OSCILLATORY BEHAVIOR FOR NONLINEAR HOMOGENEOUS NEUTRAL DIFFERENCE EQUATIONS OF SECOND ORDER WITH COEFFICIENT CHANGING SIGN

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Abstract. In this article, we obtain sufficient conditions so that all solutions of the neutral difference equation

\[ \Delta^2 (y_n - p_n L(y_{n-s})) + q_n G(y_{n-k}) = 0, \]

and all unbounded solutions of the neutral difference equation

\[ \Delta^2 (y_n - p_n L(y_{n-s})) + q_n G(y_{n-k}) - u_n H(y_{\alpha(n)}) = 0, \]

are oscillatory, where \( \Delta \) is the forward difference operator \( \Delta y_n = y_{n+1} - y_n \), \( \Delta^2 \) is the second difference operator, \( \{q_n\} \) and \( \{u_n\} \) are sequences of real numbers with \( q_n > 0, u_n \geq 0 \), and \( G, H, L \in C(R, R) \). We assume that \( \alpha(n) < n - 1 \) and it approaches \( \infty \) as \( n \to \infty \), and \( s, k \) are positive integers. Further, we assume that

\[ G(-x) = -G(x), \quad H(-x) = -H(x), \quad L(-x) = -L(x), \quad \forall x \in R \]

and

\[ xG(x) > 0, \quad xH(x) > 0, \quad xL(x) > 0 \quad \forall x > 0. \]

Some of the following assumptions are used later in this article.

(A1) There exists \( \delta > 0 \) such that for each \( x > 0 \), \( L(x) \leq \delta x \);

(A2) \( q_n > 0 \) and \( \sum_{n=n_0}^{\infty} q_n = \infty \);

(A3) \( \sum_{n=n_1}^{\infty} q_n^* = \infty \), where \( q_n^* = \min\{q_n, q_{n-s}\} \);

(A4) \( \liminf_{n \to \infty} q_n > 0 \);

(A5) \( G \) is non decreasing.

1. Introduction

In this article, we obtain sufficient conditions so that all solutions of the neutral difference equation

\[ \Delta^2 (y_n - p_n L(y_{n-s})) + q_n G(y_{n-k}) = 0, \quad n \geq n_0, \quad (1.1) \]

and all unbounded solutions of the neutral difference equation

\[ \Delta^2 (y_n - p_n L(y_{n-s})) + q_n G(y_{n-k}) - u_n H(y_{\alpha(n)}) = 0, \quad n \geq n_0 \quad (1.2) \]

are oscillatory, where \( \Delta \) is the forward difference operator \( \Delta y_n = y_{n+1} - y_n \), \( \Delta^2 y_n = \Delta(\Delta y_n) \), \( \{q_n\} \) and \( \{u_n\} \) are sequences of real numbers with \( q_n > 0, u_n \geq 0 \), and \( G, H, L \in C(R, R) \). We assume that \( \alpha(n) < n - 1 \) and it approaches \( \infty \) as \( n \to \infty \), and \( s, k \) are positive integers. Further, we assume that

\[ G(-x) = -G(x), \quad H(-x) = -H(x), \quad L(-x) = -L(x), \quad \forall x \in R \]

and

\[ xG(x) > 0, \quad xH(x) > 0, \quad xL(x) > 0 \quad \forall x > 0. \]
\( (A6) \quad \sum_{n=n_0}^{\infty} nu_n < \infty; \)
\( (A7) \quad H \) is bounded.

For the sequence \( \{p_n\} \) we state the following conditions:

\[
\begin{align*}
0 &\leq p_n \leq p, \quad (1.4) \\
0 &\leq p_n \leq 1, \quad (1.5) \\
-p &\leq p_n < 0, \quad (1.6) \\
p_n &\text{ changes sign and } -p \leq p_n \leq p, \quad (1.7) \\
1 &\leq p_n \leq p, \quad (1.8) \\
-1 &< -b \leq p_n \leq 0, \quad (1.9)
\end{align*}
\]

where \( p \) and \( b \) are positive constants.

As of now, many researchers all over the world are engaged to find necessary or sufficient conditions for oscillation or non oscillation for neutral difference equations, because of its important applications in different fields of science and technology. For the fundamentals and some recent results on the subject, one may go through the monograph \[1, 5\] and the research articles \[2, 4, 12, 14\] and the references cited there in. Sufficient conditions are found, in \[3, 4, 7, 12, 13, 14, 15, 16\], and more recently in \[2, 3\], so that every solutions of the non linear neutral difference equation

\[
\Delta^2 \left( y_n - p_n y_{n-s} \right) + q_n G(y_{n-k}) - u_n H(y_{n-r}) = f_n, \quad n \geq n_0, \quad (1.10)
\]

(or of its particular case \( u_n \equiv 0, f_n \equiv 0 \)) oscillates or tends to zero or to \( \pm \infty \) at \( \infty \). The asymptotic behavior of the solution is probably due to the presence of the forcing term \( f_n \) in \( (1.10) \).

The objective of this work is to find sufficient conditions so that all solutions of \( (1.2) \) are oscillatory under different cases of \( p_n > 0, p_n < 0 \) or \( p_n \) changing sign. For that, we had to prevent the bounded solutions of \( (1.2) \) from approaching zero by imposing a sub linear condition \( (4.4) \) or \( (4.1) \) on \( G \) as well as stop the unbounded solution of \( (1.2) \) from approaching \( \pm \infty \) by imposing a super linear condition \( (3.5) \) or \( (3.2) \) on \( G \). Then the results for \( (1.2) \) are applied to study the oscillatory behavior of the unbounded solutions of neutral difference equation

\[
\Delta^2 \left( y_n - p_n L(y_{n-s}) \right) + v_n G(y_{n-k}) = 0, \quad n \geq n_0, \quad (1.11)
\]

where \( v_n \) changes sign. Our results generalize and extend some results in \[2, 11\].

Let \( n_0 \) be a fixed nonnegative integer. Let \( \rho = \min \{ n_0 - s, n_0 - k, \inf_{n \geq n_0} \{ \alpha(n) \} \} \). By a solution of \( (1.2) \) we mean a real sequence \( \{y_n\} \) which is defined for all integers \( n \geq \rho \) and satisfies \( (1.2) \) for \( n \geq n_0 \). Clearly if the initial condition

\[
y_n = a_n \quad \text{for } \rho \leq n \leq n_0 + 1, \quad (1.12)
\]

is given then equation \( (1.2) \) has a unique solution satisfying \( (1.12) \). A non trivial solution \( \{y_n\} \) of \( (1.2) \) is said to be oscillatory if for every positive integer \( n_0 > 0 \), there exists \( n \geq n_0 \) such that \( y_n y_{n+1} \leq 0 \), otherwise \( \{y_n\} \) is said to be non-oscillatory.

2. Some lemmas

In this section, we present some lemmas to be applied in next section.
Lemma 2.1. [4, Theorem 7.6.1, page 184] Let \( \{r_n\} \) be a non negative sequence of real numbers, \( k \) a positive integer and
\[
\liminf_{n \to \infty} \sum_{i=n-k}^{n-1} r_i > \left( \frac{k}{k+1} \right)^{k+1}.
\]
(2.1)
Then the following statements are true.

(a) \( \Delta x_n + r_n x_{n-k} \leq 0 \) has no eventually positive solutions, which implies \( \Delta x_n + r_n x_{n-k} \geq 0 \) has no eventually negative solutions.

(b) \( \Delta x_n - r_n x_{n+k} \geq 0 \) has no eventually positive solutions, which implies \( \Delta x_n - r_n x_{n+k} \leq 0 \) has no eventually negative solutions.

Lemma 2.2. Suppose that (A6) and (A7) hold, and \( y_n \) is an eventually positive solution of (1.2). Then the sequence
\[
c_n = - \sum_{i=n}^{\infty} (i-n+1)u_i H(y_{\alpha(i)})
\]
(2.2)
satisfies
\[
\lim_{n \to \infty} c_n = 0, \quad c_n \leq 0, \quad \Delta c_n \geq 0,
\]
(2.3)
for \( n \) large enough, and
\[
\Delta^2 c_n = -u_n H(y_{\alpha(n)}).
\]
(2.4)
Proof. Clearly, applying \( \Delta^2 \) to (2.2), we obtain \( \Delta^2 c_n = -u_n H(y_{\alpha(n)}) \). By (A6) and (A7), \( \sum_{i=n}^{\infty} i u_i H(y_{\alpha(i)}) < \infty \). Comparing this infinite series with (2.2), we show that \( \{c_n\} \) converges absolutely to zero. The other statements follow easily. \( \square \)

Note that if \( y_n \) is eventually negative, then \( c_n \geq 0 \) and \( \Delta c_n \leq 0 \). Next, we prove an important lemma to be used later.

Lemma 2.3. Let (A1), (A6), (A7) hold, \( y_n \) be an eventually positive solution of (1.2), and \( c_n \) be defined by (2.2). Then for the sequences
\[
z_n = y_n - p_n L(y_{n-s}),
\]
(2.5)
\[
w_n = z_n + c_n
\]
(2.6)
we have the following statements:

(a) If (A2) and (A5) hold and \( p_n \) satisfy (1.4), then either \( \Delta w_n < 0 \) for large \( n \) which implies
\[
\lim_{n \to \infty} w_n = -\infty,
\]
(2.7)
or \( \Delta w_n > 0 \) for large \( n \) which implies
\[
\lim_{n \to \infty} w_n = 0,
\]
(2.8)
\( w_n < 0, \lim_{n \to \infty} \Delta w_n = 0 \).
(2.9)

(b) If in addition \( p \delta \leq 1 \), then only (2.8) and (2.9) hold.

Proof. Suppose that \( y_n \) is an eventually positive solution of (1.2). Then there exits an integer \( n_1 \geq n_0 \) such that \( y_n > 0, y_{n-s} > 0, y_{n-k} \) and \( y_{\alpha(n)} > 0 \) for \( n \geq n_1 \). Then setting \( c_n, z_n \) and \( w_n \) as in (2.2), (2.5), (2.6), and using (1.2), (2.3), (2.6), and Lemma 2.2, we obtain
\[
\Delta^2 w_n = -q_n G(y_{n-k}) \leq 0 \quad \text{for } n > n_1.
\]
(2.10)
Then $\Delta w_n$ is decreasing. Hence $\Delta w_n$ is monotonic and of single sign for $n$ large enough. It follows that either $\Delta w_n < 0$ or $\Delta w_n > 0$. If $\Delta w_n < 0$, then $w_n$ is decreasing, and using that $\Delta w_n$ is decreasing, we have
\[ \lim_{n \to \infty} \Delta w_n = -\infty. \] (2.11)
If $\Delta w_n > 0$, then $w_n$ is increasing, and using that $\Delta w_n$ is decreasing, we have
\[ \lim_{n \to \infty} \Delta w_n = \zeta \quad \text{(a finite number)}. \] (2.12)
Let us prove part (a). If (2.11) holds then clearly (2.7) follows. If (2.12) holds then, summing (2.10) from $n_2 > n_1$ to $\infty$ we obtain
\[ \sum_{n=n_2}^{\infty} q_n G(y_{n-k}) < \infty, \] (2.13)
which by using (A2) yields
\[ \liminf_{n \to \infty} y_n = 0. \] (2.14)
Then we find a subsequence $\{y_{n_k}\}$ such that $y_{n_k} \to 0$ as $k \to \infty$. Now using (A1), (A2) and Lemma 2.2 we obtain
\[ w_{n_k} < y_{n_k} + c_{n_k} \to 0 \quad \text{as } k \to \infty \] (2.15)
and
\[ w_{n_k+s} - p\delta y_{n_k} + c_{n_k+s} \to 0 \quad \text{as } k \to \infty. \] (2.16)
Since $w_n$ is monotonic, it follows that $\lim_{n \to \infty} w_n = 0$, which implies (2.8). Then (2.9) holds.

To prove part (b) of the lemma, we show that (2.7) cannot happen; therefore (2.8) and (2.9) must occur. To obtain a contradiction, let us assume that $\lim_{n \to \infty} w_n = -\infty$. Note that from (2.6) and Lemma 2.2 we have
\[ \lim_{n \to \infty} w_n = \lim_{n \to \infty} z_n; \] (2.17)
thus $\lim_{n \to \infty} z_n = -\infty$. This implies that for large $n$, there exists $\eta > 0$, however large, such that for $n \geq n_3$ implies $z_n < -\eta$ which implies by (A1) that $y_n < -\eta + p\delta y_{n-s} < y_{n-s}$. Then $y_n$ is bounded. Consequently $z_n$ and $w_n$ are bounded, which contradicts (2.7). As a result, (2.7) cannot hold and so, (2.8) holds, which implies (2.9). The proof is complete. \qed

Remark 2.4. If $y_n$ is an eventually negative solution of (1.2), then using (1.3), we observe that $x_n = -y_n$ is a positive solution of (1.2). So that all the oscillation results for the positive solutions also apply to negative solutions.

Lemma 2.5. Let $y_n$ be an eventually positive solution of (1.2), with $w_n$ as in (2.6). Then the following statements hold.

(a) If (2.7) holds, then (2.10) implies
\[ \Delta w_{n+1} + q_n G(y_{n-k}) \leq 0, \] (2.18)
which further implies
\[ \Delta z_{n+1} + q_n G(y_{n-k}) \leq 0. \] (2.19)
(b) If (2.8) holds, then (2.10) implies
\[ \Delta w_n - q_n G(y_{n-k}) \geq 0. \] (2.20)
Proof. If (2.7) holds then \( \Delta w_n < 0 \) and \( \Delta w_n^2 < 0 \). We write (2.10) as
\[
\Delta w_{n+1} + q_n G(y_{n-k}) = \Delta w_n \leq 0.
\]
Thus, (2.18) holds. From (2.6), it follows that \( \Delta w_{n+1} = \Delta z_{n+1} + \Delta c_{n+1} \). Therefore (2.18) implies \( \Delta z_{n+1} + q_n G(y_{n-k}) = -\Delta c_{n+1} \leq 0 \) by Lemma 2.2. Hence (a) is proved. Let us prove (b). If (2.8) holds then (2.9) follows as a consequence, which implies \( w_n < 0 \) and \( \Delta w_n > 0 \). Using (2.9), we write (2.10), as which implies
\[
-\Delta w_n + q_n G(y_{n-k}) = -\Delta w_{n+1} \leq 0,
\]
which implies
\[
\Delta w_n - q_n G(y_{n-k}) = \Delta w_{n+1} \geq 0.
\]
This proves of (b), and completes the proof. \( \square \)

Lemma 2.6. Let (A1), (A3), (A6), (A7) hold. Assume that there exists \( \lambda > 0 \) such that for all \( x, y \in \mathbb{R} \) with \( x + y > 0 \), we have
\[
G(x) + G(y) \geq \lambda G(x + y).
\]
Further, we assume that
\[
G(x)G(y) \geq G(xy) \quad \text{for all } x, y > 0.
\]
Let \( y_n \) be an eventually positive solution of (1.2). Define \( c_n, z_n \) and \( w_n \) as in (2.2), (2.5) and (2.6) respectively. If \( p_n \) satisfies (1.6) or (1.7), then \( \lim_{n \to \infty} w_n = 0 \). Consequently, (2.9) holds.

Proof. Suppose \( y_n \) is an eventually positive or eventually negative solution of (1.2) and \( p_n \) satisfies (1.6). From (1.2), using (2.6), (2.5), (2.2) and Lemma 2.2 we obtain (2.10). This implies \( w_n \) and \( \Delta w_n \) are monotonic and single sign. Hence, it follows that (2.17) holds and let \( \lim_{n \to \infty} z_n = \beta \). Clearly, \( z_n > 0 \) by (1.6). This implies, \( \beta \) in (2.17) cannot be in negative. If \( \beta > 0 \), then there exists a positive scalar \( \gamma \) such that \( z_n > \gamma > 0 \) for large \( n \). Clearly, \( \Delta w_n > 0 \), otherwise, \( \beta = -\infty \), a contradiction. Since \( \Delta w_n \) is decreasing, \( \lim_{n \to \infty} \Delta w_n \) exists. If \( x > y \) then using (1.3) and (2.21), we note that \( 0 < \lambda G(x-y) \leq G(x) + G(-y) = G(x) - G(y) \). Thus, (A5) holds, i.e; \( G \) is non decreasing. Then using (A5), (A1) and (1.6) in (2.5), we have
\[
z_n \leq y_n + p\delta y_{n-s}.
\]
From (2.10), by using (A3), (2.21), (2.22) and (2.23) it follows that
\[
0 \geq \Delta^2 w_n + q_n G(y_{n-k}) + G(p\delta)[\Delta^2 w_{n-s} + q_n G(y_{n-s-k})]
\]
\[
\geq \Delta^2 w_n + G(p\delta)\Delta^2 w_{n-s} + q_n^* \left[ G(y_{n-k}) + G(p\delta)G(y_{n-s-k}) \right]
\]
\[
\geq \Delta^2 w_n + G(p\delta)\Delta^2 w_{n-s} + \lambda q_n^* \left[ G(z_{n-k}) \right]
\]
\[
\geq \Delta^2 w_n + G(p\delta)\Delta^2 w_{n-s} + \lambda G(\gamma)^q_n
\]
for \( n \geq n_2 > n_1 \). Then taking summation in (2.24) from \( n_2 \) to \( l - 1 \) and then letting \( l \to \infty \), we obtain a contradiction to (A3). Thus \( \beta = \lim_{n \to \infty} w_n = 0 \), which implies (2.9).

Suppose \( p_n \) satisfies (1.7). If \( \beta > 0 \) then proceeding as above, we obtain a similar contradiction. If \( \beta < 0 \) then using (1.7), we have \( w_n \geq -p\delta y_{n-s} + c_n \). This implies \( y_n \geq \frac{c_n + y_{n-s}}{p\delta} - \frac{w_{n-s}}{p\delta} \). Then taking limit inferior on both sides of this inequality, we obtain
\[
\liminf_{n \to \infty} y_n \geq \liminf_{n \to \infty} \frac{c_n + y_{n-s}}{p\delta} + \liminf_{n \to \infty} -\frac{w_{n-s}}{p\delta} \geq -\beta/p\delta > 0.
\]
In the above we used \( \lim_{n \to \infty} c_n = 0 \) and \( \lim_{n \to \infty} w_n = \beta < 0 \). For \( -\beta/(3p\delta) = \epsilon > 0 \), we find \( n_3 > n_2 \) such that \( n > n_3 \) implies \( y_n > 2\epsilon \). As \( c_n \to 0 \), from (2.6) it follows that \( p_nL(y_{n-s}) > y_n + c_n > \epsilon > 0 \). This further implies \( p_n > \frac{\epsilon}{L(y_n)} \geq \frac{\epsilon}{y_n} > 0 \), for \( n \geq n_3 \), which contradicts that \( p_n \) changes sign. Thus \( \beta \) cannot be in negative, hence \( \lim_{n \to \infty} w_n = \beta = 0 \). Consequently (2.9) holds. Similarly, if \( y_n \) be an eventually negative solution of (1.2) then proceeding with substitution \( x_n = -y_n \) and taking note of Remark 2.4 it could be shown \( \beta = \lim_{n \to \infty} w_n = 0 \) and the proof is complete. \( \square \)

Next we have the following remark, which would be helpful in proving results concerned with neutral equation (1.1).

**Remark 2.7.** Lemmas 2.3, 2.5 and 2.6 hold for \( u_n \equiv \zeta_n \). In that case \( c_n = 0 \) and \( w_n = \zeta_n \).

The following Lemmas follow from Lemmas 2.3, 2.5, and 2.6 as a consequence of the above remark.

**Lemma 2.8.** Assume (A1) holds. Let \( y_n \) be an eventually positive solution of (1.1), and \( z_n \) be defined as in (2.5). Then
\[
\Delta^2 z_n = -q_nG(y_{n-k}) \leq 0,
\]
and the following statements hold.

(a) If (A2), (A5) hold and \( p_n \) satisfies (1.4), then either \( \Delta w_n < 0 \) for large \( n \) which implies
\[
\lim_{n \to \infty} z_n = -\infty,
\]
or \( \Delta w_n > 0 \) for large \( n \) which implies
\[
\lim_{n \to \infty} z_n = 0,
\]
\[
z_n < 0, \quad \Delta z_n > 0, \quad \lim_{n \to \infty} \Delta z_n = 0.
\]

(b) If in addition \( \delta \leq 1 \) and if \( p_n \) satisfy (1.5), then only (2.27) and (2.28) hold.

**Lemma 2.9.** If \( y_n \) is any eventually positive solution of (1.1), with \( z_n \) as in (2.5), then the following statements hold.

(a) If (2.26) holds then, (2.25) implies (2.19), i.e;
\[
\Delta z_{n+1} + q_nG(y_{n-k}) \leq 0.
\]

(b) If (2.27) holds, then (2.25) implies
\[
\Delta z_n - q_nG(y_{n-k}) \geq 0.
\]

**Lemma 2.10.** Let (A1), (A3), (2.21), and (2.22) hold, let \( y_n \) be an eventually positive or eventually negative solution of (1.1), and let \( z_n \) be as in (2.5). If \( p_n \) satisfies (1.6) or (1.7) then \( \lim_{n \to \infty} z_n = 0 \). Consequently, \( z_n < 0, \Delta z_n > 0 \) for \( y_n > 0 \) and \( z_n > 0, \Delta z_n < 0 \) for \( y_n < 0 \).
3. Main results part I

In this section, we find sufficient conditions, so that, all unbounded solutions of (1.2) oscillate.

Remark 3.1 ([Remark 4.8]). Assumption (A4) and the condition

\[ \sum_{j=1}^{\infty} q_{n_j} = \infty, \quad \text{where } q_{n_j} \text{ is any subsequence of } q_n \]  \hspace{1cm} (3.1)

are equivalent.

Theorem 3.2. Let (A1), (A4)- (A7) hold, and \( s > k + 1 \), \( \text{(1.4)} \) be satisfied. If

\[ \left| \int_{a}^{\infty} \frac{du}{G(u)} \right| < \infty, \quad \forall a \in \mathbb{R}, \]  \hspace{1cm} (3.2)

then every unbounded solution of (1.2) oscillates.

Proof. To obtain a contradiction, let \( y_n \) be an eventually positive solution of (1.2). Setting \( z_n, w_n \) and \( c_n \) as in (2.5), (2.6) and (2.2) respectively, we obtain (2.10).

Note that (A4) implies (A2). Hence, by Lemma 2.3(a), we observe that either (2.7) or (2.8) holds.

First we consider the case when (2.7) holds. Using Lemma 2.5(a), we show that (2.10) implies (2.19). From (2.7), (2.17) and Lemma 2.2, it follows that

\[ \lim_{n \to \infty} z_n = -\infty, \quad \text{which implies } \Delta z_n < 0 \]  \hspace{1cm} (3.3)

Note that \( -z_n/(p\delta) = v_n \) implies \( \Delta z_n = -p\delta \Delta v_n \). Then, substituting this expression in the above, we obtain

\[ p\delta \Delta v_{n+1} - q_n G(v_{n+s-k}) \geq 0. \]  \hspace{1cm} \text{(3.4)}

Note that \( v_n > 0, \lim_{n \to \infty} v_n = \infty \) and \( v_n \) is increasing. Dividing both sides by \( G(v_{n+s-k}) \), we obtain

\[ p\delta \frac{\Delta v_{n+1}}{G(v_{n+s-k})} \geq q_n. \]  \hspace{1cm} (3.4)

Then writing \( \Delta v_{n+1} = \int_{v_{n+1}}^{v_{n+2}} dx \), where \( v_{n+1} \leq x \leq v_{n+2} \), and using \( s - k \geq 2 \), we obtain

\[ q_n \leq p\delta \int_{v_{n+1}}^{v_{n+2}} \frac{dx}{G(x)}. \]  \hspace{1cm} \text{(3.4)}

Summing \( n_2 \) to \( l - 1 \), and then taking limit \( l \to \infty \), we obtain

\[ \sum_{n = n_2}^{\infty} q_n \leq p\delta \int_{v_{n_2+1}}^{\infty} \frac{dx}{G(x)} < \infty, \]  \hspace{1cm} (3.2)

by (3.2), which contradicts (A2).

Now we consider the case when (2.8) holds. Consequently, we obtain (2.9). Then taking summation in (2.10) from \( n_2 \) to \( \infty \) we find (2.13). As \( y_n \) is unbounded, we can find a subsequence \( \{y_{n_j}\} \) of \( \{y_n\} \) which approaches \( \infty \) as \( j \to \infty \). Then there exists \( \eta > 0 \) such that \( y_{n_j} > \eta \) for large \( j \). Then \( \sum_{j=n_3}^{\infty} q_{n_j} G(y_{n_j}) > G(\eta) \sum_{j=n_3}^{\infty} q_{n_j} \to \)}
Theorem 3.3. Let (A1), (A4)–(A7), and (3.2), $s > k + 1$, (1.8) be satisfied. Then every unbounded solution of (1.2) oscillates.

The proof of the above theorem is similar to that of theorem 3.2 we omit it.

Theorem 3.4. Let (A1), (A4)–(A7) hold. Suppose $p_n$ satisfies (1.8), $s > k + 1$, and

$$\liminf_{n \to \infty} q_n > \gamma > 0.$$  \hfill (3.5)

Suppose that

$$\liminf_{n \to \infty} \frac{\sum_{i=n-s+k+1}^{n-1} q_i}{\sum_{i=n-s+k+1}^{n-1} q_i} > \frac{p_n}{\gamma} \frac{(s-k-1)}{(s-k)}.$$  \hfill (3.6)

Then every unbounded solution of (1.2) oscillates.

Proof. To obtain a contradiction, let $y_n$ be an eventually positive solution of (1.2). Proceeding as in the proof of theorem 3.2 we show that if (2.7) holds then

$$\Delta z_{n+1} + q_n G\left(\frac{-z_{n+s}-k}{p\delta}\right) \leq 0.$$  \hfill (3.7)

Applying (3.5) to the above inequality, we obtain

$$\Delta z_{n+1} - \gamma q_n \left(\frac{z_{n+s}-k}{p\delta}\right) \leq 0.$$  \hfill (3.8)

Note that $z_n < 0$ for large $n$. Substituting $(z_{n+1}/(p\delta)) = v_n$ and $\Delta z_{n+1} = p\delta \Delta v_n$, in the above we obtain

$$\Delta v_n - \frac{\gamma}{p\delta}q_nv_{n+s-k-1} \leq 0.$$  \hfill (3.9)

Since $s - k - 1 > 0$ this is an advanced difference inequality with a negative solution $v_n$, which contradicts Lemma 2.1(b).

Next consider the case that (2.8) holds. Proceeding as in the proof of theorem 3.2 we obtain a contradiction. The proof for the case $y_n < 0$, and unbounded is similar. Thus, the proof is complete. \hfill □

Remark 3.5. Condition (3.6) implies (A2). If (3.6) holds and (A2) fails, we have $\sum_{n=1}^{\infty} q_n < \infty$ which implies

$$\frac{p\delta}{\gamma} \frac{(s-k-1)}{(s-k)} \leq \liminf_{n \to \infty} \frac{\sum_{i=n-s+k+1}^{n-1} q_i}{\sum_{i=n-s+k+1}^{n-1} q_i} \leq \limsup_{n \to \infty} \left(\sum_{i=n_1}^{n-1} q_i - \sum_{i=n_1}^{n-s+k-1} q_i\right) = 0,$$

a contradiction.

Theorem 3.6. Suppose (A1), (A4)–(A7) hold, and (1.5) and $\delta \leq 1$ are satisfied. Then every unbounded solution of (1.2) oscillates.

Proof. Let $y_n$ be an unbounded and eventually positive solution of (1.2). Setting $c_n, z_n$ and $w_n$ as in (2.2), (2.5) and (2.6) respectively, we obtain (2.10). By Lemma 2.3(b), we have $\lim_{n \to \infty} w_n = 0$. Using this, unboundedness of $y_n$ and (A4), and proceeding as in the last part of the proof of theorem 3.2 we obtain a contradiction. A similar contradiction could be obtained if $y_n$ be an eventually negative and unbounded solution of (1.2). This completes the proof. \hfill □
Theorem 3.7. Suppose (A1), (A3), (A4), (A6), (A7) hold, (1.6) or (1.7), and (2.21) and (2.22) be satisfied. Then every unbounded solution of (1.2) oscillates.

Proof. On the contrary suppose $y_n$ be an eventually positive and unbounded solution of (1.2). Setting $z_n$ and $w_n$ as in (2.5) and (2.6), we obtain (2.10). Application of Lemma 2.6 yields $\beta = \lim_{n \to \infty} w_n = 0$, $w_n < 0$ and $\Delta w_n > 0$. Then using this, unboundedness of $y_n$, (A4) and proceeding as in the last part of the proof of theorem 3.2 we obtain a contradiction. A similar contradiction could be obtain if $y_n$ be an eventually negative and unbounded solution of (1.2). This completes the proof. □

Note that the condition
\[
\lim \inf_{n \to \infty} |x_n| > 0 \quad \text{implies} \quad \lim \inf_{n \to \infty} |G(x_n)| > 0.
\]
is equivalent to
\[
\lim \inf_{u \to \pm \infty} G(u) \neq 0
\]
and note that (A5) implies (3.8). Consequently, we quote a particular case of [13, theorem 2.5, p.236] for $f_n \equiv 0$ as our next result.

Theorem 3.8. Suppose (A2), (A6), (A7) hold, and (1.9) and (3.8) are satisfied. If $L(x) = x$, then every non-oscillatory solution of (1.2) is bounded. Or equivalently every unbounded solution of (1.2) oscillates.

4. Main results part II

In this section, we find sufficient conditions so that all solutions of (1.1) oscillate under condition (A2), which is less restrictive than (A4).

Theorem 4.1. Suppose (A1), (A2), (A5) hold, and (1.4) and $s < k$ are satisfied. If
\[
\left| \int_0^{c} \frac{du}{G(u)} \right| < \infty, \quad \text{for any finite positive} \ c \in \mathbb{R},
\]
Then every bounded solution of (1.1) oscillates.

Proof. On the contrary let $y_n$ be a bounded eventually positive solution of (1.1). Setting $z_n$ as in (2.5) we obtain (2.25). Then $z_n$ is bounded and by Lemma 2.8(a), we find that (2.26) cannot hold because boundedness of $z_n$, as a result, (2.27) holds. Then (2.28) follows as a consequence which implies that $z_n < 0$ and increasing. If $p_n = 0$, then $z_n = y_n < 0$, is a contradiction. Hence $p_n > 0$. From (A1) and (1.4) it follows that $y_n - k \geq z_n + s - p \delta$. Hence, (2.25) with Lemma 2.9 (b) yields
\[
\Delta z_n - q_n G(z_{n+s-k}/(-p \delta)) \geq 0.
\]
Substituting $v_n = z_n/(-p \delta)$, which implies $-p \delta \Delta v_n = \Delta z_n$, we find that
\[
p \delta \Delta v_n + q_n G(v_{n+s-k}) \leq 0,
\]
which together with $s < k$ and $v_n$ is positive and decreasing, implies
\[
p \delta \Delta v_n + q_n G(v_n) \leq 0,
\]
Then dividing both sides of the above by $G(v_n)$ we obtain
\[
\frac{p \delta \Delta v_n}{G(v_n)} + q_n \leq 0.
\]
Then using $\Delta v_n = \int_{v_n}^{v_{n+1}} dx$ and taking $v_{n+1} \leq x \leq v_n$ we have

$$p\delta \int_{v_n}^{v_{n+1}} \frac{dx}{G(v_n)} + q_n \leq 0,$$

which implies, because of the nondecreasing character of $G$ that

$$p\delta \int_{v_n}^{v_{n+1}} \frac{dx}{G(x)} + q_n \leq 0.$$

Summing from $n = n_1$ to $l - 1$ we obtain

$$p\delta \int_{v_{n_1}}^{v_l} \frac{dx}{G(x)} + \sum_{n_1}^{l-1} q_n \leq 0.$$

As $l \to \infty$, $v_l \to 0$, and so in the limiting case, we obtain

$$\sum_{n_1}^{\infty} q_n \leq p\delta \int_{0}^{v_{n_1}} \frac{dx}{G(x)} < \infty$$

by (4.1), which contradicts (A2). The proof for the case $y_n$ being eventually negative is similar and this completes the proof. □

**Theorem 4.2.** Suppose (A1), (A2), (A5) hold, and (1.5), (4.1), $\delta \leq 1$ and $s < k$ are satisfied. Then every solution of (1.1) oscillates.

**Proof.** On the contrary, let $y_n$ be an eventually positive solution of (1.1). Setting $z_n$ as in (2.5), we obtain (2.25). Then by Lemma 2.8(b), we find that (2.27) holds. Then (2.28), follows as a consequence and $z_n < 0$. If $p_n = 0$ then $z_n = y_n < 0$ which is a contradiction. Hence from (2.25), and Lemma 2.9(b) we obtain

$$\Delta z_n - q_n G(y_n - k) \geq 0.$$  

Using $\delta \leq 1$ and (1.5), we find $y_n - k \geq \frac{y_n + s - k}{s} \geq -z_n + s - k$. Therefore,

$$\Delta z_n - q_n G(-z_n + s - k) \geq 0.$$  

Substituting $-z_n = v_n$, which implies $\Delta z_n = -\Delta v_n$, in the above, we obtain

$$\Delta v_n + q_n G(v_n + s - k) \leq 0.$$  

Then further using $s < k$ and $v_n > 0$ and decreasing, we obtain

$$\Delta v_n + q_n G(v_n) \leq 0.$$  

Dividing both sides of the above inequality, by $G(v_n)$, we obtain

$$\frac{\Delta v_n}{G(v_n)} + q_n \leq 0.$$  

Taking $v_{n+1} \leq v \leq v_n$ and using $\Delta v_n = \int_{v_n}^{v_{n+1}} dv$, we proceed as in the proof of theorem 4.1 to obtain

$$\sum_{n=n_1}^{\infty} q_n \leq \int_{0}^{v_{n_1}} \frac{dv}{G(v)} < \infty$$  

by (4.1), which contradicts (A2). The proof for the case when $y_n$ is eventually negative is similar. □
Theorem 4.3. Suppose (A1), (A2), (A5) hold, and (1.5), \( \delta \leq 1 \) and \( s < k \) are satisfied. If

\[
\liminf_{|x| \to 0} \frac{G(x)}{x} > \gamma > 0.
\]

and

\[
\liminf_{n \to \infty} \sum_{i=n-k+s}^{n-1} q_i > \frac{1}{\gamma} \left( \frac{k-s}{k-s+1} \right)^{k-s+1},
\]

then every solution of (1.1) oscillates.

Proof. As \( s < k \) and \( 0 < \delta \leq 1 \) proceeding as in the proof of the theorem 4.2, we obtain the first order delay difference inequality (4.3), which by (4.4), yields

\[
\Delta v_n + \gamma q_n v_{n-s} \leq 0
\]

which has a positive solution. This, contradicts Lemma 2.1(a). The proof for the case when \( y_n \) is eventually negative is similar. This completes the proof. \( \square \)

Theorem 4.4. Suppose (A1), (A3) hold, and (1.6), (2.21) and (2.22) are satisfied. Then every solution of (1.1) oscillates.

Proof. On the contrary, suppose \( y_n \) be an eventually positive solution of (1.1). Setting \( z_n \) as in (2.5), we obtain (2.25). Then applying Lemma 2.10, we obtain \( \beta = \lim_{n \to \infty} z_n = 0 \), which implies \( z_n < 0 \), a contradiction because \( z_n \geq 0 \) by (1.6). The proof for the case when \( y_n \) is eventually negative, is similar and thus, the proof is complete. \( \square \)

Theorem 4.5. Suppose (A1), (A3) hold, and (1.7), (2.21) and (2.22) are satisfied. Then every solution of (1.1) oscillates.

Proof. On the contrary, assume \( y_n \) be an eventually positive solution of (1.1). Setting \( z_n \) as in (2.5), we obtain (2.25). Then application of Lemma 2.10 yields \( \beta = \lim_{n \to \infty} z_n = 0 \). Consequently \( \Delta z_n > 0 \) and \( z_n < 0 \). Again, this would lead to \( p_n > \frac{L(y_n)}{L(y_{n-s})} > 0 \) for large \( n \), which is a contradiction, because \( p_n \) changes sign. For the proof of the case, when \( y_n \) is eventually negative, we may proceed with \( x_n = -y_n \) and complete the proof. \( \square \)

Theorem 4.6. Suppose (A2), (A5) hold, \( L(x) = x \), and \( p_n \) satisfies (1.9). Then every solution of (1.1) oscillates.

Proof. On the contrary assume \( y_n \) be an eventually positive solution of (1.1). Setting \( z_n \) as in (2.5), we obtain (2.25). Note that (A5) implies (3.8). Then applying Theorem 3.8 for \( u_n = 0 \), we show that \( y_n \) is bounded, which implies \( z_n \) is bounded. As \( z_n \) is monotonic, \( \lim_{n \to \infty} z_n = \beta \in \mathbb{R} \). Summing (2.25) from \( n_1 \) to \( \infty \), we obtain (2.13), which implies \( \liminf_{n \to \infty} y_n = 0 \). By [9, Lemma 2.1], we have \( \lim_{n \to \infty} z_n = 0 \). As a consequence (2.28) holds, which implies \( z_n < 0 \). However, by (1.9) we have \( z_n > 0 \), a contradiction. The proof for the case \( y_n \) < 0 is similar. Thus proof is complete. \( \square \)

Next, we give some examples to illustrate the results.

Example 4.7. Consider the neutral difference equation

\[
\Delta^2 (y_n - py_{n-4}) + 18 \left( \frac{1}{22n} + 1 - \frac{p}{16} \right) y_{n-1} - \left( \frac{72}{22n} + \frac{9}{2n} \right) H(y_{n-1}) = 0
\]
where \(|p| < 16\). Suppose \(p = \pm 2\) or \(p = \pm 1/2\). Here, \(s = 4, k = 1, q_n = 18\left(\frac{1}{2s^2} + 1 - \frac{p}{2s}\right), u_n = \left(\frac{2^s}{2s^2} + \frac{p}{2s}\right)\) and \(H(u) = u/(1 + |u|)\). Clearly, the neutral difference equation (4.6) satisfies all the conditions of Theorems 3.4, 3.6, 3.7 and 3.8. As a result, it has an unbounded solution \(y_n = 2^n(-1)^n\), which is oscillatory.

**Example 4.8.** Consider the neutral difference equation

\[
\Delta^2(y_n - by_{n-2}) + \left(9(1-b/4)(2^n + 128) + 2^{-2n}\right)G(y_{n-7}) - u_nH(y_{n-7}) = 0
\]  

where \(|b| < 4\) is suitably selected constant. Suppose \(b = \pm 1/2\). Here, \(s = 2, k = 7, u_n = \frac{2+2^n-power}{2^{n+1}(2^n-1)}, q_n = \left(9(1-b/4)(2^n + 128) + 2^{-2n}\right), G(u) = u/(1 + |u|)\) and \(H(u) = u/(2 + |u|)\). Clearly, the neutral difference equation (4.7) satisfies all the conditions of Theorems 3.6 and 3.8. Consequently, it has an unbounded solution \(y_n = 2^n(-1)^n\), which is oscillatory.

**Example 4.9.** Consider the neutral difference equation

\[
\Delta^2(y_n - pL(y_{n-3})) + \left[\frac{4\left(1 + a + p\right)}{(1 + a)(1 + \gamma)}\right]G(y_{n-5}) = 0
\]  

where \(p > 0\) is any scalar. Here \(L(x) = x/(a + |x|)\) and \(G(u) = \gamma(\gamma + |u|)\) where \(a\) and \(\gamma\) are positive constants. This neutral equation satisfies all the conditions of Theorems 4.3. As such, it has a solution \(y_n = (-1)^n\), which is oscillatory.

**Example 4.10.** Consider the neutral difference equation

\[
\Delta^2(y_n + p(-1)^nL(y_{n-5})) + 4y_{n-1}^{1/3} = 0
\]  

where \(p > 0\) is any scalar. Here \(p_n\) changes sign and satisfies (4.7). Further, \(L(x) = x/(a + |x|)\). This neutral equation satisfies all the conditions of Theorem 4.5. Hence, it has a solution \(y_n = (-1)^{3n}\), which is oscillatory.

It seems, no result in the literature, could be applied to the neutral equations (4.8)-(4.9) given in the examples above, because of the non linear term inside \(\Delta^2\).

5. **Application to Neutral Difference Equations with Oscillating Coefficients**

In this section, we find sufficient conditions so that every unbounded solution of the second order neutral difference equation (1.11) oscillates, where \(v_n\) is allowed to change sign. Let \(v^+_n = \max\{v_n, 0\}\) and \(v^-_n = \max\{-v_n, 0\}\). Then \(v_n = v^+_n - v^-_n\) and the equation (1.11) can be written as

\[
\Delta^2[y_n - p_nL(y_{n-s})] + v^+_nG(y_{n-k}) - v^-_nG(y_{n-k}) = 0.
\]  

Now we proceed as in the previous section by setting \(q_n = v^+_n, u_n = v^-_n\) and \(H(x) = G(x)\). Assumptions (A4), (A3) and (A6) become

\[
\sum_{n=n_0}^{\infty} v^+_n = \infty. \tag{5.2}
\]

\[
\liminf_{n \to \infty} v^+_n > 0. \tag{5.3}
\]

\[
\sum_{n=n_0}^{\infty} V^+_n = \infty \quad \text{where} \quad V^+_n = \min\{v^+_n, v^+_n-s\}. \tag{5.4}
\]
\[ \sum_{n=n_0}^{\infty} n v_n^- < \infty. \]  

respectively, which are feasible conditions. Therefore, the study of (1.11) reduces to the study of (5.1), which could be achieved, by following the study of (1.2) for different results in section 3. The following results for (1.11) (with \( v_n \) changing sign) follow from Theorems 3.6, 3.7 and 3.8, by replacing \( q_n \) by \( v_n^+ \), \( u_n \) by \( v_n^- \) and \( H \) by \( G \).

**Theorem 5.1.** Suppose that \((A1)\) holds with \( \delta \leq 1 \), \((A5)\) holds, \( p_n \) satisfies \((1.5)\), \( G \) is bounded, and \((5.3)\) and \((5.5)\) are satisfied. Then every unbounded solution of \((1.11)\) (with \( v_n \) changing sign) oscillates.

**Theorem 5.2.** Suppose \( p_n \) satisfies \((1.6)\) or \((1.7)\), \( G \) is bounded, \((A1)\) holds, and \((2.21)\), \((2.22)\), \((5.3)\), \((5.4)\), and \((5.5)\) are satisfied. Then every unbounded solution of \((1.11)\) (with \( v_n \) changing sign) oscillates.

**Theorem 5.3.** Suppose \( p_n \) satisfy \((1.9)\), \( G \) is bounded, \((3.8)\), \((5.2)\) and \((5.5)\) are satisfied. If \( L(x) = x \), then every unbounded solution of \((1.11)\) oscillates.

6. Final comments

Before we close this article, we would like to give our concluding remarks, which may be helpful for further research. In this paper, some oscillatory results are obtained for the neutral difference equation \((1.2)\) and \((1.1)\) by imposing different super linear conditions like \((3.5)\) or \((3.2)\), and sublinear conditions like \((4.4)\) or \((4.1)\) on \( G \). Note that the super linear condition \((3.5)\) and the sub linear condition \((4.4)\) on \( G \) include their corresponding linear case \( G(x) = x \). Authors while studying the oscillatory and asymptotic behavior of \((1.2)\) or \((1.1)\), very often find difficulty in tackling, the case of \( p_n \geq 1 \), i.e; when \((1.8)\) or \((1.4)\) are satisfied. That is why, the results \([10\), Theorems 2.6 and 2.7\] appear to be wrong, as the neutral equation

\[ \Delta^2(y_n - 4y_{n-1}) + 4^{(n+1)/3}y_{n-2}^{1/3} = 0 \]

satisfies all the conditions of the theorems, but, it admits a non oscillatory solution \( y_n = 2^n \), which tends to \( \infty \), as \( n \to \infty \), contradicting the theorems. With the super linear \( G \) with \((3.2)\) or \((3.5)\), we proved in Theorems 3.2 and 3.4 that \((A4)\) is sufficient for all unbounded solutions of \((3.2)\) to be oscillatory which is more restrictive than \((A2)\). Hence, one may extend this study to improve the results (Theorems 3.2 and 3.4) by attempting to answer the following problem.

**Problem 6.1.** Suppose that \( L(x) = x \), or \((A1)\) holds, and \( 1 \leq p_n \leq p \). Assuming \((A2)\), \((A5)\) and \((3.2)\) can we prove that every unbounded solution of \((1.1)\) oscillates?

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