Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 99, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

ASYMPTOTICALLY LINEAR AND SUPERLINEAR ELLIPTIC EQUATIONS WITH GRADIENT TERMS

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Abstract. In this article we establish the existence of solutions for elliptic problem involving a gradient term. To handle the so-called non-variational problem, we use a variational methods. We assume that the nonlinear term satisfies an asymptotically linear growth condition or a superlinear growth condition. We show the existence of at least one positive solution and one negative solution.

1. Introduction

This article concerns the existence of solutions for nonlinear elliptic equations with a gradient term,

$$\begin{gathered} -\Delta u = f(x,u,\nabla u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is bounded, smooth and open with the boundary $\partial\Omega$, $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous.

There is considerable attention on the existence of solution for nonlinear elliptic problems without the gradient term by using various variational methods. If f depends on the gradient of the solution, the problem is non-variational where the well developed critical point theory does not work. There have been quite a few works focusing on this kind of problems by using the topological degree theory; see for example, Amann-Crandall [1], Brezis-Turner [4], Pohožaev [9], Xavier [14], Yan [15]. Some innovative ideas were proposed by De Figueiredo-Girardi-Matzeu [5], based on the application of variational methods for the problem with the fixed gradient term, as well as the iterative method. The existence of solution was established while f satisfies the classical condition by Ambrosetti-Rabinowitz [2]:

(AR) there exist $\nu > 2$ and $t_0 > 0$ such that

$$0 < \nu F(x,s,\xi) \leq s f(x,s,\xi), \quad x \in \Omega,\ t \geq t_0,\ \xi \in \mathbb{R}^n,$$

where $F(x,s,\xi) = \int_0^s f(x,t,\xi)\mathrm{d}t$.

The main purpose of this article is to establish the existence of solution for (1.1) under the asymptotically linear growth condition or the superlinear growth condition. To handle the so-called non-variational problem, we follow the framework developed by De Figueiredo-Girardi-Matzeu [5].

2010 *Mathematics Subject Classification.* 35B09, 35J20, 35A01.
Key words and phrases. Positive solution; nonlinearity; gradient term; iterative method; Mountain pass theorem.

Submitted March 9, 2020. Published September 23, 2020.

It is well-known that the role of (AR) is to ensure the boundedness of the Palais-Smale sequence of the Euler-Lagrange functional. However, the asymptotically linear growth condition eliminates (AR) condition, thereby bringing some new obstacles to the argument. Besides, the nonlinearity of asymptotically linear type will compete with the spectra of the linear operator, which requests us to develop a new and different argument from the one for the superlinear case. There are some works related to asymptotically linear problems, such as Jeanjean-Tanaka [8], Stuart-Zhou [10] for second order elliptic equation, Wei-Su [13] for non-local elliptic equation, and Wei [12] for fourth-order elliptic equation etc. For more applications of this problem, we refer to Girardi-Matzeu [7] for periodic solutions of Hamiltonian system and Dong-Wei [6] for radial solutions of elliptic equation etc.

For the asymptotically linear case, we assume that the nonlinearity f satisfies the following assumptions.

- (H1) $f(x,0,\xi)=0$ for all $x\in\Omega,\ \xi\in\mathbb{R}^n$.
- (H2) The following holds uniformly for $x\in\Omega$, $\xi\in\mathbb{R}^n$:
$$\begin{aligned}0 &\le \liminf_{s\to 0}\frac{f(x,s,\xi)}{s}\le \limsup_{s\to 0}\frac{f(x,s,\xi)}{s}<\lambda_1\\ &<\liminf_{|s|\to+\infty}\frac{f(x,s,\xi)}{s}\le \limsup_{|s|\to+\infty}\frac{f(x,s,\xi)}{s}<+\infty,\end{aligned}$$
where λ_1 is the first eigenvalue of $-\Delta$ with the Dirichlet boundary condition.
- (H3) There exists $M>0$ such that for any $x\in\Omega$, $s\in\mathbb{R}$, $\xi\in\mathbb{R}^n$, it holds
$$\Big|\frac{f(x,s,\xi)}{s}\Big|\le M.$$
- (H4) f satisfies the local Lipschitz conditions: there exist constants L and K such that
$$|f(x,s_1,\xi)-f(x,s_2,\xi)|\le L|s_1-s_2|$$
for any $x\in\Omega$, $|s_1|\le\rho_1$, $|s_2|\le\rho_1$, $|\xi|\le\rho_2$, and
$$|f(x,s,\xi_1)-f(x,s,\xi_2)|\le K|\xi_1-\xi_2|$$
for any $x\in\Omega,\ |s|\le\rho_1,\ |\xi_1|\le\rho_2$ and $|\xi_2|\le\rho_2$, where $\rho_1,\ \rho_2$ are positive constants to be determined. Moreover, the Lipschitz constants L and K satisfy
$$L+\sqrt{\lambda_1}K<\lambda_1.$$

The following theorem concerns the asymptotically linear case.

Theorem 1.1. *Under hypotheses* (H1)–(H4)*, equation* (1.1) *possesses at least one positive solution and one negative solution.*

Remark 1.2. Consider
$$f(x,s,\xi)=h(s)(1+\tau g(\xi)),$$
where τ is a constant satisfying $|\tau|<1/2$, $g\in C^1(\mathbb{R}^n)$, $|g(\xi)|<1$, and
$$h(s)=\begin{cases}\lambda_1(2s+\frac{3}{2}\Lambda), & s<-\Lambda;\\ \frac{\lambda_1}{2}s, & |s|\le\Lambda;\\ \lambda_1(2s-\frac{3}{2}\Lambda), & s>\Lambda.\end{cases}$$

It is apparent that h is continuous. Then (H1)–(H4) are satisfied for τ small enough and Λ large enough.

In addition, we study the superlinear problem under the following hypotheses which are weaker than (AR).

(H5) $\lim_{s\to 0} f(x,s,\xi)/s = 0$ uniformly for $x \in \Omega$, $\xi \in \mathbb{R}^n$.

(H6) For every $l > 0$ there exists $C_1 > 0$ such that
$$F(x,s,\xi) \geq ls^2 - C_1, \quad x \in \Omega,\ s \in \mathbb{R},\ \xi \in \mathbb{R}^n,$$
where $F(x,s,\xi) = \int_0^s f(x,t,\xi)\mathrm{d}t$.

(H7) There exist constants $c_0 > 0$ and $q \in (1, 2^* - 1)$ such that for any $x \in \Omega$, $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, it holds
$$|f(x,s,\xi)| \leq c_0(1 + |s|^q),$$
where
$$2^* = \begin{cases} \frac{2n}{n-2}, & n > 2; \\ +\infty, & n \leq 2. \end{cases}$$

(H8) $\frac{f(x,s,\xi)}{|s|}$ is increasing with respect to s in $(-\infty, 0)$ and $(0, +\infty)$.

Theorem 1.3. *Under hypotheses* (H4)–(H8)*, equation* (1.1) *possesses at least one positive solution and one negative solution.*

Remark 1.4. Consider the superlinear case
$$f(x,s,\xi) = \varepsilon h(x)|s|^\alpha s g(\xi),$$
where $\varepsilon > 0$, $\alpha \in (0, 2^* - 2)$, $h \in C(\Omega)$, and $g \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Suppose that there exists a constant b such that $0 < b \leq h(x)$ and $0 < b \leq g(\xi)$. Then, for ε small enough, all assumptions of Theorem 1.3 are satisfied. There exists $\varepsilon_0 > 0$, such that for all $0 < \varepsilon < \varepsilon_0$, problem (1.1) has at least one positive solution and one negative solution.

This article is mainly motivated by De Figueiredo-Girardi-Matzeu [5], while both main results and approaches are different from the existing ones. On the one hand, unlike the assumptions in the above reference, the condition (AR) is not imposed. This means even in the superlinear case, the assumptions of this paper are slightly weaker. The asymptotically linear problem is also studied, which can be seen as an asymptotically linear version of [5]. On the other hand, we try to consider the superlinear problem in a different variational framework, including the Nehari manifold technique. Our arguments are based on some methods of nonlinear analysis. Mountain pass theorem, iterative technique and contraction mapping theorem are essential to the proofs of main results.

This article is organized as follows. In Section 2, we introduce some preliminaries and an auxiliary problem. The existence of solution for the auxiliary problem of the asymptotically linear case is established in Section 3 by means of Mountain pass theorem. Some uniform estimates are obtained to describe the property of the solution. In Section 4, we study the superlinear auxiliary problem. The Nehari manifold is defined, which transfers the nontrivial solution to the extreme point of Euler-Lagrange functional on the constraint manifold. The proofs of main results are given in Section 5, by the fixed point theorem and the iterative method.

2. Preliminaries and auxiliary problem

For any $v \in C_0^1(\Omega)$, consider the auxiliary problem

$$\begin{gathered} -\Delta u = f(x, u, \nabla v) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.1}$$

The Euler-Lagrange functional of (2.1) is

$$J_v(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \int_\Omega F(x, u, \nabla v)\mathrm{d}x,\ u \in H_0^1(\Omega).$$

It is well known that the norm

$$\|u\| = \Big(\int_\Omega |\nabla u|^2 \mathrm{d}x\Big)^{1/2}$$

is an equivalent norm in $H_0^1(\Omega)$. Denote

$$\Phi_v(u) = \int_\Omega F(x, u, \nabla v)\mathrm{d}x,$$

then

$$J_v(u) = \frac{1}{2}\|u\|^2 - \Phi_v(u).$$

Since (H3) of Theorem 1.1 or (H7) of Theorem 1.3 holds, we know that J_v is C^1, and Φ_v' is completely continuous. The weak solution of (2.1) is equivalent to the critical point of J_v.

Let λ_1 be the first eigenvalue of $-\Delta$ with the Dirichlet boundary condition and the corresponding eigenfunctions of λ_1 is denoted by φ_1. It is well known that $\lambda_1 > 0$ is simple and φ_1 is positive.

Set $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$. For any $v \in C_0^1(\Omega)$, consider the problem

$$\begin{gathered} -\Delta u = f^\pm(x, u, \nabla v) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.2}$$

where

$$f^+(x, s, \xi) = \begin{cases} f(x, s, \xi), & s \ge 0, \\ 0, & s < 0, \end{cases}$$

$$f^-(x, s, \xi) = \begin{cases} 0, & s > 0, \\ f(x, s, \xi), & s \le 0. \end{cases}$$

We define the corresponding functional $J_v^\pm : H_0^1(\Omega) \to \mathbb{R}$, by

$$J_v^\pm(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F^\pm(x, u, \nabla v)\mathrm{d}x,$$

where

$$F^\pm(x, u, v) = \int_0^u f^\pm(x, s, v)ds.$$

Denote

$$\Phi_v^\pm(u) = \int_\Omega F^\pm(x, u, \nabla v)\mathrm{d}x$$

and thus

$$J_v^\pm(u) = \frac{1}{2}\|u\|^2 - \Phi_v^\pm(u).$$

Obviously, $J_v^{\pm} \in C^1(H_0^1(\Omega), \mathbb{R})$. If u is a critical point of J_v^+ (J_v^-), then u is a weak solution of (2.2). By the weak maximum principle it follows that $u \geq 0$ (≤ 0) a.e. in Ω. Thus u is also a solution of problem (2.1). Hence, the nontrivial critical point of $J_v^+ (J_v^-)$ is actually a positive (negative) solution of (2.1).

Throughout this paper, denote by $\|\cdot\|_p$ the L^p norm in Ω.

3. Asymptotically linear case

In this section, we study (2.1) under asymptotically linear conditions. We first show that the functional $J_v^{\pm}$ has the mountain pass geometry.

Lemma 3.1. *Under the assumptions* (H1)–(H3), *$J_v^{\pm}$ is unbounded from below.*

Proof. Since (H1) holds, from (H3) it is apparent that

$$\Big|\frac{F(x,s,\xi)}{s^2}\Big| \leq \frac{M}{2}$$

for $x \in \Omega$, $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$. Then (H2) implies that there exist $\varepsilon > 0$ and $C_\varepsilon > 0$ such that

$$F^{\pm}(x,s,\xi) \geq \frac{1}{2}(\lambda_1 + \varepsilon)|s^{\pm}|^2 - C_\varepsilon, \quad x \in \Omega,\ s \in \mathbb{R},\ \xi \in \mathbb{R}^n. \tag{3.1}$$

From (3.1) it follows that

$$\begin{aligned} J_v^{\pm}(\pm t\varphi_1) &\leq \frac{1}{2}\|t\varphi_1\|^2 - \frac{1}{2}(\lambda_1+\varepsilon)\int_\Omega t^2\varphi_1^2 \mathrm{d}x + \int_\Omega C_\varepsilon \mathrm{d}x \\ &\leq \frac{t^2}{2}\|\varphi_1\|^2 - \frac{t^2}{2}(\lambda_1+\varepsilon)\|\varphi_1\|_2^2 + C_\varepsilon|\Omega| \\ &\leq \frac{1}{2}(1 - \frac{\lambda_1+\varepsilon}{\lambda_1})t^2\|\varphi_1\|^2 + C_\varepsilon|\Omega|, \end{aligned} \tag{3.2}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Then

$$\lim_{t\to+\infty} J_v^{\pm}(\pm t\varphi_1) = -\infty,$$

which completes the proof. □

Remark 3.2. Obviously, there exists $\gamma > 0$, independent of v, such that

$$J_v^{\pm}(\pm s\varphi_1) \leq 0, \quad \text{for all } s \geq \gamma.$$

Lemma 3.3. *Assume that* (H1)–(H3) *hold. Then there exist $r, R > 0$ such that*

$$J_v^{\pm}(u) \geq R, \quad \textit{if } \|u\| = r.$$

Proof. From (H1)-(H3), we can find $\varepsilon_0 > 0$ and $C_0 > 0$, such that

$$F^{\pm}(x,s,\xi) \leq \frac{1}{2}(\lambda_1 - \varepsilon_0)|s|^2 + C_0|s|^{2^*}. \tag{3.3}$$

Combining (3.3) with Poincaré inequality as well as Sobolev embedding, we have

$$\begin{aligned} J_v^{\pm}(u) &\geq \frac{1}{2}\|u\|^2 - \frac{\lambda_1-\varepsilon_0}{2}\int_\Omega |u|^2 \mathrm{d}x - C_0\int_\Omega |u|^{2^*}\mathrm{d}x \\ &\geq (\frac{1}{2} - \frac{\lambda_1-\varepsilon_0}{2\lambda_1})\|u\|^2 - C_sC_0\|u\|^{2^*}, \end{aligned} \tag{3.4}$$

where C_s is the Sobolev constant. Choosing $\|u\| = r > 0$ small enough, it follows that $J_v^{\pm}(u) \geq R > 0$. □

Lemma 3.4. *Suppose that (H2) and (H3) hold. Then every Palais-Smale sequence of $J_v^{\pm}$ has a convergent subsequence in $H_0^1(\Omega)$.*

Proof. Since Ω is bounded and (H2) and (H3) hold, it suffices to show that every (PS) sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$. We only need to prove the case of J_v^+, because the case of J_v^- is similar. Assume that $\{u_n\} \subset H_0^1(\Omega)$ is a (PS) sequence of J_v^+, i.e.,

$$J_v^+(u_n) \to c, \quad (J_v^+)'(u_n) \to 0 \quad \text{as } n \to +\infty. \tag{3.5}$$

From (H2) and (H3) we know that

$$|f^+(x, s, \xi)s| \leq C(1 + |s|^2).$$

Then (3.5) implies that for all $\varphi \in H_0^1(\Omega)$,

$$\int_\Omega \Big(\nabla u_n \cdot \nabla \varphi - f^+(x, u_n, \nabla v)\varphi\Big) \mathrm{d}x \to 0. \tag{3.6}$$

Setting $\varphi = u_n$ and using Hölder inequality we have

$$\begin{aligned} \|u_n\|^2 &= \int_\Omega f^+(x, u_n, \nabla v) u_n \mathrm{d}x + \langle (J_v^+)'(u_n), u_n \rangle \\ &\leq \int_\Omega f^+(x, u_n, \nabla v) u_n \mathrm{d}x + o(1)\|u_n\| \\ &\leq C|\Omega| + C\|u_n\|_2^2 + o(1)\|u_n\|. \end{aligned} \tag{3.7}$$

We claim that $\|u_n\|_2$ is bounded. Assume, by contradiction, that passing to a subsequence, it holds

$$\|u_n\|_2^2 \to +\infty, \quad \text{as } n \to +\infty.$$

Set $\omega_n = \frac{u_n}{\|u_n\|_2}$ and thus $\|\omega_n\|_2 = 1$. From (3.7) we know that

$$\|\omega_n\|^2 \leq o(1) + C + \frac{o(1)}{\|u_n\|_2} \cdot \frac{\|u_n\|}{\|u_n\|_2} \leq o(1) + C + o(1)\|\omega_n\|,$$

which implies that $\|\omega_n\|$ is bounded. Hence, there exists $\omega \in H_0^1(\Omega)$, $\|\omega\|_2 = 1$, such that

$$\begin{gathered} \omega_n \rightharpoonup \omega \quad \text{in } H_0^1(\Omega), \\ \omega_n \to \omega \quad \text{in } L^2(\Omega), \\ \omega_n(x) \to \omega(x) \quad \text{a.e. in } \Omega. \end{gathered}$$

From (3.6) it follows that

$$\int_\Omega \nabla \omega_n \cdot \nabla \varphi - \int_\Omega \frac{f^+(x, u_n, \nabla v)}{\|u_n\|_2} \varphi \mathrm{d}x = o(1), \ \varphi \in H_0^1(\Omega). \tag{3.8}$$

Taking $\varphi = \omega_n^-$, we have $\|\omega_n^-\| = o(1)$, which implies $\omega^-(x) = 0$ a.e. in Ω and thus $\omega(x) \geq 0$.

If $\omega(x) = 0$, from (H3) it follows that

$$\frac{|f^+(x, u_n, \nabla v)|}{\|u_n\|_2} = |\frac{f^+(x, u_n, \nabla v)}{u_n}|\omega_n \leq M\omega_n \to 0.$$

Then we have

$$\lim_{n \to +\infty} \frac{f^+(x, u_n, \nabla v)}{\|u_n\|_2} = 0.$$

If $\omega(x) > 0$, it follows $u_n = \omega_n \|u_n\|_2 \to +\infty$. Then (H2) implies that there exists $\delta > 0$, such that

$$\liminf_{n\to+\infty} \frac{f^+(x,u_n,\nabla v)}{\|u_n\|_2} = \liminf_{n\to+\infty} \frac{f^+(x,u_n,\nabla v)}{u_n}\omega_n \geq (\lambda_1+\delta)\omega.$$

Hence, from the above two cases we derive that

$$\liminf_{n\to+\infty} \frac{f^+(x,u_n,\nabla v)}{\|u_n\|_2} \geq (\lambda_1+\delta)\omega \tag{3.9}$$

for all $x \in \Omega$. Taking $\varphi = \varphi_1$ in (3.8), we obtain that

$$\begin{aligned} \lambda_1 \int_\Omega \omega\varphi_1 \mathrm{d}x &= \int_\Omega \nabla\omega \cdot \nabla\varphi_1 \mathrm{d}x \\ &= \lim_{n\to+\infty} \int_\Omega \nabla\omega_n \cdot \nabla\varphi_1 \mathrm{d}x \\ &= \lim_{n\to+\infty} \int_\Omega \frac{f^+(x,u_n,\nabla v)}{\|u_n\|_2}\varphi_1 \mathrm{d}x. \end{aligned} \tag{3.10}$$

Since $\varphi_1 > 0$, it is known from Fatou's Lemma that

$$\int_\Omega \liminf_{n\to+\infty} \frac{f^+(x,u_n,\nabla v)}{\|u_n\|_2}\varphi_1 \mathrm{d}x \leq \lim_{n\to+\infty} \int_\Omega \frac{f^+(x,u_n,\nabla v)}{\|u_n\|_2}\varphi_1 \mathrm{d}x. \tag{3.11}$$

Then, from (3.9), (3.10) and (3.11) we obtain

$$\begin{aligned} \lambda_1 \int_\Omega \omega\varphi_1 \mathrm{d}x &= \lim_{n\to+\infty} \int_\Omega \frac{f^+(x,u_n,\nabla v)}{\|u_n\|_2}\varphi_1 \mathrm{d}x \\ &\geq \int_\Omega \liminf_{n\to+\infty} \frac{f^+(x,u_n,\nabla v)}{\|u_n\|_2}\varphi_1 \mathrm{d}x \\ &\geq (\lambda_1+\delta)\int_\Omega \omega\varphi_1 \mathrm{d}x, \end{aligned}$$

which implies that $\omega \equiv 0$. However, this fact contradicts with $\|\omega_n\| = 1$ and hence $\|u_n\|_2$ is bounded. Therefore, from (3.7) we know that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. □

Lemma 3.5. *Let* (H1)–(H3) *hold. Then, for any $v \in C_0^1(\Omega)$, problem* (2.1) *has at least one positive weak solution $u_v^+ \in H_0^1(\Omega)$ and one negative weak solution $u_v^- \in H_0^1(\Omega)$.*

Proof. We define

$$\Psi^\pm = \{\psi \in C([0,1], H_0^1(\Omega)) : \psi(0) = 0,\ \psi(1) = \pm\gamma\varphi_1\},$$

where γ is given by Remark 3.2. Let

$$c_v^\pm = \inf_{\psi\in\Psi^\pm} \max_{s\in[0,1]} J_v^\pm(\psi(s)). \tag{3.12}$$

Since Lemma 3.1, Lemma 3.3 and Lemma 3.4 hold, Mountain pass theorem implies that c_v^+ (c_v^-) is a critical value of J_v^+ (J_v^-). Namely,

$$(J_v^\pm)'(u_v^\pm) = 0, \quad J_v^\pm(u_v^\pm) = \inf_{\psi\in\Psi^\pm} \max_{s\in[0,1]} J_v^\pm(\psi(s)),$$

which completes the proof. □

Now, we establish some uniform estimates for solutions $u_v^\pm$ of (2.1) obtained by Lemma 3.5.

Lemma 3.6. *Let $v \in C_0^1(\Omega)$, and* (H2) *and* (H3) *hold. Then there exists a positive constant c_0, independent of v, such that*

$$\|u_v^{\pm}\| \geq c_0$$

for all solutions $u_v^{\pm}$ of (2.1) *obtained by Lemma 3.5.*

Proof. Since $u_v^{\pm}$ is a solution of (2.1), we know

$$\int_\Omega |\nabla u_v^{\pm}|^2 \mathrm{d}x = \int_\Omega f^{\pm}(x, u_v^{\pm}, \nabla v) u_v^{\pm} \mathrm{d}x.$$

From (H2) and (H3), we know there exist positive constants ϵ and c_ϵ such that

$$|f^{\pm}(x, s^{\pm}, \xi)| \leq (\lambda_1 - \epsilon)|s^{\pm}| + c_\epsilon |s^{\pm}|^{2^*-1}, \quad \text{for any } x \in \Omega,\ s \in \mathbb{R},\ \xi \in \mathbb{R}^n.$$

Hence,

$$\int_\Omega |\nabla u_v^{\pm}|^2 \mathrm{d}x \leq (\lambda_1 - \epsilon) \int_\Omega |u_v^{\pm}|^2 \mathrm{d}x + c_\epsilon \int_\Omega |u_v^{\pm}|^{2^*} \mathrm{d}x.$$

By Poincaré inequality and Sobolev inequality, we obtain

$$(1 - \frac{\lambda_1 - \epsilon}{\lambda_1}) \|u_v^{\pm}\|^2 \leq c_\epsilon \|u_v^{\pm}\|_{2^*}^{2^*} \leq c_\epsilon c_0^{2^*} \|u_v^{\pm}\|^{2^*},$$

which implies the conclusion. □

Lemma 3.7. *Let* (H1)–(H3) *hold. Then there exists a positive constant $\overline{\rho}$, independent of v, such that*

$$\|u_v^{\pm}\| \leq \overline{\rho}$$

for all solutions $u_v^{\pm}$ obtained by Lemma 3.5.

Proof. We only give the proof for the case of J_v^+, the case of J_v^- is similar. We suppose, by contradiction, there exist subsequences $\{v_j\}$ and $\{u_{v_j}\}$, such that $\{v_j\} \subset C_0^1(\Omega)$, $\{u_{v_j}\} \subset H_0^1(\Omega)$ and

$$(J_{v_j}^+)'(u_{v_j}) = 0, \quad \|u_{v_j}\| \to +\infty \quad \text{as } j \to +\infty.$$

Then for all $\varphi \in H_0^1(\Omega)$, it holds

$$\int_\Omega \Big(\nabla u_{v_j} \cdot \nabla \varphi - f^+(x, u_{v_j}, \nabla v_j) \varphi \Big) \mathrm{d}x = 0. \tag{3.13}$$

We set $\omega_j = \frac{u_{v_j}}{\|u_{v_j}\|}$ and thus $\|\omega_j\| = 1$. Hence, there exists $\omega \in H_0^1(\Omega)$, $\|\omega\| = 1$ such that

$$\begin{gathered} \omega_j \rightharpoonup \omega \quad \text{in } H_0^1(\Omega), \\ \omega_j \to \omega \quad \text{in } L^2(\Omega), \\ \omega_j(x) \to \omega(x) \quad \text{a.e. in } \Omega. \end{gathered}$$

From (3.13) it follows

$$\int_\Omega \Big(\nabla \omega_j \cdot \nabla \varphi - \frac{f^+(x, u_{v_j}, \nabla v_j)}{\|u_{v_j}\|} \varphi \Big) \mathrm{d}x = 0. \tag{3.14}$$

Taking $\varphi = \omega_j^-$ we know $\|\omega_j^-\| = 0$, which implies $\omega(x) \geq 0$.

If $\omega(x) = 0$, from (H3) it follows that

$$\frac{|f^+(x, u_{v_j}, \nabla v_j)|}{\|u_{v_j}\|} = \Big| \frac{f^+(x, u_{v_j}, \nabla v_j)}{u_{v_j}} \Big| \omega_j \leq M \omega_j \to 0.$$

Then we have

$$\lim_{j\to+\infty}\frac{f^+(x,u_{v_j},\nabla v_j)}{\|u_{v_j}\|}=0.$$

If $\omega(x)>0$, it follows that $u_{v_j}=\omega_j\|u_{v_j}\|\to+\infty$. Then (H2) implies that there exists $\delta>0$, such that

$$\liminf_{j\to+\infty}\frac{f^+(x,u_{v_j},\nabla v_j)}{\|u_{v_j}\|}=\liminf_{j\to+\infty}\frac{f^+(x,u_{v_j},\nabla v_j)}{u_{v_j}}\omega_j\ge(\lambda_1+\delta)\omega.$$

Hence, from the above two cases,

$$\liminf_{j\to+\infty}\frac{f^+(x,u_{v_j},\nabla v_j)}{\|u_{v_j}\|}\ge(\lambda_1+\delta)\omega \tag{3.15}$$

for all $x\in\Omega$. Taking $\varphi=\varphi_1$ in (3.14), since $\varphi_1>0$, $\omega\ge0$, from (3.15) and Fatou's Lemma we derive

$$\begin{aligned}
\lambda_1\int_\Omega\omega\varphi_1\mathrm{d}x&=\int_\Omega\nabla\omega\cdot\nabla\varphi_1\mathrm{d}x\\
&=\lim_{j\to+\infty}\int_\Omega\nabla\omega_j\cdot\nabla\varphi_1\mathrm{d}x\\
&=\lim_{j\to+\infty}\int_\Omega\frac{f^+(x,u_{v_j},\nabla v_j)}{\|u_{v_j}\|}\varphi_1\mathrm{d}x\\
&\ge\int_\Omega\liminf_{j\to+\infty}\frac{f^+(x,u_{v_j},\nabla v_j)}{\|u_{v_j}\|}\varphi_1\mathrm{d}x\\
&\ge(\lambda_1+\delta)\int_\Omega\omega\varphi_1\mathrm{d}x.
\end{aligned}$$

Hence, $\omega\equiv0$, which contradicts with $\|\omega\|=1$. The proof is complete. $\square$

Lemma 3.8. *Assume that* (H1)–(H3) *hold. Then there exist positive constants* ρ_1 *and* ρ_2*, independent of* v*, such that*

$$\max_{x\in\Omega}|u_v^\pm(x)|\le\rho_1,\quad \max_{x\in\Omega}|\nabla u_v^\pm(x)|\le\rho_2.$$

Proof. Since f is continuous in all variables and $v\in C_0^1(\Omega)$, using the regularity theory we know that $u_v^\pm$ is C^2, see Brezis [3]. Hence, Sobolev embedding theorem and Lemma 3.7 imply the conclusion. $\square$

4. Nehari manifold for superlinear case

This section is devoted to the existence of critical point of J_v for superlinear case. The critical points will be obtained by means of constrained minimization. For fixed $v\in C_0^1(\Omega)$, define Nehari manifold

$$\mathcal{N}_v:=\{u\in H_0^1(\Omega)\setminus\{0\}:J_v'(u)u=0\}.$$

Lemma 4.1. *Under assumptions* (H5), (H7), (H8)*, there exists a positive constant* c_0*, independent of* v*, such that* $\|u\|\ge c_0$ *for all solutions* $u\in\mathcal{N}_v$*.*

Proof. Since $u\in\mathcal{N}_v$, we know

$$\int_\Omega|\nabla u|^2\mathrm{d}x=\int_\Omega f(x,u,\nabla v)u\mathrm{d}x.$$

From (H5) and (H7), for any $\epsilon > 0$, there exists $c_\epsilon > 0$ such that

$$|f(x,s,\xi)| \le \epsilon |s| + c_\epsilon |s|^q, \quad x\in\Omega,\ s\in\mathbb{R},\ \xi\in\mathbb{R}^n.$$

Then

$$\int_\Omega |\nabla u|^2 \mathrm{d}x \le \epsilon \int_\Omega |u|^2 \mathrm{d}x + c_\epsilon \int_\Omega |u|^{q+1}\mathrm{d}x.$$

Hence, by Poincaré inequality and Sobolev inequality we obtain

$$(1-\frac{\epsilon}{\lambda_1})\|u\|^2 \le c_\epsilon \|u\|_{q+1}^{q+1} \le c_\epsilon c_0^{q+1}\|u\|^{q+1},$$

which implies the conclusion. $\square$

Lemma 4.2. *Assume that* (H5) *and* (H7) *hold. Then*

$$\Phi_v'(u) = o(\|u\|), \quad \Phi_v(u) = o(\|u\|^2)$$

as $u \to 0$ in $H_0^1(\Omega)$.

Proof. (H5) and (H7) imply that for any given $\epsilon > 0$, there exists a positive c_ϵ such that

$$F(x,s,\xi) \le \epsilon|s|^2 + c_\epsilon |s|^{q+1}, \quad x\in\Omega,\ s\in\mathbb{R},\ \xi\in\mathbb{R}^n. \tag{4.1}$$

Then, by Hölder inequality and Sobolev inequality, it is standard to prove the lemma. $\square$

To prove the main result, we will apply the following lemma, which can be found in Szulkin and Weth [11, Theorem 12].

Lemma 4.3. *Let E be a Hilbert space and $J(u) = \frac{1}{2}\|u\| - \Phi(u)$, where*

- (i) *$\Phi'(u) = o(\|u\|)$ as $u \to 0$;*
- (ii) *$s \mapsto \Phi'(su)u/s$ is strictly increasing for all $u \neq 0$ and $s > 0$;*
- (iii) *$\Phi(su)/s^2 \to +\infty$ uniformly for u on weakly compact subset of $E \setminus \{0\}$ as $s \to +\infty$;*
- (iv) *Φ' is completely continuous.*

Then equation $J'(u) = 0$ has a ground state solution.

Lemma 4.4. *Let* (H5)–(H8) *hold. Then, for any $v \in C_0^1(\Omega)$, problem* (2.1) *has a ground state solution $u_v \in H_0^1(\Omega)$.*

Proof. It suffices to check (i)–(iv) of Lemma 4.3. Actually, Lemma 4.2 and (H8) imply (i) and (ii), respectively. For (iii), let W be a weakly compact subset of $H_0^1(\Omega)\setminus\{0\}$ and $\{u_n\}\subset W$. Passing to a subsequence, it holds

$$u_n \rightharpoonup u \in H_0^1(\Omega)\setminus\{0\}.$$

Then

$$u_n(x) \to u(x) \quad \text{a.e. in } \Omega.$$

Hence, the set $\Omega^* := \{x\in\Omega : u(x)\neq 0\}$ is a subset of Ω with positive measure. Taking $s_n \to +\infty$, we know that for $x \in \Omega^*$,

$$|s_n u_n(x)| \to +\infty, \quad \text{as } n \to +\infty.$$

Then Fatou's Lemma yields

$$\frac{\Phi_v(s_n u_n)}{s_n^2} = \int_\Omega \frac{F\big(x, s_n u_n(x), \nabla v(x)\big)}{(s_n u_n)^2} u_n^2 \mathrm{d}x \to +\infty.$$

Finally, since Ω is bounded and (H7) holds, from the compact embedding we know (iv) holds. $\square$

Remark 4.5. In the above lemma, a ground state solution is found, which is a critical point of functional J_v. From the proof of the above lemma, it can be seen that if J_v is replaced by $J_v^{\pm}$, then a ground state solution $u_v^{\pm}$ can also be obtained.

Remark 4.6. It is easy to check the following minimax characterization (see Szulkin and Weth [11]):

$$c_v := \inf_{u\in\mathcal{N}_v} J_v(u) = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \max_{s>0} J_v(su) = \inf_{u\in H_0^1(\Omega),\|u\|=1} \max_{s>0} J_v(su).$$

Lemma 4.7. *Assume that* (H5)–(H8) *hold. Then, there exists a positive constant d, such that $c_v \le d$ uniformly for $v \in C_0^1(\Omega)$.*

Proof. Because of the minimax characterization in Remark 4.6, it suffices to show that there exists $\phi \in H_0^1(\Omega)\setminus\{0\}$, such that

$$\max_{s>0} J_v(s\phi) \le d, \quad \text{uniformly for } v \in C_0^1(\Omega).$$

From (H6), for every $l > 0$ there exists $C_1 > 0$ such that

$$F(x,s,\xi) \ge ls^2 - C_1, \quad x\in\Omega, \quad s\in\mathbb{R}, \quad \xi\in\mathbb{R}^n.$$

Fix $\phi \in H_0^1(\Omega)$ with $\|\phi\| = 1$. From the above we obtain

$$\begin{aligned} J_v(s\phi) &= \tfrac{s^2}{2}\int_\Omega |\nabla\phi|^2 \mathrm{d}x - \int_\Omega F(t,s\phi,\nabla v)\mathrm{d}x &(4.2)\\ &\le \tfrac{s^2}{2}\|\phi\|^2 - \int_\Omega ls^2\phi^2\mathrm{d}x + \int_\Omega C_1\mathrm{d}x &(4.3)\\ &\le s^2\Big(\tfrac12 - l\int_\Omega \phi^2\mathrm{d}x\Big) + C_1|\Omega|. &(4.4)\end{aligned}$$

Setting $l = \frac{1}{\int_\Omega \phi^2\mathrm{d}x}$, it follows that

$$J_v(s\phi) \le -\frac12 s^2 + C_1|\Omega| \le C_1|\Omega|$$

uniformly for $v \in C_0^1(\Omega)$. □

Lemma 4.8. *There exists a positive constant $\overline{\rho}$, independent of v, such that for every ground state solution u_v given in Lemma 4.4,*

$$\|u_v\| \le \overline{\rho}.$$

Proof. By contradiction, suppose that there exist subsequences $\{v_j\} \subset C_0^1(\Omega)$ and $\{u_{v_j}\} \subset H_0^1(\Omega)$, such that $u_{v_j} \in \mathcal{N}_{v_j}$,

$$J(u_{v_j}) = \inf_{u\in\mathcal{N}_{v_j}} J(u),$$

$$\|u_{v_j}\| \to +\infty \text{ as } j \to +\infty.$$

Set $\omega_j = u_{v_j}/\|u_{v_j}\|$ and thus $\|\omega_j\| = 1$. Then, there exists $\omega \in H_0^1(\Omega)$ such that

$$\begin{gathered} \omega_j \rightharpoonup \omega \quad \text{in } H_0^1(\Omega),\\ \omega_j \to \omega \quad \text{in } L^2(\Omega),\\ \omega_j(x) \to \omega(x) \quad \text{a.e. in } \Omega. \end{gathered}$$

We claim that $\omega(x) \equiv 0$ a.e. in Ω. Denote $\Omega^* = \{x\in\Omega, \omega(x)\ne 0\}$. If $\Omega^* \ne \emptyset$, then for $x \in \Omega^*$, $|u_{v_j}(x)| \to +\infty$ as $j \to +\infty$. By (H6) we have

$$\lim_{j\to+\infty} \frac{F\big(x,u_{v_j}(x),\nabla v_j(x)\big)}{\big(u_{v_j}(x)\big)^2}\big(\omega_j(x)\big)^2 = +\infty. \tag{4.5}$$

Then Fatou's Lemma implies

$$\begin{aligned}
&\int_\Omega \lim_{j\to+\infty} \frac{F\big(x,u_{v_j}(x),\nabla v_j(x)\big)}{\big(u_{v_j}(x)\big)^2}\big(\omega_j(x)\big)^2 \mathrm{d}x \\
&\le \liminf_{j\to+\infty} \frac{1}{\|u_{v_j}\|^2}\int_\Omega F\big(x,u_{v_j}(x),\nabla v_j(x)\big)\mathrm{d}x \\
&= \lim_{j\to+\infty} \frac{1}{\|u_{v_j}\|^2}\Big(\frac{1}{2}\|u_{v_j}\|^2 - J_{v_j}(u_{v_j})\Big).
\end{aligned} \tag{4.6}$$

From the property of Nehari manifold we know that

$$J_{v_j}(u_{v_j}) = \max_{s>0} J_{v_j}(su_{v_j}).$$

Then Lemma 4.2 implies $J_{v_j}(u_{v_j}) \ge 0$. Hence, from (4.6) we obtain

$$\int_\Omega \lim_{j\to+\infty} \frac{F\big(x,u_{v_j}(x),\nabla v_j(x)\big)}{\big(u_{v_j}(x)\big)^2}\big(\omega_j(x)\big)^2 \mathrm{d}x \le \frac{1}{2},$$

which contradicts with (4.5). Therefore, Ω^* has zero measure and $\omega(t) \equiv 0$ a.e. in Ω.

Since Φ_{v_j} is weakly continuous, from Lemma 4.7 we obtain

$$d \ge J_{v_j}(u_{v_j}) \ge J_{v_j}(s\omega_j) \ge \frac{1}{2}s^2 - \Phi_{v_j}(s\omega_j) \to \frac{1}{2}s^2,$$

which is a contradiction, for s large enough. □

Lemma 4.9. *Assume that* (H5)–(H8) *hold. Then there exist positive constants* ρ_1 *and* ρ_2*, independent of* v*, such that*

$$\max_{x\in\Omega} |u_v(x)| \le \rho_1, \quad \max_{x\in\Omega} |\nabla u_v(x)| \le \rho_2$$

for all solutions u_v *obtained in Lemma 4.4.*

The proof of the above lemma is as same as the proof of Lemma 3.8.

Remark 4.10. Actually, a similar result can also be established for problem (2.2). We can find a critical point $u_v^\pm$ for functional $J_v^\pm$ and positive constants ρ_1 and ρ_2, independent of v, such that

$$\max_{x\in\Omega} |u_v^\pm(x)| \le \rho_1, \quad \max_{x\in\Omega} |\nabla u_v^\pm(x)| \le \rho_2.$$

5. Proofs of main results

In this section, we prove our main results by the iterative argument, which was established by De Figueiredo, Girardi and Matzeu [5]. Define the map

$$T^\pm : H_0^1(\Omega) \to H_0^1(\Omega), \quad T^\pm v \mapsto u_v^\pm,$$

with domain $D(T^\pm) = C_0^1(\Omega) \subset H_0^1(\Omega)$. Here $u_v^\pm$ is the solution of (2.1) given by Lemma 3.5 for the asymptotically linear case and Remark 4.5 for the superlinear case, respectively. For any $v \in C_0^1(\Omega)$, the map is well-defined, and actually, $T^\pm(C_0^1(\Omega)) \subset C_0^1(\Omega)$ because of the regularity theory. Moreover, denote

$$B_{\overline{\rho}} := \{u \in H_0^1(\Omega), \|u\| \le \overline{\rho}\},$$

where $\overline{\rho} > 0$ is the uniform bound in Lemma 3.7 for the asymptotically linear case and Lemma 4.8 for the superlinear case, respectively. Then, $T^\pm(C_0^1(\Omega)) \subset B_{\overline{\rho}}$.

Hence, $T^{\pm}(C_0^1(\Omega)) \subset B_{\overline{\rho}} \cap C_0^1(\Omega)$. Recall that a point x is a fixed point of map T, if and only if

$$x \in T(x).$$

Choosing $u_0^{\pm} \in B_{\overline{\rho}} \cap C_0^1(\Omega)$, we construct a sequence $\{u_n^{\pm}\} \subset B_{\overline{\rho}} \cap C_0^1(\Omega)$ as solutions of

$$\begin{gathered} -\Delta u_n^{\pm} = f^{\pm}(x, u_n^{\pm}, \nabla u_{n-1}^{\pm}) \quad \text{in } \Omega, \\ u_n^{\pm} = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{5.1}$$

obtained in Lemma 3.5 for asymptotically linear case and in Lemma 4.4 for superlinear case, respectively.

Proof of Theorems 1.1 and 1.3. By (5.1) for n and for $n+1$, we have

$$\begin{gathered} \int_\Omega \nabla u_n^{\pm} \cdot (\nabla u_{n+1}^{\pm} - \nabla u_n^{\pm}) \mathrm{d}x = \int_\Omega f^{\pm}(x, u_n^{\pm}, \nabla u_{n-1}^{\pm})(u_{n+1}^{\pm} - u_n^{\pm}) \mathrm{d}x, \\ \int_\Omega \nabla u_{n+1}^{\pm} \cdot (\nabla u_{n+1}^{\pm} - \nabla u_n^{\pm}) \mathrm{d}x = \int_\Omega f^{\pm}(x, u_{n+1}^{\pm}, \nabla u_n^{\pm})(u_{n+1}^{\pm} - u_n^{\pm}) \mathrm{d}x. \end{gathered}$$

According to Lemma 3.8 for the asymptotically linear case and Remark 4.10 for the superlinear case, we know that

$$\max_{x\in\Omega} |u_v^{\pm}(x)| \le \rho_1, \quad \max_{x\in\Omega} |\nabla u_v^{\pm}(x)| \le \rho_2.$$

Combining (H4) with Poincaré inequality as well as Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} \|u_{n+1}^{\pm} - u_n^{\pm}\|^2 &= \int_\Omega \Big(f^{\pm}(x, u_{n+1}^{\pm}, \nabla u_n^{\pm}) - f^{\pm}(x, u_n^{\pm}, \nabla u_{n-1}^{\pm}) \Big)(u_{n+1}^{\pm} - u_n^{\pm}) \mathrm{d}x \\ &= \int_\Omega \Big(f^{\pm}(x, u_{n+1}^{\pm}, \nabla u_n^{\pm}) - f^{\pm}(x, u_n^{\pm}, \nabla u_n^{\pm}) \Big)(u_{n+1}^{\pm} - u_n^{\pm}) \mathrm{d}x \\ &\quad + \int_\Omega \Big(f^{\pm}(x, u_n^{\pm}, \nabla u_n^{\pm}) - f^{\pm}(x, u_n^{\pm}, \nabla u_{n-1}^{\pm}) \Big)(u_{n+1}^{\pm} - u_n^{\pm}) \mathrm{d}x \\ &\le L \int_\Omega |u_{n+1}^{\pm} - u_n^{\pm}|^2 \mathrm{d}x + K \int_\Omega |\nabla u_n^{\pm} - \nabla u_{n-1}^{\pm}| |u_{n+1}^{\pm} - u_n^{\pm}| \mathrm{d}x \\ &\le \frac{L}{\lambda_1} \|u_{n+1}^{\pm} - u_n^{\pm}\|^2 + \frac{K}{\sqrt{\lambda_1}} \|u_n^{\pm} - u_{n-1}^{\pm}\| \cdot \|u_{n+1}^{\pm} - u_n^{\pm}\|. \end{aligned}$$

Hence,

$$\|u_{n+1}^{\pm} - u_n^{\pm}\| \le \frac{K\sqrt{\lambda_1}}{\lambda_1 - L} \|u_n^{\pm} - u_{n-1}^{\pm}\|.$$

From $L + \sqrt{\lambda_1} K < \lambda_1$ we know $\{u_n^{\pm}\} \subset H_0^1(\Omega)$ is a Cauchy sequence, and thus there exists $u_*^{\pm} \in H_0^1(\Omega)$ such that $u_*^{\pm} \in T^{\pm}(u_*^{\pm})$. Finally, from Lemma 3.6 for the asymptotically linear case and Lemma 4.1 for the superlinear case we know that $\|u_*^{\pm}\| \ge c_0$, which means that $u_*^{\pm}$ is a nontrivial solution. $\square$

Acknowledgements. This work was supported by the National Natural Science Foundation of China No. 11871242, and by the Natural Science Foundation of Jilin Province No. 20200201248JC.

References

[1] H. Amann, M. Crandall; On some existence theorems for semi-linear elliptic equations, *Indiana Univ. Math. J.*, **27** No. 5 (1978), 779-790.

[2] A. Ambrosetti, P. Rabinowitz; Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, **14** (1973), 349-381.

[3] H. Brezis; *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011.

[4] H. Brezis, R. Turner; On a class of superlinear elliptic problems, *Comm. Partial Differential Equations*, **2** No. 6 (1977), 601-614.

[5] D. De Figueiredo, M. Girardi, M. Matzeu; Semilinear elliptic equations with dependence on the gradient via mountain-pass techniques, *Differential Integral Equations*, **17** No. 1-2 (2004), 119-126.

[6] X. Dong, Y. Wei; Existence of radial solutions for nonlinear elliptic equations with gradient terms in annular domains, *Nonlinear Anal.*, **187** (2019), 93-109.

[7] M. Girardi, M. Matzeu; Existence of periodic solutions for some second order quasi-linear Hamiltonian systems, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, **18** No. 1 (2007), 1-9.

[8] L. Jeanjean, K. Tanaka; Singularly perturbed elliptic problems with superlinear or asymptotically linear nonlinearities, *Calc. Var. Partial Differential Equations*, **21** No. 3 (2004), 287-318.

[9] S. Pohožaev; Equations of the type $\Delta u = f(x, u, Du)$, (Russian), *Mat. Sb. (N.S.)*, **113** No.155 (1980), 324-338.

[10] C. Stuart, H. Zhou; Applying the mountain pass theorem to an asymptotically linear elliptic equation on $\mathbb{R}^N$, *Comm. Partial Differential Equations*, **24** No. 9-10 (1999), 1731-1758.

[11] A. Szulkin, T. Weth; Ground state solutions for some indefinite variational problems, *J. Funct. Anal.*, **257** No. 12 (2009), 3802-3822.

[12] Y. Wei; Multiplicity results for some fourth-order elliptic equations, *J. Math. Anal. Appl.*, **385** No. 2 (2012), 797-807, .

[13] Y. Wei, X. Su; On a class of non-local elliptic equations with asymptotically linear term, *Discrete Contin. Dyn. Syst.*, **38** No. 12 (2018), 6287-6304.

[14] J. Xavier; Some existence theorems for equations of the form $-\Delta u = f(x, u, Du)$, *Nonlinear Anal.*, **15** No. 1 (1990), 59-67.

[15] Z. Yan; A note on the solvability in $W^{2,p}(\Omega)$ for the equation $-\Delta u = f(x, u, Du)$, *Nonlinear Anal.*, **24** No. 9 (1995), 1413-1416.

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