehat{x}^r_{u-r})\,d_u(y_{\cdot-r} \otimes y)_{\cdot,t} & \text{for } r \le s \le t \le T. \end{cases} \end{aligned}$$

This mapping is well-defined because (J_1, y, J_2) is a real-valued β-Hölder continuous multiplicative functional for each $(\widehat{x}^r, y, \widehat{x}^r \otimes y) \in M^{\beta}_{1,1}(0, T)$.

Now we bound the Hölder norms of J_1 and J_2 using Proposition 4.1 and Proposition 4.2. Let $s < t \in [0, T]$. Then

- for $s < t \in [0, r)$,

$$\|J_1\|_{\beta(s,t)} \le \|\eta\|_{\beta(-r_0,0)}, \quad \|J_2\|_{2\beta(s,t)} \le \|\eta\|_{\beta(-r_0,0)} \|y\|_{\beta}, \tag{5.11}$$

- for $s < t \in [r, T]$,

$$\begin{aligned} &\|J_1\|_{\beta(s,t)} \\ &\le \|b\|_\infty (t-s)^{1-\beta} + K\|\sigma\|_\infty \|y\|_\beta \\ &\quad + K\left(\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\widehat{x}^r_{\cdot-r}\|^\lambda_{\beta(s,t)} (t-s)^{\lambda\beta}\right) \Phi_{\beta(s,t)}(\widehat{x}^r_{\cdot-r}, y_{\cdot-r})(t-s)^\beta, \\ &\|J_2\|_{2\beta(s,t)} \\ &\le \|b\|_\infty \|y\|_\beta (t-s)^{1-\beta} + K\|\sigma\|_\infty \Phi_{\beta(s,t)}(y_{\cdot-r}, y) \\ &\quad + K\left(\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\widehat{x}^r_{\cdot-r}\|^\lambda_{\beta(s,t)} (t-s)^{\lambda\beta}\right) \Phi_{\beta(s,t)}(\widehat{x}^r_{\cdot-r}, y_{\cdot-r}, y)(t-s)^\beta, \end{aligned} \tag{5.12}$$

- for $s \in [0, r)$ and $t \in [r, T]$,

$$\begin{aligned} &\|J_1\|_{\beta(s,t)} \\ &\le \|J_1\|_{\beta(s,r)} + \|J_1\|_{\beta(r,t)} \\ &\le \|\eta\|_{\beta(-r_0,0)} + \|b\|_\infty (t-r)^{1-\beta} + K\|\sigma\|_\infty \|y\|_\beta \\ &\quad + K\left(\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\widehat{x}^r_{\cdot-r}\|^\lambda_{\beta(r,t)} (t-r)^{\lambda\beta}\right) \Phi_{\beta(r,t)}(\widehat{x}^r_{\cdot-r}, y_{\cdot-r})(t-r)^\beta, \\ &\|J_2\|_{2\beta(s,t)} \\ &\le \|J_2\|_{2\beta(s,r)} + \|J_2\|_{2\beta(r,t)} \\ &\le 2\|\eta\|_{\beta(-r_0,0)} \|y\|_\beta + \|b\|_\infty \|y\|_\beta (t-r)^{1-\beta} + K\|\sigma\|_\infty \Phi_{\beta(r,t)}(y_{\cdot-r}, y) \\ &\quad + K\left(\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\widehat{x}^r_{\cdot-r}\|^\lambda_{\beta(r,t)} (t-r)^{\lambda\beta}\right) \Phi_{\beta(r,t)}(\widehat{x}^r_{\cdot-r}, y_{\cdot-r}, y)(t-r)^\beta. \end{aligned}$$

For $s < t \in [r,T]$, we set

$$(\widehat{x}^r_{\cdot-r} \otimes y_{\cdot-r})_{s,t} := (\widehat{x} \otimes y)_{s-r,t-r}\,,$$

which following [1, section 5] is a β-Hölder continuous multiplicative functional. We divide the proof into two steps.

Step 1: We find a set C^y of elements $(\widehat{x}^r, y, \widehat{x}^r \otimes y) \in M^{\beta}_{1,1}(0,T)$ such that $J(C^y) \subset C^y$. Let us recall the definitions of $\rho_{\eta,b,\sigma}$ and Λ_y from (5.8) and (5.9), respectively, and set

$$\widetilde{\Delta}_y := \big(K\rho_{\eta,b,\sigma}\Lambda_y\big)^{-1/\beta}.$$

Let C^y be the set of elements $(\widehat{x}^r, y, \widehat{x}^r \otimes y) \in M^{\beta}_{1,1}(0,T)$ satisfying the following conditions:

$$\|\widehat{x}^r\|_\infty \le M_{\eta,y}, \tag{5.13}$$

$$\sup_{0<t-s\le\widetilde{\Delta}_y} \|\widehat{x}^r\|_{\beta(s,t)} \le K\rho_{\eta,b,\sigma}(\|y\|_\beta + 1), \tag{5.14}$$

$$\sup_{0<t-s\le\widetilde{\Delta}_y} \|\widehat{x}^r \otimes y\|_{2\beta(s,t)} \le K\rho_{\eta,b,\sigma}(\|y\|_\beta + \|y\|^2_\beta + \|y_{\cdot-r} \otimes y\|_{2\beta}). \tag{5.15}$$

If we take $s,t \in [0,T]$, $s<t$ such that

$$t-s \le \widetilde{\Delta}_y, \tag{5.16}$$

then we have

$$(t-s)^\beta \le \widetilde{\Delta}_y^\beta \le \frac{1}{K\rho_{\eta,b,\sigma}(\|y\|_\beta + 1)}, \tag{5.17}$$

$$(t-s)^\beta \le \widetilde{\Delta}_y^\beta \le \frac{1}{K\rho_{\eta,b,\sigma}(\|y\|_\beta + \|y\|^2_\beta + \|y_{\cdot-r} \otimes y\|_{2\beta})}. \tag{5.18}$$

Suppose that $(\widehat{x}^r, y, \widehat{x}^r \otimes y) \in C^y$. Then using (5.14), (5.17) and (5.15), (5.18), respectively, we have

$$(t-s)^\beta \|\widehat{x}^r\|_{\beta(s,t)} \le 1, \tag{5.19}$$

$$(t-s)^\beta \|\widehat{x}^r \otimes y\|_{2\beta(s,t)} \le 1. \tag{5.20}$$

Now observe that, if $s,t \in [r,T]$ and $s<t$ satisfy (5.16), then $s-r, t-r \in [0,T]$ also satisfy this inequality. Thus,

$$(t-s)^\beta \|\widehat{x}^r_{\cdot-r}\|_{\beta(s,t)} \le 1,$$
$$(t-s)^\beta \|\widehat{x}^r_{\cdot-r} \otimes y_{\cdot-r}\|_{2\beta(s,t)} \le 1.$$

From this inequality it easily follows that

$$\begin{aligned}
&\Phi_{\beta(s,t)}(\widehat{x}^r_{\cdot-r}, y_{\cdot-r}, y)(t-s)^\beta \\
&= \Big[\|\widehat{x}^r_{\cdot-r}\|_{\beta(s,t)}\|y_{\cdot-r}\|_{\beta(s,t)}\|y\|_{\beta(s,t)} + \|y\|_{\beta(s,t)}\|\widehat{x}^r_{\cdot-r} \otimes y_{\cdot-r}\|_{2\beta(s,t)} \\
&\quad + \|\widehat{x}^r_{\cdot-r}\|_{\beta(s,t)}\|y_{\cdot-r} \otimes y\|_{2\beta(s,t)}\Big](t-s)^\beta \\
&\le \|y\|_\beta + \|y\|^2_\beta + \|y_{\cdot-r} \otimes y\|_{2\beta}.
\end{aligned} \tag{5.21}$$

Observe also that if $s \in [0,r)$ and $t \in [r,T]$ satisfy (5.16), then $t-r \le \widetilde{\Delta}_y$ and all the previous inequalities are satisfied when we change the interval (s,t) by the interval (r,t) for $t \in [r,T]$.

From expressions (5.11),(5.12) and (5.19)–(5.21), we easily get that

$$\begin{aligned} \|J_1\|_{\beta(s,t)} &\le \|\eta\|_{\beta(-r_0,0)} + \|b\|_\infty T^{1-\beta} + K\|\sigma\|_\infty \|y\|_\beta \\ &\quad + K\left(\|\sigma'\|_\infty + \|\sigma'\|_\lambda\right)\left(\|y\|_\beta + 1\right) \\ &\le K\rho_{\eta,b,\sigma}(\|y\|_\beta + 1) \end{aligned} \tag{5.22}$$

and

$$\begin{aligned} \|J_2\|_{2\beta(s,t)} &\le 2\|\eta\|_{\beta(-r_0,0)}\|y\|_\beta + \|b\|_\infty \|y\|_\beta T^{1-\beta} \\ &\quad + K\|\sigma\|_\infty(\|y\|_\beta^2 + \|y_{\cdot-r} \otimes y\|_{2\beta}) \\ &\quad + K\left(\|\sigma'\|_\infty + \|\sigma'\|_\lambda\right)\left(\|y\|_\beta + \|y\|_\beta^2 + \|y_{\cdot-r} \otimes y\|_{2\beta}\right) \\ &\le K\rho_{\eta,b,\sigma}(\|y\|_\beta + \|y\|_\beta^2 + \|y_{\cdot-r} \otimes y\|_{2\beta}) \end{aligned} \tag{5.23}$$

where $K \ge 1$.

It only remains to prove that $\|J_1\|_\infty \le M_{\eta,y}$. Set $N = \big[(T+r)\widetilde{\Delta}_y^{-1}\big]+1$ and define the partition $0 = t_0 < t_1 < \dots < t_N = T + r$, where $t_i = i\widetilde{\Delta}_y$ for $i = 0, \dots, N-1$. Estimates (5.17) and (5.22) imply

$$\sup_{u\in[t_{i-1},t_i]} |(J_1)_u| \le |(J_1)_{t_{i-1}}| + (t_i - t_{i-1})^\beta \|J_1\|_{\beta(t_{i-1},t_i)} \le |(J_1)_{t_{i-1}}| + 1.$$

Moreover,

$$\sup_{u\in[0,t_i]} |(J_1)_u| \le \sup_{u\in[0,t_{i-1}]} |(J_1)_u| + 1,$$

and iterating this inequality we finally obtain

$$\sup_{u\in[0,T]} |(J_1)_u| \le |\eta_0| + N \le |\eta_0| + T\widetilde{\Delta}_y^{-1} + 1 = M_{\eta,y}.$$

Hence, $(J_1, y, J_2) \in C^y$.

Step 2: We find a bound for the Hölder norms of $\widehat{x}^r$ and $(\widehat{x}^r \otimes y)$. We can construct a sequence of functions $\widehat{x}^{r(n)}$ and $(\widehat{x}^r \otimes y)^{(n)}$ such that, $\widehat{x}^{r(0)} = \eta_0$, $(\widehat{x}^r \otimes y)^{(0)} = 0$ and

$$\begin{gathered} \widehat{x}^{r(n)} = J_1\Big(\widehat{x}^{r(n-1)}, y, (\widehat{x}^r \otimes y)^{(n-1)}\Big), \\ (\widehat{x}^r \otimes y)^{(n)} = J_2\Big(\widehat{x}^{r(n-1)}, y, (\widehat{x}^r \otimes y)^{(n-1)}\Big). \end{gathered}$$

Notice that $\big(\widehat{x}^{r(0)}, y, (\widehat{x}^r \otimes y)^{(0)}\big) \in C^y$ and, since we have proved in Step 1 that $J\big(C^y\big) \subset C^y$, we have that $\big(\widehat{x}^{r(n)}, y, (\widehat{x}^r \otimes y)^{(n)}\big) \in C^y$ for each n. We estimate $\|\widehat{x}^{r(n)}\|_\beta$ as follows:

$$\begin{aligned} \|\widehat{x}^{r(n)}\|_\beta &\le \sup_{\substack{0\le s<t\le T \\ t-s\le\widetilde{\Delta}_y}} \frac{|\widehat{x}_t^{r(n)} - \widehat{x}_s^{r(n)}|}{(t-s)^\beta} + \sup_{\substack{0\le s<t\le T \\ t-s\ge\widetilde{\Delta}_y}} \frac{|\widehat{x}_t^{r(n)} - \widehat{x}_s^{r(n)}|}{(t-s)^\beta} \\ &\le K\rho_{\eta,b,\sigma}(\|y\|_\beta + 1) + 2\widetilde{\Delta}_y^{-\beta}\|\widehat{x}^{r(n)}\|_\infty \\ &\le K\rho_{\eta,b,\sigma}\Lambda_y(1 + 2M_{\eta,y}). \end{aligned} \tag{5.24}$$

It implies that the sequence of functions $\widehat{x}^{r(n)}$ is equicontinuous and bounded in $C^\beta(0,T)$ and the upper bound does not depend on r. So, there exists a subsequence which converges in the β'-Hölder norm if $\beta' < \beta$ and such that the upper bound of the β'-Hölder norm does not depend on r.

In a similar way we obtain the same result for $(\widehat{x}^r\otimes y)^{(n)}$. From inequality (5.20) we obtain that

$$\sup_{t_{i-1}\le s<t\le t_i}|(\widehat{x}^r\otimes y)^{(n)}_{s,t}|\le \|(\widehat{x}^r\otimes y)^{(n)}\|_{2\beta(t_{i-1},t_i)}(t_i-t_{i-1})^{2\beta}\le \widetilde{\Delta}_y^{\beta},$$

$$\sup_{0\le s<t\le T}|(\widehat{x}^r\otimes y)^{(n)}_{s,t}|\le N\widetilde{\Delta}_y^{\beta}\le T\widetilde{\Delta}_y^{\beta-1}+\widetilde{\Delta}_y^{\beta}.$$

As we did to obtain (5.24), we estimate $\|(\widehat{x}^r\otimes y)^{(n)}\|_{2\beta}$ as follows:

$$\begin{aligned}\|(\widehat{x}^r\otimes y)^{(n)}\|_{2\beta}&\le K\rho_{\eta,b,\sigma}(\|y\|_\beta^2+\|y\|_\beta+\|y_{\cdot-r}\otimes y\|_{2\beta})+T\widetilde{\Delta}_y^{-\beta-1}+\widetilde{\Delta}_y^{-\beta}\\&\le K\rho_{\eta,b,\sigma}\Lambda_y\big(2+(T+r_0)(K\rho_{\eta,b,\sigma}\Lambda_y)^{\frac{1}{\beta}}\big).\end{aligned}$$

It implies that the sequence of functions $(\widehat{x}^r\otimes y)^{(n)}$ is bounded and equicontinuous in the set of functions 2β-Hölder continuous on Δ_T, and the upper bound does not depend on r. So, there exists a subsequence which converges in the β'-Hölder norm if $\beta'<\beta$ and such that the upper bound of the β'-Hölder norm does not depend on r.

Now as n tends to infinity it is easy to see that the limit is a solution, and the limit defines a $\beta-$Hölder continuous multiplicative functional $(\widehat{x}^r,y,\widehat{x}^r\otimes y)$ and this functional satisfies estimates (5.5), (5.6) and (5.7). $\square$

Remark 5.3. In Proposition 5.2 it is proved that $\|\widehat{x}^r\|_{\beta'}\le K\rho_{\eta,b,\sigma}\Lambda_y(1+2M_{\eta,y})$. Thus we have the same bound for $\|x^r\|_{\beta'(r)}$. Moreover, using the ideas in the proof of Proposition 5.2 it is possible to prove that $\|x^r\otimes y\|_{2\beta'}$ is bounded and its bound does not depend on r.

We are also interested in the behavior of $(x^r-\widehat{x}^r)$ when r tends to zero. We can write $(x-x^r)_t$ as follows

$$\begin{aligned}(x-x^r)_t=\int_0^t\big[b(u,x_u)-b(u,x^r_u)\big]\,du+\int_0^t\big[\sigma(x_u)-\sigma(x^r_u)\big]\,dy_u\\+\int_0^t\big[\sigma(x^r_u)-\sigma(x^r_{u-r})\big]\,dy_u.\end{aligned}\tag{5.25}$$

Following the ideas in [1, Section 4], let us write $\big((x-x^r)\otimes y\big)_{s,t}$, for $s,t\in[0,T]$:

$$\begin{aligned}&\big((x-x^r)\otimes y\big)_{s,t}\\&=\int_s^t(y_t-y_u)\big[b(u,x_u)-b(u,x^r_u)\big]\,du\\&\quad+\int_s^t\big[\sigma(x_u)-\sigma(x^r_u)\big]\,d_u(y\otimes y)_{\cdot,t}+\int_s^t\big[\sigma(x^r_u)-\sigma(x^r_{u-r})\big]\,d_u(y\otimes y)_{\cdot,t}.\end{aligned}\tag{5.26}$$

It is also useful to write the following expressions:

- for $s<t\in[0,r)$,

$$(x^r-\widehat{x}^r)_t-(x^r-\widehat{x}^r)_s=\eta_{s-r}-\eta_{t-r}+\int_s^t b(u,x^r_u)\,du+\int_s^t\sigma(\eta_{u-r})\,dy_u,\tag{5.27}$$

- for $s \in [0, r)$ and $t \in [r, T]$,

$$
\begin{aligned}
(x^r - \widehat{x}^r)_t - (x^r - \widehat{x}^r)_s = \eta_{s-r} - \eta_0 + \int_{t-r}^t b(u, x_u^r)\, du - \int_0^s b(u, x_u^r)\, du \\
+ \int_{t-r}^t \sigma(x_{u-r}^r)\, dy_u - \int_0^s \sigma(\eta_{u-r})\, dy_u,
\end{aligned}
\tag{5.28}
$$

- for $s < t \in [r, T]$,

$$
\begin{aligned}
(x^r - \widehat{x}^r)_t - (x^r - \widehat{x}^r)_s = \int_s^t b(u, x_u^r)\, du - \int_{s-r}^{t-r} b(u, x_u^r)\, du \\
+ \int_s^t \sigma(x_{u-r}^r)\, dy_u - \int_{s-r}^{t-r} \sigma(x_{u-r}^r)\, dy_u.
\end{aligned}
\tag{5.29}
$$

Finally, following the ideas in [1, Section 4], we define

$$
\big((x^r - \widehat{x}^r) \otimes y\big)_{s,t} := (x^r \otimes y)_{s,t} - (\widehat{x}^r \otimes y)_{s,t},
$$

that is,

- for $s < t \in [0, r)$,

$$
\begin{aligned}
\big((x^r - \widehat{x}^r) \otimes y\big)_{s,t} = \int_s^t (y_u - y_t)\, d\eta_{u-r} + \int_s^t (y_t - y_u) b(u, x_u^r)\, du \\
+ \int_s^t \sigma(\eta_{u-r})\, d_u(y \otimes y)_{\cdot,t},
\end{aligned}
\tag{5.30}
$$

- for $s \in [0, r)$ and $t \in [r, T]$,

$$
\begin{aligned}
\big((x^r - \widehat{x}^r) \otimes y\big)_{s,t} &= \int_s^t (y_t - y_u) b(u, x_u^r)\, du + \int_s^t \sigma(x_{u-r}^r)\, d_u(y \otimes y)_{\cdot,t} \\
&\quad - (\eta_{\cdot-r} \otimes y)_{s,r} - \int_r^t (y_t - y_u) b(u - r, \widehat{x}_u^r)\, du \\
&\quad - \int_r^t \sigma(\widehat{x}_{u-r}^r)\, d_u(y_{\cdot-r} \otimes y)_{\cdot,t} - (\eta_0 - \eta_{s-r}) \otimes (y_t - y_r),
\end{aligned}
$$

- for $s < t \in [r, T]$,

$$
\begin{aligned}
&\big((x^r - \widehat{x}^r) \otimes y\big)_{s,t} \\
&= \int_s^t (y_{u+r} - y_u) b(u, x_u^r)\, du + \int_s^t \sigma(x_{u-r}^r)\, d_u\big((y - y_{\cdot-r}) \otimes y\big)_{\cdot,t} \\
&\quad + \int_s^t \big[\sigma(x_{u-r}^r) - \sigma(\widehat{x}_{u-r}^r)\big]\, d_u(y_{\cdot-r} \otimes y)_{\cdot,t}.
\end{aligned}
\tag{5.31}
$$

The following proposition gives us a result about the behavior of $(x^r - \widehat{x}^r)$ when r tends to zero.

Proposition 5.4. *Set $\beta' = \beta - \varepsilon$, where $\varepsilon > 0$ is such that $\beta - 2\varepsilon > 0$ and $\lambda > \frac{1}{\beta - \varepsilon} - 2$. Suppose that $(x, y, x \otimes y)$, $(x^r, y, x^r \otimes y)$, $(\widehat{x}^r, y, \widehat{x}^r \otimes y)$ belong to $M_{d,m}^{\beta}(0, T)$ and $(y, y, y \otimes y)$ belongs to $M_{m,m}^{\beta}(0, T)$. Assume that σ and b satisfy* (H1) *and* (H2) *respectively, and both satisfy* (H3). *Assume also that $\|\eta\|_{\beta(-r_0,0)} <$*

∞ and $\sup_{r\le r_0}\Phi_{\beta(0,r)}(\eta_{\cdot-r},y)<\infty$ and suppose that $\|(y-y_{\cdot-r})\otimes y\|_{2\beta'(r,T)}\to 0$ and $\|y_{\cdot-r}\otimes(y-y_{\cdot-r})\|_{2\beta'(r,T)}\to 0$ when r tends to zero. Then

$$
\begin{gathered}
\|x^r-\widehat{x}^r\|_\infty\le K\rho\Lambda r^{\beta'}\\
\|x^r-\widehat{x}^r\|_{\beta'}\le K\rho\Lambda r^{\varepsilon}\\
\big\|(x^r-\widehat{x}^r)\otimes y\big\|_{2\beta'}\le KM\rho^3\Lambda^3r^\varepsilon+KM\rho^3\Lambda^2\Lambda_r
\end{gathered}
$$

where $K\ge 1$, $M\ge 1$ are constants depending on $\beta,\beta',r_0,T,\sigma,y$ and

$$
\begin{aligned}
\rho=\Big(&1+3\|b\|_\infty T^{1-\beta'}+3\|\sigma\|_\infty(1+T^{\beta'})+2\|\sigma'\|_\infty(1+T^{\beta'})+3\|\sigma'\|_\infty T^{\beta'-\varepsilon}\\
&+\|\sigma'\|_\lambda\Big(2\sup_{r\le r_0}\|x^r\|_{\beta'}^\lambda+\|\eta\|_{\beta'(-r_0,0)}^\lambda\Big)T^{(\lambda+1)\beta'-\varepsilon}+\|\sigma''\|_\infty T^{\beta'}(1+T^{\beta'})\\
&+2\|\sigma''\|_\lambda\sup_{r\le r_0}\|\widehat{x}^r\|_{\beta'}^\lambda T^{(\lambda+1)\beta'}\Big)(1+T^\varepsilon),\\
\Lambda=\max\Big(&1,\|\eta\|_{\beta(-r_0,0)},\sup_{r\le r_0}\Phi_{\beta'(0,r)}(\eta_{\cdot-r},y),\Phi_{\beta(0,T)}(y,y),\\
&\sup_{r\le r_0}\Phi_{\beta'(r,T)}(y_{\cdot-r},y),\sup_{r\le r_0}\Phi_{\beta'(0,r)}(\eta_{\cdot-r},y,y),\sup_{r\le r_0}\Phi_{\beta'(0,T)}(x^r,y),\\
&\sup_{r\le r_0}\Phi_{\beta'(0,T)}(\widehat{x}^r,y)\Big)\Big(1+\sup_{r\le r_0}\|x^r\|_{\beta'(r)}\Big)\Big(1+\|y\|_\beta\Big),\\
\Lambda_r=\max\Big(&1,\sup_{r\le r_0}\|x^r\|_{\beta'}\Big)\Big(\|(y-y_{\cdot-r})\otimes y\|_{2\beta'(r,T)}\\
&+\|y_{\cdot-r}\otimes(y-y_{\cdot-r})\|_{2\beta'(r,T)}\Big).
\end{aligned}
$$

Remark 5.5. Since ρ and Λ are finite (from Proposition 5.2) and Λ_r converges to zero when r tends to zero (by hypothesis), Proposition 5.4 implies:

$$
\begin{gathered}
\|x^r-\widehat{x}^r\|_\infty\xrightarrow{r\downarrow 0}0,\\
\|x^r-\widehat{x}^r\|_{\beta'}\xrightarrow{r\downarrow 0}0,\\
\big\|(x^r-\widehat{x}^r)\otimes y\big\|_{2\beta'}\xrightarrow{r\downarrow 0}0.
\end{gathered}
$$

Proof of Proposition 5.4. We start studying the supremum norm. On the one hand, using Proposition 4.1, for $r\le r_0$, we obtain

$$
\begin{aligned}
\|x^r-\widehat{x}^r\|_{\infty(0,r)}&\le\|\eta\|_{\beta'(-r,0)}r^{\beta'}+\|b\|_\infty r+K\|\sigma\|_\infty\|y\|_{\beta'}r^{\beta'}\\
&\quad+K\Phi_{\beta'(0,r)}(\eta_{\cdot-r},y)\big(\|\sigma'\|_\infty+\|\sigma'\|_\lambda\|\eta_{\cdot-r}\|_{\beta'(0,r)}^\lambda r^{\lambda\beta'}\big)r^{2\beta'}\\
&\le\Big[\|\eta\|_{\beta(-r_0,0)}T^\varepsilon+\|b\|_\infty T^{1-\beta'}+K\|\sigma\|_\infty\|y\|_\beta T^\varepsilon\\
&\quad+K\Phi_{\beta'(0,r)}(\eta_{\cdot-r},y)\big(\|\sigma'\|_\infty+\|\sigma'\|_\lambda\|\eta\|_{\beta'(-r_0,0)}^\lambda T^{\lambda\beta'}\big)T^{\beta'}\Big]r^{\beta'}
\end{aligned}
$$

where we have used that $\|\eta\|_{\beta'(-r,0)}\le\|\eta\|_{\beta(-r_0,0)}T^\varepsilon$ and $\|y\|_{\beta'}\le\|y\|_\beta T^\varepsilon$.

On the other hand, using Proposition 4.1 we obtain

$$
\begin{aligned}
\|x^r-\widehat{x}^r\|_{\infty(r,T)}&\le\|b\|_\infty r+K\|\sigma\|_\infty\|y\|_{\beta'}r^{\beta'}\\
&\quad+K\Phi_{\beta'(0,T)}(\widehat{x}^r,y)\big(\|\sigma'\|_\infty+\|\sigma'\|_\lambda\|\widehat{x}^r\|_{\beta'(r,T)}^\lambda r^{\lambda\beta'}\big)r^{2\beta'}\\
&\le\Big[\|b\|_\infty T^{1-\beta'}+K\|\sigma\|_\infty\|y\|_\beta T^\varepsilon
\end{aligned}
$$

$$+ K\Phi_{\beta'(0,T)}(\widehat{x}^r, y)\big(\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|x^r\|_{\beta'}^\lambda T^{\lambda\beta'}\big)T^{\beta'}\Big] r^{\beta'}.$$

Hence, we have

$$\|x^r - \widehat{x}^r\|_\infty \le K\rho\Lambda r^{\beta'}. \tag{5.32}$$

Now we study the Hölder norms. Following the proof of Lemma 5.1 we easily obtain

$$\|x^r - \widehat{x}^r\|_{\beta'} \le \|x^r - \widehat{x}^r\|_{\beta'(0,r)} + \|x^r - \widehat{x}^r\|_{\beta'(r,T)}, \tag{5.33}$$

$$\begin{aligned} \|(x^r - \widehat{x}^r)\otimes y\|_{2\beta'} \le \|(x^r - \widehat{x}^r)\otimes y\|_{2\beta'(0,r)} + \|(x^r - \widehat{x}^r)\otimes y\|_{2\beta'(r,T)} \\ + \|x^r - \widehat{x}^r\|_{\beta'(0,r)}\|y\|_{\beta'}. \end{aligned} \tag{5.34}$$

So we can study the Hölder norms independently in the intervals $[0, r)$ and $[r, T]$. We deal with the Hölder norm of $(x^r - \widehat{x}^r)$. By (5.27) and Proposition 4.1 we have

$$\begin{aligned} &\|x^r - \widehat{x}^r\|_{\beta'(0,r)} \\ &\le \|\eta\|_{\beta'(-r,0)} + \|b\|_\infty r^{1-\beta'} + K\|\sigma\|_\infty \|y\|_{\beta'(0,r)} \\ &\quad + K\Phi_{\beta'(0,r)}(\eta_{\cdot-r}, y)\big(\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\eta_{\cdot-r}\|_{\beta'(0,r)}^\lambda r^{\lambda\beta'}\big)r^{\beta'} \\ &\le \Big[\|\eta\|_{\beta(-r_0,0)} + \|b\|_\infty T^{1-\beta} + K\|\sigma\|_\infty\|y\|_\beta + K\Phi_{\beta'(0,r)}(\eta_{\cdot-r}, y) \\ &\quad \times \big(\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\eta\|_{\beta'(-r_0,0)}^\lambda T^{\lambda\beta'}\big)T^{\beta'-\varepsilon}\Big] r^\varepsilon. \end{aligned} \tag{5.35}$$

In the interval $[r, T]$, observe that

$$\begin{aligned} \|x^r - \widehat{x}^r\|_{\beta'(r,T)} \le \max\Big(&\sup_{\substack{s<t\in[r,T]\\ t-s\le r}} \frac{|(x^r - \widehat{x}^r)_t - (x^r - \widehat{x}^r)_s|}{(t-s)^{\beta'}}, \\ &\sup_{\substack{s<t\in[r,T]\\ t-s\ge r}} \frac{|(x^r - \widehat{x}^r)_t - (x^r - \widehat{x}^r)_s|}{(t-s)^{\beta'}}\Big). \end{aligned} \tag{5.36}$$

On the one hand, by (5.29) and Proposition 4.1 we have

$$\begin{aligned} &\sup_{s<t\in[r,T]\, t-s\le r} \frac{|(x^r - \widehat{x}^r)_t - (x^r - \widehat{x}^r)_s|}{(t-s)^{\beta'}} \\ &\le 2\|b\|_\infty r^{1-\beta'} + 2K\|\sigma\|_\infty \|y\|_\beta r^\varepsilon \\ &\quad + 2K\Phi_{\beta'(r,T)}(\widehat{x}^r, y)\big(\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\widehat{x}^r\|_{\beta'(r,T)}^\lambda r^{\lambda\beta'}\big)r^{\beta'} \\ &\le \Big[2\|b\|_\infty T^{1-\beta} + 2K\|\sigma\|_\infty \|y\|_\beta \\ &\quad + 2K\Phi_{\beta'(0,T)}(\widehat{x}^r, y)\big(\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|x^r\|_{\beta'}^\lambda T^{\lambda\beta'}\big)T^{\beta'-\varepsilon}\Big] r^\varepsilon, \end{aligned} \tag{5.37}$$

where we have used that $\sup_{s<t\in[r,T]\, t-s\le r} \|y\|_{\beta'(s,t)} \le \|y\|_\beta r^\varepsilon$.

On the other hand, with a similar computations, by Proposition 4.1 we have

$$
\begin{aligned}
&\sup_{s<t\in[r,T]\, t-s\geq r} \frac{|(x^r-\widehat{x}^r)_t-(x^r-\widehat{x}^r)_s|}{(t-s)^{\beta'}} \\
&\leq 2\|b\|_\infty r^{1-\beta'} + 2K\|\sigma\|_\infty\|y\|_\beta T^\varepsilon r^\varepsilon \\
&\quad + 2K\Phi_{\beta'(r,T)}(\widehat{x}^r,y)\big(\|\sigma'\|_\infty+\|\sigma'\|_\lambda\|\widehat{x}^r\|^\lambda_{\beta'(r,T)}r^{\lambda\beta'}\big)r^{\beta'} \\
&\leq \Big[2\|b\|_\infty T^{1-\beta}+2K\|\sigma\|_\infty\|y\|_\beta T^\varepsilon \\
&\quad + 2K\Phi_{\beta'(0,T)}(\widehat{x}^r,y)\big(\|\sigma'\|_\infty+\|\sigma'\|_\lambda\|x^r\|^\lambda_{\beta'}T^{\lambda\beta'}\big)T^{\beta'-\varepsilon}\Big]r^\varepsilon,
\end{aligned}
\tag{5.38}
$$

where we have used that $\sup_{t\in[r,T]}\|y\|_{\beta'(t-r,t)}\leq\|y\|_\beta r^\varepsilon$. Then, by (5.33) and using (5.35), (5.36), (5.37) and (5.38) it follows that

$$
\|x^r-\widehat{x}^r\|_{\beta'}\leq K\rho\Lambda r^\varepsilon. \tag{5.39}
$$

Finally, we study the Hölder norm $\|(x^r-\widehat{x}^r)\otimes y\|_{2\beta'}$. By definition (5.30) and Proposition 4.2 we have

$$
\begin{aligned}
&\|(x^r-\widehat{x}^r)\otimes y\|_{2\beta'(0,r)} \\
&\leq \|\eta_{\cdot-r}\|_{\beta'(0,r)}\|y\|_{\beta'(0,r)}+\|y\|_{\beta'(0,r)}\|b\|_\infty r^{1-\beta'}+K\|\sigma\|_\infty\Phi_{\beta'(0,r)}(y,y) \\
&\quad + K(\|\sigma'\|_\infty+\|\sigma'\|_\lambda\|\eta_{\cdot-r}\|^\lambda_{\beta'(0,r)}r^{\lambda\beta'})\Phi_{\beta'(0,r)}(\eta_{\cdot-r},y,y)r^{\beta'} \\
&\leq \Big[\|\eta\|_{\beta(-r_0,0)}\|y\|_\beta T^\varepsilon+\|y\|_\beta\|b\|_\infty T^{1-\beta'}+K\|\sigma\|_\infty\Phi_{\beta(0,T)}(y,y)T^\varepsilon \\
&\quad + K(\|\sigma'\|_\infty+\|\sigma'\|_\lambda\|\eta\|^\lambda_{\beta'(-r_0,0)}T^{\lambda\beta'})\Phi_{\beta'(0,r)}(\eta_{\cdot-r},y,y)T^{\beta'-\varepsilon}\Big]r^\varepsilon \\
&\leq K\rho\Lambda r^\varepsilon,
\end{aligned}
\tag{5.40}
$$

where we have used that $\Phi_{\beta'(0,r)}(y,y)\leq\Phi_{\beta(0,T)}(y,y)r^{2\varepsilon}$.

Now we study the Hölder norm in the interval $[r,T]$. Let $a,b\in[r,T]$, $a<b$. Then by (5.31),

$$
\begin{aligned}
&\|(x^r-\widehat{x}^r)\otimes y\|_{2\beta'(a,b)} \\
&\leq \sup_{s<t\in[a,b]}\frac{\big|\int_s^t(y_{u+r}-y_u)b(u,x^r_u)\,du\big|}{(t-s)^{2\beta'}} \\
&\quad + \sup_{s<t\in[a,b]}\frac{\big|\int_s^t\sigma(x^r_{u-r})\,d_u\big((y-y_{\cdot-r})\otimes y\big)_{\cdot,t}\big|}{(t-s)^{2\beta'}} \\
&\quad + \sup_{s<t\in[a,b]}\frac{\big|\int_s^t\big[\sigma(x^r_{u-r})-\sigma(\widehat{x}^r_{u-r})\big]\,d_u(y_{\cdot-r}\otimes y)_{\cdot,t}\big|}{(t-s)^{2\beta'}} \\
&=: A_1+A_2+A_3.
\end{aligned}
\tag{5.41}
$$

Let us study A_1, A_2 and A_3. It is easy to see that

$$
A_1\leq\|y\|_\beta\,\|b\|_\infty T^{1-\beta'}\leq K\rho\Lambda r^\varepsilon. \tag{5.42}
$$

By Proposition 4.2 we have

$$\begin{aligned}
A_2 &\le K\|\sigma\|_\infty \,\Phi_{\beta'(a,b)}(y-y_{\cdot-r},y)\\
&\quad + K\Big(\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\widehat{x}^r\|^\lambda_{\beta'(a,b)}T^{\lambda\beta'}\Big)\,\Phi_{\beta'(a,b)}(\widehat{x}^r, y-y_{\cdot-r},y)T^{\beta'}\\
&= K\|y\|_{\beta'}\Big(\|\sigma\|_\infty + \Big(\|\sigma'\|_\infty + \|\sigma'\|_\lambda\|\widehat{x}^r\|^\lambda_{\beta'}T^{\lambda\beta'}\Big)\|\widehat{x}^r\|_{\beta'}T^{\beta'}\Big)\\
&\quad \times \|y-y_{\cdot-r}\|_{\beta'(r,T)}\\
&\quad + K\Big(\|\sigma\|_\infty + \Big(\|\sigma'\|_\infty + \|\sigma'\|_\lambda\|\widehat{x}^r\|^\lambda_{\beta'}T^{\lambda\beta'}\Big)\|\widehat{x}^r\|_{\beta'}T^{\beta'}\Big)\\
&\quad \times \|(y-y_{\cdot-r})\otimes y\|_{2\beta'(r,T)}\\
&\quad + K\|y\|_{\beta'}T^{\beta'}\Big(\|\sigma'\|_\infty + \|\sigma'\|_\lambda\|\widehat{x}^r\|^\lambda_{\beta'}T^{\lambda\beta'}\Big)\|\widehat{x}^r\otimes(y-y_{\cdot-r})\|_{2\beta'(a,b)}.
\end{aligned} \tag{5.43}$$

Now we estimate the norm $\|\widehat{x}^r\otimes(y-y_{\cdot-r})\|_{2\beta'(a,b)}$. For $s<t\in[a,b]$,

$$\begin{aligned}
\big(\widehat{x}^r\otimes(y-y_{\cdot-r})\big)_{s,t} &= \int_s^t (y_t-y_{t-r}-y_u+y_{u-r})b(u-r,\widehat{x}^r_u)\,du\\
&\quad + \int_s^t \sigma(\widehat{x}^r_{u-r})\,d_u\big(y_{\cdot-r}\otimes(y-y_{\cdot-r})\big)_{\cdot,t}.
\end{aligned}$$

So, by Proposition 4.2 and Lemma 4.8 we have

$$\begin{aligned}
&\|\widehat{x}^r\otimes(y-y_{\cdot-r})\|_{2\beta'(a,b)}\\
&\le K\Big[\big(\|\sigma'\|_\infty + \|\sigma'\|_\lambda\|\widehat{x}^r\|^\lambda_{\beta'}T^{\lambda\beta'}\big)\big(\|\widehat{x}^r\|_{\beta'}\|y\|_{\beta'} + \|\widehat{x}^r\otimes y\|_{2\beta'}\big)T^{\beta'}\\
&\quad + \|b\|_\infty T^{1-\beta'} + \|\sigma\|_\infty\|y\|_{\beta'}\Big]\|y\|_\beta r^\varepsilon\\
&\quad + K\big[\|\sigma\|_\infty + \big(\|\sigma'\|_\infty + \|\sigma'\|_\lambda\|\widehat{x}^r\|^\lambda_{\beta'}T^{\lambda\beta'}\big)\|\widehat{x}^r\|_{\beta'}T^{\beta'}\big]\\
&\quad \times \|y_{\cdot-r}\otimes(y-y_{\cdot-r})\|_{2\beta'(r,T)}.
\end{aligned} \tag{5.44}$$

Putting together (5.43) and (5.44) and inequality (4.5) we obtain

$$\begin{aligned}
A_2 &\le K\rho\Lambda r^\varepsilon + K\rho^2\Lambda^2 r^\varepsilon + K\rho^2\Lambda\Lambda_r + K\rho\Lambda_r\\
&\le K\rho^2\Lambda^2 r^\varepsilon + K\rho^2\Lambda\Lambda_r,
\end{aligned} \tag{5.45}$$

where we have used that $1\le\rho\le\rho^2$ and $1\le\Lambda\le\Lambda^2$.

Finally, by Proposition 4.7 and inequalities (5.32) and (5.39) we obtain

$$\begin{aligned}
A_3 &\le G^4_{\beta'(r,T)}(\sigma,\widehat{x}^r,\widehat{x}^r_{\cdot-r},y_{\cdot-r},y)T^{\beta'}\|x^r-\widehat{x}^r\|_\infty\\
&\quad + G^5_{\beta'(r,T)}(\sigma,\widehat{x}^r,\widehat{x}^r_{\cdot-r},y_{\cdot-r},y)T^{\beta'}\|x^r-\widehat{x}^r\|_{\beta'}\\
&\quad + G^6_{\beta'(r,T)}(\sigma,\widehat{x}^r_{\cdot-r},y)(b-a)^{\beta'}\|(x^r-\widehat{x}^r)\otimes y\|_{2\beta'(a-r,b-r)}\\
&\le K\rho^2\Lambda^2 r^\varepsilon + G^6_{\beta'(r,T)}(\sigma,\widehat{x}^r_{\cdot-r},y)(b-a)^{\beta'}\\
&\quad \times \|(x^r-\widehat{x}^r)\otimes y\|_{2\beta'(a-r,b-r)}
\end{aligned} \tag{5.46}$$

where we have used that $G^i_{\beta'(r,T)}(\sigma,\widehat{x}^r,\widehat{x}^r_{\cdot-r},y_{\cdot-r},y)T^{\beta'}\le K\rho\Lambda$ for $i=4,5$. Applying the multiplicative property, it is easy to see that

$$\begin{aligned}
&\|(x^r-\widehat{x}^r)\otimes y\|_{2\beta'(a-r,b-r)}\\
&\le \|(x^r-\widehat{x}^r)\otimes y\|_{2\beta'(a-r,a)} + \|(x^r-\widehat{x}^r)\otimes y\|_{2\beta'(a,b)} + \|x^r-\widehat{x}^r\|_{\beta'}\|y\|_{\beta'}.
\end{aligned}$$

On the one hand, by (5.39),

$$\|x^r - \widehat{x}^r\|_{\beta'}\|y\|_{\beta'} \le \|x^r - \widehat{x}^r\|_{\beta'}\|y\|_{\beta}T^{\varepsilon} \le K\rho^2\Lambda^2 r^{\varepsilon}.$$

On the other hand,

$$\|(x^r - \widehat{x}^r)\otimes y\|_{2\beta'(a-r,a)} \le K\rho^2\Lambda^2 r^{\varepsilon} + K\rho^2\Lambda\Lambda_r,$$

where the result is obtained considering separately the two cases $a \in [r, 2r)$ and $a \in [2r, T]$ and applying multiplicative property, inequalities (5.40), (5.41), (5.42), (5.45) and that $G^6_{\beta'(r,T)}(\sigma, \widehat{x}^r_{\cdot-r}, y)T^{\beta'} \le K\rho\Lambda$. Therefore,

$$\|(x^r - \widehat{x}^r)\otimes y\|_{2\beta'(a-r,b-r)} \le K\rho^2\Lambda^2 r^{\varepsilon} + K\rho^2\Lambda\Lambda_r + \|(x^r - \widehat{x}^r)\otimes y\|_{2\beta'(a,b)}.$$

Then, it follows that

$$A_3 \le G^6_{\beta'(r,T)}(\sigma, \widehat{x}^r_{\cdot-r}, y)(b-a)^{\beta'}\|(x^r - \widehat{x}^r)\otimes y\|_{2\beta'(a,b)} + K\rho^3\Lambda^3 r^{\varepsilon} + K\rho^3\Lambda^2\Lambda_r, \tag{5.47}$$

where we have used again that $G^6_{\beta'(r,T)}(\sigma, \widehat{x}^r_{\cdot-r}, y)T^{\beta'} \le K\rho\Lambda$. From inequalities (5.41), (5.42), (5.45) and (5.47) we have

$$\begin{aligned}\|(x^r - \widehat{x}^r)\otimes y\|_{2\beta'(a,b)} &\le G^6_{\beta'(r,T)}(\sigma, \widehat{x}^r_{\cdot-r}, y)(b-a)^{\beta'}\|(x^r - \widehat{x}^r)\otimes y\|_{2\beta'(a,b)}\\ &\quad + K\rho^3\Lambda^3 r^{\varepsilon} + K\rho^3\Lambda^2\Lambda_r,\end{aligned}$$

where we have used that $\rho^n \le \rho^{n+1}$ and $\Lambda^n \le \Lambda^{n+1}$ for any $n \in \mathbb{N}$. Set

$$\widetilde{\Delta} := \Big(2\sup_{r\le r_0} G^6_{\beta'(r,T)}(\sigma, \widehat{x}^r_{\cdot-r}, y)\Big)^{1/\beta'}. \tag{5.48}$$

Observe that, if a, b, $a < b$ are such that $b - a \le \widetilde{\Delta}$, then

$$\|(x^r - \widehat{x}^r)\otimes y\|_{2\beta'(a,b)} \le K\rho^3\Lambda^3 r^{\varepsilon} + K\rho^3\Lambda^2\Lambda_r. \tag{5.49}$$

Now consider a partition $r = t_0 < \cdots < t_M = T$ such that $(t_{i+1} - t_i) \le \widetilde{\Delta}$ for $i = 0\dots, M-1$. Then, using the multiplicative property iteratively, we have

$$\|(x^r - \widehat{x}^r)\otimes y\|_{2\beta'(r,T)} \le \sum_{i=0}^{M-1}\|(x^r - \widehat{x}^r)\otimes y\|_{2\beta'(t_i,t_{i+1})} + (M-1)\|x^r - \widehat{x}^r\|_{\beta'}\|y\|_{\beta'}.$$

Using (5.39) and (5.49), we obtain

$$\begin{aligned}\|(x^r - \widehat{x}^r)\otimes y\|_{2\beta'(r,T)} &\le KM\rho^3\Lambda^3 r^{\varepsilon} + KM\rho^3\Lambda^2\Lambda_r + K(M-1)\rho^2\Lambda^2 r^{\varepsilon}\\ &\le KM\rho^3\Lambda^3 r^{\varepsilon} + KM\rho^3\Lambda^2\Lambda_r.\end{aligned} \tag{5.50}$$

Finally, putting together (5.34), (5.39), (5.40) and (5.50) we have

$$\|(x^r - \widehat{x}^r)\otimes y\|_{2\beta'} \le KM\rho^3\Lambda^3 r^{\varepsilon} + KM\rho^3\Lambda^2\Lambda_r.$$

So the proof is complete. □

The following definitions will be useful for the next results:

$$\overline{G}^i_{\beta'} := \sup_{r\le r_0} G^i_{\beta'(0,T)}(\sigma, x, x^r, y) \quad i = 1, 2; \tag{5.51}$$

$$\overline{G}^3_{\beta'} := \sup_{r\le r_0} G^3_{\beta'(0,T)}(\sigma, x^r); \tag{5.52}$$

$$\overline{G}^j_{\beta'} := \sup_{r\le r_0} G^j_{\beta'(0,T)}(\sigma, x, x^r, y, y) \quad j = 4, 5; \tag{5.53}$$

$$\overline{G}^6_{\beta'} := \sup_{r\le r_0} G^6_{\beta'(0,T)}(\sigma, x^r, y). \tag{5.54}$$

The following result gives us a bound for $\|(x - x^r) \otimes y\|_{2\beta'(a,b)}$ when the interval (a, b) is small enough. Let

$$\Delta^1_{\beta'} = \big(2\overline{G}^6_{\beta'}\big)^{1/\beta'}.$$

Proposition 5.6. *Suppose that $(x, y, x\otimes y)$, $(x^r, y, x^r \otimes y)$ and $(\widehat{x}^r, y, \widehat{x}^r \otimes y)$ belong to $M^\beta_{d,m}(0, T)$, $(y, y, y \otimes y)$ belongs to $M^\beta_{m,m}(0, T)$ and $(y_{\cdot-r}, y, y_{\cdot-r} \otimes y)$ belongs to $M^\beta_{m,m}(r, T)$. Assume that σ and b satisfy* (H1) *and* (H2) *respectively. Then, for all $0 \le a < b \le T$ such that $(b - a) \le \Delta^1_{\beta'}$,*

$$\begin{aligned}
&\|(x - x^r) \otimes y\|_{2\beta'(a,b)}\\
&\le 2\big[L_N\|y\|_{\beta'}(b-a)^{1-2\beta'} + G^4_{\beta'(a,b)}(\sigma, x, x^r, y, y)\big](b-a)^{\beta'}\|x - x^r\|_{\infty(a,b)}\\
&\quad + 2G^5_{\beta'(a,b)}(\sigma, x, x^r, y, y)(b-a)^{\beta'}\|x - x^r\|_{\beta'(a,b)}\\
&\quad + 2K\rho\Lambda G^4_{\beta'(a,b)}(\sigma, x^r, \widehat{x}^r, y, y)(b-a)^{\beta'} r^{\beta'}\\
&\quad + 2K\rho\Lambda\big[G^5_{\beta'(a,b)}(\sigma, x^r, \widehat{x}^r, y, y) + M\rho^2\Lambda^2 G^6_{\beta'(a,b)}(\sigma, \widehat{x}^r, y)\big](b-a)^{\beta'} r^{\varepsilon}\\
&\quad + 2KM\rho^3\Lambda^2 G^6_{\beta'(a,b)}(\sigma, \widehat{x}^r, y)(b-a)^{\beta'}\Lambda_r.
\end{aligned}$$

Proof. The proposition is proved by applying first Propositions 4.3, 4.6 and 4.7 to definition (5.26), and then Proposition 5.4, and observing that for $a < b$ such that $(b - a) \le \Delta^1_{\beta'}$

$$G^6_{\beta'(a,b)}(\sigma, x^r, y)(b-a)^{\beta'} \le \frac{1}{2}.$$

□

6. Proof of main results

Proof of Theorem 3.1. We start by studying $\lim_{r\to0}\|x - x^r\|_\infty$. As in Lemma 5.1, we can study separately the intervals $[0, r)$ and (r, T).

First, we study the norm in the interval $[0, r)$. We apply Proposition 4.3 and Proposition 4.4 to (5.25) and obtain

$$\begin{aligned}
\|x - x^r\|_{\beta(0,r)} &\le L_N r^{1-\beta}\|x - x^r\|_{\infty(0,r)} + G^1_{\beta(0,r)}(\sigma, x, \eta_{\cdot-r}, y) r^\beta \|x - \eta_{\cdot-r}\|_{\infty(0,r)}\\
&\quad + G^2_{\beta(0,r)}(\sigma, x, \eta_{\cdot-r}, y) r^\beta \|x - \eta_{\cdot-r}\|_{\beta(0,r)}\\
&\quad + G^3_{\beta(0,r)}(\sigma, \eta_{\cdot-r}) r^\beta \|(x - \eta_{\cdot-r}) \otimes y\|_{2\beta(0,r)}.
\end{aligned}$$

Using that the supremum norm of x is bounded and the bound does not depend on r, we see that $\sup_{r\le r_0} G^i_{\beta(0,r)}(\sigma, x, \eta_{\cdot-r}, y) < \infty$ $i = 1, 2$ and $\sup_{r\le r_0} G^3_{\beta(0,r)}(\sigma, \eta_{\cdot-r}) < \infty$. So the last expression clearly approaches zero when r tends to zero.

Now we work on the interval $[r, T]$. Let $r \le a < b \le T$. Applying Propositions 4.3, 4.4, 4.5 and 5.4, we obtain

$$\begin{aligned}
\|x - x^r\|_{\beta'(a,b)} &\le \big[L_N(b-a)^{1-2\beta'} + G^1_{\beta'(a,b)}(\sigma, x, x^r, y)\big](b-a)^{\beta'}\|x - x^r\|_{\infty(a,b)}\\
&\quad + G^2_{\beta'(a,b)}(\sigma, x, x^r, y)(b-a)^{\beta'}\|x - x^r\|_{\beta'(a,b)}\\
&\quad + G^3_{\beta'(a,b)}(\sigma, x^r)(b-a)^{\beta'}\|(x - x^r) \otimes y\|_{2\beta'(a,b)}\\
&\quad + K\rho\Lambda G^1_{\beta'(a,b)}(\sigma, x^r, \widehat{x}^r, y)(b-a)^{\beta'} r^{\beta'}
\end{aligned}$$

$$+ \big[K\rho\Lambda G^2_{\beta'(a,b)}(\sigma, x^r, \widehat{x}^r, y) + K\rho^3\Lambda^3 G^3_{\beta'(a,b)}(\sigma, \widehat{x}^r)\big](b-a)^{\beta'} r^{\varepsilon}
+ K\rho^3\Lambda^2\Lambda_r G^3_{\beta'(a,b)}(\sigma, \widehat{x}^r)(b-a)^{\beta'}.$$

Set

$$\begin{aligned}
&H_r\\
&= K\rho\Lambda\big(G^1_{\beta'(0,T)}(\sigma, x^r, \widehat{x}^r, y) + 2G^3_{\beta'(0,T)}(\sigma, x^r) G^4_{\beta'(0,T)}(\sigma, x^r, \widehat{x}^r, y, y) T^{\beta'}\big) T^{\beta'} r^{\beta'}\\
&\quad + \Big[K\rho\Lambda\big(G^2_{\beta'(0,T)}(\sigma, x^r, \widehat{x}^r, y) + 2G^3_{\beta'(0,T)}(\sigma, x^r) G^5_{\beta'(0,T)}(\sigma, x^r, \widehat{x}^r, y, y) T^{\beta'}\big)\\
&\quad + K\rho^3\Lambda^3\big(G^3_{\beta'(0,T)}(\sigma, \widehat{x}^r) + 2G^3_{\beta'(0,T)}(\sigma, x^r) G^6_{\beta'(0,T)}(\sigma, \widehat{x}^r, y) T^{\beta'}\big)\Big] T^{\beta'} r^{\varepsilon}\\
&\quad + K\rho^3\Lambda^2\big(G^3_{\beta'(0,T)}(\sigma, \widehat{x}^r) + 2G^3_{\beta'(0,T)}(\sigma, x^r) G^6_{\beta'(0,T)}(\sigma, \widehat{x}^r, y) T^{\beta'}\big)\Big] T^{\beta'} \Lambda_r.
\end{aligned}$$

Observe that H_r converges to zero when r tends to zero. We select a and b such that

$$(b-a) \le \Delta^1_{\beta'} \tag{6.1}$$

and then we apply Proposition 5.6 to obtain

$$\begin{aligned}
&\|x - x^r\|_{\beta'(a,b)}\\
&\le \Big[L_N\big(2\|y\|_{\beta'} G^3_{\beta'(a,b)}(\sigma, x^r)(b-a)^{\beta'} + 1\big)(b-a)^{1-2\beta'}\\
&\quad + G^1_{\beta'(a,b)}(\sigma, x, x^r, y) + 2G^3_{\beta'(a,b)}(\sigma, x^r) G^4_{\beta'(a,b)}(\sigma, x, x^r, y, y)(b-a)^{\beta'}\Big]\\
&\quad \times (b-a)^{\beta'} \|x - x^r\|_{\infty(a,b)}\\
&\quad + \Big[G^2_{\beta'(a,b)}(\sigma, x, x^r, y) + 2G^3_{\beta'(a,b)}(\sigma, x^r) G^5_{\beta'(a,b)}(\sigma, x, x^r, y, y)\\
&\quad \times (b-a)^{\beta'}\Big](b-a)^{\beta'} \|x - x^r\|_{\beta'(a,b)}\\
&\quad + K\rho\Lambda\big(G^1_{\beta'(a,b)}(\sigma, x^r, \widehat{x}^r, y) + 2G^3_{\beta'(a,b)}(\sigma, x^r) G^4_{\beta'(a,b)}(\sigma, x^r, \widehat{x}^r, y, y)\\
&\quad \times (b-a)^{\beta'}\big)(b-a)^{\beta'} r^{\beta'}\\
&\quad + \Big[K\rho\Lambda\big(G^2_{\beta'(a,b)}(\sigma, x^r, \widehat{x}^r, y)\\
&\quad + 2G^3_{\beta'(a,b)}(\sigma, x^r) G^5_{\beta'(a,b)}(\sigma, x^r, \widehat{x}^r, y, y)(b-a)^{\beta'}\big)\\
&\quad + K\rho^3\Lambda^3\big(G^3_{\beta'(a,b)}(\sigma, \widehat{x}^r) + 2G^3_{\beta'(a,b)}(\sigma, x^r) G^6_{\beta'(a,b)}(\sigma, \widehat{x}^r, y)(b-a)^{\beta'}\big)\Big]\\
&\quad \times (b-a)^{\beta'} r^{\varepsilon}\\
&\quad + K\rho^3\Lambda^2\big(G^3_{\beta'(a,b)}(\sigma, \widehat{x}^r) + 2G^3_{\beta'(a,b)}(\sigma, x^r) G^6_{\beta'(a,b)}(\sigma, \widehat{x}^r, y)(b-a)^{\beta'}\big)\\
&\quad \times (b-a)^{\beta'} \Lambda_r\\
&\le \Big[L_N\big(2\|y\|_{\beta'} G^3_{\beta'(a,b)}(\sigma, x^r) T^{\beta'} + 1\big) T^{1-2\beta'} + G^1_{\beta'(a,b)}(\sigma, x, x^r, y)\\
&\quad + 2G^3_{\beta'(a,b)}(\sigma, x^r) G^4_{\beta'(a,b)}(\sigma, x, x^r, y, y) T^{\beta'}\Big](b-a)^{\beta'} \|x - x^r\|_{\infty(a,b)}\\
&\quad + \big[G^2_{\beta'(a,b)}(\sigma, x, x^r, y) + 2G^3_{\beta'(a,b)}(\sigma, x^r) G^5_{\beta'(a,b)}(\sigma, x, x^r, y, y) T^{\beta'}\big]\\
&\quad \times (b-a)^{\beta'} \|x - x^r\|_{\beta'(a,b)} + H_r.
\end{aligned}$$

For a and b such that

$$\big[G^2_{\beta'(a,b)}(\sigma,x,x^r,y)+2G^3_{\beta'(a,b)}(\sigma,x^r)G^5_{\beta'(a,b)}(\sigma,x,x^r,y,y)T^{\beta'}\big](b-a)^{\beta'}\leq\frac{1}{2}, \tag{6.2}$$

we obtain

$$\begin{aligned}
&\|x-x^r\|_{\beta'(a,b)}\\
&\leq 2\Big[L_N\big(2\|y\|_{\beta'}G^3_{\beta'(a,b)}(\sigma,x^r)T^{\beta'}+1\big)T^{1-2\beta'}\\
&\quad+G^1_{\beta'(a,b)}(\sigma,x,x^r,y)+2G^3_{\beta'(a,b)}(\sigma,x^r)G^4_{\beta'(a,b)}(\sigma,x,x^r,y,y)T^{\beta'}\Big]\\
&\quad\times(b-a)^{\beta'}\|x-x^r\|_{\infty(a,b)}+2H_r.
\end{aligned}\tag{6.3}$$

On the other hand, putting

$$\|x-x^r\|_{\infty(a,b)}\leq|x_a-x^r_a|+(b-a)^{\beta'}\|x-x^r\|_{\beta'(a,b)},$$

in (6.3) we obtain

$$\begin{aligned}
&\|x-x^r\|_{\infty(a,b)}\\
&\leq|x_a-x^r_a|+2\Big[L_N\big(2\|y\|_{\beta'}G^3_{\beta'(a,b)}(\sigma,x^r)T^{\beta'}+1\big)T^{1-2\beta'}\\
&\quad+G^1_{\beta'(a,b)}(\sigma,x,x^r,y)+2G^3_{\beta'(a,b)}(\sigma,x^r)G^4_{\beta'(a,b)}(\sigma,x,x^r,y,y)T^{\beta'}\Big]\\
&\quad\times(b-a)^{2\beta'}\|x-x^r\|_{\infty(a,b)}+2T^{\beta'}H_r\,.
\end{aligned}$$

For a and b, $a<b$ such that

$$\begin{aligned}
&2\Big[L_N\big(2\|y\|_{\beta'}G^3_{\beta'(a,b)}(\sigma,x^r)T^{\beta'}+1\big)T^{1-2\beta'}+G^1_{\beta'(a,b)}(\sigma,x,x^r,y)\\
&+2G^3_{\beta'(a,b)}(\sigma,x^r)G^4_{\beta'(a,b)}(\sigma,x,x^r,y,y)T^{\beta'}\Big]T^{\beta'}(b-a)^{\beta'}\leq\frac{1}{2}
\end{aligned}\tag{6.4}$$

we obtain

$$\|x-x^r\|_{\infty(a,b)}\leq 2|x_a-x^r_a|+4T^{\beta'}H_r\,,$$

and hence

$$\sup_{0\leq t\leq b}|x_t-x^r_t|\leq 2\sup_{0\leq t\leq a}|x_t-x^r_t|+4T^{\beta'}H_r\,. \tag{6.5}$$

We define now $\Delta_{\beta'}$ such that all a,b with $(b-a)\leq\Delta_{\beta'}$ fulfill (6.1), (6.2) and (6.4), that is

$$\begin{aligned}
\Delta_{\beta'}:=\Big(&16L_NT^{1-\beta'}+16\overline{G}^1_{\beta'}T^{\beta'}+4\overline{G}^2_{\beta'}\\
&+8\overline{G}^3_{\beta'}\big[4L_N\|y\|_{\beta'}T+4\overline{G}^4_{\beta'}T^{2\beta'}+\overline{G}^5_{\beta'}T^{\beta'}\big]+2\overline{G}^6_{\beta'}\Big)^{1/\beta'}.
\end{aligned}$$

Then, it is clear that (6.5) holds for all a and b such that $b-a\leq\Delta_{\beta'}$.

Now, we take a partition $0=t_0<t_1<\dots<t_M=T$ of the interval $[0,T]$ such that $(t_{i+1}-t_i)\leq\Delta_{\beta'}$. Then

$$\sup_{0\leq t\leq t_M=T}|x_t-x^r_t|\leq 2\sup_{0\leq t\leq t_{M-1}}|x_t-x^r_t|+4T^{\beta'}H_r. \tag{6.6}$$

Repeating the process M times we obtain

$$\sup_{0\leq t\leq T}|x_t-x^r_t|\leq 2^M|x_0-x^r_0|+\Big(\sum_{k=0}^{M-1}2^k\Big)4T^{\beta'}H_r=4(2^M-1)T^{\beta'}H_r$$

that clearly converges to zero when r tends to zero. Following the same arguments, obtain

$$\lim_{r\to 0}\|(x\otimes y)-(x^r\otimes y)\|_\infty = \lim_{r\to 0}\|(x-x^r)\otimes y\|_\infty .$$

□

7. Stochastic case

In this section we apply the results obtained in the deterministic case to Brownian motion paths in order to get convergence of stochastic differential equations driven by Brownian motion.

Suppose that $B=\{B_t=(B_t^1,B_t^2,\dots,B_t^m),\, t\geq 0\}$ is a m-dimensional Brownian motion. Fix a time interval $[0,T]$. Then, for $s,t\in[0,T]$ and $i,j\in\{1,\dots,m\}$, we consider the following tensor products:

$$(B^i\otimes B^j)_{s,t} := \int_s^t (B_u^i - B_s^i)\, d^\circ B_u^j - \frac{1}{2}(t-s)\mathbf{1}_{\{i=j\}}\,,$$

$$(B^i\otimes B^j_{\cdot -r})_{s,t} := \int_s^t (B_u^i - B_s^i)\, d^\circ B_{u-r}^j\,,$$

where the stochastic integral is a Stratonovich integral (see Russo and Vallois [17]). In [15] the authors show that we can choose a version $(B\otimes B_{\cdot-r})_{s,t}$ in such a way that $(B_{\cdot-r},B,B\otimes B_{\cdot-r})$ constitutes a β-Hölder continuous multiplicative functional, for a fixed $\beta\in(\frac13,\frac12)$. On the other hand, from Hu and Nualart [9] it follows that $(B,B,B\otimes B)$ is also a β-Hölder continuous multiplicative functional.

As an application of Theorem 3.1 we deduce the convergence when the delay goes to zero of the solutions for the stochastic differential delay equations

$$\begin{gathered} X^r(t)=\eta(0)+\int_0^t b(u,X_u^r)\,du+\int_0^t \sigma(X^r_{u-r})\,dB_u,\quad t\in(0,T],\\ X^r(t)=\eta(t),\quad t\in[-r,0], \end{gathered}$$

where the stochastic integral is a pathwise integral which depends on B and $(B\otimes B)$. Set $X\equiv X^0$ the solution without delay and fix $\beta\in(\frac13,\frac12)$. Then the theorem reads as follows:

Theorem 7.1. *Assume that σ and b satisfy* (H1) *and* (H2) *respectively, and both satisfy* (H3). *Assume also that $(\eta_{\cdot-r_0},B,\eta_{\cdot-r_0}\otimes B)\in M^\beta_{d,m}(0,r_0)$, $\|\eta\|_{\beta(-r_0,0)}<\infty$ and $\sup_{r\leq r_0}\Phi_{\beta(0,r)}(\eta_{\cdot-r},B)<\infty$ a.s. Then,*

$$\lim_{r\to0}\|X-X^r\|_\infty=0\quad a.s.\quad and\quad \lim_{r\to0}\|(X\otimes B)-(X^r\otimes B)\|_\infty=0\quad a.s.$$

Applying Theorem 3.1 pathwise, the proof of Theorem 7.1 is an obvious consequence of (7.2) and (7.3) of the following lemma.

Lemma 7.2. *We have*

$$\|B\otimes(B-B_{\cdot-r})\|_{2\beta'(r,T)}\to 0\quad a.s.\ when\ r\ tends\ to\ 0, \tag{7.1}$$

$$\|B_{\cdot-r}\otimes(B-B_{\cdot-r})\|_{2\beta'(r,T)}\to 0\quad a.s.\ when\ r\ tends\ to\ 0, \tag{7.2}$$

$$\|(B-B_{\cdot-r})\otimes B\|_{2\beta'(r,T)}\to 0\quad a.s.\ when\ r\ tends\ to\ 0. \tag{7.3}$$

Proof. Let us recall first that $\|B\|_\beta<\infty$ a.s. We begin estimating (7.1) when $i\neq j$ (we will consider the case $i=j$ at the end). By definition,

$$\|B\otimes(B-B_{\cdot-r})\|_{2\beta'(r,T)}$$

$$= \sup_{s,t\in[r,T]} \frac{1}{(t-s)^{2\beta'}} \Big| \int_s^t (B_u^i - B_s^i)\, d^\circ B_{u-r}^j - \int_s^t (B_u^i - B_s^i)\, d^\circ B_u^j \Big|$$

Assume first that $t-s>r$. Applying integration by parts, we have

$$\begin{aligned}
&\|B\otimes(B-B_{\cdot-r})\|_{2\beta'(r,T)}\\
&= \sup_{\substack{s,t\in[r,T]\\ t-s>r}} \frac{1}{(t-s)^{2\beta'}} \Big| (B_t^i - B_s^i)(B_{t-r}^j - B_{s-r}^j) - \int_s^t (B_{u-r}^j - B_{s-r}^j)\, d^\circ B_u^i \\
&\quad + \int_s^t (B_u^j - B_s^j)\, d^\circ B_u^i - (B_t^i - B_s^i)(B_t^j - B_s^j) \Big| \\
&\le \sup_{\substack{s,t\in[r,T]\\ t-s>r}} \frac{1}{(t-s)^{2\beta'}} \big| (B_t^i - B_s^i)(B_{t-r}^j - B_t^j) \big| \\
&\quad + \sup_{\substack{s,t\in[r,T]\\ t-s>r}} \frac{1}{(t-s)^{2\beta'}} \Big| \int_s^t (B_u^j - B_{u-r}^j)\, d^\circ B_u^i \Big| \\
&= A_1 + A_2.
\end{aligned} \tag{7.4}$$

On the one hand, by (4.4),

$$\begin{aligned}
A_1 &\le \sup_{\substack{s,t\in[r,T]\\ t-s>r}} \|B\|_{\beta'} \frac{|B_t^j - B_{t-r}^j|}{(t-s)^{\beta'}} \\
&\le \sup_{\substack{s,t\in[r,T]\\ t-s>r}} \frac{1}{(t-s)^{\beta'}} \|B\|_{\beta'} \|B - B_{\cdot-r}\|_\infty \\
&\le \sup_{\substack{s,t\in[r,T]\\ t-s>r}} \frac{r^\beta}{(t-s)^{\beta'}} \|B\|_\beta^2 T^\varepsilon \le \|B\|_\beta^2 T^\varepsilon r^\varepsilon
\end{aligned}$$

that approaches zero when r tends to zero. On the other hand, we have that $\int_s^t (B_u^j - B_{u-r}^j)\, d^\circ B_u^i$ is a continuous martingale, so it can be represented as a time-changed Brownian motion: $W_{\int_s^t (B_u^j - B_{u-r}^j)^2\, du}$, where W is a Brownian motion. Now we choose $a\in(0,\frac12)$ such that $\frac{2\beta-2\varepsilon}{2\beta+1} < a < 2\beta - 2\varepsilon$. Applying Hölder property of the Brownian motion, we have

$$\begin{aligned}
A_2 &= \sup_{\substack{s,t\in[r,T]\\ t-s>r}} \frac{1}{(t-s)^{2\beta'}} \big| W_{\int_s^t (B_u^j - B_{u-r}^j)^2\, du} \big| \\
&\le \sup_{\substack{s,t\in[r,T]\\ t-s>r}} \frac{C_{a,T}}{(t-s)^{2\beta'}} \Big| \int_s^t (B_u^j - B_{u-r}^j)^2\, du \Big|^a \\
&\le \sup_{\substack{s,t\in[r,T]\\ t-s>r}} C_{a,T} \|B\|_\beta^{2a} r^{2a\beta} (t-s)^{a-2\beta'} \\
&\le C_{a,T} \|B\|_\beta^{2a} r^{2a\beta + a - 2\beta'},
\end{aligned}$$

that clearly appraoches zero as r tends to zero.

Now assume that $t-s\le r$. By integration by part formula, we have

$$
\begin{aligned}
&\|B\otimes(B-B_{\cdot-r})\|_{2\beta'(r,T)}\\
&\le \sup_{\substack{s,t\in[r,T]\\ t-s\le r}} \frac{1}{(t-s)^{2\beta'}}\big|(B^i_t-B^i_s)(B^j_{t-r}-B^j_{s-r})\big|\\
&\quad+\sup_{\substack{s,t\in[r,T]\\ t-s\le r}} \frac{1}{(t-s)^{2\beta'}}\Big|\int_s^t (B^j_{u-r}-B^j_{s-r})\,d^\circ B^i_u\Big|\\
&\quad+\sup_{\substack{s,t\in[r,T]\\ t-s\le r}} \frac{1}{(t-s)^{2\beta'}}\Big|\int_s^t (B^i_u-B^i_s)\,d^\circ B^j_u\Big|\\
&=V_1+V_2+V_3.
\end{aligned} \tag{7.5}
$$

The first term is easy to bound. Indeed,

$$
V_1=\sup_{\substack{s,t\in[r,T]\\ t-s\le r}} \frac{|B^i_t-B^i_s|}{(t-s)^{\beta}}\cdot\frac{|B^j_{t-r}-B^j_{s-r}|}{(t-s)^{\beta}}\cdot\frac{(t-s)^{2\beta}}{(t-s)^{2\beta'}}\le \|B\|^2_\beta r^{2\varepsilon}.
$$

For the other two terms we use inequality (5.8) of Hu and Nualart [9]. It states that there exists a random variable Z such that, almost surely, for all $s,t\in[0,T]$ we have

$$
\Big|\int_s^t (B^i_u-B^i_s)\,d^\circ B^j_u\Big|\le Z|t-s|\log\frac{1}{|t-s|}.
$$

Set $M'_t=\int_s^t(B^j_{u-r}-B^j_{s-r})\,d^\circ B^i_u$. Since the process $\{M'_t,t\in[s,T]\}$ is a continuous martingale, we can follow the ideas in [9] to get that there exists a random variable Z' such that, almost surely, for all $s,t\in[0,T]$ we have

$$
\Big|\int_s^t (B^j_{u-r}-B^j_{s-r})\,d^\circ B^i_u\Big|\le Z'|t-s|\log\frac{1}{|t-s|}.
$$

Hence

$$
V_2\le Z'(t-s)^{1-2\beta'}\log\frac{1}{(t-s)}\le Z' r^{1-2\beta'}\log\frac{1}{r}
$$

and V_2 goes to zero when r tends to zero. V_3 can be studied using the same arguments.

It only remains to prove the case where $i=j$. To simplify the notation we will not write the supra-index i.

For $t-s\le r$, we apply again the integration by parts formula and we obtain that

$$
\|B\otimes(B-B_{\cdot-r})\|_{2\beta'(r,T)}\le V'_1+V'_2+V'_3+V'_4,
$$

where V'_1, V'_2 and V'_3 are the terms defined in (7.5) with $i=j$ and

$$
V'_4:=\sup_{\substack{s,t\in[r,T]\\ t-s\le r}}\frac{1}{2}|t-s|^{1-\beta'}\le\frac{1}{2}r^{1-\beta'}.
$$

So it only remains to study the terms V'_1, V'_2 and V'_3. Easily, for V'_1 we can repeat the same arguments used for V_1 and we also obtain that $V'_1\le\|B\|^2_\beta r^{2\varepsilon}$. If we focus in the second term, it can be written as

$$V_2' = \sup_{\substack{s,t\in[r,T]\\ t-s\le r}} \frac{1}{(t-s)^{2\beta'}} \Big| \int_s^t (B_{u-r} - B_{s-r}) dB_u + \frac{1}{2}\int_s^t D_u(B_{u-r} - B_{s-r}) du \Big|,$$

where D_u denotes the de Malliavin derivative. It is easy to check that this Malliavin derivative is zero. So, V_2' is now a martingale and proceeding as in the case $i \neq j$ we obtain that

$$V_2' \le C r^{1-2\beta'} \log \frac{1}{r}.$$

Finally, for the last term we have

$$V_3' \le \sup_{\substack{s,t\in[r,T]\\ t-s\le r}} \frac{1}{2(t-s)^{2\beta'}} (B_t - B_s)^2 \le \frac{1}{2} \|B\|_\beta^2 r^{2\varepsilon}.$$

Therefore, when $t - s \le r$, the three terms tend to zero when r tends to zero.

For the case $t - s > r$, by integration by parts formula we have

$$\begin{aligned} &\|B \otimes (B - B_{\cdot - r})\|_{2\beta'(r,T)} \\ &\le \sup_{\substack{s,t\in[r,T]\\ t-s> r}} \frac{1}{(t-s)^{2\beta'}} \big|(B_t - B_s)(B_{t-r} - B_t)\big| \\ &\quad + \sup_{\substack{s,t\in[r,T]\\ t-s> r}} \frac{1}{(t-s)^{2\beta'}} \Big| \int_s^t (B_u - B_{u-r})\, d^\circ B_u - \frac{1}{2}(t-s) \Big|. \end{aligned}$$

The first term is analogous to the term A_1 defined in (7.4), so it is bounded by $\|B\|_\beta^2 T^\varepsilon r^\varepsilon$.

For the second term, we can use the relation between Stratonovich and Itô integrals,

$$\int_s^t (B_u - B_{u-r})\, d^\circ B_u = \int_s^t (B_u - B_{u-r})\, dB_u + \frac{1}{2}(t-s).$$

Set $M_t'' = \int_s^t (B_u - B_{u-r})\, dB_u$. Fixed s, the process $\{M_t'', t \in [s,T]\}$ is a continuous martingale. So following the ideas used for A_2, we obtain

$$\begin{aligned} &\sup_{\substack{s,t\in[r,T]\\ t-s> r}} \frac{1}{(t-s)^{2\beta'}} \Big| \int_s^t (B_u - B_{u-r})\, d^\circ B_u - \frac{1}{2}(t-s) \Big| \\ &\le \sup_{\substack{s,t\in[r,T]\\ t-s> r}} \frac{1}{(t-s)^{2\beta'}} \Big| \int_s^t (B_u - B_{u-r})\, dB_u \Big| \\ &\le C_{a,T} \|B\|_\beta^{2a} r^{2a\beta + a - 2\beta'}, \end{aligned}$$

where $a \in (0, \frac{1}{2})$ such that $\frac{2\beta - 2\varepsilon}{2\beta + 1} < a < 2\beta - 2\varepsilon$. Thus we obtain that $\|B \otimes (B - B_{\cdot - r})\|_{2\beta'(r,T)} \to 0$ as we wish.

Inequality (7.2) can be proved with similar computations and the proof of (7.3) follows immediately from the fact that

$$\|(B - B_{\cdot - r}) \otimes B\|_{2\beta'(r,T)} \le V_2 + V_3.$$

$\square$

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