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EXISTENCE OF SOLUTION FOR A SEGMENTATION APPROACH TO THE IMPEDANCE TOMOGRAPHY PROBLEM

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Abstract. In electrical impedance tomography (EIT), image reconstruction of the conductivity distribution of a body can be calculated using measured voltages at the boundary. This is done by solving an inverse problem for an elliptic partial differential equation (PDE). In this work, we present some sensitivity results arising from the solution of the PDE. We use these to show that a segmentation approach to the EIT inverse problem has a unique solution in a suitable space using a fixed point theorem.

1. Introduction

Electrical impedance tomography (EIT) is an imaging technique proposed by Calderon [6] in recovering the spatial distribution of the conductivities in the interior of a body Ω based on the voltage and current measurements from electrodes placed around its boundary $\partial\Omega$. EIT is a non-invasive imaging technique with a wide range of applications. We can refer to the following works [12, 13, 22, 23, 24, 29, 30, 38].

The EIT consists of two sub-problems: the forward problem and the inverse problem. Suppose $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary. In the forward EIT problem, given the boundary currents $f \in L^2(\partial\Omega)$ and the conductivity distribution $\sigma \in L^\infty(\Omega)$ satisfying $\sigma(x) \geq \underline{\sigma} > 0$, for all $x \in \Omega$, the electric potential ϕ in Ω and the boundary voltage $V = \phi\big|_{\partial\Omega}$ are solved. These electrical measurements satisfy a generalized Laplace equation:

$$
\begin{gathered}
\nabla \cdot (\sigma \nabla \phi) = 0 \quad \text{in } \Omega, \\
\sigma \tfrac{\partial \phi}{\partial n} = f \quad \text{on } \partial\Omega,
\end{gathered}
\tag{1.1}
$$

where n is the outward normal direction at $\partial\Omega$. The boundary currents are chosen so that $\int_{\partial\Omega} f\, dS = 0$. This condition is imposed to satisfy the conservation of charge. Furthermore, the electric potential ϕ must satisfy $\int_{\partial\Omega} \phi\, dS = 0$. This amounts to choosing the reference voltage. The equation (1.1) can be viewed as a generalized Ohm's law and is a well-posed boundary value problem with a unique solution (up to a constant) $\phi \in H^1(\Omega)$. The partial differential equation (PDE) (1.1) is called the continuum model of EIT. We focus our study on this model. Other EIT models are discussed in [5, 8, 34].

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The inverse EIT problem or the conductivity reconstruction problem is the recovery of σ inside Ω given V and f in $\partial\Omega$. Denote

$$\tilde{L}^2(\partial\Omega) := \{f \in L^2(\partial\Omega) : \int_{\partial\Omega} f\,dS = 0\}$$

and define $\Lambda_\sigma : \tilde{L}^2(\partial\Omega) \to \tilde{L}^2(\partial\Omega)$ by

$$\Lambda_\sigma(f) = \phi\big|_{\partial\Omega}, \tag{1.2}$$

where $\phi \in H^1(\Omega)$ satisfies (1.1) and $\int_{\partial\Omega} \phi\,dS = 0$. The inverse EIT problem is the recovery of σ given Λ_σ. Although the reconstruction problem is severely ill-posed, a unique solution exists. Physically, this makes sense but to show this mathematically is not trivial. For the discussion of this result, we refer the readers to [36, 35] for the case $n \geq 3$ and to [2, 5, 25] for the case $n = 2$.

Because of its ill-posedness, the inverse EIT problem is an active research area. Hence, several approaches have been proposed to solve this problem. Different techniques are discussed in [5, 8, 20, 28, 37]. In this work, we focus on a technique proposed by Mendoza and Keeling in [27]. We assume that the conductivity σ is piecewise constant. This assumption is based on the fact that the conductivities of healthy tissues show great contrast [3, 17]. By assuming that σ is piecewise-constant, the inverse problem is treated as a segmentation problem. A segmentation technique called *"Multi-Phase Segmentation"*, proposed by Fürtinger in [15], is explored in [27]. Moreover, it is assumed that the desired conductivity can be expressed in terms of M phases, i.e., of the form

$$\sigma(x) = \sum_{m=1}^{M} \sigma_m(x)\chi_m(x), \tag{1.3}$$

where for the mth phase χ_m is the characteristic function of a subdomain $\Omega_m \subset \Omega$ and σ_m is globally smooth. In [27], the number of phases is fixed to 2, hence the method is referred to as a two-phase segmentation approach. This is possible if the subdomain Ω_1 has disjoint non-adjacent components. The subdomains Ω_1 and Ω_2 form a disjoint partition of Ω, i.e., $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega = \Omega_1 \cup \Omega_2$. The conductivity σ_2 in Ω_2 is assumed to be known and $\chi_2 = 1 - \chi_1$. Therefore, the inverse EIT problem becomes a problem of identifying σ_1 and χ_1. It is shown in [27] that σ_1 can be expressed in terms of χ_1. Given an initial guess for χ_1, an iterative algorithm is proposed. The main goal of this paper is to show that this iterative process has a unique solution given an initial guess for χ_1.

In the next section, we briefly discuss the two-phase segmentation algorithm. Then an analysis of the algorithm is carried out. We show that the algorithm can be expressed as a fixed point iteration. Finally, the existence of a fixed point in a suitable space is presented.

2. Two-phase segmentation algorithm

Let us fix $f \in \tilde{L}^2(\partial\Omega)$ and define the function $F : L^2(\Omega) \to \tilde{L}(\partial\Omega)$ by

$$F(\sigma) = \phi\big|_{\partial\Omega}, \tag{2.1}$$

where ϕ is the solution of (1.1) given σ and f. Let $\sigma^\star$ be the actual conductivity distribution in Ω. The inverse problem is to recover $\sigma^\star$ from $\Lambda_{\sigma^\star}$. Suppose $f \in \tilde{L}^2(\partial\Omega)$ and let $V^\star = \Lambda_{\sigma^\star}(f) = F(\sigma^\star)$ be the exact boundary voltage. Moreover, let $\tilde{V} \approx V^\star$ be the measured boundary voltage. Let $\tilde{\sigma}_1$ be an estimate of σ_1. To solve the EIT inverse problem, our aim is to minimize

$$\tilde{J}(\sigma_1, \chi_1) = \int_{\partial\Omega} |F(\sigma) - \tilde{V}|^2 \, dS + \int_{\Omega} \alpha |\nabla \sigma_1|^2 (\chi_1 + \epsilon) + \lambda(\sigma_1 - \tilde{\sigma}_1)^2 \, dV \tag{2.2}$$

with $\sigma = \sigma_1\chi_1 + \sigma_2(1 - \chi_1)$ and σ_2 is given. The first term of the integral is the fidelity term, the second term provides smoothness on σ_1 on Ω_1 and the third term comes from Tikohonov regularization.

Before we proceed, we first need to define the *forward solution* and the *adjoint solution* of the EIT problem.

Definition 2.1. The *forward solution* ϕ is the solution of (1.1) given $f \in \tilde{L}^2(\partial\Omega)$ and $\sigma \in L^\infty(\Omega)$. Moreover, let $\tilde{V}$ be the measured boundary voltage. We define the *adjoint solution* ϕ^* as the solution of

$$\begin{gathered} \nabla \cdot (\sigma \nabla \phi^*) = 0 \quad \text{in } \Omega, \\ \sigma \frac{\partial \phi^*}{\partial n} = F(\sigma) - \tilde{V} \quad \text{on } \partial\Omega. \end{gathered} \tag{2.3}$$

Both ϕ and ϕ^* satisfy $\int_{\partial\Omega} \phi \, dS = 0$ and $\int_{\partial\Omega} \phi^* \, dS = 0$.

Using (1.1), (2.3), and (1.3), the variational formulations of the forward and adjoint problems are

$$\int_\Omega (\sigma_1\chi_1 + \sigma_2(1 - \chi_1))\nabla\phi \cdot \nabla v \, dV = \int_{\partial\Omega} f v \, dS, \tag{2.4}$$

$$\int_\Omega (\sigma_1\chi_1 + \sigma_2(1 - \chi_1)\nabla\phi^* \cdot \nabla v \, dV = \int_{\partial\Omega} (F(\sigma_1\chi_1 + \sigma_2(1 - \chi_1)) - \tilde{V}) v \, dS, \tag{2.5}$$

for all $v \in H^1(\Omega)$. Formulations (2.4) and (2.5) have unique solutions [27] in $\tilde{H}^1(\Omega) := \{v \in H^1(\Omega) | \int_{\partial\Omega} v \, dS = 0\}$. Because of (1.3), the forward and adjoint solutions ϕ and ϕ^* are dependent on σ_1 and χ_1 alone. Thus, we have the following definition.

Definition 2.2. We define $\Phi(\sigma_1, \chi_1)$ and $\Phi^*(\sigma_1, \chi_1)$ to be the operators that map any given $\sigma_1 \in L^\infty(\Omega)$ and characteristic function χ_1 to the respective solutions ϕ and ϕ^* of (2.4) and (2.5), respectively. Equivalently, $\Phi : (\sigma_1, \chi_1) \to \phi$ and $\Phi^* : (\sigma_1, \chi_1) \to \phi^*$.

The computation of the derivative of $\tilde{J}$ is necessary to express σ_1 in terms of χ_1. For a fixed χ_1, the variational derivative of $\tilde{J}$ in (2.2) with respect to $\sigma_1 \in H^1(\Omega)$ in the direction of $\delta\sigma_1 \in H^1(\Omega)$ is given by

$$\begin{aligned} \frac{\delta \tilde{J}}{\delta \sigma_1}(\sigma_1, \chi_1; \delta\sigma_1) = &- \int_\Omega 2\chi_1 \delta\sigma_1 \nabla\phi \cdot \nabla\phi^* \, dV + \int_\Omega 2\alpha(\chi_1 + \epsilon)\nabla(\delta\sigma_1) \cdot \nabla\sigma_1 \, dV \\ &+ \int_\Omega 2\lambda(\sigma_1 - \tilde{\sigma}_1)\delta\sigma_1 \, dV. \end{aligned}$$

If we equate the above expression to 0, we conclude that the following optimality condition must be satisfied,

$$\int_\Omega [\alpha(\chi_1 + \epsilon)\nabla\sigma_1 \cdot \nabla v + \lambda(\sigma_1 - \tilde{\sigma}_1)v] \, dV = \int_\Omega (\chi_1 \nabla\phi \cdot \nabla\phi^*) v \, dV, \tag{2.6}$$

for all $v \in H^1(\Omega)$. Given χ_1 and an estimate $\tilde{\sigma}_1$, the quantity σ_1 is calculated via (2.6). We formalize this in the following definition.

Definition 2.3. We define the operator $\Sigma_1 : \chi_1 \to \sigma_1$ that maps an element $\chi_1 \in L^\infty(\Omega)$ to an element $\sigma_1 \in H^1(\Omega)$ via (2.6) and the the operator $\Sigma : \chi_1 \to \sigma$ via $\sigma(\chi_1) = \Sigma_1(\chi_1)\chi_1 + \sigma_2(1-\chi_1)$.

Definitions 2.2 and 2.3 are used to replace $\tilde{\sigma}_1$ and σ_1 in (2.6) with σ_1^k and $\sigma_1^{k+1} = \Sigma_1(\chi_1)$, respectively. Hence,

$$\begin{aligned} &\int_\Omega \alpha(\chi_1+\epsilon)\nabla\Sigma_1(\chi_1)\cdot\nabla v\,dV + \int_\Omega \lambda(\Sigma_1(\chi_1)-\sigma_1^k)v\,dV \\ &= \int_\Omega \chi_1\nabla\Phi(\sigma_1^k,\chi_1)\cdot\nabla\Phi^*(\sigma_1^k,\chi_1)v\,dV. \end{aligned} \tag{2.7}$$

Furthermore, the following operators are defined for the global conductivity,

$$\Sigma^k(\chi_1) := \sigma_1^k\chi_1 + \sigma_2(1-\chi_1), \tag{2.8}$$

$$\Sigma^{k+1}(\chi_1) := \Sigma_1(\chi_1)\chi_1 + \sigma_2(1-\chi_1), \quad (\text{with } \Sigma_1(\chi_1) = \sigma_1^{k+1}). \tag{2.9}$$

Under some assumptions, we will show that (2.7) admits a unique solution (see Lemma 3.25). Observe that the functional $\tilde{J}(\sigma_1,\chi_1)$ in (2.2) can be written as a functional $\tilde{J}(\Sigma_1(\chi_1),\chi_1)$ depending only on χ_1. To determine χ_1, we add a Total Variation (TV) regularization to (2.2) to penalize oscillations. For discussions of TV-regularization, one can refer to [31, 32, 10, 21]. Given a (sufficiently smooth) function f, its total variation is given by

$$TV(f) := \int_\Omega |\nabla f|\,dV \approx \int_\Omega \sqrt{|\nabla f|^2+\beta^2}\,dV,$$

for some $0<\beta\ll 1$ (compare, e.g., [7, 11]). To determine the optimal χ_1, our aim is to minimize the TV-regularized functional

$$\begin{aligned} J(\chi_1) = &\int_{\partial\Omega} |F(\Sigma(\chi_1)) - \tilde{V}|^2\,dS + \int_\Omega \alpha|\nabla\Sigma_1(\chi_1)|^2(\chi_1+\epsilon) \\ &+ \int_\Omega \lambda(\Sigma_1(\chi_1)-\tilde{\sigma}_1)^2 + \gamma\sqrt{|\nabla\chi_1|^2+\beta^2}\,dV, \end{aligned} \tag{2.10}$$

for $\alpha,\lambda,\gamma>0$ and $\epsilon,\beta\in(0,1)$. Thus we find an update for χ_1 that reduces the cost J. This update can be obtained using the *method of steepest descent* [33], which is given in weak form for J as follows,

$$\int_\Omega \chi_1^{k+1}v\,dV = \int_\Omega \chi_1^k v\,dV - \omega\frac{\delta J}{\delta\chi_1}(\chi_1^k;v), \tag{2.11}$$

for all $v\in H^1(\Omega)$, where $\omega\in(0,1)$ is the step size and $k\in\mathbb{N}$. Let $\chi_1^k, \delta\chi_1^k \in L^2(\Omega)$ and suppose

$$\frac{\delta\Sigma_1}{\delta\chi_1^k}(\chi_1^k;\delta\chi_1^k)\in H^1(\Omega),$$

then the variational derivative of J in (2.10) is

$$\begin{aligned} \frac{\delta J}{\delta\chi_1^k}(\chi_1^k;\delta\chi_1) = &\int_\Omega -2(\Sigma_1(\chi_1^k)-\sigma_2)\delta\chi_1^k\nabla\Phi(\sigma_1^k,\chi_1^k)\cdot\nabla\Phi^*(\sigma_1^k,\chi_1^k)\,dV \\ &+ \int_\Omega \alpha|\nabla\Sigma_1(\chi_1^k)|^2\delta\chi_1^k\,dV + \gamma\int_\Omega \frac{\nabla(\delta\chi_1^k)\cdot\nabla\chi_1^k}{\sqrt{|\nabla\chi_1^k|^2+\beta^2}}\,dV. \end{aligned} \tag{2.12}$$

Remark 2.4. Observe that (2.12) requires the calculation of $\nabla\chi_1^k$ but since χ_1^k is binary, a smooth approximation of χ_1^k is necessary. This will be discussed later. The assumption that $\frac{\delta\Sigma_1}{\delta\chi_1^k}(\chi_1^k;\delta\chi_1^k) \in H^1(\Omega)$ can be shown if χ_1^k is sufficiently smooth (see Theorem 3.28).

Instead of performing the iteration (2.11) by evaluating $\delta J/\delta\chi_1(\chi_1^k;v)$ explicitly in terms of χ_1^k, the iteration may be performed semi-implicitly by evaluating part of the variational derivative of J in (2.12) at χ_1^{k+1} as follows:

$$\int_\Omega \big[v\chi_1^{k+1} + \omega\gamma\frac{\nabla v\cdot\nabla\chi_1^{k+1}}{\sqrt{|\nabla\chi_1^k|^2+\beta^2}}\big]\,dV = \int_\Omega vG(\chi_1^k)\,dV, \tag{2.13}$$

for all $v \in H^1(\Omega)$, where

$$G(\chi_1) = \chi_1 - \omega\alpha|\nabla\Sigma_1(\chi_1)|^2 + 2\omega\big[(\Sigma_1(\chi_1)-\sigma_2)\nabla\Phi(\tilde\sigma_1,\chi_1)\cdot\nabla\Phi^*(\tilde\sigma_1,\chi_1)\big]. \tag{2.14}$$

We show later that (2.13) admits a unique solution (see Lemma 4.3).

As mentioned in Remark 2.4, a smooth approximation of χ_1 is necessary. To do this, we introduce the kernel function $\xi_\delta(x) = \frac{1}{4\pi\delta}e^{-\frac{x^2}{4\delta}}$, for some $\delta > 0$. We approximate χ_1^k using the convolution of χ_1^k and ξ_δ, i.e., we let

$$\tilde\chi_1^k := \chi_1^k * \xi_\delta = \int_{\mathbb{R}^2}\xi_\delta(x-y)\chi_1^k(y)dy. \tag{2.15}$$

The following result gives the regularity and continuity of the above mollification.

Theorem 2.5. *Let $k \in \mathbb{N}$, then $\tilde\chi_1^k$ is a real analytic function on Ω and $\tilde\chi_1^k \to \chi_1^k$ almost everywhere as $\delta \to 0$. Furthermore, suppose $0 \le \chi_1^k \le 1$, for all $x \in \Omega$. Then $0 \le \tilde\chi_1^k \le 1$* [15].

Using (2.15), we approximate (2.13) by

$$\int_\Omega \omega\gamma\frac{\nabla\tilde\chi_1^{k+1}\cdot\nabla v}{\sqrt{|\nabla\tilde\chi_1^k|^2+\beta^2}} + \tilde\chi_1^{k+1}v\,dV = \int_\Omega G(\tilde\chi_1^k)v\,dV, \quad \forall v \in H^1(\Omega). \tag{2.16}$$

Definition 2.6. We define the operator $\Theta : L^2(\Omega) \to L^2(\Omega)$ to be the solution $\tilde\chi_1^{k+1} \in L^2(\Omega)$ of (2.16) for a given $\tilde\chi_1^k \in L^2(\Omega)$.

Because the solution of (2.16) is not binary, the update for χ is obtained by performing a thresholding step.

We summarize the two-phase segmentation method in the algorithm below. The analysis of the numerical solution of the PDEs arising from this algorithm can be found in [26].

Two-phase segmentation algorithm.

(1) Given f and $\tilde V$. Choose parameters $\epsilon,\delta,\lambda,\gamma,\beta \ll 1$, $\alpha \gg 1$, and $\zeta,\omega \in (0,1)$. Select the maximum number of iterations K and the tolerance ρ. Set $k=1$ and choose the initial σ_1^k. Select an initial guess χ_1^k. The value of σ_2 is given and $\chi_2^k = 1-\chi_1^k$.

(2) Take $\tilde\chi_1^k = \chi_1^k * \xi_\delta$ and $\tilde\chi_1^{k+1} = \Theta(\tilde\chi_1^k)$. Then, χ_1 is updated by

$$\chi_1^{k+1}(x) = \begin{cases} 1, & \text{if } \tilde\chi_1^{k+1}(x) \ge \zeta,\\ 0, & \text{otherwise.}\end{cases}$$

Set $\chi_2^{k+1} = 1-\chi_1^{k+1}$.

(3) If $k = K$ or $\|\chi_1^{k+1} - \chi_1^k\|_{L^2(\Omega)} < \rho$, the algorithm terminates. Otherwise, $k \leftarrow k + 1$ and go back to step 2.

3. Analysis of the algorithm

In this section, we analyze the two-phase segmentation algorithm. We start with results obtained by assuming that $\sigma \in L^\infty(\Omega)$ and that χ_1 is a characteristic function. Because the characteristic function χ_1 is binary, we use its smooth approximation (2.15) instead. This is necessary because some of the essential results require χ_1 to have a higher regularity. This might seem like a deviation from our proposed method but we will argue that these modifications can be justified. We will then introduce a modification of the two-phase segmentation algorithm to adapt with the mollification of χ_1. In the next section, we prove that the modified version of two-phase segmentation algorithm has a fixed point via Schauder's Fixed Point Theorem.

3.1. **Preliminaries.** We already emphasized that ϕ and ϕ^* are solved using σ^k. In this section, we try to understand how a perturbation on σ^k affects ϕ and ϕ^*. Recall that σ^k depends on χ_1. Therefore, ϕ and ϕ^* depend on χ_1 as well. Working under the assumption that $\sigma \in L^\infty(\Omega)$ and χ_1 is a characteristic function, we show that ϕ and ϕ^* depend continuously on σ^k and on χ_1. We begin this section by showing that the variational forward problem and the variational adjoint problem both have unique solutions in $H^1(\Omega)$ under stated assumptions on σ. In the succeeding sections, we study the behavior of ϕ and ϕ^* when we require additional regularity of σ^k.

Theorem 3.1. *Let $\sigma^k \in L^\infty(\Omega)$ such that $0 < \underline{\sigma} \le \sigma^k(x)$ for all $x \in \Omega$ and $f \in \tilde{L}^2(\partial\Omega)$. Then (*the variational forward EIT problem*)*

$$\int_\Omega \sigma^k \nabla\phi \cdot \nabla v \, dV = \int_{\partial\Omega} \sigma^k \frac{\partial \phi}{\partial n} v \, dS, \quad \forall v \in H^1(\Omega) \tag{3.1}$$

*has a unique solution $\phi \in H^1(\Omega)$ with $\int_{\partial\Omega} \phi \, dS = 0$. Similarly, let $\tilde{V} \in \tilde{L}^2(\partial\Omega)$ be the known boundary voltage. Then (*the variational adjoint problem*)*

$$\int_\Omega \sigma^k \nabla\phi^* \cdot \nabla v \, dV = \int_{\partial\Omega} (\phi - \tilde{V}) v \, dS, \quad \forall v \in H^1(\Omega) \tag{3.2}$$

has a unique solution $\phi^* \in H^1(\Omega)$ with $\int_{\partial\Omega} \phi^* \, dS = 0$.

Proof. We define $I := \{u \in H^1(\Omega) : \int_{\partial\Omega} u \, dS = 0\}$, $a(u, v) := \int_\Omega \sigma^k \nabla u \cdot \nabla v \, dV$, and $b(v) := \int_{\partial\Omega} f v \, dS$. Clearly, a is bilinear and b is linear. Using the Cauchy-Schwarz identity, Hölder's inequality, and the definition of the H^1 norm, a is bounded, i.e.,

$$|a(u, v)| \le \|\sigma^k\|_{L^\infty(\Omega)} \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}.$$

Because $u \in I$, then $\int_{\partial\Omega} u \, dS = 0$. Therefore, using the lower bound of σ, and the generalized Friedrich's inequality [4], we obtain

$$\begin{aligned}
|a(u, u)| &\ge \underline{\sigma} \|\nabla u\|^2_{L^2(\Omega)} \\
&= \frac{\underline{\sigma}}{2} \|\nabla u\|^2_{L^2(\Omega)} + \frac{\underline{\sigma}}{2} \|\nabla u\|^2_{L^2(\Omega)} \\
&\ge \frac{\underline{\sigma}}{2} \Big(\frac{1}{C} \|u\|^2_{L^2(\Omega)} - \Big(\int_{\partial\Omega} u \, dS \Big)^2 \Big) + \frac{\underline{\sigma}}{2} \|\nabla u\|^2_{L^2(\Omega)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\underline{\sigma}}{2C}\|u\|^2_{L^2(\Omega)} + \frac{\underline{\sigma}}{2}\|\nabla u\|^2_{L^2(\Omega)} \\
&\geq \underline{\sigma}\min\big\{\frac{1}{C}, 1\big\}\|u\|^2_{H^1(\Omega)},
\end{aligned}
$$

for some $C > 0$. Finally, by the Trace Theorem [14], b is bounded. Hence, by the Lax-Milgram Theorem $\exists!\ \phi \in H^1(\Omega)$ satisfying (3.1). Similarly, there exists a unique $\phi^* \in I$ satisfying (3.2). $\square$

Corollary 3.2. *Let ϕ and ϕ^* satisfy the variational forward and the variational adjoint problem stated in the previous theorem. We have the following estimates:*

$$\|\phi\|_{H^1(\Omega)} \leq C_1\|f\|_{\tilde{L}^2(\partial\Omega)} \tag{3.3}$$

$$\|\phi^*\|_{H^1(\Omega)} \leq C_2\|(\phi - \tilde{V})\|_{\tilde{L}^2(\partial\Omega)} \tag{3.4}$$

for some $C_1, C_2 > 0$.

Note that using the Trace Theorem and the triangle inequality, ϕ^* can be further estimated by

$$\|\phi^*\|_{H^1(\Omega)} \leq C_3\Big(\|f\|_{\tilde{L}^2(\partial\Omega)} + \|\tilde{V}\|_{\tilde{L}^2(\partial\Omega)}\Big) \tag{3.5}$$

for some $C_3 > 0$. Throughout this work, we use the following notation.

Definition 3.3. We let $\delta\sigma^k$ and $\delta\chi_1$ denote perturbations of $\sigma^k \in L^\infty(\Omega)$ and $\chi_1 \in L^\infty(\Omega)$, respectively. In Definition (2.2), Φ and Φ^* are operators that map (σ_1^k, χ_1) to ϕ and ϕ^*, respectively. But because $\sigma^k = \sigma_1^k\chi_1 + \sigma_2(1 - \chi_1)$, we can make the identifications

$$\Phi(\sigma^k) = \Phi(\sigma_1^k, \chi_1), \quad \Phi^*(\sigma^k) = \Phi^*(\sigma_1^k, \chi_1).$$

Hence, $\Phi : \sigma^k \to \phi$ and $\Phi^* : \sigma^k \to \phi^*$.

Remark 3.4. Given a perturbation $\delta\sigma^k \in L^\infty(\Omega)$, how can we choose $\eta > 0$ so that the forward and the adjoint problems have unique solutions if we use $\sigma^k + \eta\delta\sigma^k$? We know that the forward and adjoint problems have unique solutions given $\sigma^k \in L^\infty$ if $\sigma^k(x) \geq \underline{\sigma} > 0$ for all $x \in \Omega$. To make sure that $\Phi(\sigma^k + \eta\delta\sigma^k)$ is unique, we can simply select η sufficiently small so that $(\sigma^k + \eta\delta\sigma^k)(x) \geq \underline{\sigma}^\tau > 0$ for all $x \in \Omega$ and $\eta \in (0, \tau)$ for some $\tau > 0$. This is possible because $\sigma^k(x) \geq \underline{\sigma} > 0$. Thus, the coercivity of the bilinear functional in the variational formulations of both the forward and the adjoint problems is guaranteed and the solvability of these problems is assured. Consequently, by (3.3) and (3.5) there exist $C_1, C_2 > 0$ such that

$$\|\Phi(\sigma^k + \eta\delta\sigma^k)\|_{H^1(\Omega)} \leq C_1\|f\|_{\tilde{L}^2(\partial\Omega)}, \tag{3.6}$$

$$\|\Phi^*(\sigma^k + \eta\delta\sigma^k)\|_{H^1(\Omega)} \leq C_2(\|f\|_{\tilde{L}^2(\partial\Omega)} + \|\tilde{V}\|_{\tilde{L}^2(\partial\Omega)}), \tag{3.7}$$

for any $\eta \in (0, \tau)$.

The result below shows how a perturbation $\delta\sigma^k$ affects ϕ and ϕ^*.

Theorem 3.5. *Let $\sigma^k, \delta\sigma^k \in L^\infty(\Omega)$. Then there exist $C_1, C_2 > 0$ such that*

$$\|\Phi(\sigma^k + \eta\delta\sigma^k) - \Phi(\sigma^k)\|_{H^1(\Omega)} \leq C_1\eta\|\delta\sigma^k\|_{L^\infty(\Omega)}, \tag{3.8}$$

$$\|\Phi^*(\sigma^k + \eta\delta\sigma^k) - \Phi^*(\sigma^k)\|_{H^1(\Omega)} \leq C_2\eta\|\delta\sigma^k\|_{L^\infty(\Omega)}, \tag{3.9}$$

for any $\eta \in (0, \tau)$, where τ is chosen according to Remark 3.4.

Proof. From (3.1), we have

$$\int_\Omega \sigma^k \nabla\Phi(\sigma^k)\cdot\nabla v\,dV = \int_\Omega fv\,dV. \tag{3.10}$$

Similarly, for $\sigma^k+\eta\delta\sigma^k$,

$$\int_\Omega (\sigma^k+\eta\delta\sigma^k)\nabla(\Phi(\sigma^k)+\delta\phi)\cdot\nabla v\,dV = \int_\Omega fv\,dV, \tag{3.11}$$

where we denote $\delta\phi := \Phi(\sigma^k+\eta\delta\sigma^k)-\Phi(\sigma^k)$. Subtracting (3.10) from (3.11), we obtain

$$\int_\Omega \sigma^k\nabla\delta\phi\cdot\nabla v\,dV = -\int_\Omega \eta\delta\sigma^k\nabla\Phi(\sigma^k+\eta\delta\sigma^k)\cdot\nabla v\,dV. \tag{3.12}$$

We define $a(u,v) := \int_\Omega \sigma^k\nabla u\cdot\nabla v\,dV$ and $b_1(v) := -\int_\Omega \eta\delta\sigma^k\nabla\Phi(\sigma^k+\eta\delta\sigma^k)\cdot\nabla v\,dV$. Clearly, a and b_1 are bilinear and linear, respectively. Recall from Theorem (3.1) that for any $u,v\in I$, $a(u,v)$ is coercive and continuous. By the Cauchy-Schwarz inequality and (3.6), b_1 is bounded. Thus, if we take $u=v=\delta\phi$, use the previous inequality, and the coercivity of $a(u,v)$ we obtain

$$\|\delta\phi\|_{H^1(\Omega)} \le \frac{1}{\bar{C}}\bar{C}_1\eta\|f\|_{\tilde{L}^2(\partial\Omega)}\|\delta\sigma^k\|_{L^\infty(\Omega)}, \tag{3.13}$$

where $\bar{C}>0$ is the coercivity constant and $\bar{C}_1$ is the constant from (3.6). This proves the first inequality. Using similar arguments, one can show (3.9). □

Definition 3.6. Recall that from (2.8), $\Sigma^k(\chi_1) := \sigma_1^k\chi_1+\sigma_2(1-\chi_1)=\sigma^k$. Therefore, ϕ and ϕ^* depend on χ_1 for a fixed σ_1^k. Hence, for brevity we denote

$$\Phi(\chi_1) := \Phi(\Sigma^k(\chi_1)) \quad\text{and}\quad \Phi^*(\chi_1) := \Phi^*(\Sigma^k(\chi_1)). \tag{3.14}$$

From here onwards, it is assumed that ϕ and ϕ^* are solved using $\sigma_1^k\chi_1+\sigma_2(1-\chi_1)$.

We now show that ϕ and ϕ^* depend continuously on χ_1.

Corollary 3.7. *Let $\chi_1\in L^\infty(\Omega)$, then $\exists\bar{C}_1,\bar{C}_2>0$ such that*

$$\|\Phi(\chi_1+\eta\delta\chi_1)-\Phi(\chi_1)\|_{H^1(\Omega)} \le \bar{C}_1\eta\|\delta\chi_1\|_{L^\infty(\Omega)}, \tag{3.15}$$

$$\|\Phi^*(\chi_1+\eta\delta\chi_1)-\Phi^*(\chi_1)\|_{H^1(\Omega)} \le \bar{C}_2\eta\|\delta\chi_1\|_{L^\infty(\Omega)}, \tag{3.16}$$

for any $\eta\in(0,\tau)$, where τ is chosen according to Remark 3.4.

Proof. We use the decomposition

$$\Sigma^k(\chi_1) = \sigma_1^k\chi_1+\sigma_2(1-\chi_1). \tag{3.17}$$

Let $\eta\in(0,\tau)$. If we use $\chi_1+\eta\delta\chi_1$ instead of χ_1, we have

$$\Sigma^k(\chi_1)+\delta\sigma^k = \Sigma^k(\chi_1)+\eta(\sigma_1^k-\sigma_2)\delta\chi_1, \tag{3.18}$$

where $\delta\sigma^k$ is the associated change in σ^k given a $\eta\delta\chi_1$ perturbation of χ_1. Subtracting (3.17) from (3.18), we obtain

$$\delta\sigma^k = \eta(\sigma_1^k-\sigma_2)\delta\chi_1. \tag{3.19}$$

The inequalities we need to show directly follow from Theorem (3.5) and

$$\|\delta\sigma^k\|_{L^\infty(\Omega)} \le \eta\|\sigma_1^k-\sigma_2\|_{L^\infty(\Omega)}\|\delta\chi_1\|_{L^\infty(\Omega)}.$$

□

3.2. **Smooth approximation of χ_1.** From (2.7), the quantity $\sigma_1^{k+1} := \Sigma_1(\chi_1)$ can be obtained via

$$\begin{aligned} &\int_\Omega \alpha(\chi_1+\epsilon)\nabla\sigma_1^{k+1}\cdot\nabla v\,dV + \int_\Omega \lambda(\sigma_1^{k+1}-\sigma_1^k)v\,dV \\ &= \int_\Omega \chi_1\nabla\Phi(\chi_1)\cdot\nabla\Phi^*(\chi_1)v\,dV. \end{aligned} \tag{3.20}$$

This equation can be interpreted as

$$a(\sigma_1^{k+1},v) = b(v), \quad \forall v\in H^1(\Omega), \tag{3.21}$$

where

$$\begin{gathered} a(\sigma_1^{k+1},v) := \int_\Omega \alpha(\chi_1+\epsilon)\nabla\sigma_1^{k+1}\cdot\nabla v\,dV + \int_\Omega \lambda\sigma_1^{k+1}v\,dV, \\ b(v) := \int_\Omega \lambda\sigma_1^k v\,dV + \int_\Omega \chi_1\nabla\Phi(\chi_1)\cdot\nabla\Phi^*(\chi_1)v\,dV. \end{gathered}$$

To guarantee solvability of (3.21), $b(v)$ must be bounded. To show this, it is necessary that $\chi_1\nabla\Phi(\chi_1)\cdot\nabla\Phi^*(\chi_1)$ be in $L^2(\Omega)$. If either $\nabla\Phi(\chi_1)$ or $\nabla\Phi^*(\chi_1)$ is in $L^\infty(\Omega)$, then $\nabla\Phi(\chi_1)\cdot\nabla\Phi^*(\chi_1)\in L^2(\Omega)$. In [9], it was shown that $\|\nabla\Phi(\chi_1)\|_{L^\infty(\Omega')}$ can be bounded by $\|\nabla\Phi(\chi_1)\|_{L^2(\Omega)}$ for some Ω' compactly embedded in Ω. This was proven under the assumption that $\sigma^k\in C^1(\bar\Omega)$. Recall that $\sigma^k = \sigma_1^k\chi_1+\sigma_2(1-\chi_1)$. Clearly, σ^k is not necessarily in $C^1(\bar\Omega)$ because χ_1 is a characteristic function. We have introduced a mollification χ_1^δ of χ_1 in (2.15) to resolve this. Thus, $\sigma_1^k\in C^\infty(\bar\Omega)$. Moreover, σ^k is not just in $C^1(\bar\Omega)$ but in $C^\infty(\bar\Omega)$ as well. This might seem like a deviation from our proposed method but technically, we can choose δ to be extremely close to 0 so that χ_1^δ is a good approximation of χ_1. We show that the mollification affects the corresponding ϕ and ϕ^* to a very small extent as long as the distance between χ_1^δ and χ_1 is small enough. But first, we need the following results.

Lemma 3.8. *Let $1\le p<\infty$ and take $1\le r\le\infty$ such that $\frac{1}{r}+1-\frac{1}{p}\in[0,1]$. Define the operator from $L^p(\Omega)$ to $L^r(\Omega)$ by*

$$T_\delta(g) := g*\xi_\delta. \tag{3.22}$$

Then T_δ is continuous and injective [15].

Lemma 3.9. *For any $g\in L^p(\Omega)$, $1\le p<\infty$, $\partial^\nu(g*\xi_\delta) = (\partial^\nu g)*\xi_\delta$, for $|\nu|\le 1$. Moreover, $\partial^\nu(g*\xi_\delta)\to\partial^\nu g$ almost everywhere as $\delta\to 0$.*

The proof of the above lemma is rather straightforward and is omitted. The second assertion follows from Lemma (2.5) (compare with [14]). For brevity, from here onwards we let

$$\chi_1^\delta := \chi_1*\xi_\delta$$

denote the mollification of χ_1. We now show how the perturbation of χ_1 affects the solution of the forward and adjoint problems.

Theorem 3.10. *For a fixed δ, the solution of the forward problem given $\chi_1\in L^\infty(\Omega)$ and the solution of the forward problem given χ_1^δ satisfy*

$$\|\Phi(\chi_1^\delta)-\Phi(\chi_1)\|_{H^1(\Omega)} \le C_1^\delta\|\chi_1^\delta-\chi_1\|_{L^\infty(\Omega)} \tag{3.23}$$

for some $C_1>0$. The same applies to the adjoint problem

$$\|\Phi^*(\chi_1^\delta)-\Phi^*(\chi_1)\|_{H^1(\Omega)} \le C_2^\delta\|\chi_1^\delta-\chi_1\|_{L^\infty(\Omega)} \tag{3.24}$$

for some $C_2 > 0$.

Proof. We proved in (3.15) that ϕ depends continuously on χ_1. Note that we proved this given the assumption that $\chi_1 \in L^\infty(\Omega)$, which implies that $\chi_1 \in L^2(\Omega)$ as well. By Lemma (3.8), we can infer that $\chi_1^\delta \in L^\infty(\Omega)$ by choosing $p = 2$ and $r = \infty$. Finally, Theorem (3.5) proves the rest of our claim. $\square$

Remark 3.11. The superscript δ in C_1^δ and C_2^δ from (3.23) and (3.24) are used to emphasize the dependence of the inequality constants on the mollification parameter δ. From here onwards, we use the same notation for all constants dependent on δ.

We know that χ_1^δ converges to χ_1 pointwise [15]. Although it does not guarantee that $\|\chi_1^\delta - \chi_1\|_{L^\infty(\Omega)}$ converges to 0, this can still be a gauge to measure the distance between the solution of the forward problem using χ_1 and the solution using χ_1^δ.

To make our modifications consistent, we find a new thresholding approach to adapt with the mollification of χ_1. First, we define a space that will be important in our succeeding computations. Let $\mathcal{B}(\Omega)$ be the Borel $\sigma-$algebra over Ω and let $\mu(\cdot)$ denote the Lebesgue measure. For any $A_1, A_2 \in \mathcal{B}(\Omega)$, we define the *symmetric difference* of A_1 and A_2 to be

$$A_1 \triangle A_2 := (A_1 \backslash A_2) \cup (A_2 \backslash A_1).$$

Definition 3.12. Let the distance $d : \mathcal{B}(\Omega) \times \mathcal{B}(\Omega) \to \mathbb{R} \cup \{\infty\}$ be $d(A_1, A_2) = \mu(A_1 \triangle A_2)$. We now define $\mathcal{M}(\Omega) = (\mathcal{B}(\Omega), d)/\ker(d)$.

The space $\mathcal{M}(\Omega)$ is, in fact, a metric space [19, 15]. In our algorithm, we did a thresholding on Θ in order to get an update for χ_1. In the following definition, we modify this thresholding to make it coherent with the mollification of χ_1.

Definition 3.13. Let $z \in L^2(\Omega) \backslash H^1(\Omega)$. We define $H : L^2(\Omega) \to \mathcal{M}(\Omega)$ by

$$H(g) = \{x \in \Omega : ((g^\delta - \zeta + \delta z) * \xi_\delta)(x) \geq 0\}, \tag{3.25}$$

for some $\zeta \in (0, 1)$ and $g^\delta = g * \xi_\delta$. Moreover, define $M : \mathcal{M}(\Omega) \to L^2(\Omega)$ as the map that assigns $\omega \in \mathcal{M}(\Omega)$ to its characteristic function χ_ω, that is,

$$M(\omega) = \chi_\omega. \tag{3.26}$$

Observe that if we let $\delta \to 0$, then

$$(g_\delta - \zeta + \delta z) * \xi_\delta \to g - \zeta. \tag{3.27}$$

See [15]. This means that as $\delta \to 0$, $\mathcal{H}(g)$ becomes the set of $x \in \Omega$ for which $g(x) \geq \zeta$. Using the above-mentioned functions T_δ, H, M, and Θ (Definition 2.6), we can modify the two-phase segmentation algorithm into

$$\chi_1^{k+1} = (T_\delta \circ M \circ H \circ \Theta \circ T_\delta)(\chi_1^k). \tag{3.28}$$

The main goal of this article is to show that (3.28) has a fixed point.

Remark 3.14. It is again emphasized that the introduction of $z \in L^2(\Omega) \backslash H^1(\Omega)$ in (3.25) and the modification of Algorithm (1) as a result of the mollification of χ_1 are purely technical devices and are used only for theoretical purposes. This might seem like a deviation from Algorithm (1) but as shown in (3.23), (3.24) and (3.27), these changes are justified.

3.3. **Gradient of the functional J.** This section is devoted to showing the validity of the explicit formulation of the gradient of J in (2.10). As shown in the computation of (2.12), it is sufficient to show that

$$\frac{\delta\Sigma_1}{\delta\chi_1}(\chi_1;\delta\chi_1) \in H^1(\Omega).$$

This is not necessarily true for an arbitrary characteristic function χ_1. Hence, we use χ_1^δ instead of χ_1 so that we are dealing with a smooth function rather than a characteristic function. Thus, we are now solving σ_1^{k+1} using the equation

$$\int_\Omega \alpha(\chi_1^\delta+\epsilon)\nabla\sigma_1^{k+1}\cdot\nabla v\,dV + \int_\Omega \lambda(\sigma_1^{k+1}-\sigma_1^k)v\,dV = \int_\Omega \chi_1^\delta\nabla\phi\cdot\nabla\phi^* v\,dV, \tag{3.29}$$

for all $v \in H^1(\Omega)$. Before we start our calculations, we first make few assumptions. These will be used throughout our analysis.

Let $\alpha, \underline{\sigma}, \lambda, \delta > 0$. We assume that $\sigma_1^k \in C^\infty(\bar\Omega)$ such that $\Sigma^k(\chi_1^\delta) \geq \underline{\sigma} > 0$. We also assume that $\partial\Omega$ is sufficiently smooth. (3.30)

The assumption that $\sigma_1^k \in C^\infty(\bar\Omega)$ might seem like a strong assumption but if we can show that $\sigma_1^{k+1} \in C^\infty(\bar\Omega)$ as well, then this assumption makes sense. Moreover, the initial guess for σ_1 in our algorithm can be chosen to be constant throughout Ω so that it is in $C^\infty(\bar\Omega)$.

We now investigate what happens to ϕ and ϕ^* if we use the above assumption.

Lemma 3.15. *Under assumption* (3.30)*,*

$$\|\nabla\phi\|_{L^\infty(\Omega)} \leq C^\delta\|\phi\|_{H^1(\Omega)}, \tag{3.31}$$

for some $C^\delta > 0$ (compare with [9]*). In fact, $\phi \in C^\infty(\bar\Omega)$.*

Proof. By assumption (3.30) and Lemma (2.5), $\sigma^k \in C^\infty(\bar\Omega)$. Let $l \geq 1$. Obviously, $\sigma^k \in C^l(\bar\Omega)$. Therefore, using standard regularity estimates (see, e.g., [16]), we obtain

$$\|\phi\|_{H^{l+2}(\Omega)} \leq C_1\|\phi\|_{H^1(\Omega)}, \tag{3.32}$$

for some $C_1 > 0$. Furthermore, by the Sobolev imbedding theorem [14], we have

$$\|\phi\|_{C^{l,\gamma}(\bar\Omega)} \leq C_2\|\phi\|_{H^{l+2}(\Omega)}. \tag{3.33}$$

By the definition of $\|\cdot\|_{C^{l,\gamma}(\bar\Omega)}$, the embedding

$$C^{1,\gamma}(\bar\Omega) \hookrightarrow C^1(\bar\Omega) \tag{3.34}$$

is continuous (see [18]). If we compare (3.32), (3.33), and (3.34), we can deduce that there exists $C > 0$ such that

$$\|\nabla\phi\|_{L^\infty(\Omega)} \leq C\|\nabla\phi\|_{H^1(\Omega)}, \tag{3.35}$$

which completes the first part of the proof. Moreover, because $l \geq 1$, it follows that $\phi(\chi_1^\delta) \in C^\infty(\bar\Omega)$. $\square$

Lemma 3.16. *Under assumption* (3.30)*, there exists $C^\delta > 0$ such that*

$$\|\nabla\phi^*\|_{L^\infty(\Omega)} \leq C^\delta\|\phi^*\|_{H^1(\Omega)}. \tag{3.36}$$

Furthermore, $\phi^* \in C^\infty(\bar\Omega)$.

The proof of this theorem is similar to that of the previous theorem. We now analyze the dependence of ϕ and ϕ^* on χ_1^δ. From here onwards, we denote

$$\delta\chi_1^\delta := \delta\chi_1 * \xi_\delta$$

so that $(\chi_1 + \eta\delta\chi_1) * \xi_\delta = \chi_1^\delta + \eta\delta\chi_1^\delta$.

Suppose we replace χ_1^δ with $\chi_1^\delta + \eta\delta\chi_1^\delta$. Again by Remark 3.4, η should be taken from the set $(0, \tau)$ where τ is chosen so that $\sigma^k + \eta\delta\sigma^k \geq \underline{\sigma}^\tau > 0$. Therefore, from (3.6), (3.7), (3.14), (3.31), and (3.36), the following estimates hold

$$\|\nabla\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta)\|_{L^\infty(\Omega)} \leq C_1\|f\|_{\tilde{L}^2(\partial\Omega)}, \tag{3.37}$$

$$\|\nabla\Phi^*(\chi_1^\delta + \eta\delta\chi_1^\delta)\|_{L^\infty(\Omega)} \leq C_2\Big(\|f\|_{\tilde{L}^2(\partial\Omega)} + \|\tilde{V}\|_{\tilde{L}^2(\partial\Omega)}\Big), \tag{3.38}$$

for some $C_1, C_2 > 0$ and for all $\eta \in (0, \tau)$.

In Corollary 3.7, we have shown that ϕ and ϕ^* depend continuously on χ_1. Observe that this theorem holds with any χ_1 whose value is between 0 and 1. Therefore, this also holds when we use χ_1^δ instead because $0 \leq \chi_1^\delta \leq 1$ as proven in Lemma (2.5). We state this in the following lemma.

Lemma 3.17. *Under assumption* (3.30)*, there exists $C_1^\delta, C_2^\delta > 0$ such that*

$$\|\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta) - \Phi(\chi_1^\delta)\|_{H^1(\Omega)} \leq C_1^\delta\eta\|\delta\chi_1\|_{L^2(\Omega)}, \tag{3.39}$$

$$\|\Phi^*(\chi_1^\delta + \eta\delta\chi_1^\delta) - \Phi^*(\chi_1^\delta)\|_{H^1(\Omega)} \leq C_2^\delta\eta\|\delta\chi_1\|_{L^2(\Omega)}, \tag{3.40}$$

for any $\eta \in (0, \tau)$, where τ is chosen according to Remark 3.4.

We define

$$\psi(\chi_1^\delta) := \nabla\Phi(\chi_1^\delta) \cdot \nabla\Phi^*(\chi_1^\delta). \tag{3.41}$$

Observe that the right-hand side of (3.29) includes $\psi(\chi_1^\delta)$. In the next lemma, we show that ψ depends continuously on χ_1. This will be necessary when we analyze the solution of (3.29). Note that $\nabla\Phi^*(\chi_1^\delta) \in L^2(\Omega)$ by Corollary 3.2 and $\nabla\Phi(\chi_1^\delta) \in L^\infty(\Omega)$ by (3.31). Therefore, by Hölder's inequality,

$$\psi(\chi_1^\delta) \in L^2(\Omega). \tag{3.42}$$

This means that ψ is a map from $\chi_1 \in L^2(\Omega)$ to $\nabla\Phi(\chi_1^\delta) \cdot \nabla\Phi^*(\chi_1^\delta) \in L^2(\Omega)$. In the following lemma, we prove that this mapping is continuous.

Lemma 3.18. *Under assumption* (3.30)*, there exists $C^\delta \geq 0$ such that*

$$\|\psi(\chi_1^\delta + \eta\delta\chi_1^\delta) - \psi(\chi_1^\delta)\|_{L^2(\Omega)} \leq C^\delta\eta\|\delta\chi_1\|_{L^2(\Omega)}, \tag{3.43}$$

for any $\eta \in (0, \tau)$, where τ is chosen according to Remark 3.4.

Proof. Adding and subtracting $\nabla\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta) \cdot \nabla\Phi^*(\chi_1^\delta)$ to $\psi(\chi_1^\delta + \eta\delta\chi_1^\delta) - \psi(\chi_1^\delta)$, we obtain

$$\psi(\chi_1^\delta + \eta\delta\chi_1^\delta) - \psi(\chi_1^\delta) =: A_1(\chi_1^\delta; \delta\chi_1^\delta) + A_2(\chi_1^\delta; \delta\chi_1^\delta),$$

where

$$\begin{gathered}
A_1(\chi_1^\delta; \delta\chi_1^\delta) = \nabla\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta) \cdot \nabla\Phi^*(\chi_1^\delta + \eta\delta\chi_1^\delta) - \nabla\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta) \cdot \nabla\Phi^*(\chi_1^\delta), \\
A_2(\chi_1^\delta; \delta\chi_1^\delta) = \nabla\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta) \cdot \nabla\Phi^*(\chi_1^\delta) - \nabla\Phi(\chi_1^\delta) \cdot \nabla\Phi^*(\chi_1^\delta).
\end{gathered}$$

Thus

$$\begin{aligned}
&\|\psi(\chi_1^\delta + \eta\delta\chi_1^\delta) - \psi(\chi_1^\delta)\|_{L^2(\Omega)} \\
&\leq \|A_1(\chi_1; \delta\chi_1)\|_{L^2(\Omega)} + \|A_2(\chi_1; \delta\chi_1)\|_{L^2(\Omega)}.
\end{aligned} \tag{3.44}$$

We can estimate $A_1(\chi_1^\delta;\delta\chi_1^\delta)$ using the Hölder's inequality, (3.37), and (3.40). Thus,

$$\|A_1(\chi_1^\delta;\delta\chi_1^\delta)\|_{L^2(\Omega)} \le C_1\eta\|f\|_{\tilde{L}^2(\partial\Omega)}\|\delta\chi_1\|_{L^2(\Omega)},$$

for some $C_1 > 0$. Similarly, using the Hölder's inequality, (3.36), and (3.39), we obtain

$$\|A_2(\chi_1^\delta;\delta\chi_1^\delta)\|_{L^2(\Omega)} \le C_2\eta\|\nabla\Phi^*(\chi_1^\delta)\|_{L^\infty(\Omega)}\|\delta\chi_1\|_{L^2(\Omega)},$$

for some $C_2 > 0$. The estimates for $A_1(\chi_1^\delta;\delta\chi_1^\delta)$ and $A_2(\chi_1^\delta;\delta\chi_1^\delta)$, together with (3.44), complete the proof. $\square$

Recall that the goal of this section is to validate the explicit formulation of the gradient of $J(\chi_1)$ by showing that $\frac{\delta\Sigma_1}{\delta\chi_1}(\chi_1^\delta;\delta\chi_1^\delta) \in H^1(\Omega)$. To accomplish this, we first need to show that the derivatives of both Φ and Φ^* with respect to χ_1 converge in $H^1(\Omega)$. We start by looking for candidates for the derivatives.

Lemma 3.19. *Under assumption* (3.30)*, there exists $D_\phi(\chi_1;\delta\chi_1) \in H^1(\Omega)$ satisfying*

$$\int_\Omega \sigma^k\nabla D_\phi(\chi_1;\delta\chi_1)\cdot\nabla v\,dV = \int_\Omega(\sigma_1^k-\sigma_2)\delta\chi_1^\delta\nabla\Phi(\chi_1^\delta)\cdot\nabla v\,dV \tag{3.45}$$

with $\int_{\partial\Omega} D_\phi(\chi_1;\delta\chi_1)\,dS = 0$, for all $v \in H^1(\Omega)$, such that $\int_{\partial\Omega} v\,dS = 0$.

Proof. Let $u, v \in H^1(\Omega)$ such that $\int_{\partial\Omega} u\,dS = \int_{\partial\Omega} v\,dS = 0$. We define

$$a(u,v) := \int_\Omega \sigma^k\nabla u\cdot\nabla v\,dV, \quad b(v) := \int_\Omega(\sigma_1^k-\sigma_2)\delta\chi_1^\delta\nabla\Phi(\chi_1^\delta)\cdot\nabla v\,dV.$$

From the proof of Theorem (3.1), a is bilinear, coercive and bounded. Obviously, b is linear. We wish to employ the Lax-Milgram theorem so it is sufficient to show that b is bounded. Indeed, using the Cauchy-Schwarz inequality and the Hölder's inequality we obtain

$$|b(v)| \le \|(\sigma_1^k-\sigma_2)\delta\chi_1^\delta\|_{L^\infty(\Omega)}\|\nabla\Phi(\chi_1^\delta)\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}.$$

The right-hand side of the last inequality is bounded because of Corollary (3.2) and the fact that $(\sigma_1^k-\sigma_2)\delta\chi_1^\delta \in C^\infty(\bar\Omega)$. $\square$

Lemma 3.20. *Under assumption* (3.30)*, there exists $D_{\phi^*}(\chi_1;\delta\chi_1) \in H^1(\Omega)$ satisfying*

$$\begin{aligned}&\int_\Omega \sigma^k\nabla D_{\phi^*}(\chi_1;\delta\chi_1)\cdot\nabla v\,dV\\ &= \int_\Omega(\sigma_1^k-\sigma_2)\delta\chi_1^\delta\nabla\Phi^*(\chi_1^\delta)\cdot\nabla v\,dV + \int_{\partial\Omega} D_\phi(\chi_1;\delta\chi_1)v\,dV,\end{aligned} \tag{3.46}$$

with $\int_{\partial\Omega} D_{\phi^*}(\chi_1;\delta\chi_1)\,dS = 0$, for all $v \in H^1(\Omega)$, such that $\int_{\partial\Omega} v\,dS = 0$.

The proof of this lemma is similar to the proof of the previous lemma; we omit it. Now we prove that $D_\phi(\chi_1;\delta\chi_1)$ and $D_{\phi^*}(\chi_1;\delta\chi_1)$ in the last two lemmas are the derivatives of Φ and Φ^* at χ_1 in the direction of $\delta\chi_1$, respectively. We state and prove this in the following lemma.

Lemma 3.21. *Under assumption* (3.30)*, we have*

$$\lim_{\eta\to0}\Big\|\frac{\Phi(\chi_1^\delta+\eta\delta\chi_1^\delta)-\Phi(\chi_1^\delta)}{\eta} - D_\phi(\chi_1^\delta;\delta\chi_1^\delta)\Big\|_{H^1(\Omega)} = 0, \tag{3.47}$$

$$\lim_{\eta\to 0}\Big\|\frac{\Phi^*(\chi_1^\delta+\eta\delta\chi_1^\delta)-\Phi^*(\chi_1^\delta)}{\eta}-D_{\phi^*}(\chi_1^\delta;\delta\chi_1^\delta)\Big\|_{H^1(\Omega)}=0. \tag{3.48}$$

Thus, we can make the identifications

$$\frac{\delta\Phi}{\delta\chi_1}(\chi_1^\delta;\delta\chi_1^\delta)=D_\phi(\chi_1;\delta\chi_1),\quad \frac{\delta\Phi^*}{\delta\chi_1}(\chi_1^\delta;\delta\chi_1^\delta)=D_{\phi^*}(\chi_1;\delta\chi_1).$$

Furthermore, because $D_\phi(\chi_1;\delta\chi_1), D_{\phi^*}(\chi_1;\delta\chi_1)\in H^1(\Omega)$ *we have*

$$\frac{\delta\Phi}{\delta\chi_1}(\chi_1^\delta;\delta\chi_1^\delta),\ \frac{\delta\Phi^*}{\delta\chi_1}(\chi_1^\delta;\delta\chi_1^\delta)\in H^1(\Omega),$$

with

$$\int_{\partial\Omega}\frac{\delta\Phi}{\delta\chi_1}(\chi_1^\delta;\delta\chi_1^\delta)\,dS=\int_{\partial\Omega}\frac{\delta\Phi^*}{\delta\chi_1}(\chi_1^\delta;\delta\chi_1^\delta)\,dS=0.$$

Proof. From (3.1), we have $\int_\Omega\sigma^k\nabla\Phi(\chi_1^\delta)\cdot\nabla v\,dV=\int_\Omega fv\,dV$, where $\sigma^k=\Sigma^k(\chi_i^\delta)$. Then we obtain

$$\int_\Omega\sigma^k\nabla\Phi(\chi_1^\delta)\cdot\nabla v\,dV=\int_\Omega fv\,dV. \tag{3.49}$$

Similarly, for $\chi_1^\delta+\eta\delta\chi_1^\delta$,

$$\int_\Omega[\sigma^k+\eta(\sigma_1^k-\sigma_2)\delta\chi_1^\delta]\nabla\Phi(\chi_1^\delta+\eta\delta\chi_1^\delta)\cdot\nabla v\,dV=\int_\Omega fv\,dV. \tag{3.50}$$

Subtracting (3.49) from (3.50), we obtain

$$\begin{aligned}&\int_\Omega\sigma^k\nabla(\Phi(\chi_1^\delta+\eta\delta\chi_1^\delta)-\Phi(\chi_1^\delta))\cdot\nabla v\,dV\\&=\int_\Omega\eta(\sigma_1^k-\sigma_2)\delta\chi_1^\delta\nabla\Phi(\chi_1^\delta+\eta\delta\chi_1^\delta)\cdot\nabla v\,dV.\end{aligned} \tag{3.51}$$

Dividing by η, we have

$$\begin{aligned}&\int_\Omega\sigma^k\nabla\Big(\frac{\Phi(\chi_1^\delta+\eta\delta\chi_1^\delta)-\Phi(\chi_1^\delta)}{\eta}\Big)\cdot\nabla v\,dV\\&=\int_\Omega(\sigma_1^k-\sigma_2)\delta\chi_1^\delta\nabla\Phi(\chi_1^\delta+\eta\delta\chi_1^\delta)\cdot\nabla v\,dV.\end{aligned} \tag{3.52}$$

Recall from (3.45) that

$$\int_\Omega\sigma^k\nabla D_\phi(\chi_1^\delta;\delta\chi_1^\delta)\cdot\nabla v\,dV=\int_\Omega(\sigma_1^k-\sigma_2)\delta\chi_1^\delta\nabla\Phi(\chi_1^\delta)\cdot\nabla v\,dV. \tag{3.53}$$

Subtracting (3.52) and (3.53), we obtain

$$\int_\Omega\sigma^k\nabla\Big(\frac{\Phi(\chi_1^\delta+\eta\delta\chi_1^\delta)-\Phi(\chi_1^\delta)}{\eta}-D_\phi(\chi_1^\delta;\delta\chi_1^\delta)\Big)\cdot\nabla v\,dV=:A(v) \tag{3.54}$$

where

$$A(v)=\int_\Omega(\sigma_1^k-\sigma_2)\delta\chi_1^\delta\nabla[\Phi(\chi_1^\delta+\eta\delta\chi_1^\delta)-\Phi(\chi_1^\delta)]\cdot\nabla v\,dV,$$

which can be estimated using the Cauchy-Schwarz inequality and (3.15):

$$|A(v)|\le C_1\eta\|(\sigma_1^k-\sigma_2)\delta\chi_1^\delta\|_{L^\infty(\Omega)}\|\delta\chi_1\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)}. \tag{3.55}$$

It is worth noting that $(\sigma_1^k - \sigma_2)\delta\chi_1^\delta \in L^\infty(\Omega)$ and $\delta\chi_1 \in L^2(\Omega)$ so that the right-hand side of the above inequality is bounded. Now observe that $a(u,v) := \int_\Omega \sigma^k \nabla u \cdot \nabla v \, dV$ is coercive as demonstrated in the proof of Theorem (3.1), i.e.,

$$|a(u,u)| \ge \bar{C}\|u\|^2_{H^1(\Omega)}, \tag{3.56}$$

for some $\bar{C} > 0$, for any $u \in H^1(\Omega)$ such that $\int_{\partial\Omega} u \, dS = 0$. Hence, the left hand side of (3.54) is bounded from above if we set

$$u = \frac{\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta) - \Phi(\chi_1^\delta)}{\eta} - D_\phi(\chi_1^\delta; \delta\chi_1^\delta).$$

Using this fact and comparing (3.54) and (3.55), we obtain the estimate

$$\begin{aligned} &\bar{C}\|\frac{\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta) - \Phi(\chi_1^\delta)}{\eta} - D_\phi(\chi_1^\delta; \delta\chi_1^\delta)\|_{H^1(\Omega)} \\ &\le C_1\eta\|(\sigma_1^k - \sigma_2)\delta\chi_1^\delta\|_{L^\infty(\Omega)}\|\delta\chi_1\|_{L^2(\Omega)}. \end{aligned}$$

Therefore, taking the limit of the last equality as $\eta \to 0$, we obtain

$$\lim_{\eta\to 0}\|\frac{\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta) - \Phi(\chi_1^\delta)}{\eta} - D_\phi(\chi_1^\delta; \delta\chi_1^\delta)\|_{H^1(\Omega)} = 0. \tag{3.57}$$

The rest of the proof is similar to show the convergence of the derivative of Φ^* with respect to χ_1 in $H^1(\Omega)$. The last statements of the lemma can be inferred directly from the last two lemmas. $\square$

Now that we have shown that the derivatives of both Φ and Φ^* converge in $H^1(\Omega)$, the next step is to show that ψ in (3.41) has a derivative with respect to χ_1 which converges in $L^2(\Omega)$. We first prove the following lemma.

Lemma 3.22. *Under assumption* (3.30)*, we let*

$$\delta\phi := \Phi(\chi_1^\delta + \eta\delta\chi_1^\delta) - \Phi(\chi_1^\delta). \tag{3.58}$$

Then there exists $C^\delta > 0$ such that

$$\begin{aligned} &\|\nabla\delta\phi\|_{L^\infty(\Omega)} \\ &\le C^\delta\big\{\|\nabla\delta\phi\|_{L^2(\Omega)} + \eta\|\nabla\cdot((\sigma_1^k - \sigma_2)\delta\chi_1^\delta\nabla\phi(\chi_1^\delta + \eta\delta\chi_1^\delta))\|_{H^1(\Omega)}\big\}, \end{aligned} \tag{3.59}$$

for any $\eta \in (0,\tau)$, where τ is chosen according to Remark 3.4.

Proof. Set

$$\delta\sigma^k = \Sigma^k(\chi_1^\delta + \eta\delta\chi_1^\delta) - \Sigma^k(\chi_1^\delta). \tag{3.60}$$

Then $\delta\phi$ and $\delta\sigma$ satisfy

$$\begin{gathered} \nabla\cdot(\Sigma^k(\chi_1^\delta)\nabla(\delta\phi)) = -\nabla\cdot(\delta\sigma^k\nabla\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta)) \quad \text{on } \Omega, \\ \sigma\frac{\partial(\delta\phi)}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

By assumption (3.30) and since $\delta\chi_1^\delta$ is a mollification of $\delta\chi_1$, we deduce that $\nabla\cdot(\delta\sigma^k\nabla\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta)) \in H^1(\Omega)$. Because $\chi_1^\delta, \sigma_1^k \in C^\infty(\bar\Omega)$, then $\Sigma^k(\chi_1^\delta) \in C^\infty(\bar\Omega)$. Thus, $\sigma^k(\chi_1^\delta) \in C^1(\bar\Omega)$. Using standard regularity estimate (see, e.g., [16]),

$$\|\delta\phi\|_{H^3(\Omega)} \le C_1(\|\delta\phi\|_{H^1(\Omega)} + \|\nabla\cdot(\delta\sigma^k\nabla\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta))\|_{H^1(\Omega)}), \tag{3.61}$$

for some $C_1 > 0$. Furthermore, by the Sobolev imbedding theorem, we have

$$\|\delta\phi\|_{C^{1,\gamma}(\bar\Omega)} \le C_2\|\delta\phi\|_{H^3(\Omega)}. \tag{3.62}$$

By the definition of $\|\cdot\|_{C^{l,\gamma}(\bar{\Omega})}$, the embedding

$$C^{1,\gamma}(\bar{\Omega}) \hookrightarrow C^{1}(\bar{\Omega}) \tag{3.63}$$

is continuous (see, e.g., [18]). If we compare (3.61), (3.62), and (3.63), we can deduce that $\exists \bar{C} > 0$ such that

$$\|\nabla\delta\phi\|_{L^\infty(\Omega)} \le C_2\Big(\|\delta\phi\|_{H^1(\Omega)} + \|\nabla\cdot(\delta\sigma^k\nabla\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta))\|_{H^1(\Omega)}\Big). \tag{3.64}$$

Observe that $\delta\sigma^k = \eta(\sigma_1^k - \sigma_2)\delta\chi_1^\delta$. Therefore,

$$\begin{aligned} &\|\nabla\cdot(\delta\sigma^k\nabla\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta))\|_{H^1(\Omega)} \\ &= \eta\|\nabla\cdot((\sigma_1^k - \sigma_2)\delta\chi_1^\delta\nabla\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta))\|_{H^1(\Omega)}. \end{aligned} \tag{3.65}$$

Recall that all solutions of the forward problem have zero boundary integral. Thus,

$$\int_{\partial\Omega}\delta\phi\,dS = \int_{\partial\Omega}\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta) - \Phi(\chi_1^\delta)\,dS = 0 - 0 = 0.$$

Using this and the generalized Friedrich's inequality, we obtain

$$\begin{aligned} \|\nabla\delta\phi\|^2_{L^2(\Omega)} &= \frac{1}{2}\|\nabla\delta\phi\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla\delta\phi\|^2_{L^2(\Omega)} \\ &\ge \frac{C_3}{2}\|\delta\phi\|^2_{L^2(\Omega)} - \frac{1}{2}\Big(\int_{\partial\Omega}\delta\phi\,dS\Big)^2 + \frac{1}{2}\|\nabla\delta\phi\|^2_{L^2(\Omega)} \\ &= \frac{C_3}{2}\|\delta\phi\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla\delta\phi\|^2_{L^2(\Omega)} \\ &\ge \min\Big\{\frac{C_3}{2}, \frac{1}{2}\Big\}\|\delta\phi\|^2_{H^1(\Omega)} \end{aligned} \tag{3.66}$$

for some $C_3 > 0$. Using (3.66) and (3.65), (3.64) becomes

$$\begin{aligned} \|\nabla\delta\phi\|_{L^\infty(\Omega)} &\le C_2\big\{\|\delta\phi\|_{H^1(\Omega)} + \|\nabla\cdot(\delta\sigma^k\nabla\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta))\|_{H^1(\Omega)}\big\} \\ &\le C_2\Big\{\frac{1}{\sqrt{\min\{\frac{C_3}{2}, \frac{1}{2}\}}}\|\nabla\delta\phi\|_{L^2(\Omega)} \\ &\quad + \eta\|\nabla\cdot((\sigma_1^k - \sigma_2)\delta\chi_1^\delta\nabla\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta))\|_{H^1(\Omega)}\Big\} \\ &\le 2C_2\max\Big\{\frac{1}{\sqrt{\min\{\frac{C_3}{2}, \frac{1}{2}\}}}, 1\Big\}\Big\{\|\nabla\delta\phi\|_{L^2(\Omega)} \\ &\quad + \eta\|\nabla\cdot((\sigma_1^k - \sigma_2)\delta\chi_1^\delta\nabla\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta))\|_{H^1(\Omega)}\Big\}. \end{aligned}$$

Because of (3.58) and (3.60), our claim immediately follows from the above inequality. □

Note that

$$\nabla\Phi(\chi_1^\delta) \in L^\infty(\bar{\Omega}), \nabla\Phi^*(\chi_1^\delta) \in L^\infty(\Omega), \quad \nabla\frac{\delta\Phi}{\delta\chi_1^\delta}(\chi_1^\delta;\delta\chi_1^\delta), \nabla\frac{\delta\Phi^*}{\delta\chi_1^\delta}(\chi_1^\delta;\delta\chi_1^\delta) \in L^2(\Omega).$$

Therefore, $\nabla\Phi(\chi_1^\delta)\cdot\nabla\frac{\delta\Phi^*}{\delta\chi_1}(\chi_1^\delta;\delta\chi_1^\delta) + \nabla\frac{\delta\Phi}{\delta\chi_1}(\chi_1^\delta;\delta\chi_1^\delta)\cdot\nabla\Phi^*(\chi_1^\delta) \in L^2(\Omega)$. We show in the next lemma that this is in fact the derivative of ψ defined in (3.41).

Lemma 3.23. *Under assumption* (3.30)*, we have*

$$\lim_{\eta\to 0}\|\frac{\psi(\chi_1^\delta+\eta\delta\chi_1^\delta)-\psi(\chi_1^\delta)}{\eta}-\frac{\delta\psi}{\delta\chi_1}(\chi_1^\delta;\delta\chi_1^\delta)\|_{L^2(\Omega)}=0 \tag{3.67}$$

with

$$\frac{\delta\psi}{\delta\chi_1}(\chi_1^\delta;\delta\chi_1^\delta):=\nabla\Phi(\chi_1^\delta)\cdot\nabla\frac{\delta\Phi^*}{\delta\chi_1}(\chi_1^\delta;\delta\chi_1^\delta)+\nabla\frac{\delta\Phi}{\delta\chi_1}(\chi_1^\delta;\delta\chi_1^\delta)\cdot\nabla\Phi^*(\chi_1^\delta).$$

Proof. For any perturbation $\delta\chi_1$ of χ_1, we have

$$\begin{aligned}
&\frac{\psi(\chi_1^\delta+\eta\delta\chi_1^\delta)-\psi(\chi_1^\delta)}{\eta}\\
&=\frac{\nabla\Phi(\chi_1^\delta+\eta\delta\chi_1^\delta)\cdot\nabla\Phi^*(\chi_1^\delta+\eta\delta\chi_1^\delta)}{\eta}-\frac{\nabla\Phi(\chi_1^\delta)\cdot\nabla\Phi^*(\chi_1^\delta)}{\eta}\\
&=\nabla\Phi(\chi_1^\delta+\eta\delta\chi_1^\delta)\cdot\frac{\nabla\Phi^*(\chi_1^\delta+\eta\delta\chi_1^\delta)-\nabla\Phi^*(\chi_1^\delta)}{\eta}\\
&\quad+\frac{\nabla\Phi(\chi_1^\delta+\eta\delta\chi_1^\delta)-\nabla\Phi(\chi_1^\delta)}{\eta}\cdot\nabla\Phi^*(\chi_1^\delta).
\end{aligned}\tag{3.68}$$

To continue with our proof, we first perform some convenient calculations. By adding and subtracting a term, the following is obtained:

$$\begin{aligned}
&\nabla\Phi(\chi_1^\delta+\eta\delta\chi_1^\delta)\cdot\frac{\nabla\Phi^*(\chi_1^\delta+\eta\delta\chi_1^\delta)-\nabla\Phi^*(\chi_1^\delta)}{\eta}\\
&-\nabla\Phi(\chi_1^\delta)\cdot\nabla\frac{\delta\Phi^*}{\delta\chi_1^\delta}(\chi_1^\delta;\delta\chi_1^\delta):=A_1+A_2,
\end{aligned}\tag{3.69}$$

where

$$\begin{aligned}
A_1&=\nabla\Phi(\chi_1^\delta+\eta\delta\chi_1^\delta)\cdot\frac{\nabla\Phi^*(\chi_1^\delta+\eta\delta\chi_1^\delta)-\nabla\Phi^*(\chi_1^\delta)}{\eta}\\
&\quad-\nabla\Phi(\chi_1^\delta+\eta\delta\chi_1^\delta)\cdot\nabla\frac{\delta\Phi^*}{\delta\chi_1}(\chi_1^\delta;\delta\chi_1^\delta),\\
A_2&=\nabla\Phi(\chi_1^\delta+\eta\delta\chi_1^\delta)\cdot\frac{\delta\nabla\Phi^*}{\delta\chi_1}(\chi_1^\delta;\delta\chi_1^\delta)-\nabla\Phi(\chi_1^\delta)\cdot\nabla\frac{\delta\Phi^*}{\delta\chi_1}(\chi_1^\delta;\delta\chi_1^\delta).
\end{aligned}$$

Using the Hölder's inequality and (3.37), we obtain

$$\|A_1\|_{L^2(\Omega)}\le C_1\|f\|_{\tilde{L}^2(\partial\Omega)}\|\frac{\nabla\Phi^*(\chi_1+\eta\delta\chi_1)-\nabla\Phi^*(\chi_1)}{\eta}-\nabla\frac{\delta\Phi^*}{\delta\chi_1}(\chi_1^\delta;\delta\chi_1^\delta)\|_{L^2(\Omega)},$$

for some $C_1>0$. Recall from (3.39) that

$$\|\Phi(\chi_1^\delta+\eta\delta\chi_1^\delta)-\Phi(\chi_1^\delta)\|_{H^1(\Omega)}\le C_2\eta\|\delta\chi_1\|_{L^2(\Omega)},\tag{3.70}$$

for some $C_2>0$. Using the Cauchy-Schwarz inequality, (3.48), and (3.59), we obtain

$$\begin{aligned}
\|A_2\|_{L^2(\Omega)}&\le C_3\eta\big\{C_4\|\delta\chi_1\|_{L^2(\Omega)}+\|\nabla\cdot([\sigma_1^k-\sigma_2]\delta\chi_1^\delta\nabla\Phi(\chi_1^\delta+\eta\delta\chi_1^\delta))\|_{H^1(\Omega)}\big\}\\
&\quad\times\|\nabla\frac{\delta\Phi^*}{\delta\chi_1}(\chi_1^\delta;\delta\chi_1^\delta)\|_{H^1(\Omega)},
\end{aligned}$$

for some $C_3,C_4>0$. Define

$$A_3:=\frac{\nabla\Phi(\chi_1+\eta\delta\chi_1)-\nabla\Phi(\chi_1)}{\eta}\cdot\nabla\Phi^*(\chi_1)-\nabla\frac{\delta\Phi}{\delta\chi_1}(\chi_1;\delta\chi_1)\cdot\nabla\Phi^*(\chi_1).\tag{3.71}$$

Hence,

$$\frac{\psi(\chi_1^\delta + \eta\delta\chi_1^\delta) - \psi(\chi_1^\delta)}{\eta} = \sum_{i=1}^{3} A_i.$$

Using Hölder's inequality and (3.36),

$$\begin{aligned}
&\|A_3\|_{L^2(\Omega)} \\
&\le \|\frac{\nabla\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta) - \nabla\Phi(\chi_1^\delta)}{\eta} - \nabla\frac{\delta\Phi}{\delta\chi_1}(\chi_1^\delta;\delta\chi_1^\delta)\|_{L^2(\Omega)}\|\nabla\Phi^*(\chi_1^\delta)\|_{L^\infty(\Omega)}.
\end{aligned}$$

Comparing (3.68), (3.69) and (3.71) we obtain

$$\|\frac{\psi(\chi_1^\delta + \eta\delta\chi_1^\delta) - \psi(\chi_1^\delta)}{\eta} - \frac{\delta\psi}{\delta\chi_1}(\chi_1^\delta;\delta\chi_1^\delta)\|_{L^2(\Omega)} \le \sum_{i=1}^{3}\|A_i\|,$$

using the triangle inequality. Now we only need to show that all the terms on the right-hand side of this inequality converge to 0 as η goes to 0. From the estimate of the L^2-norm of A_2 above, we can see that $\|A_2\|_{L^2(\Omega)} \to 0$. From (3.47) and (3.48), we can deduce that

$$\begin{gathered}
\lim_{\eta\to 0}\|\frac{\nabla\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta) - \nabla\Phi(\chi_1^\delta)}{\eta} - \nabla\frac{\delta\Phi}{\delta\chi_1^\delta}(\chi_1^\delta;\delta\chi_1^\delta)\|_{L^2(\Omega)} = 0, \\
\lim_{\eta\to 0}\|\frac{\nabla\Phi^*(\chi_1^\delta + \eta\delta\chi_1^\delta) - \nabla^*\Phi(\chi_1^\delta)}{\eta} - \nabla\frac{\delta\Phi^*}{\delta\chi_1^\delta}(\chi_1^\delta;\delta\chi_1^\delta)\|_{L^2(\Omega)} = 0.
\end{gathered}$$

These imply that $\|A_1\|_{L^2(\Omega)}, \|A_3\|_{L^2(\Omega)} \to 0$. □

We now use our results on Φ and Φ^* to study Σ_1. We first show that under assumption (3.30), (3.29) has a unique solution σ_1^{k+1}. We then proceed with finding the regularity of the said solution. Observe that σ_1^{k+1} depends on ϕ and ϕ^*. Hence, we can investigate how the mollification of χ_1 affects σ_1^{k+1}. We show that σ_1^{k+1} continuously depends on χ_1. Furthermore, we prove that $\frac{\delta\Sigma_1}{\delta\chi_1}(\chi_1^\delta;\delta\chi_1^\delta) \in H^1(\Omega)$. We start by equipping $H^1(\Omega)$ with a suitable norm.

Proposition 3.24. *Under assumption* (3.30) *we define*

$$|v|^2_{H^1(\Omega)} := \alpha\int_\Omega(\chi+\epsilon)|\nabla v|^2\,dV + \theta\lambda\int v^2\,dV,$$

where $\chi(x) \in [0,1]$ for all $x \in \Omega$, and let $\|\cdot\|_{H^1(\Omega)}$ be the standard $H^1(\Omega)$ norm. Then $|\cdot|_{H^1(\Omega)}$ and $\|\cdot\|_{H^1(\Omega)}$ are equivalent.

Proof. Observe that

$$\begin{aligned}
\min\{\alpha\epsilon,\lambda\}\|v\|^2_{H^1(\Omega)} &\le \alpha\int_\Omega \epsilon|\nabla v|^2\,dV + \theta\int_\Omega |v|^2\,dV \\
&\le \alpha\int_\Omega(\chi+\epsilon)|\nabla v|^2\,dV + \lambda\int_\Omega |v|^2\,dV = |v|^2_{H^1(\Omega)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|v|^2_{H^1(\Omega)} &= \alpha\int_\Omega(\chi+\epsilon)|\nabla v|^2\,dV + \lambda\int_\Omega |v|^2\,dV \\
&\le \alpha\int_\Omega(1+\epsilon)|\nabla v|^2\,dV + \lambda\int_\Omega |v|^2\,dV
\end{aligned}$$

$$\leq \max\{\alpha(1+\epsilon),\lambda\}\,\|v\|^2_{H^1(\Omega)}.$$

□

We now establish the regularity of σ_1^{k+1} .

Lemma 3.25. *Under assumption* (3.30)*, the variational formulation*

$$\begin{aligned}&\int_\Omega \alpha(\chi_1^\delta+\epsilon)\nabla\sigma_1^{k+1}\cdot\nabla v\,dV+\int_\Omega \lambda(\sigma_1^{k+1}-\sigma_1^k)v\,dV\\&=\int_\Omega \chi_1^\delta\nabla\Phi(\chi_1^\delta)\cdot\nabla\Phi^*(\chi_1^\delta)v\,dV\end{aligned}\tag{3.72}$$

for all $v\in H^1(\Omega)$, has a unique solution $\sigma_1^{k+1}\in H^1(\Omega)$. Furthermore,

$$\|\sigma_1^{k+1}\|_{H^1(\Omega)}\leq\frac{2\sqrt{\mu(\Omega)}\max\{C_1^\delta,C_2^\delta\}}{\min\{\alpha\epsilon,\lambda\}}\tag{3.73}$$

where $C_1^\delta=\|\nabla\Phi(\chi_1^\delta)\|_{L^\infty(\Omega)}\|\nabla\Phi^*(\chi_1^\delta)\|_{L^\infty(\Omega)}$ and $C_2^\delta=\lambda\|\sigma_1^k\|_{L^\infty(\Omega)}$.

Proof. Let $u,v\in H^1(\Omega)$ and define $a(u,v)=\int_\Omega\alpha(\chi_1^\delta+\epsilon)\nabla u\cdot\nabla v\,dV+\int_\Omega\lambda uv\,dV$ and $b(v)=\int_\Omega(\chi_1^\delta)\nabla\Phi(\chi_1^\delta)\cdot\nabla\Phi^*(\chi_1^\delta)v\,dV+\int_\Omega\lambda\sigma_1^k v\,dV$. It is obvious that a is bilinear and b is linear. Using the Cauchy-Schwarz inequality, one can prove that $a(u,v)$ is continuous. We can also easily show that $a(u,v)$ is coercive using the previous proposition:

$$\begin{aligned}|a(u,u)|&=\int_\Omega\alpha(\chi_1^\delta+\epsilon)|\nabla u|^2V+\int_\Omega\lambda u^2\,dV\\&=|u|^2_{H^1(\Omega)}\geq\min\{\alpha\epsilon,\theta\}\|u\|^2_{H^1(\Omega)}.\end{aligned}$$

Furthermore, the continuity of $b(v)$ can be proven using the Cauchy-Schwarz inequality, the bounds in (3.31), and (3.36). Hence, by Lax-Milgram Theorem there is a unique $\sigma_1^k\in H^1(\Omega)$ satisfying (3.72) for all $v\in H^1(\Omega)$. The $H^1(\Omega)$ bound for σ_1^{k+1} directly follows. □

Remark 3.26. Given any perturbation $\delta\chi_1^\delta$ and $\eta>0$, we have to make sure that the quantity $\Sigma_1(\chi_1^\delta+\eta\delta\chi_1^\delta)$ is well-defined. As shown in the proof of Lemma (3.25), replacing χ_1^δ with $\chi_1^\delta+\eta\delta\chi_1^\delta$ cannot be done with just any η. To make sure that the bilinear functional a is coercive, $\chi_1^\delta+\eta\delta\chi_1^\delta+\epsilon$ must be positive. Since $\chi_1^\delta+\epsilon>0$, we can choose η small enough so that $\chi_1^\delta+\eta\delta\chi_1^\delta+\epsilon>0$ is satisfied. Hence, similar to Remark 3.4, we take η from the set $(0,\bar\tau)$ for $\bar\tau$ sufficiently small so that $\Sigma_1(\chi_1^\delta+\eta\delta\chi_1^\delta)$ makes sense. Therefore, combining (3.37), (3.38) and (3.73), there exists $C>0$ such that

$$\|\Sigma_1(\chi_1^\delta+\eta\delta\chi_1^\delta)\|_{H^1(\Omega)}<\infty,\tag{3.74}$$

for any $\eta\in(0,\hat\tau)$, where $\hat\tau=\min\{\tau,\bar\tau\}$.

Because χ_1^δ is a mollification of χ_1, it is real analytic. Then $\chi_1^\delta+\epsilon\in C^\infty(\bar\Omega)$. Moreover, from Lemma (3.15), $\nabla\Phi(\chi_1^\delta),\nabla\Phi^*(\chi_1^\delta)\in C^\infty(\bar\Omega)$. This means that $\chi_1^\delta\nabla\Phi(\chi_1^\delta)\cdot\nabla\Phi^*(\chi_1^\delta)$, $\chi_1^\delta+\epsilon$, λ are all in $C^\infty(\bar\Omega)$. Then using standard regularity estimates [16] on

$$\begin{gathered}-\alpha\nabla\cdot[(\chi_1^\delta+\epsilon)\nabla\sigma_1^{k+1}]+\lambda(\sigma_1^{k+1}-\sigma_1^k)=\chi_1^\delta\nabla\Phi(\chi_1^\delta)\cdot\nabla\Phi^*(\chi_1^\delta)\quad\text{on }\Omega,\\\frac{\partial\sigma_1^{k+1}}{\partial n}=0\quad\text{on }\partial\Omega,\end{gathered}\tag{3.75}$$

gives us $\sigma_1^{k+1} \in C^\infty(\bar{\Omega})$. Consequently, $\|\nabla\Sigma_1(\chi_1^\delta)\|_{L^\infty(\Omega)} < \infty$. Furthermore,

$$\|\nabla\Sigma_1(\chi_1^\delta + \eta\delta\chi_1^\delta)\|_{L^\infty(\Omega)} < \infty, \tag{3.76}$$

for any $\eta \in (0, \hat{\tau})$, where $\hat{\tau} = \min\{\tau, \bar{\tau}\}$, τ and $\bar{\tau}$ are chosen according to Remark 3.4 and Remark 3.26, respectively. Before we can show that $\frac{\delta\Sigma_1}{\delta\chi_1}(\chi_1^\delta; \delta\chi_1^\delta) \in H^1(\Omega)$, we first need to find a candidate derivative.

Lemma 3.27. *Under assumption* (3.30)*, there exists $D_\sigma(\chi_1; \delta\chi_1) \in H^1(\Omega)$ such that*

$$\begin{aligned}&\int_\Omega \alpha(\chi_1^\delta + \epsilon)\nabla D_\sigma(\chi_1^\delta; \delta\chi_1^\delta) \cdot \nabla v \, dV + \int_\Omega \lambda D_\sigma(\chi_1^\delta; \delta\chi_1^\delta) v \, dV \\ &= \int_\Omega \chi_1^\delta \frac{\delta\psi}{\delta\chi_1}(\chi_1^\delta; \delta\chi_1^\delta) v + \delta\chi_1^\delta \nabla\Phi(\chi_1^\delta) \cdot \nabla\Phi^*(\chi_1^\delta) v - \alpha\delta\chi_1^\delta \nabla(\Sigma_1(\chi_1^\delta)) \cdot \nabla v \, dV,\end{aligned} \tag{3.77}$$

for all $v \in H^1(\Omega)$.

The proof of the above lemma is similar to that of Lemma 3.19; we omit it. We now have all the necessary tools to show that $\frac{\delta\Sigma_1}{\delta\chi_1}(\chi_1^\delta; \delta\chi_1^\delta) \in H^1(\Omega)$. We prove that this is exactly $D_\sigma(\chi_1^\delta; \delta\chi_1^\delta)$ computed in the previous lemma.

Theorem 3.28. *Under assumption* (3.30)*, there exists $C^\delta > 0$ such that*

$$\|\Sigma_1(\chi_1^\delta + \eta\delta\chi_1^\delta) - \Sigma_1(\chi_1^\delta)\|_{H^1(\Omega)} \le C^\delta \eta \|\delta\chi_1\|_{L^2(\Omega)} \tag{3.78}$$

for any $\eta \in (0, \hat{\tau})$, where $\hat{\tau} = \min\{\tau, \bar{\tau}\}$, τ and $\bar{\tau}$ are both chosen according to Remark 3.4 and Remark 3.26, respectively. Also,

$$\lim_{\eta\to 0} \Big\| \frac{\Sigma_1(\chi_1^\delta + \eta\delta\chi_1^\delta) - \Sigma_1(\chi_1^\delta)}{\eta} - D_\sigma(\chi_1^\delta; \delta\chi_1^\delta) \Big\| = 0. \tag{3.79}$$

Furthermore, we make the identification $\frac{\delta\Sigma_1}{\delta\chi_1}(\chi_1^\delta; \delta\chi_1^\delta) = D_\sigma(\chi_1^\delta; \delta\chi_1^\delta) \in H^1(\Omega)$.

Proof. Let $v \in H^1(\Omega)$ and $\eta > 0$. From (3.72), we have

$$\begin{aligned}&\int_\Omega \alpha(\chi_1^\delta + \epsilon)\nabla\Sigma_1(\chi_1^\delta) \cdot \nabla v + \lambda(\Sigma_1(\chi_1^\delta) - \sigma_1^k) v \, dV \\ &= \int_\Omega \chi_1^\delta \nabla\Phi(\chi_1^\delta) \cdot \nabla\Phi^*(\chi_1^\delta) v \, dV.\end{aligned} \tag{3.80}$$

Similarly, if we replace χ_1^δ in (3.72) with $\chi_1^\delta + \eta\delta\chi_1^\delta$, we have

$$\begin{aligned}&\int_\Omega \alpha(\chi_1^\delta + \eta\delta\chi_1^\delta + \epsilon)\nabla(\Sigma_1(\chi_1^\delta + \eta\delta\chi_1^\delta)) \cdot \nabla v \, dV \\ &+ \int_\Omega \lambda(\Sigma_1(\chi_1^\delta + \eta\delta\chi_1^\delta) - \sigma_1^k) v \, dV =: A(v),\end{aligned} \tag{3.81}$$

where

$$A(v) = \int_\Omega (\chi_1^\delta + \eta\delta\chi_1^\delta)\nabla\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta) \cdot \nabla\Phi^*(\chi_1^\delta + \delta\chi_1^\delta) v \, dV.$$

Subtracting (3.80) from (3.81), we obtain

$$A_1(v) + A_2(v) = B_1(v) + B_2(v) + B_3(v), \tag{3.82}$$

where

$$A_1(v) := \int_\Omega \alpha(\chi_1^\delta + \epsilon)\nabla[\Sigma_1(\chi_1^\delta + \eta\delta\chi_1^\delta) - \Sigma_1(\chi_1^\delta)] \cdot \nabla v \, dV,$$

$$
\begin{aligned}
A_2(v) &:= \int_\Omega \lambda(\Sigma_1(\chi_1^\delta + \eta\delta\chi_1^\delta) - \Sigma_1(\chi_1^\delta))v\, dV,\\
B_1(v) &:= \int_\Omega \chi_1^\delta[\nabla\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta)\cdot\nabla\Phi^*(\chi_1^\delta + \eta\delta\chi_1^\delta) - \nabla\Phi(\chi_1^\delta)\cdot\nabla\Phi^*(\chi_1^\delta)]v\, dV,\\
B_2(v) &:= \int_\Omega \eta\delta\chi_1^\delta\nabla\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta)\cdot\nabla\Phi^*(\chi_1^\delta + \eta\delta\chi_1^\delta)v\, dV,\\
B_3(v) &:= -\int_\Omega \alpha\eta\delta\chi_1^\delta\nabla(\Sigma_1(\chi_1^\delta + \eta\delta\chi_1^\delta))\cdot\nabla v\, dV,
\end{aligned}
$$

for all $v \in H^1(\Omega)$. We define $a(\Sigma_1(\chi_1^\delta + \eta\delta\chi_1^\delta) - \Sigma_1(\chi_1^\delta), v) := A_1(v) + A_2(v)$ and $b(v) := B_1(v) + B_2(v) + B_3(v)$, for all $v \in H^1(\Omega)$. In Lemma (3.25), we have shown already that a is bilinear, coercive, and continuous. Clearly, b is linear. Now we only need to show that b is continuous. So we need to estimate $B_1(v)$, $B_2(v)$, and $B_3(v)$. For all these, we use the Cauchy-Schwarz inequality. To show continuity of $B_1(v)$, we also use (3.43). For $B_2(v)$ and $B_3(v)$, one uses the Cauchy-Schwarz inequality, Hölder's inequality, and Young's inequality for convolutions to show continuity. It is worth noting that $\|\nabla\Phi(\chi_1^\delta + \eta\delta\chi_1^\delta)\|_{L^\infty(\Omega)}, \|\nabla\Phi^*(\chi_1^\delta + \eta\delta\chi_1^\delta)\|_{L^\infty(\Omega)}$, and the quantity $\|\Sigma_1(\chi_1^\delta + \eta\delta\chi_1^\delta)\|_{H^1(\Omega)}$ are independent of η as shown in (3.37), (3.38), and (3.74). Combining these implies that $b(v)$ is bounded.

We make the substitution $u = v = \Sigma_1(\chi_1^\delta + \eta\delta\chi_1^\delta) - \Sigma_1(\chi_1^\delta)$. Furthermore, using the coercivity of a and the boundedness of b, we conclude that

$$
\|\Sigma_1(\chi_1^\delta + \eta\delta\chi_1^\delta) - \Sigma_1(\chi_1^\delta)\|_{H^1(\Omega)} \le C\eta\|\delta\chi_1\|_{L^2(\Omega)}. \tag{3.83}
$$

Hence, the proof of our first statement is complete.

Subtracting (3.77) from (3.82), we obtain

$$
D_1(v) + D_2(v) = E_1(v) + E_2(v) + E_3(v), \tag{3.84}
$$

where

$$
\begin{aligned}
D_1(v) &= \int_\Omega \alpha(\chi_1^\delta + \epsilon)\nabla\big[\frac{\Sigma_1(\chi_1^\delta + \eta\delta\chi_1^\delta) - \Sigma_1(\chi_1^\delta)}{\eta} - D_\sigma(\chi_1^\delta; \delta\chi_1^\delta)\big]\cdot\nabla v\, dV,\\
D_2(v) &= \int_\Omega \lambda\big[\frac{\Sigma_1(\chi_1^\delta + \eta\delta\chi_1^\delta) - \Sigma_1(\chi_1^\delta)}{\eta} - D_\sigma(\chi_1^\delta; \delta\chi_1^\delta)\big]v\, dV,\\
E_1(v) &= \int_\Omega \chi_1^\delta\big[\frac{\psi(\chi_1^\delta + \eta\delta\chi_1^\delta) - \psi(\chi_1^\delta)}{\eta} - \frac{\delta\psi}{\delta\chi_1^\delta}(\chi_1^\delta; \delta\chi_1^\delta)\big]v\, dV,\\
E_2(v) &= \int_\Omega \delta\chi_1^\delta\big[\psi(\chi_1^\delta + \eta\delta\chi_1^\delta) - \psi(\chi_1^\delta)\big]v\, dV,\\
E_3(v) &= -\int_\Omega \alpha\delta\chi_1^\delta\nabla\big[\Sigma_1(\chi_1^\delta + \eta\delta\chi_1^\delta) - \Sigma_1(\chi_1^\delta)\big]\cdot\nabla v\, dV,
\end{aligned}
$$

and ψ is the function defined in (3.41).

Using the Cauchy-Schwarz inequality, Hölder's inequality, (3.43), and (3.83), we can estimate $E_1(v)$, $E_2(v)$, and $E_3(v)$. Indeed,

$$
|E_1(v)| \le \|\frac{\psi(\chi_1^\delta + \eta\delta\chi_1^\delta) - \psi(\chi_1^\delta)}{\eta} - \frac{\delta\psi}{\delta\chi_1}(\chi_1^\delta; \delta\chi_1^\delta)\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}.
$$

Similarly,

$$
|E_2(v)| \le C_2\eta\|\delta\chi_1^\delta\|_{L^\infty(\Omega)}\|\delta\chi_1\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)},
$$

for some $C_2 > 0$. Finally,

$$\begin{aligned}|E_3(v)| &\le \alpha\|\delta\chi_1^\delta\|_{L^\infty(\Omega)}\|\nabla[\Sigma_1(\chi_1^\delta+\eta\delta\chi_1^\delta)-\Sigma_1(\chi_1^\delta)]\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}\\ &\le C\eta\|\delta\chi_1^\delta\|_{L^\infty(\Omega)}\|\delta\chi_1\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)}.\end{aligned}$$

If we make the substitution

$$v = \frac{\Sigma_1(\chi_1^\delta+\eta\delta\chi_1^\delta)-\Sigma_1(\chi_1^\delta)}{\eta} - D_\sigma(\chi_1^\delta;\delta\chi_1^\delta),$$

then by Proposition 3.24, we obtain

$$\begin{aligned}|D_1(v)+D_2(v)| &= \Big|\frac{\Sigma_1(\chi_1^\delta+\eta\delta\chi_1^\delta)-\Sigma_1(\chi_1^\delta)}{\eta} - D_\sigma(\chi_1^\delta;\delta\chi_1^\delta)\Big|^2_{H^1(\Omega)}\\ &\ge \min\{\alpha\epsilon,\lambda\}\Big\|\frac{\Sigma_1(\chi_1^\delta+\eta\delta\chi_1^\delta)-\Sigma_1(\chi_1^\delta)}{\eta} - D_\sigma(\chi_1^\delta;\delta\chi_1^\delta)\Big\|^2_{H^1(\Omega)}.\end{aligned}$$

Hence, the above inequality and (3.84) imply

$$\begin{aligned}&\min\{\alpha\epsilon,\lambda\}\Big\|\frac{\Sigma_1(\chi_1^\delta+\eta\delta\chi_1^\delta)-\Sigma_1(\chi_1^\delta)}{\eta} - D_\sigma(\chi_1^\delta;\delta\chi_1^\delta)\Big\|_{H^1(\Omega)}\\ &\le |E_1(v)|+|E_2(v)|+|E_3(v)|.\end{aligned}$$

Taking the limit as $\eta \to 0$, it is clear that $|E_2(v)|+|E_3(v)| \to 0$.

Lastly, from (3.67), we have $|E_1(v)| \to 0$, which then implies our second statement. The last statement follows immediately from the previous lemma. □

Now that we have established that $\frac{\delta\Sigma_1}{\delta\chi_1}(\chi_1^\delta;\delta\chi_1^\delta) \in H^1(\Omega)$, the gradient of the functional J shown in (2.14) is justified.

4. Existence of a fixed point

In (3.28), the update for χ_1 was introduced. In this section, we show that this update has a fixed point. In other words, we show that

$$\Upsilon(\chi_1) := (T_\delta \circ M \circ H \circ \Theta \circ G \circ T_\delta)(\chi_1) \tag{4.1}$$

has a fixed point on some suitable space. We use the following fixed point theorem to prove this.

Theorem 4.1 (Schauder fixed point). *Let K be a convex subset of $L^2(\Omega)$ and suppose $\Upsilon : K \to L^2(\Omega)$ is continuous. Suppose $\Upsilon(K)$ is a compact subset of K. Then Υ has a fixed point in K (see* [15]*).*

Thus, it is necessary to show that Υ is continuous on a convex subset K of $L^2(\Omega)$ and that $\Upsilon(K)$ is compact in K. The previous section justified the calculated formulation of the function G defined in (2.14). We now show that

$$G(\chi_1) = \chi_1 - \omega[-2(\Sigma_1(\chi_1)-\sigma_2)\psi(\chi_1) + \alpha|\nabla\Sigma_1(\chi_1)|^2] \tag{4.2}$$

is continuous.

Lemma 4.2. *Under assumption* (3.30)*,*

$$\lim_{\eta\to 0}\|G(\chi_1^\delta+\eta\delta\chi_1^\delta) - G(\chi_1^\delta)\|_{L^2(\Omega)} = 0. \tag{4.3}$$

Proof. Denote

$$A_1(\chi_1^\delta) := 2\omega[\Sigma_1(\chi_1^\delta + \eta\delta\chi_1^\delta) - \sigma_2]\psi(\chi_1^\delta),$$
$$A_2(\chi_1^\delta) := -\alpha\omega|\nabla\Sigma_1(\chi_1^\delta)|^2.$$

Thus, using (4.2), the triangle inequality, and Young's inequality for a convolution, we obtain

$$\begin{aligned} &\|G(\chi_1^\delta + \eta\delta\chi_1^\delta) - G(\chi_1^\delta)\|_{L^2(\Omega)} \\ &\le \eta\|\xi_\delta\|_{L^1(\Omega)}\|\delta\chi_1\|_{L^2(\Omega)} + \|A_1(\chi_1^\delta + \eta\delta\chi_1^\delta) - A_1(\chi_1^\delta)\|_{L^2(\Omega)} \\ &\quad + \|A_2(\chi_1^\delta + \eta\delta\chi_1^\delta) - A_2(\chi_1^\delta)\|_{L^2(\Omega)}. \end{aligned} \tag{4.4}$$

Adding and subtracting $2\omega[\Sigma_1(\chi_1^\delta + \eta\delta\chi_1^\delta) - \sigma_2]\psi(\chi_1^\delta)$ to $A_1(\chi_1^\delta + \eta\delta\chi_1^\delta) - A_1(\chi_1^\delta)$ and using the triangle inequality, we obtain

$$\|A_1(\chi_1^\delta + \eta\delta\chi_1^\delta) - A_1(\chi_1^\delta)\|_{L^2(\Omega)} \le \|B_1(\chi_1;\delta\chi_1)\|_{L^2(\Omega)} + \|B_2(\chi_1;\delta\chi_1)\|_{L^2(\Omega)},$$

where

$$\begin{aligned} B_1(\chi_1^\delta;\delta\chi_1^\delta) &= -2\omega[\Sigma_1(\chi_1^\delta + \eta\delta\chi_1^\delta) - \sigma_2]\psi(\chi_1^\delta + \eta\delta\chi_1^\delta) \\ &\quad + 2\omega[\Sigma_1(\chi_1^\delta + \eta\delta\chi_1^\delta) - \sigma_2]\psi(\chi_1^\delta), \\ B_2(\chi_1^\delta;\delta\chi_1^\delta) &= -2\omega[\Sigma_1(\chi_1^\delta + \eta\delta\chi_1^\delta) - \sigma_2]\psi(\chi_1^\delta) + 2\omega[\Sigma_1(\chi_1^\delta) - \sigma_2]\psi(\chi_1^\delta). \end{aligned}$$

By using Hölder's inequality, (3.76), (3.43), (3.31), (3.36), and (3.78), we can estimate $B_1(\chi_1^\delta;\delta\chi_1^\delta)$ and $B_2(\chi_1^\delta;\delta\chi_1^\delta)$ as follows:

$$\|B_1(\chi_1^\delta;\delta\chi_1^\delta)\|_{L^2(\Omega)} \le 2C_1\omega\eta\|[\Sigma_1(\chi_1^\delta + \eta\delta\chi_1^\delta) - \sigma_2]\|_{L^\infty(\Omega)}\|\delta\chi_1\|_{L^2(\Omega)}, \tag{4.5}$$

for some $C_1 > 0$ and

$$\|B_2(\chi_1^\delta;\delta\chi_1^\delta)\|_{L^1(\Omega)} \le C_2\eta\|\delta\chi_1\|_{L^2(\Omega)}\|\nabla\Phi(\chi_1^\delta)\|_{L^\infty(\Omega)}\|\nabla\Phi^*(\chi_1^\delta)\|_{L^2(\Omega)}, \tag{4.6}$$

for some $C_2 > 0$. Also, $A_2(\chi_1^\delta + \eta\delta\chi_1^\delta) - A_2(\chi_1^\delta)$ can be estimated using (3.76) and (3.78):

$$\begin{aligned} &\|A_2(\chi_1^\delta + \eta\delta\chi_1^\delta) - A_2(\chi_1^\delta)\|_{L^2(\Omega)} \\ &\le \alpha\omega C_3\eta\|\delta\chi_1\|_{L^2(\Omega)}\big\{\|\Sigma_1(\chi_1^\delta\eta\delta\chi_1^\delta)\|_{L^\infty(\Omega)} + \|\Sigma_1(\chi_1^\delta)\|_{L^\infty(\Omega)}\big\}, \end{aligned} \tag{4.7}$$

for some $C_3 > 0$. Comparing (4.4), (4.5), (4.6), and (4.7) implies the existence of a $C > 0$ such that

$$\|G(\chi_1^\delta + \eta\delta\chi_1^\delta) - G(\chi_1^\delta)\|_{L^2(\Omega)} \le C\eta\|\delta\chi_1\|_{L^2(\Omega)},$$

for any $\eta \in (0, \hat{\tau})$, where $\hat{\tau} = \min\{\tau, \bar{\tau}\}$, τ and $\bar{\tau}$ are both chosen according to Remark 3.4 and Remark 3.26, respectively. Taking the limit of the above inequality as $\eta \to 0$ gives us our desired result. □

Now that we have shown the continuity of the function G, we prove the continuity of the operator Θ (see Definition 2.6). Before proving continuity, we first show that given χ_1^δ, (2.16) has a solution in $H^1(\Omega)$. Note that because $\chi_1^\delta \in C^\infty(\Omega)$, $\nabla\chi_1^\delta$ is bounded in Ω for a fixed δ. Hence,

$$\sqrt{|\nabla\chi_1^\delta|^2 + \beta^2} \le \sqrt{\|\nabla\chi_1^\delta\|^2_{L^\infty(\Omega)} + \beta^2} =: \bar{K} < \infty.$$

Or equivalently,

$$\frac{1}{\sqrt{|\nabla\chi_1^\delta|^2 + \beta^2}} \ge \frac{1}{\bar{K}}. \tag{4.8}$$

Obviously, for any $\beta > 0$, $\sqrt{|\nabla\chi_1^\delta|^2+\beta^2} \geq \beta$, which can be expressed as

$$\frac{1}{\sqrt{|\nabla\chi_1^\delta|^2+\beta^2}} \leq \frac{1}{\beta}. \tag{4.9}$$

For $u, v \in H^1(\Omega)$, we define

$$\begin{aligned} a(u,v) &= \int_\Omega \omega\gamma \frac{\nabla u\cdot\nabla v}{\sqrt{|\nabla\chi_1^\delta|^2+\beta^2}} + uv\,dV, \qquad (4.10)\\ b(v) &= \int_\Omega G(\chi_1^\delta)v\,dV. \end{aligned}$$

Clearly, a and b are bilinear and linear, respectively. Observe that using the Cauchy-Schwarz inequality and (4.9), we obtain

$$|a(u,v)| \leq 2\max\{\frac{\omega\gamma}{\beta},1\}\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)},$$

for all $v \in H^1(\Omega)$, which makes a continuous. It can also be shown using the Cauchy-Schwarz inequality that $b(v)$ is also continuous. Using (4.9), a can be proven to be coercive. Therefore, using the Lax-Milgram theorem, there exists a unique $\theta \in H^1(\Omega)$ satisfying $a(\theta, v) = b(v)]$ for all $v \in H^1(\Omega)$.

Lemma 4.3. *Under assumption* (3.30)*, there exists a unique $\theta \in H^1(\Omega)$ satisfying*

$$\int_\Omega \omega\gamma \frac{\nabla\theta\cdot\nabla v}{\sqrt{|\nabla\chi_1^\delta|^2+\beta^2}} + \theta v\,dV = \int_\Omega G(\chi_1^\delta)v\,dV, \tag{4.11}$$

for any $v \in H^1(\Omega)$ provided that $\frac{\partial v}{\partial n} = 0$ on $\partial\Omega$. Furthermore, there exists $C > 0$ such that

$$\|\theta\|_{H^1(\Omega)} \leq \frac{1}{\min\{\frac{\omega\gamma}{K},1\}}\|G(\chi_1^\delta)\|_{L^2(\Omega)}.$$

The above lemma tells us that given χ_1, a mollification can be performed to obtain a unique solution $\theta \in H^1(\Omega)$ to (4.11). Hence, we can think of $\Theta(\chi_1^\delta)$ as a function that maps an element $\chi_1 \in L^2(\Omega)$ to an element $\theta \in H^1(\Omega)$. Note that given a perturbation $\delta\chi_1^\delta$ of χ_1^δ, $\Theta(\chi_1^\delta + \eta\delta\chi_1^\delta)$ is well-defined for any $0 < \eta < \infty$ because for coercivity we just need $\sqrt{\|\nabla(\chi_1^\delta+\eta\delta\chi_1^\delta)\|^2_{L^\infty(\Omega)}+\beta^2}$ to be finite. Since $\chi_1^\delta + \eta\delta\chi_1^\delta \in C^\infty(\bar\Omega)$, this is not a problem. From the definition of G and the inequalities (3.37), (3.38), and (3.76), we can infer that

$$\|\Theta(\chi_1^\delta+\eta\delta\chi_1^\delta)\|_{H^1(\Omega)} \leq C\|G(\chi_1^\delta+\eta\delta\chi_1^\delta)\|_{L^2(\Omega)} < \infty, \tag{4.12}$$

for some $C > 0$. We now prove that this map is continuous.

Lemma 4.4. *Under assumption* (3.30)*,*

$$\lim_{\eta\to 0}\|\Theta(\chi_1^\delta+\eta\delta\chi_1^\delta) - \Theta(\chi_1^\delta)\|_{H^1(\Omega)} = 0.$$

Proof. By the previous lemma, note that $\Theta(\chi_1^\delta)$ satisfies

$$\int_\Omega \omega\gamma \frac{\nabla\Theta(\chi_1^\delta)\cdot\nabla v}{\sqrt{|\nabla\chi_1^\delta|^2+\beta^2}} + \Theta(\chi_1^\delta)v\,dV = \int_\Omega G(\chi_1^\delta)v\,dV. \tag{4.13}$$

Similarly, $\Theta(\chi_1^\delta+\eta\delta\chi_1^\delta)$ satisfies

$$\begin{aligned}&\int_\Omega \omega\gamma\frac{\nabla\Theta(\chi_1^\delta+\eta\delta\chi_1^\delta)\cdot\nabla v}{\sqrt{|\nabla(\chi_1^\delta+\eta\delta\chi_1^\delta)|^2+\beta^2}}+\Theta(\chi_1^\delta+\eta\delta\chi_1^\delta)v\,dV\\&=\int_\Omega G(\chi_1^\delta+\eta\delta\chi_1^\delta)v\,dV.\end{aligned}\tag{4.14}$$

Subtracting (4.13) from (4.14), we obtain

$$A_1(v)+A_2(v)=B(v),\tag{4.15}$$

with

$$\begin{gathered}A_1(v):=\int_\Omega\omega\gamma\Big[\frac{\nabla\Theta(\chi_1^\delta+\eta\delta\chi_1^\delta)\cdot\nabla v}{\sqrt{|\nabla\chi_1^\delta+\eta\delta\chi_1^\delta|^2+\beta^2}}-\frac{\nabla\Theta(\chi_1^\delta)\cdot\nabla v}{\sqrt{|\nabla\chi_1^\delta|^2+\beta^2}}\Big]\,dV,\\A_2(v):=\int_\Omega[\Theta(\chi_1^\delta+\eta\delta\chi_1^\delta)-\Theta(\chi_1^\delta)]v\,dV,\\B(v):=\int_\Omega[G(\chi_1^\delta+\eta\delta\chi_1^\delta)-G(\chi_1^\delta)]v\,dV.\end{gathered}$$

We subtract and add the term

$$\int_\Omega\omega\gamma\frac{\nabla\Theta(\chi_1^\delta+\eta\delta\chi_1^\delta)\cdot\nabla v}{\sqrt{|\nabla\chi_1^\delta|^2+\beta^2}}\,dV$$

to $A_1(v)$ to obtain

$$A_1(v)=A_3(v)+A_4(v),\tag{4.16}$$

with

$$\begin{gathered}A_4(v):=\int_\Omega\omega\gamma D(\eta)\frac{\nabla\Theta(\chi_1^\delta+\eta\delta\chi_1^\delta)\cdot\nabla v}{\sqrt{|\nabla(\chi_1^\delta)|^2+\beta^2}},\\A_3(v):=\int_\Omega\omega\gamma\frac{[\nabla\Theta(\chi_1^\delta+\eta\delta\chi_1^\delta)-\nabla\Theta(\chi_1^\delta)]\cdot\nabla v}{\sqrt{|\nabla\chi_1^\delta|^2+\beta^2}}\,dV,\\D(\eta):=\frac{1}{\sqrt{|\nabla(\chi_1^\delta+\eta\delta\chi_1^\delta)|^2+\beta^2}}-\frac{1}{\sqrt{|\nabla\chi_1^\delta|^2+\beta^2}}.\end{gathered}$$

From (4.15) and (4.16), we have

$$A_3(v)+A_2(v)=B(v)-A_4(v).\tag{4.17}$$

From the definition of the bilinear functional a in (4.10), we deduce that

$$A_3(v)+A_2(v)=a(\Theta(\chi_1^\delta+\eta\delta\chi_1^\delta)-\Theta(\chi_1^\delta),v).$$

From the coercivity of a we can show that

$$\begin{aligned}&|(A_3+A_2)(\Theta(\chi_1^\delta+\eta\delta\chi_1^\delta)-\Theta(\chi_1^\delta))|\\&\ge\min(\frac{\omega\gamma}{\bar K},1)\|\Theta(\chi_1^\delta+\eta\delta\chi_1^\delta)-\Theta(\chi_1^\delta)\|^2_{H^1(\Omega)}.\end{aligned}\tag{4.18}$$

On the other hand, using the Cauchy-Schwarz inequality, we obtain

$$|B(v)|\le\|G(\chi_1^\delta+\eta\delta\chi_1^\delta)-G(\chi_1^\delta)\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)}.\tag{4.19}$$

Moreover, using the Cauchy-Schwarz inequality, Hölder's inequality, and (4.9), we have

$$|A_4(v)|\le\frac{\omega\gamma}{\beta}\|D(\eta)\|_{L^\infty(\Omega)}\|\Theta(\chi_1^\delta+\eta\delta\chi_1^\delta)\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.\tag{4.20}$$

Comparing (4.17), (4.18), (4.19), and (4.20), we obtain

$$\begin{aligned}
&\min\{\frac{\omega\gamma}{\bar{K}},1\}\|\Theta(\chi_1^\delta+\eta\delta\chi_1^\delta)-\Theta(\chi_1^\delta)\|_{H^1(\Omega)}^2\\
&\le \|G(\chi_1^\delta+\eta\delta\chi_1^\delta)-G(\chi_1^\delta)\|_{L^2(\Omega)}+\frac{\omega\gamma}{\beta}\|D(\eta)\|_{L^\infty(\Omega)}\|\Theta(\chi_1^\delta+\eta\delta\chi_1^\delta)\|_{H^1(\Omega)}.
\end{aligned}$$

From (4.12), the right-hand side of the above inequality is bounded for any $\eta\in(0,\hat{\tau})$, where $\hat{\tau}=\min\{\tau,\bar{\tau}\}$, τ and $\bar{\tau}$ are both chosen according to Remark 3.4 and Remark 3.26, respectively. Obviously, $\lim_{\eta\to 0}\|D(\eta)\|_{L^\infty(\Omega)}=0$. The above inequality, together with (4.3), establishes our claim. □

We now establish that the function H defined in (3.25) is continuous. The details of the proof can be found in [15].

Lemma 4.5. *Let $z\in L^2(\Omega)\backslash H^1(\Omega)$ and suppose $\{g_n\}_{n=1}^\infty\subset L^2(\Omega)$ converge to g in $L^2(\Omega)$. Then*

$$\lim_{n\to\infty}\mu(H(g)\triangle H(g_n))=0, \tag{4.21}$$

where

$$H(g)=\{x\in\Omega:((g^\delta-\zeta+\delta z)*\xi_\delta)(x)\ge 0\},$$

for some $\zeta\in(0,1)$ and $g^\delta=g*\xi_\delta$. In other words, H is continuous in $L^2(\Omega)$ (see [15]).

In our next computations, we prove the continuity of the function M defined in (3.26). Recall that M maps elements of $\mathcal{M}(\Omega)$ to their corresponding characteristic functions. We now try to find a suitable space for these characteristic functions. Intuitively, convergence of these characteristic functions is dependent upon the convergence of their associated supports. We choose $L^2(\Omega)$ to be the space of the characteristic functions and select $\mathcal{M}(\Omega)$ to be the space of their associated supports. Recall that $\mathcal{M}(\Omega)$ is a metric space equipped with the measure of the symmetric difference. The following lemma proves how these two spaces are related.

Lemma 4.6. *Let $\hat{\chi}$ and χ be characteristic functions on Ω whose supports are given by $\Omega_{\hat{\chi}}$ and Ω_χ, respectively. Then*

$$\mu(\Omega_{\hat{\chi}}\triangle\Omega_\chi)=\|\hat{\chi}-\chi\|_{L^2(\Omega)}^2.$$

Proof. Because χ and $\hat{\chi}$ are characteristic functions, we have

$$\Omega_{\hat{\chi}}\backslash\Omega_\chi=\{x:x\in\Omega_{\hat{\chi}}\wedge x\notin\Omega_\chi\}=\{x:\hat{\chi}(x)=1\wedge\chi(x)=0\}.$$

Similarly,

$$\Omega_\chi\backslash\Omega_{\hat{\chi}}=\{x:\chi(x)=1\wedge\hat{\chi}(x)=0\}.$$

Thus, from the definition of symmetric difference and the fact that $\Omega_{\hat{\chi}}\backslash\Omega_\chi$ and $\Omega_\chi\backslash\Omega_{\hat{\chi}}$ are disjoint sets, we obtain

$$\begin{aligned}
\mu(\Omega_{\hat{\chi}}\triangle\Omega_\chi)&=\mu((\Omega_{\hat{\chi}}\backslash\Omega_\chi)\cup(\Omega_\chi\backslash\Omega_{\hat{\chi}}))\\
&=\mu(\{x:\hat{\chi}(x)=1\wedge\chi(x)=0\})+\mu(\{x:\chi(x)=1\wedge\hat{\chi}(x)=0\})\\
&=\int_\Omega\hat{\chi}(1-\chi)\,dV+\int_\Omega\chi(1-\hat{\chi})\,dV\\
&=\|\hat{\chi}-\chi\|_{L^2(\Omega)}^2.
\end{aligned}$$

□

Now that we have established a mode of convergence for the characteristic functions and their associated sets, we can prove that M is continuous.

Lemma 4.7. *Suppose* $\{\omega_n\}_{n=1}^{\infty} \subset \mathcal{M}(\Omega)$ *such that* $\omega_n \to \omega$ *in* $\mathcal{M}(\Omega)$*, that is,* $\lim_{n\to\infty} \mu(\omega_n \triangle \omega) = 0$*. Then*

$$\lim_{n\to\infty} \|M(\omega_n) - M(\omega)\|_{L^2(\Omega)} = 0,$$

where $M : \mathcal{M}(\Omega) \to L^2(\Omega)$ *is a function that maps* ω *to its corresponding characteristic function, that is,* $M(\omega) = \chi^{\omega}$*. In other words,* M *is continuous on* $\mathcal{M}(\Omega)$*.*

Proof. We denote

$$M(\omega_n) =: \chi^n \text{ and } M(\omega) := \chi^{\omega}.$$

Then by Lemma (4.6),

$$\lim_{n\to\infty} \|M(\omega_n) - M(\omega)\|^2_{L^2(\Omega)} = \lim_{n\to\infty} \|\chi^n - \chi^{\omega}\|^2_{L^2(\Omega)} = \lim_{n\to\infty} \mu(\omega_n \triangle \omega) = 0.$$

□

We have shown continuity of G, Θ, H and M. Finally, we can prove that Υ has a fixed point.

Theorem 4.8. *Under assumption* (3.30) *we let* $z \in L^2(\Omega)\backslash H^1(\Omega)$*. Then the function* $\Upsilon : L^2(\Omega) \to L^2(\Omega)$ *defined by*

$$\Upsilon(\chi_1) := (T_\delta \circ M \circ H \circ \Theta \circ G \circ T_\delta)(\chi_1)$$

has a fixed point in the set

$$K := \{\chi_1 \in L^2(\Omega) : 0 \le \chi_1 \le 1 \text{ a.e. } \Omega\}. \tag{4.22}$$

Proof. We employ the Schauder Fixed Point Theorem. We first need to show that K is a convex set in $L^2(\Omega)$. Let $\chi_1, \bar{\chi}_1 \in K$ and $\lambda \in (0,1)$. Obviously, $\lambda\chi_1 + (1-\lambda)\bar{\chi}_1 \in L^2(\Omega)$. We only need to show that $0 \le \lambda\chi_1 + (1-\lambda)\bar{\chi}_1 \le 1$. Because $\lambda, 1-\lambda > 0$ then $\lambda\chi_1 + (1-\lambda)\bar{\chi}_1 \ge 0$. Furthermore, $\lambda\chi_1 + (1-\lambda)\bar{\chi}_1 \le \lambda + 1 - \lambda = 1$. Thus, $\lambda\chi_1 + (1-\lambda)\bar{\chi}_1 \in K$ and K is convex in $L^2(\Omega)$.

We show next that Υ is continuous. The functions G, Θ, H, and M are continuous as proven in Lemma 4.2, Lemma 4.4, Lemma 4.5, and Lemma 4.7, respectively. From Lemma 3.8, T_δ is continuous as well by choosing $p = 2$, $r = 2$, and $q = 1$. Because composition of continuous functions is continuous, Υ is continuous.

We only need to show that $\Upsilon(K) \subset K$ and that $\Upsilon(K)$ is compact in K. Recall that $(M \circ H \circ \Theta \circ G)(\chi_1^{\delta})$ is a characteristic function. Thus,

$$0 \le (M \circ H \circ \Theta \circ G)(\chi_1^{\delta}) \le 1.$$

By Theorem 2.5,

$$0 \le (T_\delta \circ M \circ H \circ \Theta \circ G)(\chi_1^{\delta}) \le 1,$$

and so $\Upsilon(\chi_1) \in K$. Let $\bar{\chi}_1$ be an arbitrary element of K. Let us denote $\omega := (H \circ \Theta \circ G)(\bar{\chi}_1^{\delta})$, $\chi^{\omega} := M(\omega)$, and $\chi_\delta^{\omega} := T_\delta(\chi^{\omega})$. By Lemma 3.9, Hölder's inequality, and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
|\nabla\chi_\delta^{\omega}(x)| &= |\int_\Omega \nabla\xi_\delta(x-y)\chi^{\omega}(y)dy| \\
&\le \|\chi^{\omega}\|_{L^\infty(\Omega)} \int_\Omega |\nabla\xi_\delta(x-y)|dy \\
&\le \sqrt{\mu(\Omega)}\|\nabla\xi_\delta\|_{L^2(\Omega)}.
\end{aligned}$$

Hence,

$$\|\nabla\chi_\delta^\omega\|_{L^2(\Omega)}^2 = \int_\Omega |\int_\Omega \nabla\xi_\delta(x-y)\chi^\omega(y)dy|^2dx \le \mu(\Omega)^2\|\nabla\xi_\delta\|_{L^2(\Omega)}^2.$$

From Lemma 2.5, χ_δ^ω is real analytic and so $\chi_\delta^\omega \in H^1(\Omega)$. For a fixed δ, we compute the $H^1(\Omega)$ norm of χ_δ^ω using Young's inequality for convolutions and the Hölder's inequality:

$$\begin{aligned}
\|\Upsilon(\bar\chi_1)\|_{H^1(\Omega)}^2 &= \|\chi_\delta^\omega\|_{H^1(\Omega)}^2\\
&= \|\chi_\delta^\omega\|_{L^2(\Omega)}^2 + \|\nabla\chi_\delta^\omega\|_{L^2(\Omega)}^2\\
&\le \|\chi^\omega * \xi_\delta\|_{L^2(\Omega)}^2 + \mu(\Omega)^2\|\nabla\xi_\delta\|_{L^2(\Omega)}^2\\
&\le \|\chi^\omega\|_{L^1(\Omega)}^2\|\xi_\delta\|_{L^2(\Omega)}^2 + \mu(\Omega)^2\|\nabla\xi_\delta\|_{L^2(\Omega)}^2\\
&\le \|\chi^\omega\|_{L^\infty(\Omega)}^2\mu(\Omega)^2\|\xi_\delta\|_{L^2(\Omega)}^2 + \mu(\Omega)^2\|\nabla\xi_\delta\|_{L^2(\Omega)}^2\\
&\le \mu(\Omega)^2\big(\|\xi_\delta\|_{L^2(\Omega)}^2 + \|\nabla\xi_\delta\|_{L^2(\Omega)}^2\big).
\end{aligned}$$

Since $\bar\chi_1$ is arbitrary, any sequence $\{\Upsilon(\chi_1^n)\}_{n=1}^\infty$ is bounded in the $H^1(\Omega)$ norm for a fixed δ. Because Ω is bounded, $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ and $\{\Upsilon(\chi_1^n)\}_{n=1}^\infty$ has a convergent subsequence [1]. Therefore $\Upsilon(K)$ is compact. Using Schauder Fixed Point theorem, Υ_1 has a fixed point on K. □

The fixed point is attained given any arbitrary $\chi_1 \in L^2(\Omega)$ such that $0 \le \chi_1 \le 1$ and $\bar\sigma_1$ in $C^\infty(\Omega)$, which can be chosen to be a constant. The introduction of a mollifier was used to guarantee the existence of the fixed point. Note that a fixed point is guaranteed for any arbitrary $\delta > 0$.

5. Conclusion

The EIT problem is the image reconstruction of the conductivity distribution of a body Ω given current and electrical potential data on the boundary $\partial\Omega$. In [27], a two-phase segmentation algorithm was proposed in reconstructing conductivity distribution in EIT. The algorithm arised from the minimization of a functional which depends on the conductivity distribution $\sigma = \sigma_1\chi_1 + \sigma_2(1-\chi_1)$. The value of σ_2 is fixed and known while σ_1 is expressed in terms of χ_1. Hence, the functional depends on χ_1 alone. An iterative algorithm using the method of steepest descent is then explored. Moreover, the algorithm is summarized using a composition of several functions of χ_1. By introducing a mollification on χ_1, continuity of these functions was shown. Finally, the existence of a fixed point of the proposed method was proved using the Schauder Fixed Point theorem.

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