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REPRESENTATION OF SOLUTIONS OF A SECOND ORDER DELAY DIFFERENTIAL EQUATION

KEE QIU, JINRONG WANG

Abstract. In this article, we study an inhomogeneous second order delay differential equation on the fractal set $\mathbb{R}^{\alpha n}$ $(0 < \alpha \leq 1)$, based on the theory of local calculus. We introduce delay cosine and sine type matrix functions and give their properties on the fractal set. We give the representation of solutions to second order differential equations with pure delay and two delays.

1. Introduction

In 2003, Khusainov and Shuklin [5] introduced the useful notation of delayed exponential matrix functions, which is used to represent solutions of linear autonomous time-delay systems with permutation matrices. Khusainov and Diblík [4] transferred this idea for solving the Cauchy problem for an oscillating system with second order and pure delay, by constructing special delayed matrix of cosine and sine type. These pioneer works led to many new results in integer and fractional order differential equations with delays and discrete delayed system; see [1, 2, 3, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 24, 25].

In 2012, Yang [20] transferred the standard calculus to local calculus on a fractal set, which is utilized in various non-differentiable problems that appear in complex systems of real-world phenomena. Furthermore, the non-differentiability occurring in science and engineering was modeled by the local fractional ordinary or partial differential equations [19, 21, 23]. As an effective research tool for continuous non-differentiable function, local fractional calculus has attracted a lot of attention, see [22].

In light of the above mentioned theory of local fractional calculus and delayed matrix of cosine and sine type on real set, we shall introduce the notation of delayed matrix of cosine and sine type on the fractal set $\mathbb{R}^{\alpha n}$ $(0 < \alpha \leq 1)$. The potential applications of the delayed cosine and sine type matrix function on a fractal set will be effective for homogeneous or inhomogeneous delay differential equation ona fractal set with constant matrix coefficients. In this article, we use two new special matrix functions to derive the representation of the solution to the following second

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order inhomogeneous delay differential equations on a fractal set:

$$\begin{aligned} y^{(2\alpha)}(x)+A^2y(x-\tau)=f(x), \quad y(x)\in\mathbb{R}^{\alpha n},\ x\geq 0,\ \tau>0,\\ y(x)\equiv\phi(x), \quad y^{(\alpha)}(x)\equiv\phi^{(\alpha)}(x), \quad -\tau\leq x\leq 0, \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} y^{(2\alpha)}(x)+A^2y(x-\tau_1)+B^2y(x-\tau_2)=f(x),\\ y(x)\in\mathbb{R}^{\alpha n},\ x\geq 0,\ \tau_1,\tau_2>0,\\ y(x)=\phi(x), \quad y^{(\alpha)}(x)=\phi^{(\alpha)}(x), \quad -\tau\leq x\leq 0, \end{aligned} \tag{1.2}$$

where $y^{(n\alpha)}(x)$ is the $n\alpha$-local fractional derivative on the fractal set $\mathbb{R}^{\alpha n}$ $(0<\alpha\leq 1)$, and $f: R_0^+\to\mathbb{R}^{\alpha n}$ is a given function, the matrices $A=(a_{ij}^{\alpha})_n$ and $B=(b_{ij}^{\alpha})_n$ are permutable constant matrices on a fractal set with $\det A\neq 0$ and $\det B\neq 0$, and $\phi(x)$ is an arbitrary twice local continuously differentiable vector function on the fractal set, i.e., $\phi\in C_{2\alpha}([-\tau,0],\mathbb{R}^{\alpha n})$.

Following the approach in [4, 5, 13], the main contribution of this article is deriving the representation of (1.1) and (1.2) involving special matrix functions on the fractal set. Section 2 introduces the concepts of matrix functions called delay cosine and sine type on a fractal set, and gives their properties. Section 3 gives the representation of solution to (1.1). The final section gives the representation of solution to (1.2).

2. Preliminaries

We recall some basic definitions of local fractional calculus from [20, 22]. Let $\mathbb{R}^\alpha$ $(0<\alpha\leq 1)$ be α-type set of the real line. If $a^\alpha, b^\alpha, c^\alpha\in\mathbb{R}^\alpha$, then

(i) $a^\alpha+b^\alpha\in\mathbb{R}^\alpha$, $a^\alpha b^\alpha\in\mathbb{R}^\alpha$.
(ii) $a^\alpha+b^\alpha=b^\alpha+a^\alpha=(a+b)^\alpha=(b+a)^\alpha$ and $(a-b)^\alpha=a^\alpha-b^\alpha$.
(iii) $a^\alpha+(b^\alpha+c^\alpha)=(a+b)^\alpha+c^\alpha$.
(iv) $a^\alpha b^\alpha=b^\alpha a^\alpha=(ab)^\alpha=(ba)^\alpha$.
(v) $a^\alpha(b^\alpha c^\alpha)=(a^\alpha b^\alpha)c^\alpha$.
(vi) $a^\alpha(b^\alpha+c^\alpha)=a^\alpha b^\alpha+a^\alpha c^\alpha$.
(vii) $a^\alpha+0^\alpha=0^\alpha+a^\alpha=a^\alpha$ and $a^\alpha 1^\alpha=1^\alpha a^\alpha=a^\alpha$.

Definition 2.1. A function $f:\mathbb{R}\to\mathbb{R}^\alpha$ is called local fractional continuous at $x=x_0$, if for each $\varepsilon>0$, there exists $\delta>0$ such that

$$|f(x)-f(x_0)|<\varepsilon^\alpha$$

holds whenever $|x-x_0|<\delta$, where $\varepsilon,\delta\in\mathbb{R}$. If $f(x)$ is local fractional continuous in the domain (a,b), then, we denote $f(x)\in C_\alpha(a,b)$.

Definition 2.2. Suppose that $f\in C_\alpha(a,b)$, $0<\alpha\leq 1$, and that for $\delta>0$ and $0<|x-x_0|<\delta$, the limit

$$D^{(\alpha)}f(x_0)=\frac{d^\alpha f(x)}{dx^\alpha}\Big|_{x=x_0}=\lim_{x\to x_0}\frac{\Gamma(1+\alpha)(f(x)-f(x_0))}{(x-x_0)^\alpha},$$

exists and is finite. Then $D^{(\alpha)}f(x_0)$ is said to be the local fractional derivative of f of order α at $x=x_0$. It is convenient to denote the local fractional derivative as $f^{(\alpha)}(x_0)$.

Let $f(u,x)$ be defined in a domain $\wp$ of the ux-plane. The local fractional partial derivative operator of $f(u,x)$ of order α with respect to u in a domain $\wp$ is defined by

$$f^{(\alpha)}(u_0,x)=\frac{\partial^\alpha f(u,x)}{\partial u^\alpha}\Big|_{u=u_0}=\lim_{u\to u_0}\frac{\Gamma(1+\alpha)(f(u,x)-f(u_0,x))}{(u-u_0)^\alpha}.$$

Similarly, the local fractional partial derivative operator of $f(u,x)$ of higher order $n\alpha$ with respect to u in a domain $\wp$ is defined by

$$f^{(n\alpha)}(u_0,x)=\frac{\partial^{n\alpha} f(u,x)}{\partial u^\alpha}\Big|_{u=u_0}=\overbrace{\frac{\partial^\alpha}{\partial u^\alpha}\dots\frac{\partial^\alpha}{\partial u^\alpha}}^{n\ times}f(u,x)\Big|_{u=u_0},$$

where n is a positive integer.

Definition 2.3. Let $f\in C_\alpha[a,b]$. Then the local fractional integral of function f of order α is defined by

$${}_aI_b^{(\alpha)}f(x)=\frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)(dt)^\alpha=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t_j\to 0}\sum_{j=0}^{N-1}f(t_j)(\Delta t_j)^\alpha,$$

where $\Delta t_j=t_{j+1}-t_j$ with $a=t_0<t_1<\dots<t_{N-1}<t_N=b,[t_j,t_{j+1}]$ is a partition of the interval $[a,b]$. Note that ${}_aI_a^{(\alpha)}f(x)=0$ and ${}_aI_b^{(\alpha)}f(x)=-{}_bI_a^{(\alpha)}f(x)$ if $a<b$.

Now we introduce the concepts matrix functions called delay cosine and sine type on the fractal set $\mathbb{R}^{\alpha n}$ $(0<\alpha\le 1)$.

Definition 2.4. The delayed cosine type matrix function is deifined as

$$\cos_\tau(Ax^\alpha):=\begin{cases}\Theta, & -\infty<x<-\tau,\\ I & -\tau\le x<0,\\ I-A^2\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+A^4\frac{(x-\tau)^{4\alpha}}{\Gamma(1+4\alpha)}+\dots & \\ +(-1)^kA^{2k}\frac{(x-(k-1)\tau)^{2k\alpha}}{\Gamma(1+2k\alpha)}, & (k-1)\tau\le x<k\tau,\ k\in\mathbb{N},\end{cases}$$

and the delayed sine type matrix function as

$$\sin_\tau(Ax^\alpha):=\begin{cases}\Theta, & -\infty<x<-\tau,\\ A\frac{(x+\tau)^\alpha}{\Gamma(1+\alpha)}, & -\tau\le x<0,\\ A\frac{(x+\tau)^\alpha}{\Gamma(1+\alpha)}-A^3\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\dots & \\ +(-1)^kA^{2k+1}\frac{(x-(k-1)\tau)^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)}, & (k-1)\tau\le x<k\tau,\ k\in\mathbb{N},\end{cases}$$

where $A=(a_{ij}^\alpha)_n$ is a constant matrix on the fractal set, Θ is the null matrix and I is the identity matrix. Moreover, $\mathbb{N}$ denotes the set of all nonnegative integers.

Next, we introduce two functions via an analogous delayed sine and cosine type matrix functions on the fractal set, which are tools for solving differential equation with two delays.

Definition 2.5. We define $U_{\tau_1,\tau_2}^{A,B}(x),V_{\tau_1,\tau_2}^{A,B}(x):\mathbb{R}\to L(\mathbb{R}^{\alpha n})$ as follows:

$$U_{\tau_1,\tau_2}^{A,B}(x)=\sum_{\substack{i,j\ge 0\\ i\tau_1+j\tau_2\le x}}(-1)^{i+j}C_{i+j}^iA^{2i}B^{2j}\frac{(x-i\tau_1-j\tau_2)^{2(i+j)\alpha}}{\Gamma(1+2(i+j)\alpha)},$$

$$V_{\tau_1,\tau_2}^{A,B}(x)=\sum_{\substack{i,j\geq 0\\ i\tau_1+j\tau_2\leq x}}(-1)^{i+j}C_{i+j}^{i}A^{2i}B^{2j}\frac{(x-i\tau_1-j\tau_2)^{(2(i+j)+1)\alpha}}{\Gamma(1+(2(i+j)+1)\alpha)},$$

where $\tau_1,\tau_2>0, A, B$ are $n\times n$ constant matrixes on the fractal set $\mathbb{R}^{\alpha n}$, by definition $U_{\tau_1,\tau_2}^{A,B}(x)=0$, $V_{\tau_1,\tau_2}^{A,B}(x)=0$ if $x<0$.

Some properties of $U_{\tau_1,\tau_2}^{A,B}(x), V_{\tau_1,\tau_2}^{A,B}(x)$ are established in Lemma 2.12 below. Now, we give some properties associated with the local fractional derivatives and the local fractional integrals on the fractal set, see [20, 22].

Lemma 2.6. (i) *Suppose that* $g^{(\alpha)}(x)=f(x)\in C_\alpha[a,b]$, *then*

$${}_aI_b^{(\alpha)}f(x)=g(b)-g(a).$$

(ii) *Suppose that* $f,g\in C_\alpha[a,b]$, *and* $,g\in D_\alpha(a,b)$, *then*

$${}_aI_b^{(\alpha)}(f(x)g^{(\alpha)}(x))=f(x)g(x)\Big|_a^b-{}_aI_b^{(\alpha)}(f^{(\alpha)}(x)g(x)).$$

(iii) *Suppose that* $f\in C_\alpha[a,b]$, *then*

$$\begin{gathered}\frac{d^\alpha}{dx^\alpha}\int_a^x f(\xi)(d\xi)^\alpha=\Gamma(1+\alpha)f(x),\quad x\in(a,b),\\ \frac{d^\alpha}{dx^\alpha}\int_a^{u(x)} f(\xi)(d\xi)^\alpha=\Gamma(1+\alpha)f(u(x))(u'(x))^\alpha,\end{gathered}$$

for $x\in[a,b]$ *and* $u\in C_1[a,b]$.

(iv) *Suppose that* $f(u,x)\in C_\alpha([a,b],[c,d])$, $\frac{\partial^\alpha}{\partial u^\alpha}f\in C_\alpha([a,b],[c,d])$, *then*

$$\phi(u)=\frac{1}{\Gamma(1+\alpha)}\int_a^b f(u,x)(dx)^\alpha$$

is a local fractional derivative on $[a,b]$, *and*

$$\frac{d^\kappa}{du^\kappa}\phi(u)=\frac{1}{\Gamma(1+\alpha)}\int_a^b\frac{d^\kappa f(u,x)}{du^\kappa}(dx)^\alpha,\quad 0<\kappa\leq 1.$$

(v) *Suppose that* $f(u,x)\in C_\alpha([a,b],[c,d])$, $\frac{\partial^\alpha}{\partial u^\alpha}f\in C_\alpha([a,b],[c,d])$, $c(u),d(u)\in C_1[a,b]$, $c\leq c(u)\leq d$, $c\leq d(u)\leq d$ *for any* $u\in[a,b]$, *then*

$$\phi(u)=\frac{1}{\Gamma(1+\alpha)}\int_{c(u)}^{d(u)} f(u,x)(dx)^\alpha$$

is a local fractional derivative on $[a,b]$, *and*

$$\begin{aligned}\frac{d^\kappa}{du^\kappa}\phi(u)=&\frac{1}{\Gamma(1+\alpha)}\int_{c(u)}^{d(u)}\frac{d^\kappa f(u,x)}{du^\kappa}(dx)^\alpha\\ &+f(u,d(u))(d'(u))^\alpha-f(u,c(u))(c'(u))^\alpha,\quad 0<\kappa\leq 1.\end{aligned}$$

Lemma 2.7. *We have*

$$\begin{gathered}\frac{d^\alpha x^{k\alpha}}{dx^\alpha}=\frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)}x^{(k-1)\alpha},\\ \frac{1}{\Gamma(1+\alpha)}\int_a^b x^{k\alpha}(dx)^\alpha=\frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}(b^{(k+1)\alpha}-a^{(k+1)\alpha}),\quad k>0.\end{gathered}$$

Lemma 2.8. *Suppose that $f(x), g(x) \in D_\alpha(a,b)$, $\lambda, \gamma \in \mathbb{R}$. The local fractional differentiation rules of non-differentiable functions defined on fractal set are listed as follows:*

(i) $(\lambda f(x) \pm \gamma g(x))^{(\alpha)} = \lambda f^{(\alpha)}(x) + \gamma g^{(\alpha)}(x)$.
(ii) $(f(x)g(x))^{(\alpha)} = f^{(\alpha)}(x)g(x) + f(x)g^{(\alpha)}(x)$.
(iii) $(f(x)/g(x))^{(\alpha)} = (f^{(\alpha)}(x)g(x) - f(x)g^{(\alpha)}(x))/g^2(x)$, *provided* $g(x) \neq 0$.

Suppose that $g(x) = f(u(x))$, and $f^{(\alpha)}(u)$ and $u'(x)$ exist. Then

$$g^{(\alpha)}(x) = f(u(x))^{(\alpha)} = f^{(\alpha)}(u)(u'(x))^\alpha.$$

Lemma 2.9. *Suppose that $f(x), g(x) \in C_\alpha[a,b]$, $\lambda, \gamma \in \mathbb{R}$. The local fractional integral rules of non-differentiable functions defined on a fractal set are listed as follows:*

(i) ${}_aI_b^{(\alpha)}(\lambda f(x) \pm \gamma g(x)) = \lambda {}_aI_b^{(\alpha)} f(x) + \gamma {}_aI_b^{(\alpha)} g(x)$.
(ii) ${}_aI_b^{(\alpha)} f(x) = {}_aI_c^{(\alpha)} f(x) + {}_cI_b^{(\alpha)} f(x)$, *provided* $a < c < b$.

It should be noted that the fractional derivative in the following represent the one-side derivative in nodes $x = k\tau, k = 0, 1, 2, \dots$ and $x = \tau_1, \tau_2$.

Lemma 2.10. *For delayed cosine type matrix function $\cos_\tau(Ax^\alpha)$, one has*

$$\begin{aligned} \Big(\cos_\tau(Ax^\alpha)\Big)^{(\alpha)} &= -A\sin_\tau(A(x-\tau)^\alpha), \\ \Big(\cos_\tau(Ax^\alpha)\Big)^{(2\alpha)} &= -A^2\cos_\tau(A(x-\tau)^\alpha). \end{aligned} \tag{2.1}$$

In other words, the delayed cosine type matrix function is a solution of differential equation of the second order with pure delay on fractal set

$$y^{(2\alpha)}(x) + A^2 y(x-\tau) = 0,$$

subject to initial value condition $y(x) = I$, $-\tau \le x \le 0$.

Proof. Let A and τ are fixed. Firstly, for arbitrary $x \in (-\infty, -\tau)$, $\cos_\tau(Ax^\alpha) = \sin_\tau(A(x-\tau)^\alpha) = \cos_\tau(A(x-\tau)^\alpha) = \Theta$. Obviously, (2.1) holds.

Secondly, $\cos_\tau(Ax^\alpha) = I$, $\sin_\tau(A(x-\tau)^\alpha) = \Theta$, $\cos_\tau(A(x-\tau)^\alpha) = \Theta$, which reduces to $\big(\cos_\tau(Ax^\alpha)\big)^{(\alpha)} = I^{(\alpha)} = \Theta = \sin_\tau(A(x-\tau)^\alpha)$ and $\big(\cos_\tau(Ax^\alpha)\big)^{(2\alpha)} = I^{(2\alpha)} = \Theta = \cos_\tau(A(x-\tau)^\alpha)$ for arbitrary $x \in [-\tau, 0)$, then (2.1) holds.

Finally, for an arbitrary $x : (k-1)\tau \le x < k\tau)$, we have

$$\begin{aligned} \Big(\cos_\tau(Ax^\alpha)\Big)^{(\alpha)} = \Big(I &- A^2\frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + A^4\frac{(x-\tau)^{4\alpha}}{\Gamma(1+4\alpha)} \\ &+ \dots + (-1)^k A^{2k}\frac{(x-(k-1)\tau)^{2k\alpha}}{\Gamma(1+2k\alpha)}\Big)^{(\alpha)}, \end{aligned} \tag{2.2}$$

applying Lemmas 2.7 and 2.8, we have

$$\begin{aligned} &\Big(\cos_\tau(Ax^\alpha)\Big)^{(\alpha)} \\ &= -A^2\frac{x^\alpha}{\Gamma(1+\alpha)} + A^4\frac{(x-\tau)^{3\alpha}}{\Gamma(1+3\alpha)} + \dots + (-1)^k A^{2k}\frac{(x-(k-1)\tau)^{(2k-1)\alpha}}{\Gamma(1+(2k-1)\alpha)} \\ &= -A\Big(A\frac{x^\alpha}{\Gamma(1+\alpha)} - A^3\frac{(x-\tau)^{3\alpha}}{\Gamma(1+3\alpha)} + \dots + (-1)^{k-1}A^{2k-1}\frac{(x-(k-1)\tau)^{(2k-1)\alpha}}{\Gamma(1+(2k-1)\alpha)}\Big) \end{aligned}$$

$$= -A\sin_\tau(A(x-\tau)^\alpha).$$

Then

$$\begin{aligned}
&\Big(\cos_\tau(Ax^\alpha)\Big)^{(2\alpha)}\\
&= \Big(\Big(\cos_\tau(Ax^\alpha)\Big)^{(\alpha)}\Big)^{(\alpha)}\\
&= -A\Big(\sin_\tau(A(x-\tau)^\alpha)\Big)^{(\alpha)}\\
&= -A\Big(A\frac{x^\alpha}{\Gamma(1+\alpha)} - A^3\frac{(x-\tau)^{3\alpha}}{\Gamma(1+3\alpha)} + \dots + (-1)^{k-1}A^{2k-1}\frac{(x-(k-1)\tau)^{(2k-1)\alpha}}{\Gamma(1+(2k-1)\alpha)}\Big)^{(\alpha)}\\
&= -A\Big(A - A^3\frac{(x-\tau)^{2\alpha}}{\Gamma(1+2\alpha)} + \dots + (-1)^{k-1}A^{2k-1}\frac{(x-(k-1)\tau)^{2(k-1)\alpha}}{\Gamma(1+2(k-1)\alpha)}\Big)\\
&= -A^2\Big(I - A^2\frac{(x-\tau)^{2\alpha}}{\Gamma(1+2\alpha)} + \dots + (-1)^{k-1}A^{2(k-1)}\frac{(x-(k-1)\tau)^{2(k-1)\alpha}}{\Gamma(1+2(k-1)\alpha)}\Big)\\
&= -A^2\cos_\tau(A(x-\tau)^\alpha).
\end{aligned}$$

This completes the proof. $\square$

Remark 2.11. Using a method similar to the one in the proof of Lemma 2.10, the following rule of fractional differentiation is true for the sine type matrix function.

$$\Big(\sin_\tau(Ax^\alpha)\Big)^{(\alpha)} = A\cos_\tau(Ax^\alpha), \quad \Big(\sin_\tau(Ax^\alpha)\Big)^{(2\alpha)} = -A^2\sin_\tau(A(x-\tau)^\alpha).$$

In this case, the delayed sine type matrix function is a solution of differential system of the second order with pure delay on fractal set

$$y^{(2\alpha)}(x) + A^2y(x-\tau) = 0,$$

that satisfies the initial conditions $y(x) = A\frac{(x+\tau)^\alpha}{\Gamma(1+\alpha)}$ for $-\tau \le x \le 0$.

Lemma 2.12. *Let $\tau_1, \tau_2 > 0$, $A = (a_{ij}^\alpha)_n$, $B = (b_{ij}^\alpha)_n$ be permutable constant matrices on fractal set with* $\det A \neq 0, \det B \neq 0$. *Then both $U^{A,B}_{\tau_1,\tau_2}(x)$ and $V^{A,B}_{\tau_1,\tau_2}(x)$ satisfy*

$$y^{(2\alpha)}(x) + A^2y(x-\tau_1) + B^2y(x-\tau_2) = 0. \tag{2.3}$$

for any $x \in \mathbb{R}$.

Proof. (i) If $\tau := \tau_1 = \tau_2$, then

$$\begin{aligned}
U^{A,B}_{\tau_1,\tau_2}(x) &= \sum_{\substack{i,j\ge 0\\ i\tau_1+j\tau_2\le x}} (-1)^{i+j}C^i_{i+j}A^{2i}B^{2j}\frac{(x-i\tau_1-j\tau_2)^{2(i+j)\alpha}}{\Gamma(1+2(i+j)\alpha)}\\
&= \sum_{\substack{k\ge 0\\ k\tau\le x}}\sum_{\substack{i,j\ge 0\\ i+j=k}} (-1)^{i+j}C^i_{i+j}A^{2i}B^{2j}\frac{(x-(i+j)\tau)^{2(i+j)\alpha}}{\Gamma(1+2(i+j)\alpha)}\\
&= \sum_{\substack{k\ge 0\\ k\tau\le x}} (-1)^k(A^2+B^2)^k\frac{(x-k\tau)^{2k\alpha}}{\Gamma(1+2k\alpha)}\\
&= \cos_\tau\Big(\sqrt{A^2+B^2}(x-\tau)^\alpha\Big),
\end{aligned}$$

where $k = i + j$. Using Lemma 2.10, we have $\cos_\tau\left(\sqrt{A^2+B^2}(x-\tau)^\alpha\right)$ is a solution of $y^{(2\alpha)}(x) + (A^2+B^2)y(x-\tau) = 0$, i.e.

$$\left(U^{A,B}_{\tau_1,\tau_2}(x)\right)^{(2\alpha)} + A^2 U^{A,B}_{\tau_1,\tau_2}(x-\tau_1) + B^2 U^{A,B}_{\tau_1,\tau_2}(x-\tau_2) = 0.$$

(ii) If $\tau_1 \neq \tau_2$, suppose that $\tau_1 < \tau_2$. Firstly, we suppose that $x < \tau_1$, so that

$$U^{A,B}_{\tau_1,\tau_2}(x) = I, \quad U^{A,B}_{\tau_1,\tau_2}(x-\tau_1) = 0, \quad U^{A,B}_{\tau_1,\tau_2}(x-\tau_2) = 0,$$

since $i\tau_1 + j\tau_2 \le x < \tau_1$, Definition 2.5 indicates that $i = 0$ and $j = 0$. Thus, (2.3) holds.

Secondly, we suppose that $\tau_1 \le x < \tau_2$, i.e., $x - \tau_2 < 0$, then

$$U^{A,B}_{\tau_1,\tau_2}(x-\tau_2) = 0,$$

and

$$U^{A,B}_{\tau_1,\tau_2}(x) = \sum_{\substack{i\ge 0\\ i\tau_1\le x}} (-1)^i A^{2i} \frac{(x-i\tau_1)^{2i\alpha}}{\Gamma(1+2i\alpha)} = \cos_\tau(A(x-\tau_1)^\alpha),$$

since $i\tau_1 + j\tau_2 \le x < \tau_2$ in the Definition 2.5 indicates that $j = 0$, using Lemma 2.10, we obtain (2.3).

Finally, we suppose that $x \ge \tau_2$. It suffices to note that

$$U^{A,B}_{\tau_1,\tau_2}(x) := I + \omega_1(x) + \omega_2(x) + \omega_3(x), \tag{2.4}$$

where

$$\omega_1(x) = \sum_{\substack{i\ge 1\\ i\tau_1\le x}} (-1)^i A^{2i} \frac{(x-i\tau_1)^{2i\alpha}}{\Gamma(1+2i\alpha)} = -A^2\frac{(x-\tau_1)^{2\alpha}}{\Gamma(1+2\alpha)} + A^4\frac{(x-2\tau_1)^{4\alpha}}{\Gamma(1+4\alpha)} - \dots,$$

$$\omega_2(x) = \sum_{\substack{j\ge 1\\ j\tau_2\le x}} (-1)^j B^{2j} \frac{(x-j\tau_2)^{2j\alpha}}{\Gamma(1+2j\alpha)} = -B^2\frac{(x-\tau_2)^{2\alpha}}{\Gamma(1+2\alpha)} + B^4\frac{(x-2\tau_2)^{4\alpha}}{\Gamma(1+4\alpha)} - \dots,$$

$$\omega_3(x) = \sum_{\substack{i,j\ge 1\\ i\tau_1+j\tau_2\le x}} (-1)^{i+j} C^i_{i+j} A^{2i}B^{2j} \frac{(x-i\tau_1-j\tau_2)^{2(i+j)\alpha}}{\Gamma(1+2(i+j)\alpha)}.$$

Calculating the second local fractal derivative of $\omega_1(x)$, we have

$$\begin{aligned}
\omega_1^{(2\alpha)}(x) &= \sum_{\substack{i\ge 1\\ i\tau_1\le x}} (-1)^i A^{2i} \frac{(x-i\tau_1)^{2(i-1)\alpha}}{\Gamma(1+2(i-1)\alpha)}\\
&= -A^2 \sum_{\substack{i\ge 1\\ i\tau_1\le x}} (-1)^{i-1} A^{2(i-1)} \frac{(x-\tau_1-(i-1)\tau_1)^{2(i-1)\alpha}}{\Gamma(1+2(i-1)\alpha)}\\
&= -A^2 \sum_{\substack{i\ge 0\\ i\tau_1\le x-\tau_1}} (-1)^i A^{2i} \frac{(x-\tau_1-i\tau_1)^{2i\alpha}}{\Gamma(1+2i\alpha)}\\
&= -A^2 - A^2 \sum_{\substack{i\ge 1\\ i\tau_1\le x-\tau_1}} (-1)^i A^{2i} \frac{(x-\tau_1-i\tau_1)^{2i\alpha}}{\Gamma(1+2i\alpha)}\\
&= -A^2 - A^2\omega_1(x-\tau_1).
\end{aligned}$$

Analogously, we have

$$\omega_2^{(2\alpha)}(x) = -B^2 - B^2\omega_2(x-\tau_2).$$

By using the properties of binomial numbers $C_{n+1}^m = C_n^m + C_n^{m-1}$ and $C_n^k = C_n^{n-k}$, for $n, m \geq 1$, we find that

$$\begin{aligned}
&\omega_3^{(2\alpha)}(x) \\
&= \sum_{\substack{i,j\geq 1\\ i\tau_1+j\tau_2\leq x}} (-1)^{i+j} C_{i+j}^{i} A^{2i}B^{2j}\frac{(x-i\tau_1-j\tau_2)^{2(i+j-1)\alpha}}{\Gamma(1+2(i+j-1)\alpha)} \\
&= \sum_{\substack{i,j\geq 1\\ i\tau_1+j\tau_2\leq x}} (-1)^{i+j} C_{i+j-1}^{i-1} A^{2i}B^{2j}\frac{(x-i\tau_1-j\tau_2)^{2(i+j-1)\alpha}}{\Gamma(1+2(i+j-1)\alpha)} \\
&\quad + \sum_{\substack{i,j\geq 1\\ i\tau_1+j\tau_2\leq x}} (-1)^{i+j} C_{i+j-1}^{i} A^{2i}B^{2j}\frac{(x-i\tau_1-j\tau_2)^{2(i+j-1)\alpha}}{\Gamma(1+2(i+j-1)\alpha)} \\
&= \sum_{\substack{i,j\geq 1\\ i\tau_1+j\tau_2\leq x}} (-1)^{i+j} C_{i-1+j}^{i-1} A^{2i}B^{2j}\frac{(x-\tau_1-(i-1)\tau_1-j\tau_2)^{2(i-1+j)\alpha}}{\Gamma(1+2(i-1+j)\alpha)} \\
&\quad + \sum_{\substack{i,j\geq 1\\ i\tau_1+j\tau_2\leq x}} (-1)^{i+j} C_{i+j-1}^{j-1} A^{2i}B^{2j}\frac{(x-i\tau_1-j\tau_2)^{2(i+j-1)\alpha}}{\Gamma(1+2(i+j-1)\alpha)} \\
&= \sum_{\substack{i,j\geq 1\\ i\tau_1+j\tau_2\leq x}} (-1)^{i+j} C_{i-1+j}^{i-1} A^{2i}B^{2j}\frac{(x-\tau_1-(i-1)\tau_1-j\tau_2)^{2(i-1+j)\alpha}}{\Gamma(1+2(i-1+j)\alpha)} \\
&\quad + \sum_{\substack{i,j\geq 1\\ i\tau_1+j\tau_2\leq x}} (-1)^{i+j} C_{i+j-1}^{j-1} A^{2i}B^{2j}\frac{(x-\tau_2-i\tau_1-(j-1)\tau_2)^{2(i+j-1)\alpha}}{\Gamma(1+2(i+j-1)\alpha)} \\
&= -A^2 \sum_{\substack{i,j\geq 1\\ i\tau_1+j\tau_2\leq x}} (-1)^{i-1+j} C_{i-1+j}^{i-1} A^{2(i-1)}B^{2j}\frac{(x-\tau_1-(i-1)\tau_1-j\tau_2)^{2(i-1+j)\alpha}}{\Gamma(1+2(i-1+j)\alpha)} \\
&\quad - B^2 \sum_{\substack{i,j\geq 1\\ i\tau_1+j\tau_2\leq x}} (-1)^{i+j-1} C_{i+j-1}^{j-1} A^{2i}B^{2(j-1)}\frac{(x-\tau_2-i\tau_1-(j-1)\tau_2)^{2(i+j-1)\alpha}}{\Gamma(1+2(i+j-1)\alpha)}.
\end{aligned}$$

We now replace $i-1$ by i in the first sum and $j-1 \to j$ in the second sum above, then we have

$$\begin{aligned}
\omega_3^{(2\alpha)}(x) = -A^2 &\sum_{\substack{i\geq 0, j\geq 1\\ i\tau_1+j\tau_2\leq x-\tau_1}} (-1)^{i+j} C_{i+j}^{i} A^{2i}B^{2j}\frac{(x-\tau_1-i\tau_1-j\tau_2)^{2(i+j)\alpha}}{\Gamma(1+2(i+j)\alpha)} \\
- B^2 &\sum_{\substack{i\geq 1, j\geq 0\\ i\tau_1+j\tau_2\leq x-\tau_2}} (-1)^{i+j} C_{i+j}^{j} A^{2i}B^{2j}\frac{(x-\tau_2-i\tau_1-j\tau_2)^{2(i+j)\alpha}}{\Gamma(1+2(i+j)\alpha)}.
\end{aligned}$$

Further, we split the first sum into $i=0$ and $i\geq 1$ and the second sum into $j=0$ and $j\geq 1$, then

$$\begin{aligned}\omega_3^{(2\alpha)}(x) = -A^2 & \sum_{\substack{j\geq 1\\ j\tau_2\leq x-\tau_1}} (-1)^j B^{2j}\frac{(x-\tau_1-j\tau_2)^{2j\alpha}}{\Gamma(1+2j\alpha)}\\
&- A^2\sum_{\substack{i,j\geq 1\\ i\tau_1+j\tau_2\leq x-\tau_1}} (-1)^{i+j}C_{i+j}^i A^{2i}B^{2j}\frac{(x-\tau_1-i\tau_1-j\tau_2)^{2(i+j)\alpha}}{\Gamma(1+2(i+j)\alpha)}\\
&- B^2\sum_{\substack{i\geq 1\\ i\tau_1\leq x-\tau_2}} (-1)^i A^{2i}\frac{(x-\tau_2-i\tau_1)^{2i\alpha}}{\Gamma(1+2i\alpha)}\\
&- B^2\sum_{\substack{i,j\geq 1\\ i\tau_1+j\tau_2\leq x-\tau_2}} (-1)^{i+j}C_{i+j}^j A^{2i}B^{2j}\frac{(x-\tau_2-i\tau_1-j\tau_2)^{2(i+j)\alpha}}{\Gamma(1+2(i+j)\alpha)}\\
&= -A^2\omega_2(x-\tau_1)-A^2\omega_3(x-\tau_1)-B^2\omega_1(x-\tau_2)-B^2\omega_3(x-\tau_2).\end{aligned}$$

Substituting the formulas for $\omega_1^{(2\alpha)}(x)$, $\omega_2^{(2\alpha)}(x)$, $\omega_3^{(2\alpha)}(x)$ and calculating the second fractal derivative both sides of (2.4), we obtain

$$\begin{aligned}\Big(U_{\tau_1,\tau_2}^{A,B}(x)\Big)^{(2\alpha)} &= -A^2\Big(I+\omega_1(x-\tau_1)+\omega_2(x-\tau_1)+\omega_3(x-\tau_1)\Big)\\
&\quad - B^2\Big(I+\omega_1(x-\tau_2)+\omega_2(x-\tau_2)+\omega_3(x-\tau_2)\Big)\\
&= -A^2U_{\tau_1,\tau_2}^{A,B}(x-\tau_1)-B^2U_{\tau_1,\tau_2}^{A,B}(x-\tau_2).\end{aligned}$$

Thus, we arrive at the relation. Further, we proceed by analogy with $V_{\tau_1,\tau_2}^{A,B}(x)$. Statement holds with $V_{\tau_1,\tau_2}^{A,B}(x)$ instead of $U_{\tau_1,\tau_2}^{A,B}(x)$. Therefore, we have the results. □

3. Solutions of differential equation with pure delay on fractal set

We study the linear homogeneous differential delay equations on fractal sets,

$$\begin{gathered}y^{(2\alpha)}(x)+A^2y(x-\tau)=0,\quad y(x)\in\mathbb{R}^{\alpha n},\ x\geq 0,\ \tau>0,\\
y(x)=\phi(x), y^{(\alpha)}(x)=\phi^{(\alpha)}(x),\quad -\tau\leq x\leq 0.\end{gathered}\tag{3.1}$$

Theorem 3.1. *Suppose that the matrix $A=(a_{ij}^\alpha)_n$ is a constant matrix on a fractal set with $\det A\neq 0$, and $\phi(x)\in C_{2\alpha}([-\tau,0],\mathbb{R}^{\alpha n})$. Then the solution $y(x)$ of (3.1) can be expressed as*

$$\begin{aligned}y(x) &= (\cos_\tau(Ax^\alpha))\phi(-\tau)+A^{-1}(\sin_\tau(Ax^\alpha))\phi^{(\alpha)}(-\tau)\\
&\quad +\frac{A^{-1}}{\Gamma(1+\alpha)}\int_{-\tau}^0 \sin_\tau(A(x-\tau-s)^\alpha)\phi^{(2\alpha)}(s)(ds)^\alpha.\end{aligned}\tag{3.2}$$

Proof. We seek for a solution of (3.1) in the form

$$\begin{aligned}y(x) &= (\cos_\tau(Ax^\alpha))c_1+(\sin_\tau(Ax^\alpha))c_2\\
&\quad +\frac{1}{\Gamma(1+\alpha)}\int_{-\tau}^0 \sin_\tau(A(x-\tau-s)^\alpha)z^{(2\alpha)}(s)(ds)^\alpha,\end{aligned}\tag{3.3}$$

where c_1, c_2 are unknown constant vectors on $\mathbb{R}^{\alpha n}$ and $z(x) : [-\tau, +\infty) \to \mathbb{R}^{\alpha n}$ is an unknown twice continuously differentiable vector function. From Lemma 2.10 and Remark 2.11 , i.e., due to linearity, we know that (3.3) is a solution of (3.1) for arbitrary c_1, c_2 and vector function $z(x) \in C_{2\alpha}([-\tau, 0], \mathbb{R}^{\alpha n})$. Now we try to fix the constants c_1, c_2 and the vector function $z(x)$ in such manner that the initial conditions $y(x) = \phi(x), y^{(\alpha)}(x) = \phi^{(\alpha)}(x), -\tau \le x \le 0$, are satisfied.

We use (3.3) to represent the first initial condition $y(x) = \phi(x)$, $-\tau \le x \le 0$, i.e.,

$$\begin{aligned}&(\cos_\tau(Ax^\alpha))c_1 + (\sin_\tau(Ax^\alpha))c_2\\&+ \frac{1}{\Gamma(1+\alpha)} \int_{-\tau}^{0} \sin_\tau(A(x-\tau-s)^\alpha) z^{(2\alpha)}(s)(ds)^\alpha = \phi(x).\end{aligned}$$

This leads to

$$c_1 + c_2 A\frac{(x+\tau)^\alpha}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(1+\alpha)} \int_{-\tau}^{0} \sin_\tau(A(x-\tau-s)^\alpha) z^{(2\alpha)}(s)(ds)^\alpha = \phi(x), \tag{3.4}$$

where

$$(\cos_\tau(Ax^\alpha)) = I, \quad (\sin_\tau(Ax^\alpha)) = A\frac{(x+\tau)^\alpha}{\Gamma(1+\alpha)}.$$

Since

$$\sin_\tau(A(x-\tau-s)^\alpha) = \begin{cases} 0, & x \le s \le 0,\\ A\frac{(x-s)^\alpha}{\Gamma(1+\alpha)}, & -\tau \le s \le x,\end{cases}$$

it follows that

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)} \int_{-\tau}^{0} \sin_\tau(A(x-\tau-s)^\alpha) z^{(2\alpha)}(s)(ds)^\alpha\\&= \frac{A}{\big(\Gamma(1+\alpha)\big)^2} \int_{-\tau}^{x} (x-s)^\alpha z^{(2\alpha)}(s)(ds)^\alpha.\end{aligned} \tag{3.5}$$

Using (i) and (ii) of Lemma 2.6 and Lemma 2.7 to the right-hand side of (3.5), i.e., using local fractional integration by parts and local fractional derivative for the right of (3.5), it is necessary to verify that

$$\begin{aligned}&\frac{A}{\big(\Gamma(1+\alpha)\big)^2} \int_{-\tau}^{x} (x-s)^\alpha z^{(2\alpha)}(s)(ds)^\alpha\\
&= \frac{A}{\Gamma(1+\alpha)}\Big((x-s)^\alpha z^{(\alpha)}(s)\Big|_{-\tau}^{x} - \frac{1}{\Gamma(1+\alpha)}\int_{-\tau}^{x} z^{(\alpha)}(s)((x-s)^\alpha)^{(\alpha)}(ds)^\alpha\Big)\\
&= \frac{A}{\Gamma(1+\alpha)}\Big((x-s)^\alpha z^{(\alpha)}(s)\Big|_{-\tau}^{x} + \frac{1}{\Gamma(1+\alpha)}\int_{-\tau}^{x} z^{(\alpha)}(s)(ds)^\alpha \Gamma(1+\alpha)\Big)\\
&= \frac{A}{\Gamma(1+\alpha)}\Big(-(x+\tau)^\alpha z^{(\alpha)}(-\tau) + \int_{-\tau}^{x} z^{(\alpha)}(s)(ds)^\alpha\Big)\\
&= -\frac{A}{\Gamma(1+\alpha)}(x+\tau)^\alpha z^{(\alpha)}(-\tau) + \frac{A}{\Gamma(1+\alpha)}\int_{-\tau}^{x} z^{(\alpha)}(s)(ds)^\alpha\\
&= -\frac{A}{\Gamma(1+\alpha)}(x+\tau)^\alpha z^{(\alpha)}(-\tau) + Az(s)\Big|_{-\tau}^{x}\\
&= -\frac{A}{\Gamma(1+\alpha)}(x+\tau)^\alpha z^{(\alpha)}(-\tau) + Az(x) - Az(-\tau).\end{aligned}$$

submitting this and (3.5) into (3.4), it follows that

$$c_1 + c_2 A\frac{(x+\tau)^\alpha}{\Gamma(1+\alpha)} - \frac{A}{\Gamma(1+\alpha)}(x+\tau)^\alpha z^{(\alpha)}(-\tau) + Az(x) - Az(-\tau) = \phi(x). \quad (3.6)$$

Let us rewrite the above equalityin the form

$$\Big(c_1 - Az(-\tau)\Big) + \Big(c_2 - z^{(\alpha)}(-\tau)\Big)\frac{A(x+\tau)^\alpha}{\Gamma(1+\alpha)} + Az(x) = \phi(x). \quad (3.7)$$

Applying Lemmas 2.7 and 2.8 to both sides of (3.7) and paying attention to the second initial condition $y^{(\alpha)}(x) = \phi^{(\alpha)}(x)$, $-\tau \le x \le 0$, we have

$$A\Big(c_2 - z^{(\alpha)}(-\tau)\Big) + Az^{(\alpha)}(x) = \phi^{(\alpha)}(x). \quad (3.8)$$

In this case, a combination of (3.7) and (3.8), one has

$$c_1 = \phi(-\tau), c_2 = A^{-1}\phi^{(\alpha)}(-\tau), \qquad z(x) = A^{-1}\phi(x),$$

since $\det(A) \neq 0$. Putting c_1, c_2 and $z(x)$ into (3.3), we obtain (3.2). $\square$

Remark 3.2. To obtain some alternative conclusions, with the assumptions in Theorem 3.1, one can apply integration by parts via Lemma 2.6. We have

$$\begin{aligned}
&\frac{A^{-1}}{\Gamma(1+\alpha)}\int_{-\tau}^{0} \sin_\tau(A(x-\tau-s)^\alpha)\phi^{(2\alpha)}(s)(ds)^\alpha \\
&= A^{-1}\frac{1}{\Gamma(1+\alpha)}\int_{-\tau}^{0} \sin_\tau(A(x-\tau-s)^\alpha)(d\phi^{(\alpha)}(s))^\alpha \\
&= A^{-1}\Big(\sin_\tau(A(x-\tau-s)^\alpha)\phi^{(\alpha)}(s)\Big|_{-\tau}^{0} \\
&\quad + \frac{1}{\Gamma(1+\alpha)}\int_{-\tau}^{0} \phi^{(\alpha)}(s)A\cos_\tau(A(x-\tau-s)^\alpha)(ds)^\alpha\Big) \\
&= A^{-1}\sin_\tau(A(x-\tau)^\alpha)\phi^{(\alpha)}(0) - A^{-1}\sin_\tau(Ax^\alpha)\phi^{(\alpha)}(-\tau) \\
&\quad + \frac{1}{\Gamma(1+\alpha)}\int_{-\tau}^{0} A\cos_\tau((x-\tau-s)^\alpha)(d\phi(s))^\alpha \\
&= A^{-1}\sin_\tau(A(x-\tau)^\alpha)\phi^{(\alpha)}(0) - A^{-1}\sin_\tau(Ax^\alpha)\phi^{(\alpha)}(-\tau) \\
&\quad + \cos_\tau(A(x-\tau-s)^\alpha)\phi(s)\Big|_{-\tau}^{0} \\
&\quad - \frac{A}{\Gamma(1+\alpha)}\int_{-\tau}^{0} \phi(s)\sin_\tau((x-2\tau-s)^\alpha)(ds)^\alpha \\
&= A^{-1}\sin_\tau(A(x-\tau)^\alpha)\phi^{(\alpha)}(0) - A^{-1}\sin_\tau(Ax^\alpha)\phi^{(\alpha)}(-\tau) \\
&\quad + \cos_\tau(A(x-\tau)^\alpha)\phi(0) \\
&\quad - \cos_\tau(Ax^\alpha)\phi(-\tau) - \frac{A}{\Gamma(1+\alpha)}\int_{-\tau}^{0} \sin_\tau((x-2\tau-s)^\alpha)\phi(s)(ds)^\alpha.
\end{aligned}$$

This implies that the conclusion of Theorem 3.1 can be expressed as

$$\begin{aligned}
y(x) = &(\cos_\tau(A(x-\tau)^\alpha))\phi(0) + A^{-1}(\sin_\tau(A(x-\tau)^\alpha))\phi^{(\alpha)}(0) \\
&- \frac{A}{\Gamma(1+\alpha)}\int_{-\tau}^{0} \sin_\tau(A(x-2\tau-s)^\alpha)\phi(s)(ds)^\alpha.
\end{aligned}$$

To end this section, we consider the inhomogeneous differential delay system on a fractal set

$$\begin{gathered} y^{(2\alpha)}(x) + A^2 y(x-\tau) = f(x), \quad y(x) \in \mathbb{R}^{\alpha n},\ x \geq 0,\ \tau > 0, \\ y(x) = 0, \quad -\tau \leq x \leq 0. \end{gathered} \tag{3.9}$$

Theorem 3.3. *Suppose that the matrix $A = (a_{ij}^{\alpha})_n$ is a constant matrix on a fractal set with $\det A \neq 0$, and $f : \mathbb{R}_0^+ \to \mathbb{R}^{\alpha n}$ is a given function. Then the solution $y_0(x)$ of the inhomogeneous equation (3.9) can be expressed as*

$$y_0(x) = \frac{A^{-1}}{\Gamma(1+\alpha)} \int_0^x \sin_\tau(A(x-\tau-s)^\alpha) f(s)(ds)^\alpha.$$

Proof. We will try to seek a particular solution $y_0(x)$ of the inhomogeneous equation (3.9), employing the method of variation of an arbitrary constant in the form

$$y_0(x) = \frac{1}{\Gamma(1+\alpha)} \int_0^x \sin_\tau(A(x-\tau-s)^\alpha) C(s)(ds)^\alpha,$$

where $C(s), 0 \leq s \leq x$, is an unknown function. Local fractional differentiating the function $y_0(x)$, we obtain

$$\begin{aligned} &y_0^{(\alpha)}(x) \\ &= \frac{1}{\Gamma(1+\alpha)} \int_0^x A\cos_\tau(A(x-\tau-s)^\alpha) C(s)(ds)^\alpha + \sin_\tau(A(x-\tau-s)^\alpha) C(s)\Big|_{s=x} \\ &= \frac{A}{\Gamma(1+\alpha)} \int_0^x \cos_\tau(A(x-\tau-s)^\alpha) C(s)(ds)^\alpha, \end{aligned}$$

and

$$\begin{aligned} &y_0^{(2\alpha)}(x) \\ &= \frac{A}{\Gamma(1+\alpha)} \int_0^x \Big((A\sin_\tau(A(x-2\tau-s)^\alpha) C(s)\Big)(ds)^\alpha \\ &\quad + A\cos_\tau(A(x-\tau-s)^\alpha) C(s)\Big|_{s=x} \\ &= -\frac{A^2}{\Gamma(1+\alpha)} \int_0^x \sin_\tau(A(x-2\tau-s)^\alpha) C(s)(ds)^\alpha + A\cos_\tau(A(-\tau)^\alpha) C(x) \\ &= -\frac{A^2}{\Gamma(1+\alpha)} \int_0^x \sin_\tau(A(x-2\tau-s)^\alpha) C(s)(ds)^\alpha + AC(x), \end{aligned}$$

since $\sin_\tau(A(x-2\tau-s)^\alpha) = 0$, when $x-\tau \leq s \leq x$. Hence,

$$y_0^{(2\alpha)}(x) = -\frac{A^2}{\Gamma(1+\alpha)} \int_0^{x-\tau} \sin_\tau(A(x-2\tau-s)^\alpha) C(s)(ds)^\alpha + AC(x).$$

Substituting $y_0^{(2\alpha)}(x)$ and $y_0(x-\tau)$ into system (3.9), we obtain

$$\begin{aligned} &-\frac{A^2}{\Gamma(1+\alpha)} \int_0^{x-\tau} \sin_\tau(A(x-2\tau-s)^\alpha) C(s)(ds)^\alpha + AC(x) \\ &+ \frac{A^2}{\Gamma(1+\alpha)} \int_0^{x-\tau} \sin_\tau(A(x-2\tau-s)^\alpha) C(s)(ds)^\alpha = f(x). \end{aligned}$$

Since $\det A \neq 0$, we obtain $C(x) = A^{-1} f(x)$. Thus, we arrive at the results in Theorem 3.3. $\square$

As we know, the solution of system (1.1) is the sum of solution of homogeneous problem (3.1) and a particular solution of (3.9). Therefore, collecting the results of Theorem 3.1, Remark 3.2 and Theorem 3.3, we obtain the following results.

Corollary 3.4. *Solution of* (1.1) *can be represented in the form*

$$y(x)=\begin{cases}\phi(x), & -\tau\le x\le 0,\\ (\cos_\tau(Ax^\alpha))\phi(-\tau) \\ +\frac{A^{-1}}{\Gamma(1+\alpha)}\int_{-\tau}^{0}\sin_\tau(A(x-\tau-s)^\alpha)\phi^{(2\alpha)}(s)(ds)^\alpha \\ +A^{-1}(\sin_\tau(Ax^\alpha))\phi^{(\alpha)}(-\tau) \\ +\frac{A^{-1}}{\Gamma(1+\alpha)}\int_0^x\sin_\tau(A(x-\tau-s)^\alpha)f(s)(ds)^\alpha, & x\ge 0,\end{cases}$$

or

$$y(x)=\begin{cases}\phi(x), & -\tau\le x\le 0,\\ (\cos_\tau(A(x-\tau)^\alpha))\phi(0) \\ -\frac{A}{\Gamma(1+\alpha)}\int_{-\tau}^{0}\sin_\tau(A(x-2\tau-s)^\alpha)\phi(s)(ds)^\alpha \\ +A^{-1}(\sin_\tau(A(x-\tau)^\alpha))\phi^{(\alpha)}(0) \\ +\frac{A^{-1}}{\Gamma(1+\alpha)}\int_0^x\sin_\tau(A(x-\tau-s)^\alpha)f(s)(ds)^\alpha, & x\ge 0.\end{cases}$$

4. Solutions of differential equation with two delays on a fractal set

In this section, we deduce the representation of a solution of system (1.2) by using matrix functions $U^{A,B}_{\tau_1,\tau_2}(x), V^{A,B}_{\tau_1,\tau_2}(x)$ which is counterpart of formulas in Corollary 3.4.

Theorem 4.1. *Suppose that the matrix $A=(a^\alpha_{ij})_n, B=(b^\alpha_{ij})_n$ are permutable constant matrix on fractal set with $\det A\neq 0$, $\det B\neq 0$. Let $\tau_1,\tau_2>0$, $\tau:=\max\{\tau_1,\tau_2\}$, $\phi\in C_\alpha([-\tau,0],\mathbb{R}^{\alpha n})$, and $f:[0,\infty)\to\mathbb{R}^{\alpha n}$ be a given function. Then the solution $y(x)$ of* (1.2) *has the form*

$$y(x)=\begin{cases}\phi(x), & -\tau\le x\le 0,\\ U(x)\phi(0)+V(x)\phi^{(\alpha)}(0) \\ -A^2\frac{1}{\Gamma(1+\alpha)}\int_{-\tau_1}^{0}V(x-\tau_1-s)\phi(s)(ds)^\alpha \\ -B^2\frac{1}{\Gamma(1+\alpha)}\int_{-\tau_2}^{0}V(x-\tau_2-s)\phi(s)(ds)^\alpha \\ +\frac{1}{\Gamma(1+\alpha)}\int_0^x V(x-s)f(s)(ds)^\alpha, & x\ge 0,\end{cases}\tag{4.1}$$

where $U(x)=U^{A,B}_{\tau_1,\tau_2}(x)$, $V(x)=V^{A,B}_{\tau_1,\tau_2}(x)$.

Proof. The main steps on the proof are as follows:

Step1: we show that Theorem 4.1 hold by using Lemma 2.12 and Corollary 3.4 if $\tau_1=\tau_2$.

Step2: let $\tau_1<\tau_2$, we show that $y(x)$ satisfies the initial value condition on $[-\tau,0]$ and $y(0)=\phi(0)$, $y^{(\alpha)}(0)=\phi^{(\alpha)}(0)$ from the form of $y(x)$ and the calculating of local fractal derivative of $y(x)$.

Step3: we show that $y(x)$ is a solution of system (1.2) from the following three cases because $x\ge 0$: $0\le x<\tau_1$, $\tau_1\le x<\tau_2$ and $x\ge\tau_2$.

The detailed proof process is as below:

(i) We consider only the case $\tau_1 \neq \tau_2$ because if $\tau_1 = \tau_2$, then one can use Lemma 2.12 and Corollary 3.4 to show that Theorem 4.1 holds.

(ii) We show that $y(x)$ satisfies the initial value condition on $[-\tau, 0]$ and $y(0) = \phi(0)$, $y^{(\alpha)}(0) = \phi^{(\alpha)}(0)$. Due to the form of $y(x)$, if $x \geq 0$, suppose that $x < min\{\tau_1, \tau_2\}$, then we have

$$U(x) = I, \quad V(x) = \frac{x^\alpha}{\Gamma(1+\alpha)},$$

$$V(x-\tau_i-s) = \begin{cases} 0, & s \in [x-\tau_i, 0], \\ \frac{(x-\tau_i-s)^\alpha}{\Gamma(1+\alpha)}, & s \in [-\tau_i, x-\tau_i], \end{cases} \qquad V(x-s) = \frac{(x-s)^\alpha}{\Gamma(1+\alpha)},$$

for $s \in [0, x]$, imply that $x - s \in [0, x] \subset [0, \min\{\tau_1, \tau_2\}]$. After some calculation, we obtain

$$\begin{aligned} & y(x) \\ & = \phi(0) + \frac{x^\alpha}{\Gamma(1+\alpha)}\phi^{(\alpha)}(0) - \frac{A^2}{\Gamma(1+\alpha)} \int_{-\tau_1}^{x-\tau_1} \frac{(x-\tau_1-s)^\alpha}{\Gamma(1+\alpha)}\phi(s)(ds)^\alpha \\ & \quad - \frac{B^2}{\Gamma(1+\alpha)} \int_{-\tau_2}^{x-\tau_2} \frac{(x-\tau_2-s)^\alpha}{\Gamma(1+\alpha)}\phi(s)(ds)^\alpha + \frac{1}{\Gamma(1+\alpha)} \int_0^x \frac{(x-s)^\alpha}{\Gamma(1+\alpha)} f(s)(ds)^\alpha. \end{aligned}$$

Calculating local fractal derivative of $y(x)$, from Lemmas 2.6, 2.7, 2.8, 2.9, we have

$$\begin{aligned} y^{(\alpha)}(x) = \phi^{(\alpha)}(0) & - \frac{A^2}{\Gamma(1+\alpha)} \int_{-\tau_1}^{x-\tau_1} \phi(s)(ds)^\alpha \\ & - \frac{B^2}{\Gamma(1+\alpha)} \int_{-\tau_2}^{x-\tau_2} \phi(s)(ds)^\alpha + \frac{1}{\Gamma(1+\alpha)} \int_0^x f(s)(ds)^\alpha. \end{aligned}$$

Let $x \to 0^+$, then

$$\lim_{x\to 0^+} y(x) = \phi(0), \quad \lim_{x \to 0^+} y^{(\alpha)}(x) = \phi^{(\alpha)}(0).$$

Obviously, $y(x)$ satisfies the initial condition $[-\tau, 0]$, which completes the proof for this case.

(iii) Now we show that $y(x)$ is a solution of system (1.2). Since $\tau_1 \neq \tau_2$, let $\tau_1 < \tau_2$. Firstly, if $0 \leq x < \tau_1$, then

$$y^{(2\alpha)}(x) = -A^2\phi(x-\tau_1) - B^2\phi(x-\tau_2) + f(x),$$

at the sometime, $\phi(x-\tau_1) = y(x-\tau_1), \phi(x-\tau_2) = y(x-\tau_2)$ while $0 \leq x < \tau_1 < \tau_2$, we have $x - \tau_1 < 0, x - \tau_2 < 0$. We find that $y(x)$ is a solution of system (1.2).

Secondly, if $\tau_1 \leq x < \tau_2$, then for any $s \in [x-\tau_2, 0]$, $V(x-\tau_2-s) = 0$, and we have

$$\begin{aligned} y(x) = U(x)\phi(0) + V(x)\phi^{(\alpha)}(0) & - \frac{A^2}{\Gamma(1+\alpha)} \int_{-\tau_1}^{0} V(x-\tau_1-s)\phi(s)(ds)^\alpha \\ & - \frac{B^2}{\Gamma(1+\alpha)} \int_{-\tau_2}^{x-\tau_2} V(x-\tau_2-s)\phi(s)(ds)^\alpha \\ & + \frac{1}{\Gamma(1+\alpha)} \int_0^x V(x-s)f(s)(ds)^\alpha. \end{aligned}$$

Calculating the local fractal derivative of $y(x)$, from Lemma 2.6 and the properties of $U(x)=U^{A,B}_{\tau_1,\tau_2}(x), V(x)=V^{A,B}_{\tau_1,\tau_2}(x)$, we have

$$\begin{aligned}
&y^{(\alpha)}(x)\\
&= U^{(\alpha)}(x)\phi(0)+V^{(\alpha)}(x)\phi^{(\alpha)}(0)-\frac{A^2}{\Gamma(1+\alpha)}\int_{-\tau_1}^{0}V^{(\alpha)}(x-\tau_1-s)\phi(s)(ds)^\alpha\\
&\quad-\frac{B^2}{\Gamma(1+\alpha)}\int_{-\tau_2}^{x-\tau_2}V^{(\alpha)}(x-\tau_2-s)\phi(s)(ds)^\alpha\\
&\quad-B^2V(x-\tau_2-(x-\tau_2))\phi(x-\tau_2)((x-\tau_2)')^\alpha\\
&\quad+B^2V(x-\tau_2-(-\tau_2))\phi(-\tau_2)((-\tau_2)')^\alpha\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^x V^{(\alpha)}(x-s)f(s)(ds)^\alpha+V(x-x)f(x)(x')^\alpha\\
&\quad-V(x-0)f(0)(0')^\alpha\\
&= U^{(\alpha)}(x)\phi(0)+V^{(\alpha)}(x)\phi^{(\alpha)}(0)-\frac{A^2}{\Gamma(1+\alpha)}\int_{-\tau_1}^{0}V^{(\alpha)}(x-\tau_1-s)\phi(s)(ds)^\alpha\\
&\quad-\frac{B^2}{\Gamma(1+\alpha)}\int_{-\tau_2}^{x-\tau_2}V^{(\alpha)}(x-\tau_2-s)\phi(s)(ds)^\alpha-B^2V(0)\phi(x-\tau_2)\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^x V^{(\alpha)}(x-s)f(s)(ds)^\alpha+V(0)f(x)
\end{aligned}$$

Clearly, $V(0)=0$ from Definition 2.5, thus

$$\begin{aligned}
&y^{(\alpha)}(x)\\
&= U^{(\alpha)}(x)\phi(0)+V^{(\alpha)}(x)\phi^{(\alpha)}(0)-\frac{A^2}{\Gamma(1+\alpha)}\int_{-\tau_1}^{0}V^{(\alpha)}(x-\tau_1-s)\phi(s)(ds)^\alpha\\
&\quad-\frac{B^2}{\Gamma(1+\alpha)}\int_{-\tau_2}^{x-\tau_2}V^{(\alpha)}(x-\tau_2-s)\phi(s)(ds)^\alpha\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^x V^{(\alpha)}(x-s)f(s)(ds)^\alpha\\
&= U^{(\alpha)}(x)\phi(0)+V^{(\alpha)}(x)\phi^{(\alpha)}(0)-\frac{A^2}{\Gamma(1+\alpha)}\int_{-\tau_1}^{0}U(x-\tau_1-s)\phi(s)(ds)^\alpha\\
&\quad-\frac{B^2}{\Gamma(1+\alpha)}\int_{-\tau_2}^{x-\tau_2}U(x-\tau_2-s)\phi(s)(ds)^\alpha+\frac{1}{\Gamma(1+\alpha)}\int_0^x U(x-s)f(s)(ds)^\alpha,
\end{aligned}$$

because $V^{(\alpha)}(x)=U(x)$ follows from Definition 2.5. Using a method similar to the calculation of $y^{(\alpha)}(x)$, we have

$$\begin{aligned}
y^{(2\alpha)}(x)&=U^{(2\alpha)}(x)\phi(0)+V^{(2\alpha)}(x)\phi^{(\alpha)}(0)\\
&\quad-\frac{A^2}{\Gamma(1+\alpha)}\int_{-\tau_1}^{0}U^{(\alpha)}(x-\tau_1-s)\phi(s)(ds)^\alpha\\
&\quad-\frac{B^2}{\Gamma(1+\alpha)}\int_{-\tau_2}^{x-\tau_2}U^{(\alpha)}(x-\tau_2-s)\phi(s)(ds)^\alpha-B^2U(0)\phi(x-\tau_2)\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^x U^{(\alpha)}(x-s)f(s)(ds)^\alpha+U(0)f(x).
\end{aligned}$$

Applying Lemma 2.12 to $y^{(2\alpha)}(x)$ and noticing that $U(x-\tau_2)=0$, $V(x-\tau_2)=0$ because $x<\tau_2$ and $U(0)=1$, $V^{(\alpha)}(x)=U(x)$ from Definition 2.5, we have

$$
\begin{aligned}
&y^{(2\alpha)}(x)\\
&=\Big(-A^2U(x-\tau_1)-B^2U(x-\tau_2)\Big)\phi(0)\\
&\quad+\Big(-A^2V(x-\tau_1)-B^2V(x-\tau_2)\Big)\phi^{(\alpha)}(0)\\
&\quad-\frac{A^2}{\Gamma(1+\alpha)}\int_{-\tau_1}^{0}V^{(2\alpha)}(x-\tau_1-s)\phi(s)(ds)^{\alpha}\\
&\quad-\frac{B^2}{\Gamma(1+\alpha)}\int_{-\tau_2}^{x-\tau_2}V^{(2\alpha)}(x-\tau_2-s)\phi(s)(ds)^{\alpha}\\
&\quad-B^2\phi(x-\tau_2)+\frac{1}{\Gamma(1+\alpha)}\int_0^x V^{(2\alpha)}(x-s)f(s)(ds)^{\alpha}+f(x)\\
&=-A^2U(x-\tau_1)\phi(0)-A^2V(x-\tau_1)\phi^{(\alpha)}(0)\\
&\quad-\frac{A^2}{\Gamma(1+\alpha)}\int_{-\tau_1}^{0}\Big(-A^2V(x-2\tau_1-s)-B^2V(x-\tau_1-\tau_2-s)\Big)\phi(s)(ds)^{\alpha}\\
&\quad-\frac{B^2}{\Gamma(1+\alpha)}\int_{-\tau_2}^{x-\tau_2}\Big(-A^2V(x-\tau_1-\tau_2-s)\\
&\quad-B^2V(x-2\tau_2-s)\Big)\phi(s)(ds)^{\alpha}-B^2\phi(x-\tau_2)\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^x\Big(-A^2V(x-\tau_1-s)-B^2V(x-\tau_2-s)\Big)f(s)(ds)^{\alpha}+f(x).
\end{aligned}
$$

Since $V(x)=0$ for $x<0$, we have $V(x-\tau_1-\tau_2-s)=0$, $V(x-2\tau_2-s)=0$, $V(x-\tau_2-s)=0$ if $\tau_1\le x<\tau_2$, and we have

$$
\begin{aligned}
&y^{(2\alpha)}(x)\\
&=-A^2U(x-\tau_1)\phi(0)-A^2V(x-\tau_1)\phi^{(\alpha)}(0)\\
&\quad+\frac{A^4}{\Gamma(1+\alpha)}\int_{-\tau_1}^{0}V(x-2\tau_1-s)\phi(s)(ds)^{\alpha}\\
&\quad+\frac{A^2B^2}{\Gamma(1+\alpha)}\int_{-\tau_2}^{x-\tau_2}V(x-\tau_1-\tau_2-s)\phi(s)(ds)^{\alpha}\\
&\quad-B^2\phi(x-\tau_2)-\frac{A^2}{\Gamma(1+\alpha)}\int_0^x V(x-\tau_1-s)f(s)(ds)^{\alpha}+f(x)\\
&=-A^2\Big(U(x-\tau_1)\phi(0)+V(x-\tau_1)\phi^{(\alpha)}(0)\\
&\quad-\frac{A^2}{\Gamma(1+\alpha)}\int_{-\tau_1}^{0}V(x-2\tau_1-s)\phi(s)(ds)^{\alpha}\\
&\quad-\frac{B^2}{\Gamma(1+\alpha)}\int_{-\tau_2}^{x-\tau_1-\tau_2}V(x-\tau_1-\tau_2-s)\phi(s)(ds)^{\alpha}\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^{x-\tau_1}V(x-\tau_1-s)f(s)(ds)^{\alpha}\Big)\\
&\quad-B^2\phi(x-\tau_2)+f(x)
\end{aligned}
$$

$$
\begin{aligned}
&= -A^2\Big(U(x-\tau_1)\phi(0)+V(x-\tau_1)\phi^{(\alpha)}(0)\\
&\quad -\frac{A^2}{\Gamma(1+\alpha)}\int_{-\tau_1}^{0} V(x-2\tau_1-s)\phi(s)(ds)^\alpha\\
&\quad -\frac{B^2}{\Gamma(1+\alpha)}\int_{-\tau_2}^{0} V(x-\tau_1-\tau_2-s)\phi(s)(ds)^\alpha\\
&\quad +\frac{1}{\Gamma(1+\alpha)}\int_{0}^{x-\tau_1} V(x-\tau_1-s)f(s)(ds)^\alpha\Big)\\
&\quad -B^2\phi(x-\tau_2)+f(x)\\
&= -A^2y(x-\tau_1)-B^2\phi(x-\tau_2)+f(x).
\end{aligned}
$$

Note $\phi(x-\tau_2)=y(x-\tau_2)$ if $x<\tau_2$. We have $x-\tau_2<0$ and $y(x)$ is a solution of system (1.2).

Finally, if $x\geq\tau_2$, then we obtain

$$
\begin{aligned}
y(x) &= U(x)\phi(0)+V(x)\phi^{(\alpha)}(0)-\frac{A^2}{\Gamma(1+\alpha)}\int_{-\tau_1}^{0} V(x-\tau_1-s)\phi(s)(ds)^\alpha\\
&\quad -\frac{B^2}{\Gamma(1+\alpha)}\int_{-\tau_2}^{0} V(x-\tau_2-s)\phi(s)(ds)^\alpha\\
&\quad +\frac{1}{\Gamma(1+\alpha)}\int_{0}^{x} V(x-s)f(s)(ds)^\alpha.
\end{aligned}
$$

Calculating the local fractal derivative of $y(x)$, and using $V^{(\alpha)}(x)=U(x)$ and Lemma 2.12, we obtain

$$
\begin{aligned}
&y^{(\alpha)}(x)\\
&= U^{(\alpha)}(x)\phi(0)+V^{(\alpha)}(x)\phi^{(\alpha)}(0)-\frac{A^2}{\Gamma(1+\alpha)}\int_{-\tau_1}^{0} V^{(\alpha)}(x-\tau_1-s)\phi(s)(ds)^\alpha\\
&\quad -\frac{B^2}{\Gamma(1+\alpha)}\int_{-\tau_2}^{0} V^{(\alpha)}(x-\tau_2-s)\phi(s)(ds)^\alpha+\frac{1}{\Gamma(1+\alpha)}\int_{0}^{x} V^{(\alpha)}(x-s)f(s)(ds)^\alpha\\
&= U^{(\alpha)}(x)\phi(0)+V^{(\alpha)}(x)\phi^{(\alpha)}(0)-\frac{A^2}{\Gamma(1+\alpha)}\int_{-\tau_1}^{0} V^{(\alpha)}(x-\tau_1-s)\phi(s)(ds)^\alpha\\
&\quad -\frac{B^2}{\Gamma(1+\alpha)}\int_{-\tau_2}^{0} V^{(\alpha)}(x-\tau_2-s)\phi(s)(ds)^\alpha+\frac{1}{\Gamma(1+\alpha)}\int_{0}^{x} U(x-s)f(s)(ds)^\alpha.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&y^{(2\alpha)}(x)\\
&= U^{(2\alpha)}(x)\phi(0)+V^{(2\alpha)}(x)\phi^{(\alpha)}(0)-\frac{A^2}{\Gamma(1+\alpha)}\int_{-\tau_1}^{0} V^{(2\alpha)}(x-\tau_1-s)\phi(s)(ds)^\alpha\\
&\quad -\frac{B^2}{\Gamma(1+\alpha)}\int_{-\tau_2}^{0} V^{(2\alpha)}(x-\tau_2-s)\phi(s)(ds)^\alpha\\
&\quad +\frac{1}{\Gamma(1+\alpha)}\int_{0}^{x} U^{(\alpha)}(x-s)f(s)(ds)^\alpha+U(0)f(x)\\
&= U^{(2\alpha)}(x)\phi(0)+V^{(2\alpha)}(x)\phi^{(\alpha)}(0)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{A^2}{\Gamma(1+\alpha)} \int_{-\tau_1}^{0} \Big(- A^2 V(x-2\tau_1 - s) - B^2 V(x-\tau_1-\tau_2-s) \Big) \phi(s)(ds)^\alpha \\
&- \frac{B^2}{\Gamma(1+\alpha)} \int_{-\tau_2}^{0} \Big(- A^2 V(x-\tau_1-\tau_2 - s) - B^2 V(x-2\tau_2-s) \Big) \phi(s)(ds)^\alpha \\
&+ \frac{1}{\Gamma(1+\alpha)} \int_{0}^{x} U^{(2\alpha)}(x-s) f(s)(ds)^\alpha + f(x) \\
={}& \Big(- A^2 U(x-\tau_1) - B^2 U(x-\tau_2) \Big) \phi(0) \\
&+ \Big(- A^2 V(x-\tau_1) - B^2 V(x-\tau_2) \Big) \phi^{(\alpha)}(0) \\
&+ \frac{A^4}{\Gamma(1+\alpha)} \int_{-\tau_1}^{0} V(x-2\tau_1-s)\phi(s)(ds)^\alpha \\
&+ \frac{A^2B^2}{\Gamma(1+\alpha)} \int_{-\tau_1}^{0} V(x-\tau_1-\tau_2-s)\phi(s)(ds)^\alpha \\
&+ \frac{A^2B^2}{\Gamma(1+\alpha)} \int_{-\tau_2}^{0} V(x-\tau_1-\tau_2-s)\phi(s)(ds)^\alpha \\
&+ \frac{B^4}{\Gamma(1+\alpha)} \int_{-\tau_2}^{0} V(x-2\tau_2-s)\phi(s)(ds)^\alpha \\
&- \frac{A^2}{\Gamma(1+\alpha)} \int_{0}^{x} V(x-\tau_1-s) f(s)(ds)^\alpha \\
&- \frac{B^2}{\Gamma(1+\alpha)} \int_{0}^{x} V(x-\tau_2-s) f(s)(ds)^\alpha + f(x) \\
={}& -A^2 \Big(U(x-\tau_1)\phi(0) + V(x-\tau_1)\phi^{(\alpha)}(0) \\
&- \frac{A^2}{\Gamma(1+\alpha)} \int_{-\tau_1}^{0} V(x-2\tau_1-s)\phi(s)(ds)^\alpha \\
&- \frac{B^2}{\Gamma(1+\alpha)} \int_{-\tau_2}^{0} V(x-\tau_1-\tau_2-s)\phi(s)(ds)^\alpha \\
&+ \frac{1}{\Gamma(1+\alpha)} \int_{0}^{x-\tau_1} V(x-\tau_1-\tau_2) f(s)(ds)^\alpha \Big) \\
&- B^2 \Big(U(x-\tau_2)\phi(0) + V(x-\tau_2)\phi^{(\alpha)}(0) \\
&- \frac{A^2}{\Gamma(1+\alpha)} \int_{-\tau_1}^{0} V(x-\tau_1-\tau_2-s)\phi(s)(ds)^\alpha \\
&- \frac{B^2}{\Gamma(1+\alpha)} \int_{-\tau_2}^{0} V(x-2\tau_2-s)\phi(s)(ds)^\alpha \\
&+ \frac{1}{\Gamma(1+\alpha)} \int_{0}^{x-\tau_2} V(x-\tau_2-s) f(s)(ds)^\alpha \Big) + f(x) \\
={}& -A^2 y(x-\tau_1) - B^2 y(x-\tau_2) + f(x).
\end{aligned}
$$

From above, we can see that (4.1) is a solution of system (1.2). The proof is complete. □

Concluding remarks. From the delayed cosine and sine type matrix function on the fractal set $\mathbb{R}^{\alpha n}(0 < \alpha \leq 1)$ corresponding to second order inhomogeneous delay differential equations with permutable constant matrix coefficients, we provide a representation of a solution to the second order inhomogeneous delay differential equations with pure delay and two delays. It is worth mentioning that although there are many continued contributions in a linear discrete/differential systems with pure delay with permutable matrices, no results were obtained for such systems with non permutable matrices on fractal set $\mathbb{R}^{\alpha n}(0 < \alpha \leq 1)$. A representation of a solution to delay discrete/differential systems with non permutable matrices on fractal set $\mathbb{R}^{\alpha n}(0 < \alpha \leq 1)$ is open at present, worthy our further study.

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Kee Qiu
School of Mathematics and Big Data, Guizhou Education University, Guiyang, Guizhou 550018, China.
School of Mathematics and Information Sciences, Guangzhou University, Guangzhou 510006, China
Email address: qke456@sina.com

Jinrong Wang (corresponding author)
Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, China.
School of Mathematical Sciences, Qufu Normal University, Qufu 273165, Shandong, China
Email address: jrwang@gzu.edu.cn