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ASYMPTOTIC BEHAVIOR FOR A NON-AUTONOMOUS MODEL OF NEURAL FIELDS WITH VARIABLE EXTERNAL STIMULI

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Abstract. In this work we consider the class of nonlocal non-autonomous evolution problems in a bounded smooth domain Ω in $\mathbb{R}^N$

$$\partial_t u(t,x) = -a(t)u(t,x) + b(t)\int_{\mathbb{R}^N} J(x,y)f(t,u(t,y))\,dy - h + S(t,x), \quad t \geq \tau$$
$$u(\tau,x) = u_\tau(x),$$

with $u(t,x) = 0$ for $t \geq \tau$ and $x \in \mathbb{R}^N \backslash \Omega$. Under appropriate assumptions we study the asymptotic behavior of the evolution process, generated by this problem in a suitable Banach space. We prove results on existence, uniqueness and smoothness of the solutions and on the existence of pullback attractor for the evolution process. We also prove a continuous dependence of the evolution process with respect to the external stimuli function present in the model. Furthermore, using the continuous dependence of the evolution process, we prove the upper semicontinuity of pullback attractors with respect to the external stimuli function. We finish this article with a small discussion about the model and about a biological interpretation of the result on the continuous dependence of neuronal activity with respect to the external stimuli function.

1. Introduction

Neural field equations describe the spatio-temporal evolution of variables such as synaptic or firing rate activity in populations of neurons. The neural field model has already been well analyzed in the literature (see [1, 4, 5, 7, 11, 13, 14, 15, 16, 21, 25, 26, 28, 33, 32]). Although this model has been used to working memory model, it arises also in cognitive development of infants, (see [29, 31]), and in timing sensory integration for robot simulation of autistic behavior (see [3]).

As in [1], we will denote by $u(t,x)$ the membrane potential of a neuron located at position x, and time t, which we are assuming as a differentiable function of t, and $J(x,y)$ will denote the average intensity of connections from neurons located at place y to those at place x.

We also assume that the pulse emission rate of neurons at position x, and time t, depends on t and $u(x,t)$, that is, it is given by $f(t,u(t,x))$. The activity $f(t,u(t,y))$ of neurons at y causes an increase in the potential $u(t,x)$ at x, through the connections $J(x,y)$, such that the rate of emission of pulses is proportional to $J(x,y)f(t,u(t,x))$. We also assume that the potential $u(t,x)$ decays, with

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speed $0 < \alpha(t) < \alpha_0$, to a constant $-h$ (which we call the threshold of the field), and that it increases proportionally to the sum of all the stimuli arriving with speed $b(t)$ at the neurons. Then, denoting by $S(x,t)$ the intensity of the sum of applied external stimuli at x at time t, and writing $a(t) = 1/\alpha(t)$ we have the following non-autonomous evolution equation

$$\partial_t u(t,x) = -a(t)u(t,x) + b(t)\int_{\mathbb{R}^N} J(x,y)f(t,u(t,y))\,dy - h + S(t,x). \tag{1.1}$$

Here we consider that the rate of the intensity of neuronal potential varies explicitly accordingly to time. Thus, we expect to have a more realistic model in (1.1), when compared to what happens in the brain, since the potential action of the electric impulses of the neuronal membrane is a consequence of the inversion of the polarity inside the membrane, which is not necessarily constant.

Note that, when $a(t) = b(t) = 1/\lambda$, for any $t \in \mathbb{R}$, for some constant $\lambda > 0$, and $f(t,x) = f(x)$, equation (1.1) becomes

$$\lambda\partial_t u(t,x) = -u(t,x) + \int_{\mathbb{R}^N} J(x,y)f(u(t,y))\,dy - \lambda h + \lambda S(t,x).$$

In particular, if $a(t) = b(t) = 1$, for all $t \in \mathbb{R}$ and $S(t,x) = h$, equation (1.1) becomes

$$\partial_t u(t,x) = -u(t,x) + \int_{\mathbb{R}^N} J(x,y)f(t,u(t,y))\,dy.$$

Therefore, equation (1.1) generalizes the models studied in [1, 2, 5, 11, 13, 14, 15, 16, 18, 25, 26, 28, 31, 33].

Below we introduce the notation, terminology and some additional hypotheses, which are already well known in the literature, (see, for example [1, 2, 5, 11, 17, 21]).

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain modelling the geometric configuration of the network, $u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a function modelling the mean membrane potential, $u(t,x)$ be the potential of a patch of tissue located at position $x \in \Omega$ at time $t \in \mathbb{R}$ and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a time dependent transfer function. We say that a neuron at a point x is active at time t if $f(t,u(t,x)) > 0$. In what follows, $b : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$0 < b(t) \le b_0 < \infty,$$

and it denotes the increasing speed of the potential function $u(t,x)$. Since the decreasing speed of the potential function $u(t,x)$ satisfies $0 < \alpha(t) < \alpha_0$, we can assume that there exist positive constants a_- and a_0 such that

$$0 < a_- \le a(t) \le a_0 < \infty.$$

Let us also denote the integrable function $J : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ as the connection between locations, that is, $J(x,y)$ is the strength of the connections of neuronal activity at location y on the activity of the neuron at location x. The strength of the connection is assumed to be symmetric, that is $J(x,y) = J(y,x)$, for any $x, y \in \mathbb{R}^N$ and that

$$\int_{\mathbb{R}^N} J(x,y)dy = \int_{\mathbb{R}^N} J(x,y)dx = 1.$$

Under the above conditions, we study the following non-autonomous model for neural fields

$$
\begin{gathered}
\partial_t u(t,x) = -a(t)u(t,x) + b(t)Kf(t,u(t,y))\,dy - h + S(t,x), \quad t \geqslant \tau,\ x \in \Omega, \\
u(\tau,x) = u_\tau(x), \quad x \in \Omega, \\
u(t,x) = 0, \quad t \geqslant \tau,\ x \in \mathbb{R}^N \backslash \Omega,
\end{gathered} \tag{1.2}
$$

where the integral operator, with symmetric kernel, K is given, for all $v \in L^1(\mathbb{R}^N)$, by

$$
Kv(x) := \int_{\mathbb{R}^N} J(x,y)v(y)\,dy.
$$

Also we will assume that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies some growth conditions, as presented along the Section 2, and that $S : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is continuous at variable t and $S(t,\cdot) \in L^p(\Omega)$, for all $t \in \mathbb{R}$.

We aim to study the asymptotic behavior of the evolution process associated to the Cauchy problem (1.2) under an appropriate Banach space, as well as obtain some biological conclusion. Then, using the same techniques employed in [5, 17], we prove results on existence, uniqueness and smoothness of the solutions, and we also prove the existence of pullback attractors for the evolution process associated to (1.2), which is a more general model than the models analyzed in previous published works on the subject. In addition, we prove a continuous dependence of the solutions with respect to the external stimuli function S, concluding mathematically that the neuronal activity depends continuously on the sum of external stimuli involved in the neuronal system. This suggests the need for intensive therapies to stimulate people with poor neuronal activity as, in some cases, people with autism or other neurological disorders. Furthermore, using the result of continuous dependence of the evolution process, we also prove the upper semicontinuity of pullback attractors with respect to function S.

This article is organized as follows. In Section 2, under the growth conditions (2.7), (2.9), (2.11) and (2.14), on the function f, we prove that (1.2) generates a $\mathcal{C}^1$ evolution process in the phase space

$$
X_p = \{u \in L^p(\mathbb{R}^N) : u(x) = 0 \text{ for } x \in \mathbb{R}^N \backslash \Omega\} \tag{1.3}
$$

with the induced norm, satisfying the "variation of constants formula"

$$
u(t,x) = \begin{cases} e^{-(A(t)-A(\tau))}u_\tau(x) + \int_\tau^t e^{-(A(t)-A(s))}b(s)Kf(s,u(s,\cdot))(x)\,ds \\ + \int_\tau^t e^{-(A(t)-A(s))}[S(s,x) - h]ds, & x \in \Omega, \\ 0, & x \in \mathbb{R}^N \backslash \Omega, \end{cases}
$$

where $A(\xi) = \int_0^\xi a(\eta)d\eta$, for any $\xi \geq \tau$. In Section 3, we prove existence of a pullback attractor in the phase space X_p. Section 4 is dedicated to continuity with respect to the external stimuli function S. In Subsection 4.1 we study the continuity of the process with respect to the function S, and in Subsection 4.2 we use this result to prove an upper semicontinuity of the pullback attractors. Finally, in Section 5, we conclude presenting a brief discussion about the model and about a biological interpretation of the result on the continuous dependence of neuronal activity with respect to the external stimuli function.

2. Flow generated by the model problem

In this section we show the existence of global solution for problem (1.2) and that it generates a $\mathcal{C}^1$ evolution process in an appropriate Banach space. For more details on the process evolution (or infinite-dimensional non-autonomous dynamical systems) see, for example [8, 10, 23, 22] and for finite-dimensional non-autonomous dynamical systems, see [9]. See also [5, 30] for related works.

2.1. **Well posedness.** In this subsection, under suitable growth condition on the nonlinearity f, we show the well posedness of problem (1.2) in the phase space X_p, for $1 \leq p \leq \infty$, given by

$$X_p = \{u \in L^p(\mathbb{R}^N) : u(x) = 0, \text{for } x \in \mathbb{R}^N \backslash \Omega\}$$

with the induced norm. It is easy to see that the Banach space X_p is canonically isometric to $L^p(\Omega)$, then we usually identify the two spaces, without further comment. For simplicity, we use the same notation for a function defined on the whole $\mathbb{R}^N$ and also for its restriction on Ω wherever we believe the intention is clear in the context. To obtain well posedness of (1.2) in X_p, we consider the Cauchy problem

$$\begin{aligned} \frac{du}{dt} &= F(t,u),\ t > \tau, \\ u(\tau) &= u_\tau, \end{aligned} \tag{2.1}$$

where the map $F : \mathbb{R} \times X_p \to X_p$ is defined by

$$F(t,u)(x) = \begin{cases} -a(t)u(x) + b(t)Kf(t,u)(x) - h + S(t,x), & \text{if } t \in \mathbb{R},\ x \in \Omega, \\ 0, & \text{if } t \in \mathbb{R},\ x \in \mathbb{R}^N \backslash \Omega, \end{cases} \tag{2.2}$$

where

$$Kf(t,u)(x) := \int_{\mathbb{R}^N} J(x,y) f(t,u(y))dy. \tag{2.3}$$

The map K is well defined as a bounded linear operator in various function spaces, depending on the properties assumed for J; for example, with J satisfying the hypotheses stated in the introduction, K is well defined in X_p as shown in the lemma below, which was proved in [17].

Lemma 2.1. *Let K be defined by* (2.3) *and $\|J\|_r := \sup_{x\in\Omega} \|J(x,\cdot)\|_{L^r(\Omega)}$, $1 \leq r \leq \infty$. If $u \in L^p(\Omega)$ with $1 \leq p \leq \infty$, then $Ku \in L^\infty(\Omega)$, and*

$$|Ku(x)| \leq \|J\|_q \|u\|_{L^p(\Omega)} \quad \text{for all } x \in \Omega, \tag{2.4}$$

where $1 \leq q \leq \infty$ is the conjugate exponent of p. Moreover,

$$\|Ku\|_{L^p(\Omega)} \leq \|J\|_1 \|u\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)}. \tag{2.5}$$

If $u \in L^1(\Omega)$, then $Ku \in L^p(\Omega)$, $1 \leq p \leq \infty$, and

$$\|Ku\|_{L^p(\Omega)} \leq \|J\|_p \|u\|_{L^1(\Omega)}. \tag{2.6}$$

The following definition is well known in the theory of ODEs in Banach spaces and it can be found in [5].

Definition 2.2. If E is a normed space, and $I \subset \mathbb{R}$ is an interval, we say that a function $F : I \times E \to E$ is *locally Lipschitz continuous (or simply locally Lipschitz)* with respect to the second variable if, for any $(t_0, x_0) \in I \times E$, there exists a constant

C and a rectangle $R = \{(t,x) \in I \times E : |t-t_0| < b_1, \|x-x_0\| < b_2\}$ such that, if (t,x) and (t,y) belong to R, then

$$\|F(t,x) - F(t,y)\| \leq C\|x-y\|.$$

We say that F is *Lipschitz continuous on bounded sets with respect to the second variable* if the rectangle R in the previous definition can be chosen as any bounded rectangle in $\mathbb{R} \times E$.

Remark 2.3. If the normed space E is locally compact the definitions of locally Lipschitz continuous and Lipschitz continuous on bounded sets are equivalent.

Now, proceeding as in [5, 17], we prove that the map F, given in (2.2), is well defined under appropriate growth conditions on f and it is locally Lipschitz continuous (see Proposition 2.5 below, which generalizes [5, Proposition 3.3] and [17, Proposition 2.4]).

Lemma 2.4. *Let us assume the same hypotheses stated in Lemma 2.1 hold, and that the function f satisfies the growth condition*

$$|f(t,x)| \leq C_1(t)(1+|x|^p), \quad \text{for any } (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{2.7}$$

with $1 \leq p < \infty$ and $C_1 : \mathbb{R} \to \mathbb{R}$ is a locally bounded function. Then the function F given by (2.2) is well defined on $\mathbb{R} \times X_p$. If, for any $t \in \mathbb{R}$, the function $f(t,\cdot)$ is locally bounded, then F is well defined on $\mathbb{R} \times L^\infty(\Omega)$.

Proof. Suppose $1 \leq p < \infty$. Given $u \in L^p(\Omega)$, denoting the function $f(t,u)(x) = f(t,u(x))$ by $f(t,u)$ and using (2.7), it easy to see that, for each $t \in \mathbb{R}$

$$\|f(t,u)\|_{L^1(\Omega)} \leq C_1(t)(|\Omega| + \|u\|^p_{L^p(\Omega)}). \tag{2.8}$$

Thus, using (2.6) and (2.8), it follows that

$$\begin{aligned}
&\|F(t,u)\|_{L^p(\Omega)} \\
&\leq a_0\|u\|_{L^p(\Omega)} + b_0\|Kf(t,u)\|_{L^p(\Omega)} + \|S(t,\cdot)\|_{L^p(\Omega)} + \|h\|_{L^p(\Omega)} \\
&\leq a_0\|u\|_{L^p(\Omega)} + b_0\|J\|_p\|f(t,u)\|_{L^1(\Omega)} + \|S(t,\cdot)\|_{L^p(\Omega)} + h|\Omega|^{1/p} \\
&\leq a_0\|u\|_{L^p(\Omega)} + b_0\|J\|_p(C_1(t)|\Omega| + C_1(t)\|u\|^p_{L^p(\Omega)}) + \|S(t,\cdot)\|_{L^p(\Omega)} + h|\Omega|^{1/p} \\
&\leq a_0\|u\|_{L^p(\Omega)} + b_0C_1(t)\|J\|_p|\Omega| + b_0C_1(t)\|J\|_p\|u\|^p_{L^p(\Omega)} + \|S(t,\cdot)\|_{L^p(\Omega)} + h|\Omega|^{1/p}.
\end{aligned}$$

Since $S(t,\cdot) \in L^p(\Omega)$, it follows immediately that F is well defined in the space $L^p(\Omega)$ for $1 \leq p < \infty$. If $p = \infty$ the result easily follows from (2.4). $\square$

Proposition 2.5. *Under the hypotheses of Lemma 2.4, if a and b are continuous functions and f and S are continuous functions with respect to the first variable, then F is also continuous on the first variable. Moreover if*

$$|f(t,x) - f(t,y)| \leq C_2(t)(1+|x|^{p-1} + |y|^{p-1})|x-y|, \tag{2.9}$$

for any $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, $t \in \mathbb{R}$, and for some strictly positive function $C_2 : \mathbb{R} \to \mathbb{R}$, then, for any $1 \leq p < \infty$, the function F is locally Lipschitz continuous on bounded sets with respect to the second variable. If $p = \infty$, this is true if f is locally Lipschitz fuction with respect to the second variable.

Proof. Suppose that $f(t, x)$ is continuous at t. Then for any $(t, u) \in \mathbb{R} \times X_p$, we obtain

$$\|f(t, u) - f(t+\xi, u)\|_{L^1(\Omega)} \le \int_\Omega |f(t, u(x)) - f(t+\xi, u(x))|\,dx \tag{2.10}$$

for a small $\xi \in \mathbb{R}$. From (2.7), it follows that the integrand in (2.10) is bounded by $2C(1+|u(x)|^p)$, where C is a bound for $C(t)$ in a neighborhood of t, and it goes to 0 as $\xi \to 0$. Hence, using Lebesgue dominated convergence theorem, it follows that $\|f(t, u) - f(t+\xi, u)\|_{L^1(\Omega)} \to 0$ as $\xi \to 0$. Thus, using (2.5) and (2.8), we obtain

$$\begin{aligned}
&\|F(t+\xi, u) - F(t, u)\|_{L^p(\Omega)}\\
&\le |a(t) - a(t+\xi)|\|u\|_{L^p(\Omega)} + |b(t+\xi) - b(t)|\|K(f(t+\xi, u)\|_{L^p(\Omega)}\\
&\quad + |b(t)|\|K(f(t+\xi, u) - f(t, u))\|_{L^p(\Omega)} + \|S(t+\xi, \cdot) - S(t, \cdot)\|_{L^p(\Omega)}\\
&\le |a(t) - a(t+\xi)|\|u\|_{L^p(\Omega)} + |b(t+\xi) - b(t)|\|J\|_p C_1(t)(|\Omega| + \|u\|^p_{L^p(\Omega)})\\
&\quad + |b(t)|\|J\|_p\|f(t+\xi, u) - f(t, u)\|_{L^1(\Omega)} + \|S(t+\xi, \cdot) - S(t, \cdot)\|_{L^p(\Omega)}
\end{aligned}$$

which approaches 0 as $\xi \to 0$, proving the continuity of F in t.

Now assume that

$$|f(t, x) - f(t, y)| \le C_2(t)(1 + |x|^{p-1} + |y|^{p-1})|x - y|,$$

for some $1 < p < \infty$, where $C_2 : \mathbb{R} \to \mathbb{R}$ is a strictly positive function. Then, for u and v belonging to $L^p(\Omega)$, using Hölder inequality, see [6], we obtain

$$\begin{aligned}
&\|f(t, u) - f(t, v)\|_{L^1(\Omega)}\\
&\le \int_\Omega C_2(t)(1 + |u(x)|^{p-1} + |v(x)|^{p-1})|u - v|\,dx\\
&\le C_2(t)\Big[\int_\Omega (1 + |u(x)|^{p-1} + |v(x)|^{p-1})^q dx\Big]^{1/q}\Big[\int_\Omega |u(x) - v(x)|^p dx\Big]^{1/p}\\
&\le C_2(t)\Big[\|1\|_{L^q(\Omega)} + \|u^{p-1}\|_{L^q(\Omega)} + \|v^{p-1}\|_{L^q(\Omega)}\Big]\|u - v\|_{L^p(\Omega)}\\
&\le C_2(t)\big[|\Omega|^{1/q} + \|u\|^{p/q}_{L^p(\Omega)} + \|v\|^{p/q}_{L^p(\Omega)}\big]\|u - v\|_{L^p(\Omega)},
\end{aligned}$$

where q is the conjugate exponent of p.

Using (2.6) once again and the hypotheses on f, it follows that

$$\begin{aligned}
&\|F(t, u) - F(t, v)\|_{L^p(\Omega)}\\
&\le a_0\|u - v\|_{L^p(\Omega)} + b_0\|K(f(t, u) - f(t, v))\|_{L^p(\Omega)}\\
&\le a_0\|u - v\|_{L^p(\Omega)} + b_0\|J\|_p\|f(t, u) - f(t, v)\|_{L^1(\Omega)}\\
&\le \Big(a_0 + b_0 C_2(t)\|J\|_p\big[|\Omega|^{1/q} + \|u\|^{p/q}_{L^p(\Omega)} + \|v\|^{p/q}_{L^p(\Omega)}\big]\Big)\|u - v\|_{L^p(\Omega)},
\end{aligned}$$

showing that F is Lipschitz on bounded sets of $L^p(\Omega)$ as claimed.

If $p = 1$, the proof is similar. Suppose finally that $\|u\|_{L^\infty(\Omega)} \le R$, $\|v\|_{L^\infty(\Omega)} \le R$ and let M be the Lipschitz constant of f in the interval $[-R, R] \subset \mathbb{R}$. Then

$$|f(t, u(x)) - f(t, v(x))| \le M|u(x) - v(x)|, \quad \text{for any } x \in \Omega,$$

and this allows us to conclude that

$$\|f(t, u) - f(t, v)\|_{L^\infty(\Omega)} \le M\|u - v\|_{L^\infty(\Omega)}.$$

Thus, by (2.5) we have that

$$\|F(t, u) - F(t, v)\|_{L^\infty(\Omega)} \le a_0\|u - v\|_{L^\infty(\Omega)} + b_0\|K(f(t, u) - f(t, v))\|_{L^\infty(\Omega)}$$

$$\leq (a_0 + b_0 M\|J\|_1)\,\|u-v\|_{L^\infty(\Omega)}.$$

This completes the proof. □

Using Proposition 2.5 and well known results of ODEs in Banach spaces [12] it follows that the initial value problem (2.1) has a unique local solution for any initial condition in X_p. For the existence of a global solution, we use [24, Theorem 5.6.1].

Proposition 2.6. *Under same hypotheses in Proposition 2.5, if there exists a constant $k_1 \in \mathbb{R}$, independent of t, such that f satisfies the dissipative condition*

$$\limsup_{|x|\to\infty} \frac{|f(t,x)|}{|x|} < k_1. \tag{2.11}$$

Then problem (2.1) *has a unique globally defined solution for any initial condition in X_p, which is given for $t \geq \tau$, by the "variation of constants formula"*

$$u(t,x) = \begin{cases} e^{-(A(t)-A(\tau))}u_\tau(x) + \int_\tau^t e^{-(A(t)-A(s))} b(s)Kf(s,u(s,\cdot))(x)\,ds \\ + \int_\tau^t e^{-(A(t)-A(s))}[-h+S(s,x)]ds, & x\in\Omega, \\ 0, & x\in\Omega^c, \end{cases} \tag{2.12}$$

where $A(\xi) = \int_0^\xi a(\eta)d\eta$, for any $\xi \geq \tau$, and $\Omega^c = \mathbb{R}^N \backslash \Omega$.

Proof. The existence and uniqueness of local solutions for (2.1), in X_p, follow from Proposition 2.5 and the well-known results in [12]. The variation of constants formula (2.12) can be easily verified by direct derivation. Now, using condition (2.11), it follows that

$$|f(t,x)| \leq k_2(t) + k_1|x|, \quad \text{for any } (t,x)\in\mathbb{R}\times\mathbb{R}^N, \tag{2.13}$$

for some continuous and strictly positive function $k_2 : \mathbb{R} \to \mathbb{R}$.

If $1 \leq p < \infty$, using (2.5) and (2.13), we obtain the estimate

$$\|Kf(t,u)\|_{L^p(\Omega)} \leq \|f(t,u)\|_{L^p(\Omega)} \leq k_2(t)|\Omega|^{1/p} + k_1\|u\|_{L^p(\Omega)}.$$

For $p = \infty$, using the same arguments (or passing to the limit $p \to \infty$ in the previous inequality), we have

$$\|Kf(t,u)\|_{L^\infty(\Omega)} \leq k_2(t) + k_1\|u\|_{L^\infty(\Omega)}\,.$$

Now defining the function $g : [t_0,\infty)\times\mathbb{R}^+ \to \mathbb{R}^+$ by

$$g(t,r) = b_0|\Omega|^{1/p}k_2(t) + \|S\|_p + h|\Omega|^{1/p} + (k_1+a_0)r,$$

it follows that problem (2.1) satisfies the hypothesis of [24, Theorem 5.6.1] and the existence of a global solution follows immediately. □

2.2. **Smoothness of the evolution process.** In this subsection we show that problem (1.2) generates a $\mathcal{C}^1$ flow in the phase space X_p.

Proposition 2.7. *Assume the same hypotheses of Proposition 2.6 hold and that the function f is continuously differentiable with respect to the second variable and $\partial_2 f$ satisfies the growth condition*

$$|\partial_2 f(t,x)| \leq C_1(t)(1+|x|^{p-1}), \quad \textit{for any } (t,x)\in\mathbb{R}\times\mathbb{R}^N, \tag{2.14}$$

for $1 \le p < \infty$. Then $F(t,\cdot)$ is continuously Frechét differentiable on X_p with derivative

$$DF(t,u)v(x) = \begin{cases} -a(t)v(x) + b(t)K(\partial_2 f(t,u)v)(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^N \backslash \Omega. \end{cases}$$

Proof. Using that f is continuously differentiable in the second variable, by a simple computation, it follows that the Gateaux's derivative of $F(t,\cdot)$ is

$$DF(t,u)v(x) := \begin{cases} -a(t)v(x) + b(t)K(\partial_2 f(t,u)v)(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^N \backslash \Omega, \end{cases}$$

where $(\partial_2 f(t,u)v)(x) := \partial_2 f(t,u(x)) \cdot v(x)$. Note that the operator $D_2F(t,u)$ is a linear operator on X_p.

Let $u \in L^p(\Omega)$, with $1 \le p < \infty$. Then, if q is the conjugate exponent of p, it is easy to see that

$$\|\partial_2 f(t,u)\|_{L^q(\Omega)} \le C_1(t)\big(|\Omega|^{1/q} + \|u\|^{p-1}_{L^p(\Omega)}\big). \tag{2.15}$$

From this estimate and Hölder's inequality, it follows that

$$\|\partial_2 f(t,u) \cdot v\|_{L^1(\Omega)} \le C_1(t)(|\Omega|^{1/q} + \|u\|^{p-1}_{L^p(\Omega)})\|v\|_{L^p(\Omega)}.$$

Hence, by estimate (2.6), we conclude that

$$\begin{aligned} \|DF(t,u)\cdot v\|_{L^p(\Omega)} &\le \|a_0 v\|_{L^p(\Omega)} + b_0\|K(\partial_2 f(t,u)v)\|_{L^p(\Omega)} \\ &\le \|a_0 v\|_{L^p(\Omega)} + b_0 C_1(t)\|J\|_p \|\partial_2 f(t,u)v\|_{L^1(\Omega)} \\ &\le \|a_0 v\|_{L^p(\Omega)} + b_0 C_1(t)\|J\|_p\big(|\Omega|^{1/q} + \|u\|^{p-1}_{L^p(\Omega)}\big)\|v\|_{L^p(\Omega)} \\ &= [a_0 + b_0 C_1(t)\|J\|_p\big(|\Omega|^{1/q} + \|u\|^{p-1}_{L^p(\Omega)}\big)]\|v\|_{L^p(\Omega)}, \end{aligned}$$

that is, $DF(t,u)$ is a bounded operator. In the case $p = \infty$, it follows that for each $u \in L^\infty(\Omega)$, $|\partial_2 f(t,u)|$ is bounded by $C_2(t)$. Hence

$$\|\partial_2 f(t,u)v\|_{L^\infty(\Omega)} \le C_2(t)\|v\|_{L^\infty(\Omega)}.$$

Thus

$$\begin{aligned} \|DF(t,u)\cdot v\|_{L^\infty(\Omega)} &\le a_0\|v\|_{L^\infty} + b_0\|K(\partial_2 f(t,u)v)\|_{L^\infty(\Omega)} \\ &\le a_0\|v\|_{L^\infty} + b_0\|J\|_1\|\partial_2 f(t,u)v\|_{L^\infty(\Omega)} \\ &\le a_0\|v\|_{L^\infty} + b_0 C_2(t)\|J\|_1\|v\|_{L^\infty(\Omega)} \\ &= (a_0 + b_0 C_2(t)\|J\|_1)\|v\|_{L^\infty(\Omega)}, \end{aligned}$$

which results, also in this case, in the boundedness of $DF(t,u)$.

Now, suppose that u_1, u_2 and v belong to $L^p(\Omega)$, $1 \le p < \infty$. Using (2.6) and Hölder's inequality it follows that

$$\begin{aligned} \|(DF(t,u_1) - DF(t,u_2))v\|_{L^p(\Omega)} &\le b_0\|K[(\partial_2 f(t,u_1) - \partial_2 f(t,u_2))v]\|_{L^p(\Omega)} \\ &\le b_0\|J\|_p\|(\partial_2 f(t,u_1) - \partial_2 f(t,u_2))v\|_{L^1(\Omega)} \\ &\le b_0\|J\|_p\|\partial_2 f(t,u_1) - \partial_2 f(t,u_2)\|_{L^q(\Omega)}\|v\|_{L^p(\Omega)} \\ &= b_0\|J\|_p\|\partial_2 f(t,u_1) - \partial_2 f(t,u_2)\|_{L^q(\Omega)}\|v\|_{L^p(\Omega)}\,. \end{aligned}$$

Then to prove continuity of the derivative, $DF(t,\cdot)$, it is sufficient to show that

$$\|\partial_2 f(t,u_1) - \partial_2 f(t,u_2)\|_{L^q(\Omega)} \to 0$$

as $\|u_1 - u_2\|_{L^p(\Omega)} \to 0$. On the other hand, by (2.14), it follows that

$$|\partial_2 f(t,u_1)(x) - \partial_2 f(t,u_2)(x)|^q \le [C_1(t)(2 + |u_1(x)|^{p-1} + |u_2(x)|^{p-1})]^q.$$

A simple computation shows that the right-hand-side of this inequality is integrable. Then the result follows from Lebesgue convergence theorem.

In the case $p = \infty$, the continuity of DF follows from (2.5) and from the continuity of $\partial_2 f(t,u)$. Therefore, it follows from [27, Proposition 2.8] that $F(t,\cdot)$ is Fréchet differentiable with continuous derivative in X_p. $\square$

Thanks to Proposition 2.7 and well known results in [12, 20], we have the following result.

Corollary 2.8. *Assume the hypotheses of Proposition 2.7 hold. Then, for each $t \in \mathbb{R}$ and $u_\tau \in X_p$, the unique solution of* (2.1) *with initial condition u_τ exists for all $t \ge \tau$, and the solution $(t,\tau,x) \mapsto u(t,x) = u(t;\tau,x,u_\tau)$ (defined by* (2.12)*) gives rise to a family of nonlinear $\mathcal{C}^1$ process on X_p, given by*

$$T(t,\tau)u_\tau(x) := u(t,x),\ t \ge \tau \in \mathbb{R}.$$

3. Existence of a pullback attractor

In this section we prove the existence of a pullback attractor $\{\mathcal{A}(t); t \in \mathbb{R}\}$ in X_p for the evolution process $\{T(t,\tau); t \ge \tau, \tau \in \mathbb{R}\}$ for $1 \le p < \infty$, generalizing, among others, [17, Theorem 3.2] and [5, Theorem 4.2].

Lemma 3.1. *Assume that the hypotheses of Proposition 2.7 hold with the constant k_1 in* (2.11) *satisfying $k_1 b_0 < a_-$. Let*

$$R_\delta(t) = \frac{1}{a_- - k_1 b_0}(1+\delta)[b_0 k_2(t)|\Omega|^{1/p} + \|S(t,\cdot)\|_{L^p(\Omega)}], \tag{3.1}$$

where k_2 is derived from (2.13) *and δ is any positive constant. Then the ball, centered at the origin with radius $R_\delta(t)$, in the space $L^p(\Omega)$, $1 \le p < \infty$, which we denote by $\mathcal{B}(0, R_\delta(t))$, pullback absorbs bounded subsets of X_p at time $t \in \mathbb{R}$ with respect to the process $T(\cdot,\cdot)$ generated by* (2.1).

Proof. If $u(t,x)$ is the solution of (2.1) with initial condition $u_\tau \in X_p$, for $1 \le p < \infty$, then

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |u(t,x)|^p dx \\
&= \int_\Omega p|u(t,x)|^{p-1}\operatorname{sgn}(u(t,x))u_t(t,x)dx \\
&= -pa(t)\int_\Omega |u(t,x)|^p dx + pb(t)\int_\Omega |u(t,x)|^{p-1}\operatorname{sgn}(u(t,x))Kf(t,u(t,x))dx \\
&\quad + p\int_\Omega |u(t,x)|^{p-1}\operatorname{sgn}(u(t,x))S(t,x)dx - ph\int_\Omega |u(t,x)|^{p-1}dx.
\end{aligned} \tag{3.2}$$

Thus, if q is the conjugate exponent of p, by Hölder's inequality, estimate (2.5), and condition (2.11), we have

$$\begin{aligned}
&\int_\Omega |u(t,x)|^{p-1}\operatorname{sgn}(u(t,x))Kf(t,u(t,x))\,dx \\
&\le \Big(\int_\Omega |u(t,x)|^{q(p-1)}dx\Big)^{1/q}\Big(\int_\Omega |Kf(t,u(t,x))|^p dx\Big)^{1/p} \\
&\le \Big(\int_\Omega |u(t,x)|^p dx\Big)^{1/q}\|J\|_1\|f(t,u(t,\cdot))\|_{L^p(\Omega)} \\
&\le \|u(t,\cdot)\|^{p-1}_{L^p(\Omega)}\big(k_1\|u(t,\cdot)\|_{L^p(\Omega)}+k_2(t)|\Omega|^{1/p}\big),
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\int_\Omega |u(t,x)|^{p-1}\operatorname{sgn}(u(t,x))S(t,x)\,dx \\
&\le \Big(\int_\Omega |u(t,x)|^{q(p-1)}dx\Big)^{1/q}\Big(\int_\Omega |S(t,x)|^p dx\Big)^{1/p} \\
&\le \Big(\int_\Omega |u(t,x)|^p dx\Big)^{1/q}\|S(t,\cdot)\|_{L^p(\Omega)} \\
&\le \|u(t,\cdot)\|^{p-1}_{L^p(\Omega)}\|S(t,\cdot)\|_{L^p(\Omega)}\,.
\end{aligned} \tag{3.4}$$

Hence, using (3.3) and (3.4) in (3.2), we obtain

$$\begin{aligned}
&\frac{d}{dt}\|u(t,\cdot)\|^p_{L^p(\Omega)} \\
&\le -pa(t)\|u(t,\cdot)\|^p_{L^p(\Omega)} + pb(t)\|u(t,\cdot)\|^{p-1}_{L^p(\Omega)}\big(k_1\|u(t,\cdot)\|_{L^p(\Omega)}+k_2(t)|\Omega|^{1/p}\big) \\
&\quad + p\|u(t,\cdot)\|^{p-1}_{L^p(\Omega)}\|S(t,\cdot)\|_{L^p(\Omega)} - ph|\Omega|^{1/p}\|u(t,\cdot)\|^{p-1}_{L^p(\Omega)}.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{d}{dt}\|u(t,\cdot)\|^p_{L^p(\Omega)} &\le -a_-p\|u(t,\cdot)\|^p_{L^p(\Omega)} \\
&\quad + pb_0\|u(t,\cdot)\|^{p-1}_{L^p(\Omega)}\Big(k_1\|u(t,\cdot)\|_{L^p(\Omega)}+k_2(t)|\Omega|^{1/p}\Big) \\
&\quad + p\|S(t,\cdot)\|_{L^p(\Omega)}\|u(t,\cdot)\|^{p-1}_{L^p(\Omega)} \\
&= -a_-p\|u(t,\cdot)\|^p_{L^p(\Omega)} + pb_0k_1\|u(t,\cdot)\|^p_{L^p(\Omega)} \\
&\quad + pb_0|\Omega|^{1/p}k_2(t)\|u(t,\cdot)\|^{p-1}_{L^p(\Omega)} + p\|S(t,\cdot)\|_{L^p(\Omega)}\|u(t,\cdot)\|^{p-1}_{L^p(\Omega)} \\
&= p\|u(t,\cdot)\|^p_{L^p(\Omega)}\big[-a_- + k_1b_0 + \frac{(b_0k_2(t)|\Omega|^{1/p}+\|S(t,\cdot)\|_{L^p(\Omega)})}{\|u(t,\cdot)\|^p_{L^p(\Omega)}}\big].
\end{aligned}$$

Writing $\varepsilon = a_- - k_1b_0 > 0$, since

$$\|u(t,\cdot)\|_{L^p(\Omega)} \ge \frac{1}{\varepsilon}(1+\delta)\big(b_0k_2(t)|\Omega|^{1/p}+\|S(t,\cdot)\|_{L^p(\Omega)}\big),$$

we obtain

$$\frac{d}{dt}\|u(t,\cdot)\|^p_{L^p(\Omega)} \le p\|u(t,\cdot)\|^p_{L^p(\Omega)}\big(-\varepsilon+\frac{\varepsilon}{1+\delta}\big) = -\frac{\delta p}{(1+\delta)}\varepsilon\|u(t,\cdot)\|^p_{L^p(\Omega)}.$$

Therefore,

$$\begin{aligned}\|u(t,\cdot)\|^p_{L^p(\Omega)} &\le e^{-\frac{\delta p}{(1+\delta)}\varepsilon(t-\tau)}\|u_\tau\|^p_{L^p(\Omega)} \\ &= e^{-\frac{\delta p}{(1+\delta)}(a_- - k_1 b_0)(t-\tau)}\|u_\tau\|^p_{L^p(\Omega)}.\end{aligned} \tag{3.5}$$

Thus, the result follows immediately. □

Theorem 3.2. *In addition to the conditions of Lemma 3.1, suppose that $C_1(t)$ and $k_2(t)$ are non-decreasing functions and*

$$\|J_x\|_{L^p(\Omega)} = \sup_{x\in\Omega}\|\partial_x J(x,\cdot)\|_{L^q(\Omega)} < \infty, \quad \|\partial_x S\|_p = \sup_{t\in\mathbb{R}_+}\|\partial_x S(t,\cdot)\|_{L^p(\Omega)} < \infty.$$

Then there exists a pullback attractor $\{\mathcal{A}(t); t\in\mathbb{R}\}$ for the process $\{T(t,\tau); t\ge\tau, \tau\in\mathbb{R}\}$ generated by (2.1) *in $X_p = L^p(\Omega)$ and the "section" $\mathcal{A}(t)$ of the pullback attractor $\mathcal{A}(\cdot)$ of $T(\cdot,\cdot)$ is contained in the ball centered at the origin with radius $R_\delta(t)$ defined in* (3.1)*, in $L^p(\Omega)$, for any $\delta>0$, $t\in\mathbb{R}$ and $1\le p<\infty$.*

Proof. From Theorem 2.6 it follows that, for each initial value $u(\tau,\cdot)\in X_p$ and initial time $\tau\in\mathbb{R}$, the process generated by (2.1) has a unique solution, which we can to write, for $x\in\Omega$, as

$$T(t,\tau)u(\tau,x) = T_1(t,\tau)u(\tau,x) + T_2(t,\tau)u(\tau,x),$$

where

$$\begin{gathered}T_1(t,\tau)u(\tau,x) := e^{-(A(t)-A(\tau))}u(\tau,x),\\ T_2(t,\tau)u(\tau,x) := \int_\tau^t e^{-(A(t)-A(s))}b(s)[Kf(s,u(s,x)) + S(s,x) - h]ds.\end{gathered}$$

Now, we use [8, Theorem 2.37] to prove that $T(\cdot,\cdot)$ is pullback asymptotically compact. For this, let $u\in B$ be a bounded subset of X_p. Without loss of generality, we suppose that B is contained in the ball centered at the origin of radius $r>0$. Then, for $t\ge\tau$, we have

$$\|T_1(t,\tau)u\|_{L^p(\Omega)} \le re^{-(A(t)-A(\tau))} \le re^{-a_- t}e^{a_0\tau} = \sigma(t,\tau)\to 0, \quad t\to\infty.$$

Using (3.5), it follows that $\|u(t,\cdot)\|_{L^p(\Omega)}\le M$, for $t\ge\tau$, where M is given in (3.6) below

$$M = M(t) = \max\Big\{r, \frac{2[b_0 k_2(t)|\Omega|^{1/p} + \|S(t,\cdot)\|_{L^p(\Omega)}]}{a_- - k_1 b_0}\Big\} > 0. \tag{3.6}$$

Then, using (2.8), we have

$$\|f(t,u)\|_{L^1(\Omega)} \le C_1(t)(|\Omega| + \|u\|^p_{L^p(\Omega)}) \le C_1(t)(|\Omega| + M(t)^p).$$

Since

$$\partial_x(T_2(t,\tau)u(\tau,x)) = \int_\tau^t e^{-(A(t)-A(s))}[b(s)\frac{\partial}{\partial x}Kf(t,u)(t,x) + \frac{\partial S}{\partial x}(s,x)]ds.$$

proceeding as in (2.6) (with J_x replacing J) and using (2.8), it follows that

$$\|\partial_x(Kf(t,u))\|_{L^p(\Omega)} \le \|J_x\|_{L^p(\Omega)}b_0\|f(t,u)\|_{L^1(\Omega)} \le C_1(t)\|J_x\|_{L^p(\Omega)}(|\Omega| + M(t)^p).$$

Thus, since C_1 and k_2 are non-decreasing, we obtain

$$\begin{aligned}&\|\partial_x(T_2(t,\tau)u)\|_{L^p(\Omega)}\\ &\le \int_\tau^t e^{-(A(t)-A(s))}\Big(b(s)\|\partial_x Kf(s,u(s,\cdot))\|_{L^p(\Omega)} + \|\partial_x S(s,\cdot)\|_{L^p(\Omega)}\Big)ds\end{aligned}$$

$$
\begin{aligned}
&\le \int_\tau^t e^{-(A(t)-A(s))}\Big(b_0 C_1(s)\|J_x\|_{L^p(\Omega)}|\Omega|^{1/p} + M(s)^p + \|\partial_x S\|_p\Big)\,ds \\
&\le \int_\tau^t e^{-(A(t)-A(s))}\Big(b_0 C_1(t)\|J_x\|_{L^p(\Omega)}|\Omega|^{1/p} + M(t)^p + \|\partial_x S\|_p\Big)\,ds \\
&\le C_1(t)\|J_x\|_p \frac{1}{a_0}[e^{(a_0-a_-)t} - e^{-a_- t}e^{a_0\tau}]\big(|\Omega|^{1/p} + M(t)^p\big) \\
&\quad + C_1(t)\frac{1}{a_0}[e^{(a_0-a_-)t} - e^{-a_- t}e^{a_0\tau}]\|\partial_x S\|_p \\
&\le C_1(t)\|J_x\|_p \frac{1}{a_0}e^{(a_0-a_-)t}(|\Omega| + M(t)^p) + C_1(t)\frac{1}{a_0}e^{(a_0-a_-)t}\|\partial_x S\|_p \\
&= \frac{C_1(t)\|J_x\|_p(|\Omega| + M(t)^p) + \|\partial_x S\|_p}{a_0}e^{(a_0-a_-)t}.
\end{aligned}
$$

Hence, for any $u \in B$ and $t > \tau$, the value of $\|\frac{\partial}{\partial x}T_2(t,\tau)u\|_{L^p(\Omega)}$ is bounded by a constant (independent of $u \in B$). Then $T_2(t,\tau)u$ belongs to a ball in the space $W^{1,p}(\Omega)$ for all $u \in B$. Hence, by the Sobolev embedding theorem, it follows that $T_2(t,\tau)$ is a compact operator, for any $t > \tau$.

Therefore, using Lemma 3.1 and [8], there exists the pullback attractor $\{\mathcal{A}(t); t \in \mathbb{R}\}$ and each "section" $\mathcal{A}(t)$ of the pullback attractor $\mathcal{A}(\cdot)$ is the pullback ω-limit set of any bounded subset of X_p containing the ball centered at the origin with radius R_δ, given in (3.1), for any $\delta > 0$. Since the ball centered at the origin with radius R_δ pullback absorbs bounded subsets of X_p, it also follows that the set $\mathcal{A}(t)$ is contained in the ball centered at the origin of X_p and of radius

$$
R(t) = \frac{1}{a_- - k_1 b_0}[b_0 k_2(t)|\Omega|^{1/p} + \|S\|_p]
$$

for any $t \in \mathbb{R}$ and $1 \le p < \infty$. □

4. Continuity with respect to parameter S

A natural question to examine at this point is the depedence of the process with respect to parameters that arise in the equation. In this section we prove the continuity of the process with respect to a external stimuli function and we use this result to prove the upper semicontinuity of the pullback attractors.

4.1. **Continuity of the process with respect to external stimuli.**

From now on we denote by $T_S(t,\tau)$ the family of processes associated with the family of problems

$$
\begin{gathered}
\partial_t u_S(t,x) = -a(t)u_S(t,x) + b(t)Kf(t,u_S(t,x)) + S(t,x),\ t \ge \tau, \quad x \in \Omega, \\
u_S(\tau,x) = u_\tau(x), \quad x \in \Omega, \\
u_S(t,x) = 0, \quad t \ge \tau,\ x \in \mathbb{R}^N\backslash\Omega.
\end{gathered} \tag{4.1}
$$

In this subsection we prove the continuous dependence of the process with respect to the stimuli function S at $S_0 \in \Sigma$, where

$$
\Sigma = \{S : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R},\ \|S\|_p = \sup_{t\in\mathbb{R}_+}\|S(t,\cdot)\|_{L^p(\Omega)} < \infty\}.
$$

More precisely we have the following result.

Theorem 4.1. *In addition to the hypotheses of Theorem 3.2, suppose that the function C_2 given in (2.9) is non-decreasing. Then, if $T_S(\cdot,\cdot)$ denotes the process generated by the problem (4.1), for $S \in \Sigma$, we have that $T_S(t,\tau)u_\tau$ converges uniformly to $T_{S_0}(t,\tau)u_\tau$ in X_p, as $\|S - S_0\|_p \to 0$, for $t \in [\tau, L]$, and any $L > \tau$.*

Proof. Let $L > \tau$ and $u_S(t,x) = T_S(t,\tau)u_\tau(x)$ be the solution of (4.1) for $t \in [\tau, L]$, given by (2.12). Then, for $x \in \Omega$, and

$$\begin{aligned} u_S(t,x) - u_{S_0}(t,x) &= \int_\tau^t e^{-(A(t)-A(s))} b(s)[K(f(s,u_S(s,x)) - f(s,u_{S_0}(s,x)))]ds \\ &\quad + \int_\tau^t e^{-(A(t)-A(s))}[S(s,x) - S_0(s,x)]ds \end{aligned}$$

Thus, for $x \in \Omega$, using (2.6), we obtain

$$\begin{aligned} &\|u_S(t,\cdot) - u_{S_0}(t,\cdot)\|_{L^p(\Omega)} \\ &\le \int_\tau^t e^{-(A(t)-A(s))} b_0 \|J\|_p \|f(s,u_S(s,\cdot)) - f(s,u_{S_0}(s,\cdot))\|_{L^1(\Omega)} ds \\ &\quad + \int_\tau^t e^{-(A(t)-A(s))} \|S(s,\cdot) - S_0(s,\cdot)\|_{L^p(\Omega)} ds. \end{aligned}$$

By (2.9) it follows that

$$\begin{aligned} \|u_S(t,\cdot) - u_{S_0}(t,\cdot)\|_{L^p(\Omega)} &\le \int_\tau^t e^{-(A(t)-A(s))} b_0 \|J\|_p C_2(s) \Big[|\Omega|^{1/q} + \|u_S(s,\cdot)\|_{L^p(\Omega)}^{p/q} \\ &\quad + \|u_{S_0}(s,\cdot)\|_{L^p(\Omega)}^{p/q}\Big] \|u_S(s,\cdot) - u_{S_0}(s,\cdot)\|_{L^p(\Omega)} ds \\ &\quad + \int_\tau^t e^{-(A(t)-A(s))} \sup_{s\in\mathbb{R}} \|S(s,\cdot) - S_0(s,\cdot)\|_{L^p(\Omega)} ds. \end{aligned}$$

Let $B \subset X_p$ be a bounded subset (for example a ball of radius ρ) such that $u_S(t,\cdot) \in B$ for all $S \in \Sigma$ and $t \in [\tau, L]$. Then

$$\begin{aligned} &e^{A(t)} \|u_S(t,\cdot) - u_{S_0}(t,\cdot)\|_{L^p(\Omega)} \\ &\le \int_\tau^t b_0 \|J\|_p C_2(s) \big[|\Omega|^{1/q} + 2\rho^{p/q}\big] e^{A(s)} \|u_S(s,\cdot) - u_{S_0}(s,\cdot)\|_{L^p(\Omega)} ds \\ &\quad + \int_\tau^t e^{A(s)} \|S - S_0\|_p ds. \end{aligned}$$

Using the Gronwall Generalized inequality [19], we obtain

$$e^{A(t)} \|u_S(t,\cdot) - u_{S_0}(t,\cdot)\|_{L^p(\Omega)} \le \Big(\int_\tau^t e^{A(s)} \|S - S_0\|_p ds\Big) e^{\int_\tau^t b_0 \|J\|_p C_2(s)[|\Omega|^{1/q} + 2\rho^{p/q}]ds}.$$

Hence, for $t \in [\tau, L]$, it follows that

$$\begin{aligned} &\|u_S(t,\cdot) - u_{S_0}(t,\cdot)\|_{L^p(\Omega)} \\ &\le \Big(\int_\tau^t e^{-(A(t)-A(s))} \|S - S_0\|_p ds\Big) e^{\int_\tau^t b_0 \|J\|_p C_2(s)[|\Omega|^{1/q} + 2\rho^{p/q}]ds} \\ &\le \frac{e^{(a_0 - a_-)t}}{a_0} e^{\int_\tau^t b_0 \|J\|_p C_2(s)[|\Omega|^{1/q} + 2\rho^{p/q}]ds} \|S - S_0\|_p. \end{aligned}$$

The result follows. ◻

4.2. **Upper semicontinuity of the pullback attractors.** In this subsection $\{\mathcal{A}_S(t); t \in \mathbb{R}\}$ denotes the pullback attractor for the process $T_S(\cdot,\cdot)$ in X_p, for $1 \le p < \infty$. Using Theorem 4.1, we prove that the family of pullback attractors $\{\mathcal{A}_S(t); t \in \mathbb{R}\}_{S\in\Sigma}$ is upper-semicontinuous at $S_0 \in \Sigma$, i.e., we show that

$$\lim_{t\to\infty} \operatorname{dist}_H(\mathcal{A}_S(t), \mathcal{A}_{S_0}(t)) = 0,$$

where $\operatorname{dist}_H(\cdot,\cdot)$ denotes the Hausdorff semi-distance.

Theorem 4.2. *Under the hypotheses of Theorem 4.1 the family of pullback attractors $\{\mathcal{A}_S(t); t \in \mathbb{R}\}_{S\in\Sigma}$ is upper semicontinuous at $S_0 \in \Sigma$.*

Proof. Note that, from Theorem 3.2, it follows that

$$\cup_{S\in\Sigma}\mathcal{A}_S(t) \subset B(0,R),$$

where $R = R(t) = \frac{1}{a_- - k_1 b_0}[b_0 k_2(t)|\Omega|^{1/p} + p\|S\|_p]$. Let us fix $\varepsilon > 0$ and $t \in \mathbb{R}$. Thus choose $\tau \in \mathbb{R}$, $\tau \le t$, such that

$$\operatorname{dist}_H(T_{S_0}(t,\tau)B(0,R), \mathcal{A}_{S_0}(t)) < \frac{\varepsilon}{2}.$$

Now, by Theorem 4.1, it follows that there exists $\delta > 0$ such that, for $\|S - S_0\|_p < \delta$, we have

$$\sup_{a_S\in\mathcal{A}_S(\tau)} \operatorname{dist}(T_S(t,\tau)a_S, T_{S_0}(t,\tau)a_S) < \frac{\varepsilon}{2}.$$

Then, for $\|S - S_0\|_p < \delta$, using the invariance of the pullback attractors, we obtain

$$\begin{aligned}
&\operatorname{dist}_H(\mathcal{A}_S(t), \mathcal{A}_{S_0}(t)) \\
&\le \operatorname{dist}_H(T_S(t,\tau)\mathcal{A}_S(\tau), T_{S_0}(t,\tau)\mathcal{A}_S(\tau)) + \operatorname{dist}_H(T_{S_0}(t,\tau)\mathcal{A}_S(\tau), T_{S_0}(t,\tau)\mathcal{A}_{S_0}(\tau)) \\
&= \sup_{a_S\in\mathcal{A}_S(\tau)} \operatorname{dist}_H(T_S(t,\tau)a_S, T_{S_0}(t,\tau)a_S) + \operatorname{dist}_H(T_{S_0}(t,\tau)\mathcal{A}_S(\tau), \mathcal{A}_{S_0}(t)) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

□

5. Discussions and biological interpretation

As we saw in the introduction, equation (1.1) generalizes the model studied in [1], which is already well known in the literature, because we consider that the rate in the intensity of neuronal potential is explicitly time dependent, while in [1] this rate was considered constant. We expect to have a more realistic model when compared to what happens in the brain, since this behavior is due to variations of polarity inside the membrane, which is not necessarily constant. Furthermore, in Proposition 2.6 and Corollary 2.8, we are not considering that the synaptic connectivity function $J(x,y)$ is smooth, as occurs for example in [1, 2, 5, 13]. For these results, we assume $J \in L^1(\mathbb{R}^N)$, leaving the model closer to real situation of mild autism, where simple breaks in the synaptic connections occurs. Thus, we hope that the results on global existence and smoothness of solutions, given in Proposition 2.6 and Corollary 2.8 contribute to future research.

In Theorem 4.1 we show that the neuronal activity depends continuously on the sum of the external stimuli involved in the process. This reinforces the importance of appropriate continuous stimulation for a good neural activity, especially in individuals suffering from neurological disorder, as occurs in cases of cerebral paralysis and in some cases of autism.

Finally, we expect that the mathematical results presented in Theorem 3.2 and Theorem 4.2 will contribute to other mathematical properties associated with the dynamics of this model and that other biological conclusions may be possible.

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References

[1] Amari, S.; *Dynamics of pattern formation in lateral-inhibition type neural fields*, Biol. Cybernetics, **27** (1977), 77–87.

[2] Amari, S.; *Dynamics stability of formation of cortical maps*, In: M. A. Arbib and S. I. Amari (Eds), Dynamic Interactions in Neural Networks: Models and Data, Springer-Verlag, New York, (1989) 15–34.

[3] Barakova, E. I. B.; *Timing sensory integration for robot simulation of autistic behavior*, Robotics and automation Magazine, **16** (2012), 1-8,

[4] Beurle, R. L.; *Properties of a mass of cells capable of regenerating pulses*, Philosophical Transactions of the Royal Society London B, **240** (1956), 55–94.

[5] Bezerra, F. D.; Pereira, A. L.; Da Silva, S. H.; *Existence, regularity and upper semicontinuity of pullback attractors for the evolution process associated to a neural field model*, Electronic Journal of Qualitative Theory of Differential equations, **2017** (2017), 1-18.

[6] Brezis H.; *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.

[7] Carroll, S. R.; Bressloff, P.; *Symmetric Bifurcations in a Neural Field Model for Encoding the Direction of Spatial Contrast Gradients*, *Journal on Applied Dynamical Systems*, **17** no. 1 (2018), 1–51.

[8] Carvalho, A. N.; Langa, J. A.; Robinson, J. C.; *Attractors for infinite-dimensional non-autonomous dynamical systems.* Applied Mathematical Sciences 182. Springer-Verlag, New York, 2012.

[9] Castillo, S.; Pinto, M.; Torres, R.; *Asymptotic formulae for solutions to impulsive differential equations with piecewise constant argument of generalized type*, Electron. J. Differential Equations, **2019** no. 40 (2019), 1-22.

[10] Chepyzhov, V. V.; Vishik, M. I.; *Attractors for Equations of Mathematical Physics*, in: Colloquium Publications, **49**, America Mathematical Society, 2002.

[11] Coombes, S.; Lord, G. J.; Owen, M. R.; *Waves and bumps in neuronal networks with axo-dendritic synaptic interactions.* Physica D, **178** (2003), 219–241.

[12] Daleckii, J. L.; Krein, M. G.; *Stability of Solutions of Differential Equations in Banach Spaces.* American Mathematical Society, Providence, Rhode Island, 1974.

[13] Da Silva, S. H.; Pereira, A. L.; *Global attractors for neural fields in a weighted space.* Matemática Contemporanea, **36** (2009), 139–153.

[14] Da Silva, S. H.; *Existence and upper semicontinuity of global attractors for neural fields in an unbounded domain.* Electronic Journal of Differential Equations, **2010** no. 138 (2010), 1–12.

[15] Da Silva, S. H.; *Existence and upper semicontinuity of global attractors for neural network in a bounded domain.* Differential Equations and Dynamical Systems, **19** no. 1-2 (2011), 87–96.

[16] Da Silva, S. H.; *Properties of an equation for neural fields ina bounded domain.* Electronic Journal of Differential Equations, **2012** no. 42 (2012), 1–9.

[17] Da Silva, S. H.; Pereira, A. L.; *Asymptotic behavior for a nonlocal model of neural fields.* São Paulo Journal of Math. Sci., **9** (2015), 181–194.

[18] Ermentrout, G. B.; Jalics, J. Z. Rubin, J. E.; *Stimulus-driven travelling solutions in continuum neuronal models with general smooth firing rate functions.* SIAM, J. Appl. Math., **70** (2010), 3039–3064.

[19] Hale, J. K.; *Ordinary differential equations.* Pure and Applied Mathematics. A Series of Texts and monographsLecture, V. XXI, Krieger Publishing Company, Florida, 1981.

[20] Henry, D.; *Geometric Theory of Semilinear Parabolic Equations.* Lecture Notes in Mathematics n. 840, Springer-Verlag, 1981.

[21] Kishimoto, K.; Amari, S.; *Existence and stability of local excitations in homogeneous neural fields*, Journal Mathematical Biology, **7** (1979), 303–318.

[22] Kloeden, P. E.; Schmalfuß, B.; *Asymptotic behaviour of non-autonomous difference inclusions*, Systems Control Lett., **33**, (1998) 275–280.

[23] Kloeden, P. E.; *Pullback attractors in nonautonomous difference equations.* J. Differ. Equations Appl., **6** no. 1 (2000), 33–52.

[24] Ladas, G. E.; Lakshmikantham, V.; *Differential equations in abstract spaces*, Academic Press, New York, 1972.

[25] Laing, C. R.; Troy, W. C.; Gutkin, B.; Ermentrout, G. B.; *Multiple Bumps in a neuronal model of Working Memory.* SIAM Journal on Applied Mathematics, **63** no. 1 (2002), 206–225.

[26] Pinto, D. J.; Ermentrout, G. B.; *Spatially structured activity in synaptically coupled neuronal networks: I traveling fronts and pulses.* SIAM Journal on Applied Mathematics, **62** (2001), 206–225.

[27] Rall, L. B.; *Nonlinear Functional Analysis and Applications.* Academic Press, New York-London, 1971.

[28] Rukatmatakul, S.; Yimprayoon, P.; *Traveling wave front solutions in lateral-excitatory neuronal networks.* Songklanakarin J. Sci. Tecnol., **30** no. 3 (2008), 313–321.

[29] Sandamirskaya, Y.; *Dynamic neural fields as a step toward cognitive neuromorphic architectures.* Frontiers in Neuroscience, **30** no. 3, (2008), 313–321.

[30] Sell, G. R.; *Non-autonomous differential equations and dynamical systems*, Trans. Amer. Math. Soc., **127** (1967), 241–283.

[31] Thelen, E.; Schoner, G.; Scheier, C.; Smith, L. B.; *The dynamics of embodiment: a field theory of infant perseverative reaching,* Behavioral and Brain Sciences, **24** (2001), 1–34.

[32] Wilson, H. R.; Cowan, J. D.; *Excitatory and inhibitory interactions in localized populations of model neurons.* Biophys. J., **12** (1972), 1–24.

[33] Zhang, L.; *Existence, uniqueness and exponential stability of traveling wave solutions of some integral differential equations arising from neuronal networks*, Journal of Differential Equations, **197** (2004), 162–196.

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