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SOLUTIONS TO VISCOUS BURGERS EQUATIONS WITH TIME DEPENDENT SOURCE TERM

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Abstract. We study the existence and uniqueness of weak solutions for a Cauchy problem of a viscous Burgers equation with a time dependent reaction term involving Dirac measure. After applying a Hopf like transformation, we investigate the associated two initial boundary value problems by assuming a common boundary. The existence of the boundary data is shown with the help of Abel's integral equation. We then derive explicit representation of the boundary function. Also, we prove that the solutions of associated initial boundary value problems converge uniformly to a nonzero constant on compact sets as t approaches ∞

1. Introduction

This article concerns the existence, uniqueness and regularity of solutions to the Burgers equation with time dependent point source,

$$u_t + uu_x - u_{xx} = \frac{2}{1+t}\delta(x), \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

subject to the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{1.2}$$

where $u_0 \in W^{1,\infty}(\mathbb{R}) \cap C^1(\mathbb{R}) \cap L^1(\mathbb{R})$. In the literature, the study on viscous Burgers equation with source terms

$$u_t + u\,u_x = u_{xx} + f(x,t), \quad x \in \mathbb{R},\ t > 0, \tag{1.3}$$

has obtained much recognition because of extended importance in numerous fields of science, technology and biology [1, 17]. Construction of explicit solutions for viscous Burgers equation with inhomogeneous terms and large time behavior of these solutions are discussed by several authors. Eule and Friedrich [8] discussed the solutions of externally forced Burgers equation

$$u_t + u\,u_x = u_{xx} + xG(t), \tag{1.4}$$

by considering $G(t)$ to be constant in the first case and stochastic white noise force in the other case. They examined the problem in relation with stretched vortices

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in hydrodynamics flows. Salas [16] examined a specific case of (1.4) and derived the n-shock wave solutions with the help of traveling wave method via generalized Hopf-Cole transformation. He connected the problem of solving (1.4) with the Riccati and heat equations. Also, several explicit solutions of (1.4) were listed in that paper.

Kloosterziel [11] investigated the solutions for linear heat equation

$$v_t = v_{xx}, \quad x \in \mathbb{R},\ t > 0, \tag{1.5}$$

subject to the initial data

$$v(x,0) = v_0(x), \quad x \in \mathbb{R}, \tag{1.6}$$

where v_0 is any square integrable function with respect to the exponentially growing weight function $e^{\frac{x^2}{2}}$. Using similarity transformation, the author constructed the following self-similar solutions of (1.5),

$$v_n(x,t) = \frac{1}{(2\pi)^{1/4}(2^n\, n!)^{1/2}(1+2t)^{\frac{1+n}{2}}} e^{\frac{x^2}{2(1+2t)}} H_n\Big(\frac{x}{\sqrt{2(1+2t)}}\Big), \tag{1.7}$$

where H_n is Hermite polynomial of order n. An interesting feature of the solutions to (1.7) is that the set of functions $\{v_n(x,0)\}$ is a complete orthonormal system for the Hilbert space $L^2(\mathbb{R}, e^{\frac{x^2}{2}})$ and hence any function $v_0 \in L^2(\mathbb{R}, e^{\frac{x^2}{2}})$ can be expanded as an infinite series in terms of $\{v_n(x,0)\}$. Hence, the solution of (1.5)-(1.6) is represented as an infinite series of self-similar solutions (1.7). Another feature of the constructed self similar solutions (1.7) is that the decay rate is obtained easily which, in turn, gives the large time asymptotes to the solution of (1.5)-(1.6). Ding-Jiu-He [6] constructed the explicit solutions of non-homogeneous Burgers equation

$$u_t + u\, u_x = \mu u_{xx} + kx, \quad x \in \mathbb{R},\ t > 0, \tag{1.8}$$

subject to the initial data

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{1.9}$$

where $\mu > 0$ and k is constant. The authors imposed two conditions on the initial function u_0 that it is locally integrable and $\int_0^x u_0(y)\, dy = o(x^2)$ as $|x| \to \infty$. They applied Hopf transformation to reduce (1.8)-(1.9) to the linear differential equation and then represented the solution of resulting linear differential equation in terms of Fourier-Hermite series. They proved that the solution $u(x,t)$ of the initial value problem (1.8)-(1.9) behaves like $\sqrt{k}x$ for large time t. However, Chidella and Yadav [4] noticed that bounded and compactly supported initial functions u_0 do not satisfy the conditions imposed by Ding-Jiu-He [6] and so considered the nonhomogeneous Burgers equation (1.8)-(1.9) with an assumption on u_0 that

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2} - \int^x u_0(r)dr} dx < \infty.$$

Using the Hopf transformation and standard transformations of Polyanin and Nazaikinskii [12], they reduced the initial value problem (1.8)-(1.9) to an initial value problem for heat equation and then used the results of Kloosterziel [11] to express the solution of the heat equation in terms of the self-similar solutions of the heat equation. Buyukasik and Pashaev [3] discussed the shock wave solutions, triangular wave solutions, N-wave solutions and rational function solutions for the Burgers equation (1.8). Rao and Yadav [15] considered a non-homogeneous Burgers equation

$$u_t + u\,u_x = u_{xx} + \frac{kx}{(2\beta t+1)^2}, \quad x \in \mathbb{R},\ t > 0, \tag{1.10}$$

subject to the unbounded initial data and expressed the solutions in terms of the self similar solutions of a linear partial differential equation with variable coefficients. They obtained the large time behavior of the solution of the nonhomogeneous Burgers equation. Rao and Yadav [14] investigated solutions for (1.10) by assuming that the initial data is compactly supported and bounded. Engu-Ahmed-Murugan [7] proved the existence of a solution for the initial value problem of a nonhomogeneous Burgers equation and expressed the solution in terms of Hermite polynomials. Their analysis mainly depends on Hopf-Cole transformation and method of variation of parameters. The authors have also given the rates of convergence of an approximate solution to the true solution of the initial value problem. In regards to construction of fundamental solutions of evolutionary equations, we refer to Pskhu [13] and the references there in.

However, investigating the solutions for viscous Burgers equation with source term involving the Dirac delta measure becomes complicated as the linearization process of the viscous Burgers equation with the source term leads to two different linear partial differential equations on the two upper quarter planes. Further, considering the nontrivial initial condition with the nonhomogeneous viscous Burgers equation increases the complexity more. Chung-Kim-Slemrod [5] studied the existence, uniqueness and asymptotic behavior of solutions to a Cauchy problem for the viscous Burgers equation with Dirac delta measure as source term.

To understand the viscous Burgers equation with time dependent point source involving dirac function, we consider an initial value problem for the non homogeneous viscous Burgers equation (1.2) where δ is the Dirac delta function concentrated at $x = 0$. We use the Hopf transformation for linearization. Linearized partial differential equation consists of Heaviside function and so one needs to study the problem on two upper quarter planes separately with common boundary along the positive t-axis. With the help of Abel's integral equation, we intend to establish the existence and uniqueness of the common boundary data of the linearized partial differential equations. We then look for the explicit representation of the boundary function. In view of the integrals involved in representation of the boundary function, we seek the asymptotic behavior of it for large time t. Using this asymptotic behavior, asymptotic behavior of the solution of the linearized partial differential equation is established. Making use of inverse Hopf-Cole tranformation, existence, uniqueness and regularity of the solutions to the non homogeneous viscous Burgers equation is discussed. Eventually, the convergence of the solutions to zero on compact intervals is obtained.

The rest of this article is organized as follows. Section 2 deals with the linearization of the Cauchy problem and then the existence, uniqueness of the common boundary data of the resulting two initial-boundary value problems. This section also discusses the explicit representation of the common boundary data and its asymptotic behavior. In section 3, existence and uniqueness of the global weak solutions for the Cauchy problem is discussed. Further, asymptotic behavior the solutions is investigated.

2. Burgers equation with inhomogeneous term

Consider the Burgers equation with time dependent point source given by (1.1) subject to the initial condition (1.2) where $u_0 \in W^{1,\infty}(\mathbb{R}) \cap C^1(\mathbb{R}) \cap L^1(\mathbb{R})$. The Hopf-Cole transformation [2, 10], is given by

$$\theta(x,t) = \exp\Big(-\frac{1}{2}\int_{-\infty}^{x} u_0(y)dy\Big). \tag{2.1}$$

Then $\theta(x,0) =: \theta_0(x) \in W^{2,\infty}(\mathbb{R}) \cap C^2(\mathbb{R})$ and the Cauchy problem (1.1)-(1.2) reduces to

$$\theta_t - \theta_{xx} + \frac{H(x)}{(1+t)}\theta = 0\,, \quad x \in \mathbb{R}, \quad t > 0, \tag{2.2}$$

$$\theta(x,0) = \theta_0(x), \quad x \in \mathbb{R}, \tag{2.3}$$

where $H(x)$ is the Heaviside function. The above Cauchy problem is split into two problems, namely,

$$\begin{gathered} L_t - L_{xx} = 0, \quad x < 0,\ t > 0, \\ L(x,0) = \theta_0(x), \quad x < 0, \\ L(0,t) = g(t), \quad t > 0, \end{gathered} \tag{2.4}$$

and

$$\begin{gathered} R_t - R_{xx} = -\frac{R}{(1+t)}, \quad x \geq 0,\, t > 0, \\ R(x,0) = \theta_0(x), \quad x \geq 0, \\ R(0,t) = g(t), \quad t > 0. \end{gathered} \tag{2.5}$$

It is to be noted that the same boundary function $g(t)$ is taken for both the initial-boundary value problems (2.4) and (2.5) to assume that $\theta(x,t)$ is continuous on the positive t axis. Further, we assume temporarily that $g(t)$ is continuously differentiable on $[0,\infty)$ and will be calculated after showing the existence of $g(t)$.

Consider the transformation [12]

$$w(x,t) = (1+t)R(x,t). \tag{2.6}$$

Then the initial-boundary value problem (2.5) reduces to the associated initial-boundary value problem for the heat equation

$$\begin{gathered} w_t - w_{xx} = 0, \quad x \geq 0,\ t > 0, \\ w(x,0) = \theta_0(x), \quad x \geq 0, \\ w(0,t) = (1+t)g(t), \quad t > 0. \end{gathered} \tag{2.7}$$

Solving (2.7) and retracing $R(x,t)$ using (2.6), we obtain

$$\begin{aligned} R(x,t) = \frac{1}{1+t}\Big[&\frac{1}{2\sqrt{\pi t}}\int_0^\infty \theta_0(\xi)\big[e^{\frac{-(\xi-x)^2}{4t}} - e^{\frac{-(\xi+x)^2}{4t}}\big]d\xi \\ &+ \int_0^t \big(g(\tau)(1+\tau)\big)' \operatorname{erfc}\Big(\frac{x}{2\sqrt{t-\tau}}\Big)d\tau + g(0)\operatorname{erfc}\Big(\frac{x}{2\sqrt{t}}\Big)\Big], \end{aligned} \tag{2.8}$$

where $\operatorname{erfc}(x)$ is the complementary error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-y^2}\,dy.$$

Similarly, solving the initial-boundary value problem (2.4), we obtain

$$\begin{aligned} L(x,t) = \frac{-1}{2\sqrt{\pi t}} \int_0^\infty \theta_0(-\xi)\big[e^{\frac{-(\xi-x)^2}{4t}} - e^{\frac{-(\xi+x)^2}{4t}}\big]d\xi \\ + \int_0^t g'(\tau)\operatorname{erfc}\Big(\frac{-x}{2\sqrt{t-\tau}}\Big)d\tau + g(0)\operatorname{erfc}\Big(\frac{-x}{2\sqrt{t}}\Big). \end{aligned} \tag{2.9}$$

We impose a constraint on $\theta(x,t)$ that space derivatives exist along the positive t-axis and are equal. That is,

$$R_x(0^+,t) = L_x(0^-,t), \quad t > 0. \tag{2.10}$$

Calculating $R_x(x,t)$ and $L_x(x,t)$ from (2.8) and (2.9) respectively and putting $x = 0$, we find that

$$\begin{aligned} R_x(0,t) &= \frac{1}{(1+t)2\sqrt{\pi t^3}} \int_0^\infty \xi\theta_0(\xi)e^{\frac{-\xi^2}{4t}}d\xi \\ &\quad - \frac{1}{(1+t)\sqrt{\pi}} \int_0^t \big(g(\tau)(1+\tau)\big)' \frac{1}{\sqrt{t-\tau}}d\tau - \frac{g(0)}{(1+t)\sqrt{\pi t}}, \\ L_x(0,t) &= \frac{-1}{2\sqrt{\pi t^3}} \int_0^\infty \xi\theta_0(-\xi)e^{\frac{-\xi^2}{4t}}d\xi + \frac{1}{\sqrt{\pi}}\int_0^t \frac{g'(\tau)}{\sqrt{t-\tau}}d\tau - \frac{g(0)}{\sqrt{\pi t}}. \end{aligned}$$

Hence, using (2.10), we obtain the integral equation

$$\begin{aligned} \int_0^t \frac{(3t-\tau+2)}{2(1+t)}\frac{g'(\tau)}{\sqrt{t-\tau}}d\tau = \frac{1}{4}\int_0^\infty \xi\Big[\frac{\theta_0(\xi)}{1+t} + \theta_0(-\xi)\Big]\frac{e^{\frac{-\xi^2}{4t}}}{\sqrt{t^3}}d\xi \\ - \frac{g(0)(3t+2)}{2\sqrt{t}(1+t)}. \end{aligned} \tag{2.11}$$

In the view of the above integral equation, the following theorem discusses the existence and uniqueness of boundary condition $g(t)$.

Theorem 2.1. *For $\theta_0 \in W^{2,\infty}(\mathbb{R}) \cap C^2(\mathbb{R})$ such that $g(0) = \theta_0(0)$, there exists unique continuous bounded function $g(t)$ satisfying (2.11).*

Proof. The integral equation (2.11) can be written in Abel's integral equation form,

$$\frac{1}{\sqrt{\pi}} \int_0^t \frac{K(t,\tau)}{\sqrt{t-\tau}}\upsilon(\tau)d\tau = F(t), \quad \text{for all } t > 0, \tag{2.12}$$

with

$$\upsilon(\tau) = g'(\tau), \quad K(t,\tau) = \frac{(3t-\tau+2)}{2(1+t)},$$

and

$$F(t) = \frac{1}{4\sqrt{\pi}}\int_0^\infty \xi\big[\frac{\theta_0(\xi)}{1+t} + \theta_0(-\xi)\big]\frac{e^{\frac{-\xi^2}{4t}}}{\sqrt{t^3}}d\xi - \frac{g(0)(3t+2)}{2\sqrt{\pi t}(1+t)}. \tag{2.13}$$

It can be observed that $K(t,\tau)$ is continuous and $K(t,t) = 1$ for $0 \le \tau \le t < \infty$. Further, $\frac{\partial K}{\partial t} = -\frac{1}{2}[\frac{1-\tau}{(1+t)^2}]$ is bounded for $0 \le \tau \le t < \infty$. Note that $F(0)$ cannot be obtained directly from (2.13). Hence, integrating by parts in (2.13) simplifies to

$$F(t) = \frac{1}{2\sqrt{\pi}}\Big[2\int_0^\infty e^{-\eta^2}\Big(\frac{\theta_0'(2\sqrt{t}\eta)}{1+t} - \theta_0'(-2\sqrt{t}\eta)\Big)d\eta - \frac{2\sqrt{t}}{1+t}\theta_0(0)\Big], \tag{2.14}$$

for the choice of $\theta_0(0) = g(0)$. Then we notice that $F(0) = 0$ and

$$F'(t) = \frac{2}{\sqrt{\pi t}}\Big[2\int_0^\infty e^{-\eta^2}\Big(\frac{\eta\theta_0''(2\sqrt{t}\eta)}{(1+t)} - \frac{\sqrt{t}\theta_0'(2\sqrt{t}\eta)}{(1+t)^2} + \eta\theta_0''(2\sqrt{t}\eta)\Big)d\eta - \frac{(1-t)\theta_0}{(1+t)^2}\Big].$$

The condition on θ_0 and simplification of the above equation lead to $|F'(t)| \le \frac{m}{\sqrt{t}}$ for some $m > 0$ depending on θ_0. We define

$$D^{1/2}F = \frac{1}{\sqrt{\pi}}\frac{d}{dt}\int_0^t \frac{F(s)}{\sqrt{t-s}}ds = \frac{1}{\sqrt{\pi}}\int_0^t \frac{F'(s)}{\sqrt{t-s}}ds, \tag{2.15}$$

which yields

$$|D^{1/2}F| \le \frac{C}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{s}\sqrt{t-s}}ds = C\sqrt{\pi}.$$

i.e., $D^{1/2}F$ is continuous and bounded for all $t > 0$. Using standard results [9], there exists a unique continuous bounded solution $\upsilon(t)$ for Abel's equation (2.12). By defining $g(t) := \theta_0(0) + \int_0^t \upsilon(\tau)d\tau$, we conclude that $g(t)$ satisfies all the desired properties in the statement. $\square$

Therefore, the solution of (2.2)-(2.3) is well defined and is given by

$$\theta(x,t) = \begin{cases} R(x,t), & x \ge 0,\\ L(x,t), & x < 0, \end{cases} \tag{2.16}$$

where $R(x,t)$ and $L(x,t)$ are given in (2.8) and (2.9) respectively.

Using the results given in [9], we state the bounds for υ as follows,

$$|\upsilon(t)| \le e^{2Mt} \parallel D^{1/2}F(t) \parallel_{L^\infty(0,t)} \le \gamma e^{2Mt},$$

where $M = sup_{t\le\tau}\left|\frac{\partial K}{\partial t}(t,\tau)\right|$ and for some number $\gamma > 0$. Having shown the existence and uniqueness for $g(t)$, we derive explicit expression for it.

Explicit representation of $g(t)$. By reorganizing Abel's integral equation (2.12) and then applying integration by parts lead to the classical Abel's integral equation form [9],

$$\frac{1}{\sqrt{\pi}}\int_0^t \Big[\frac{3}{2}\int_0^\tau \upsilon(s)ds + (2\tau+2)\upsilon(\tau)\Big]\frac{1}{\sqrt{t-\tau}}d\tau = 2(1+t)F(t). \tag{2.17}$$

Multiplying both sides of (2.17) by $\frac{1}{\sqrt{\pi}\sqrt{y-t}}$, where $0 < t < y < \infty$ and integrating from 0 to y, we obtain

$$\frac{1}{\pi}\int_0^y \frac{1}{\sqrt{y-t}}\int_0^t \Big[\frac{3}{2}\int_0^\tau \upsilon(s)ds + (2\tau+2)\upsilon(\tau)\Big]\frac{1}{\sqrt{t-\tau}}d\tau\, dt = \int_0^y \frac{2(1+t)F(t)}{\sqrt{\pi}\,\sqrt{y-t}}dt.$$

Changing the order of integration and rearranging yields

$$\begin{aligned}&\frac{1}{\pi}\int_0^y \Big(\int_\tau^y \frac{1}{\sqrt{y-t}}\frac{1}{\sqrt{t-\tau}}\,dt\Big)\Big[\frac{3}{2}\int_0^\tau \upsilon(s)ds + (2\tau+2)\upsilon(\tau)\Big]d\tau\\ &= \frac{1}{\sqrt{\pi}}\int_0^y \frac{2(1+t)F(t)}{\sqrt{y-t}}dt.\end{aligned}$$

Simplifying the integral in the left side of the above equation directs to

$$\int_0^y \Big[\frac{3}{2}\int_0^\tau \upsilon(s)ds + (2\tau+2)\upsilon(\tau)\Big]d\tau = \frac{1}{\sqrt{\pi}}\int_0^y \frac{2(1+t)F(t)}{\sqrt{y-t}}dt.$$

Differentiating with respect to t, we obtain

$$\frac{3}{2}\int_0^t \upsilon(s)ds + (2t+2)\upsilon(t) = \frac{1}{\sqrt{\pi}}\frac{d}{dt}\int_0^t \frac{2(1+\tau)F(\tau)}{\sqrt{t-\tau}}d\tau. \tag{2.18}$$

Substitution of $\upsilon(t) = g'(t)$ and simplification leads to

$$\frac{d}{dt}\Big((t+1)^{3/4}g(t)\Big) = \frac{3\ g(0)}{4\ (t+1)^{1/4}} + \frac{1}{\sqrt{\pi}(t+1)^{1/4}}\frac{d}{dt}\int_0^t \frac{(1+\tau)F(\tau)}{\sqrt{t-\tau}}\,d\tau. \tag{2.19}$$

Using $F(t)$ given in (2.13), we evaluate the second term on the right side of the above equation which yields

$$\frac{1}{\sqrt{\pi}(t+1)^{1/4}}\frac{d}{dt}\int_0^t \frac{(1+\tau)F(\tau)}{\sqrt{t-\tau}}d\tau = \frac{B'(t)}{\pi(t+1)^{1/4}} - \frac{3\ g(0)}{4(t+1)^{1/4}},$$

where

$$B(t) = \int_0^t\int_0^\infty \frac{\eta\left[\theta_0(2\sqrt{\tau}\eta) + (1+\tau)\theta_0(-2\sqrt{\tau}\eta)\right]e^{-\eta^2}}{\sqrt{\tau}\sqrt{t-\tau}}\,d\eta\,d\tau. \tag{2.20}$$

Substituting (2.20) in (2.19) yields

$$\frac{d}{dt}\Big((t+1)^{3/4}g(t)\Big) = \frac{B'(t)}{\pi(t+1)^{1/4}}.$$

Integrating the above equation leads to

$$g(t) = \frac{1}{(t+1)^{3/4}}\Big[\theta_0(0) - \frac{B(0)}{\pi}\Big] + \frac{B(t)}{\pi(t+1)} + \frac{1}{4\pi(t+1)^{3/4}}\int_0^t \frac{B(r)}{(r+1)^{\frac{5}{4}}}dr, \tag{2.21}$$

where $B(t)$ is given in (2.20), which is the unique explicit equation for the boundary condition $g(t)$.

An example. Consider the Cauchy problem (1.1)-(1.2) with trivial initial data $u_0 \equiv 0$ to have a look at the large time behavior of the boundary condition $g(t)$ easily. In this case, we obtain $\theta_0 \equiv 1$ and $B(t)$ is given by

$$\begin{aligned} B(t) &= \int_0^t\int_0^\infty \frac{\eta\big[1+(1+\tau)\big]e^{-\eta^2}}{\sqrt{\tau}\sqrt{t-\tau}}d\eta\,d\tau \\ &= \frac{1}{2}\Big[\int_0^t \frac{2}{\sqrt{\tau}\sqrt{t-\tau}}d\tau + \int_0^t \frac{\tau}{\sqrt{\tau}\sqrt{t-\tau}}d\tau\Big] \\ &= \frac{\pi}{4}(t+4). \end{aligned}$$

Hence, equation (2.21) reduces to

$$g(t) = \frac{2}{3(t+1)^{3/4}} + \frac{1}{3}.$$

Thus $g(t)$ approaches to $1/3$ as t appraoches ∞.

Considering the Cauchy problem (1.1)-(1.2), it is observed that the large time behavior of $g(t)$ will remain same as that of $g(t)$ concerned to the trivial initial data case.

Asymptotic behavior of $g(t)$**.** Since $u_0 \in L^1(\mathbb{R})$, we have

$$\theta_0(x) \to k \quad \text{as} \quad x \to \infty, \tag{2.22}$$

for some real constant k. It is easy to observe that $\theta_0(x) \to 1$ as $x \to -\infty$.

Lemma 2.2. *Let $g(t)$ be the boundary condition as in* (2.21)*. Then with condition* (2.22)*, we have*

$$\lim_{t\to\infty} g(t) = \frac{1}{3}. \tag{2.23}$$

Proof. Let $\tau = \gamma t$ in $B(t)$ given in (2.21). Then

$$\begin{aligned}
B(t) &= \int_0^\infty \int_0^1 \frac{\eta\big[\theta_0(2\eta\sqrt{\gamma t}) + (1+\gamma t)\theta_0(-2\eta\sqrt{\gamma t})\big]e^{-\eta^2}}{\sqrt{\gamma t}\sqrt{t-\gamma t}} t\, d\gamma\, d\eta \\
&= \int_0^\infty \int_0^1 \frac{\eta\big[\theta_0(2\eta\sqrt{\gamma t}) + \theta_0(-2\eta\sqrt{\gamma t})\big]e^{-\eta^2}}{\sqrt{\gamma}\sqrt{1-\gamma}} d\gamma\, d\eta \\
&\quad + \int_0^\infty \int_0^1 \frac{\eta\gamma t\theta_0(-2\eta\sqrt{\gamma t})e^{-\eta^2}}{\sqrt{\gamma}\sqrt{1-\gamma}}\, d\gamma\, d\eta \\
&=: I_1 + I_2.
\end{aligned}$$

Since u_0 is continuous and essentially bounded, there exists a real $M > 0$ such that $|\theta_0(2\eta\sqrt{\gamma t}) + \theta_0(-2\eta\sqrt{\gamma t})| \le M$ for $2\eta\sqrt{\gamma t} \in \mathbb{R}$. Further, $\frac{M\eta e^{-\eta^2}}{\sqrt{\gamma}\sqrt{1-\gamma}}$ is summable over $[0,\infty] \times [0,1]$. Thus, by dominated convergence theorem, the condition (2.22) yields

$$\begin{gathered}
\lim_{t\to\infty} I_1 = \int_0^\infty \eta\, e^{-\eta^2} \int_0^1 \frac{k+1}{\sqrt{\gamma}\sqrt{1-\gamma}}\, d\gamma\, d\eta = \frac{(k+1)\pi}{2}, \\
\lim_{t\to\infty} \frac{I_2}{t} = \int_0^\infty \eta\ e^{-\eta^2} \int_0^1 \frac{\sqrt{\gamma}}{\sqrt{1-\gamma}}\, d\gamma\, d\eta = \frac{\pi}{4}.
\end{gathered}$$

Using the above values, we obtain

$$\lim_{t\to\infty} \frac{B(t)}{(1+t)} = \lim_{t\to\infty} \frac{I_1}{(1+t)} + \lim_{t\to\infty} \frac{I_2}{t} \frac{1}{(1+\frac{1}{t})} = \frac{\pi}{4}.$$

It can be seen that

$$\lim_{t\to\infty} \frac{1}{4\pi(t+1)^{3/4}} \int_0^t \frac{B(r)}{(r+1)^{\frac{5}{4}}} dr = \lim_{t\to\infty} \frac{1}{3\pi} \frac{B(t)}{(1+t)} = \frac{1}{12}.$$

Using these estimates in (2.21) as $t \to \infty$, we complete the proof. □

3. Global weak solutions

Definition 3.1. A function $u(x,t)$ defined in $\mathbb{R} \times (0,\infty)$ is said to be a (global) weak solution if $u \in L^2(\mathbb{R} \times (0,\infty)) \cap W^{1,\infty}(\mathbb{R} \times (0,\infty))$ and u satisfies (1.1) in the sense of distributions. i.e.,

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} (u\phi_t + \frac{1}{2}u^2\phi_x - u_x\phi_x)\, dx\, dt + \int_0^\infty \frac{2}{1+t}\phi(0,t)\, dt \\
&- \int_{\mathbb{R}} u_0(x)\, \phi(x,0)\, dx = 0,
\end{aligned} \tag{3.1}$$

for all test functions $\phi \in C_0^\infty(\mathbb{R} \times [0,\infty))$.

Theorem 3.2. *For the initial data $\theta_0 \in W^{2,\infty}(\mathbb{R}) \cap C^2(\mathbb{R})$, the Cauchy problem* (2.2)-(2.3) *admits a positive solution. i.e., $\theta(x,t) > 0$ for all $x \in \mathbb{R}$ and $t > 0$.*

Proof. In view of the fact that $u_0 \in L^1(\mathbb{R})$ and (2.1), we obtain $\theta_0(x) > 0$ for all $x \in \mathbb{R}$. Considering the maximum principle, it is sufficient to show that $\theta(0,t) = g(t) > 0, \forall\, t > 0$. On the contrary, assume that $\theta(0,t) = g(t) \leq 0$ for some $t > 0$. This implies that there exists a point $q > 0$ which satisfies $g(t)$ and $g(\sigma)$ is non-negative for $\sigma \leq q$. By rearranging the kernel in (2.11),

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty \xi\big[\theta_0(\xi) + (1+t)\theta_0(-\xi)\big]\frac{e^{\frac{-\xi^2}{4t}}}{\sqrt{t^3}}d\xi - g(0)\big[3\sqrt{t} + \frac{2}{\sqrt{t}}\big] \\
&= \int_0^t \sqrt{(t-\sigma)}g'(\sigma)d\sigma + 2(t+1)\int_0^t \frac{g'(\sigma)}{\sqrt{t-\sigma}}d\sigma.
\end{aligned} \tag{3.2}$$

Note that

$$\begin{gathered}
\int_0^q \sqrt{(t-\sigma)}g'(\sigma)d\sigma = -g(0)\sqrt{t} + \int_0^q \frac{g(\sigma)}{2}\frac{1}{\sqrt{t-\sigma}}d\sigma, \\
\int_0^q \frac{g'(\sigma)}{\sqrt{t-\sigma}}d\sigma = -\frac{g(0)}{\sqrt{t}} - \int_0^q \frac{g(\sigma)}{2}\frac{1}{(t-\sigma)^{3/2}}d\sigma.
\end{gathered}$$

Splitting the right side of (3.2) and substituting above we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty \xi\big[\theta_0(\xi) + (1+t)\theta_0(-\xi)\big]\frac{e^{\frac{-\xi^2}{4t}}}{\sqrt{t^3}}d\xi + \int_0^q \frac{2+t+\sigma}{2(t-\sigma)^{3/2}}g(\sigma)d\sigma \\
&= \int_q^t \frac{3t-\sigma+2}{\sqrt{t-\sigma}}g'(\sigma)d\sigma.
\end{aligned} \tag{3.3}$$

It is clear that the first term of the above equation admits a non-negative lower bound. Using $0 \leq \sigma \leq q < t$, we obtain

$$0 < \frac{2+t}{t^{3/2}} \leq \frac{2+t+\sigma}{(t-\sigma)^{3/2}}$$

which in turn provides

$$\lim_{t\to q}\int_0^q \frac{2+t+\sigma}{(t-\sigma)^{3/2}}g(\sigma)d\sigma \geq \frac{2+q}{q^{3/2}}\int_0^q g(\sigma)d\sigma > 0.$$

Hence, it is proved that left side of (3.3) admits a positive lower bound as $t \to q$, whereas the right side vanishes as $t \to q$, which is a contradiction. $\square$

Theorem 3.3. *With the initial data u_0 and the solution $\theta(x,t)$ of* (2.2)-(2.3)*, there exists a unique weak solution of* (1.1)-(1.2) *given by*

$$u(x,t) = -2\frac{\theta_x(x,t)}{\theta(x,t)}. \tag{3.4}$$

Moreover, the solution $u(x,t)$ is in $C^\infty(\mathbb{R}\setminus\{0\}\times(0,\infty))$.

Proof. Let us prove the existence of a weak solution of (1.1)-(1.2). It is known that

$$\theta_{xx} = \theta_t + \frac{\theta}{(1+t)}, \quad \text{for} \quad x \geq 0.$$

Hence, we obtain

$$u_x(0^+,t) = \frac{-2}{\theta(0^+,t)}\Big[\theta_t(0^+,t) + \frac{\theta(0^+,t)}{(1+t)} - \frac{\theta_x^2(0^+,t)}{\theta(0^+,t)}\Big],$$

$$u_x(0^-,t) = \frac{2}{\theta(0^-,t)}\Big[\theta_t(0^-,t) - \frac{\theta_x^2(0^-,t)}{\theta(0^-,t)}\Big].$$

Thus using the above equations one can obtain

$$u_x(0^+,t) - u_x(0^-,t) + \frac{2}{(1+t)} = \frac{2}{g(t)}\big[\theta_t(0^-,t) - \theta_t(0^+,t)\big]. \tag{3.5}$$

Let U and V be the domains of $R(x,t)$ and $L(x,t)$ respectively. Integrating $u\,\phi_t$ by parts gives

$$\begin{aligned}
\iint_U u\,\phi_t\,dx\,dt &= -\iint_U u_t\,\phi\,dx\,dt - \int_{\tau=0}^{\infty} u(\tau,0)\,\phi(\tau,0)\,d\tau,\\
\iint_V u\,\phi_t\,dx\,dt &= -\iint_V u_t\,\phi\,dx\,dt - \int_{\tau=-\infty}^{0} u(\tau,0)\,\phi(\tau,0)\,d\tau.
\end{aligned}$$

Similarly, integrating $u_{xx}\phi$ by parts, we obtain

$$\begin{aligned}
\iint_U u_{xx}\,\phi\,dx\,dt &= -\iint_U u_x\,\phi_x\,dx\,dt - \int_{t=0}^{\infty} u_x(0^+,t)\,\phi(0,t)\,dt,\\
\iint_V u_{xx}\,\phi\,dx\,dt &= -\iint_V u_x\,\phi_x\,dx\,dt + \int_{t=0}^{\infty} u_x(0^-,t)\,\phi(0,t)\,dt.
\end{aligned}$$

Further, integrating $\frac{u^2}{2}\phi_x$ by parts on U and V provides

$$\int_0^\infty\int_{\mathbb{R}}\Big(\frac{u^2}{2}\Big)\phi_x\,dx\,dt = -\int_0^\infty\int_{\mathbb{R}}\Big(\frac{u^2}{2}\Big)_x\phi\,dx\,dt.$$

Hence, for $\phi\in C_c^\infty(\mathbb{R}\times[0,\infty))$, we obtain

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}}\Big[u\phi_t + \frac{u^2}{2}\phi_x - u_x\phi_x\Big]\,dx\,dt + \int_0^\infty \frac{2}{(1+t)}\phi(0,t)\,dt\\
&+\int_{\mathbb{R}} u_0(x)\phi(x,0)dx = -2\int_{t=0}^{\infty}\frac{\theta_t(0^+,t)-\theta_t(0^-,t)}{g(t)}\phi(0,t)dt.
\end{aligned} \tag{3.6}$$

The dominated convergence theorem and then integration by parts reduce the right hand side expression of above equation to zero.

Let us prove uniqueness. Let u and v be two solutions of (3.1), and put $w=u-v$. Then, we obtain

$$\int_0^\infty\int_{\mathbb{R}}\big(w\phi_t + \frac{1}{2}(u+v)w\phi_x - w_x\phi_x\big)dx\,dt = 0, \tag{3.7}$$

for all test functions $\phi\in C_0^\infty(\mathbb{R}\times[0,\infty))$.

For a fixed $T>0$, we define $\phi = w(x,t)H(T-t)H(x)$ in the domain $\{0\le x<\infty, t>0\}$ and $\phi = w(x,t)H(T-t)H_1(x)$, where $H_1(x) = 1-H(x)$, in $\{\infty < x < 0, t>0\}$. Note that the defined function ϕ in both domains is not a test function. However, we can use this ϕ in (3.7) using usual approximation techniques as $C_0^\infty(\mathbb{R}\times[0,\infty))$ is dense in $H_0^1(\mathbb{R}\times[0,\infty))$.

For $0\le x<\infty$, the weak derivatives of ϕ with respect to t is $w_t(x,t)-w(x,t)\delta(t-T)$ and the weak derivative of ϕ with respect to x is $w_x(x,t)+w(x,t)\delta(x)$. Similarly for $-\infty<x<0$, the weak derivatives of ϕ with respect to t is $w_t(x,t)-w(x,t)\delta(t-T)$ and the weak derivative of ϕ with respect to x is $w_x(x,t)-w(x,t)\delta(x)$.

For $\phi = w(x,t)H(T-t)H_1(x)$, integral equation (3.7) turns out to be

$$\begin{aligned}
&\int_0^T \int_{-\infty}^0 -w\, w_t\, dx\, dt + \int_{-\infty}^0 w^2(x,t)\, dx \\
&- \frac{1}{2}\Big[\int_0^T \int_{-\infty}^0 (u+v) w\, w_x\, dx\, dt - \int_0^T (w\,(u+v)w)(0,t)\, dt\Big] \\
&+ \int_0^T \int_{-\infty}^0 w_x^2\, dx\, dt - \int_0^T (w_x w)(0,t)\, dt = 0.
\end{aligned} \tag{3.8}$$

Similarly for $\phi = w(x,t)H(T-t)H(x)$, integral equation (3.7) yields

$$\begin{aligned}
&\int_0^T \int_0^\infty -w\, w_t\, dx\, dt + \int_0^\infty w^2(x,t)\, dx \\
&- \frac{1}{2}\Big[\int_0^T \int_0^\infty (u+v) w\, w_x\, dx\, dt + \int_0^T (w\,(u+v)w)(0,t)\, dt\Big] \\
&+ \Big[\int_0^T \int_0^\infty w_x^2\, dx\, dt + \int_0^T (w_x w)(0,t)\, dt\Big] = 0.
\end{aligned} \tag{3.9}$$

Also using that $w(x,0) = 0$, we deduce

$$\begin{aligned}
-\int_0^T \int_{\mathbb{R}} w\, w_t\, dx\, dt + \int_{\mathbb{R}} w^2(x,t) dx &= \frac{-1}{2} \int_{\mathbb{R}} \int_0^T \partial_t(w^2)\, dt\, dx + \|w(\cdot\,,T)\|^2 \\
&= \frac{1}{2} \|w(\cdot\,,T)\|_2^2.
\end{aligned} \tag{3.10}$$

Hence, equations (3.8)-(3.9) with the above equation lead to

$$\begin{aligned}
&\frac{1}{2}\|w(\cdot,T)\|_2^2 + \int_0^T \int_{\mathbb{R}} w_x^2\, dx\, dt + \int_0^T \big[(w_x w)(0^+,t) - (w_x w)(0^-,t)\big] dt \\
&= \frac{1}{2}\Big[\int_0^T \int_{\mathbb{R}} (u+v) w\, w_x\, dx\, dt + \int_0^T \big[((u+v)w^2)(0^+,t) - ((u+v)w^2)(0^-,t)\big] dt\Big],
\end{aligned}$$

which implies

$$\begin{aligned}
\|w(\cdot,T)\|_2^2 + 2\int_0^T \|w_x(\cdot\,,t)\|_2^2\, dt &\le \frac{1}{2}\int_0^T \int_{\mathbb{R}} \|(u+v)(t)\|_\infty |w(x,t)\| w_x(t)|\, dx\, dt \\
&\le \frac{1}{2}\int_0^T \|(u+v)(t)\|_\infty \|w\|_2 \|w_x\|_2\, dt \\
&\le \frac{1}{4}\int_0^T \|(u+v)(t)\|_\infty^2 \|w(x,t)\|_2^2 dt + \frac{1}{4}\int_0^T \|w_x\|_2^2 dt \\
&\le \frac{M_0}{4}\int_0^T \|w(\cdot,t)\|_2^2\, dt + 2\int_0^T \|w_x(\cdot,t)\|_2^2\, dt.
\end{aligned}$$

Hence,

$$\|w(\cdot,T)\|_2^2 \le \frac{M_0}{4}\int_0^T \|w(\cdot,t)\|_2^2\, dt.$$

Using Gronwall's inequality, we conclude that $w(x,T) = 0$ a.e. for all $T > 0$.

Let us prove smoothness of solutions. Since $\theta(x,t)$ is positive solution of the heat equation for $x < 0$ and $(1+t)\theta(x,t)$ is also a positive solution of heat equation for $x > 0$, the solution $u(x,t)$ of the Cauchy problem (1.1)-(1.2) given in (3.4) is well-defined and is smooth on the domain $\mathbb{R} \setminus \{0\} \times (0,\infty)$. $\square$

Theorem 3.4. *$xR(x,t)$ and $xR_x(x,t)$ are uniformly convergent on compact sets and their limits are*

$$\lim_{t\to\infty} x\,R(x,t) = \frac{x}{3}, \quad \lim_{t\to\infty} x\,R_x(x,t) = 0. \tag{3.11}$$

Proof. First, we prove that $R(x,t)$ converges to $1/3$ uniformly on compact sets. Assume $0 \le x \le A$ for some $A > 0$. Let $g(t)$ be bounded by M. Integrating the second term in (2.8) by parts, we have

$$\begin{aligned} R(x,t) = \frac{1}{1+t}\Big[&\frac{1}{\sqrt{\pi}}\int_{\frac{-x}{2\sqrt{t}}}^{\infty} \theta_0(2\sqrt{t}\eta + x)e^{-\eta^2}\,d\eta \\ &- \frac{1}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{t}}}^{\infty} \theta_0(2\sqrt{t}\eta - x)e^{-\eta^2}\,d\eta\Big] \\ &+ \frac{x}{2\sqrt{\pi}(1+t)}\int_0^t \frac{g(t-\tau)(1+t-\tau)e^{\frac{-x^2}{4\tau}}}{\tau^{3/2}}\,d\tau. \end{aligned} \tag{3.12}$$

It is seen that the first term vanishes uniformly as $t \to \infty$ and hence ignore it. Then we consider

$$|R(x,t) - \frac{1}{3}| = \Big|\frac{x}{2\sqrt{\pi}(1+t)}\int_0^t \frac{g(t-\tau)(1+t-\tau)e^{\frac{-x^2}{4\tau}}}{\tau^{3/2}}\,d\tau - \frac{2}{\sqrt{\pi}}\int_0^{\infty}\frac{1}{3}e^{-\eta^2}\,d\eta\Big|.$$

Expanding the second term of (3.12) and changing the variable, the above expression turns out to be

$$\begin{aligned} |R(x,t) - \frac{1}{3}| \le \frac{2}{\sqrt{\pi}}\int_0^{\infty}\big|g(t-\tau) - \frac{1}{3}\big|e^{-\eta^2}\,d\eta + \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{t}}} |g(t-\tau)|e^{-\eta^2}\,d\eta \\ + \frac{x}{2\sqrt{\pi}}\int_0^t \frac{|g(t-\tau)|e^{\frac{-x^2}{4\tau}}}{(1+t)\sqrt{\tau}}\,d\tau. \end{aligned} \tag{3.13}$$

Note that the second and third term of the above equation admits a uniform bound and vanishes uniformly as t tends to ∞. i.e., one can obtain

$$\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{t}}} |g(t-\tau)|e^{-\eta^2}\,d\eta \le \frac{2\,M}{\sqrt{\pi}}\int_0^{\frac{A}{2\sqrt{t}}} e^{-\eta^2}\,d\eta \le M\,\mathrm{erf}\Big(\frac{A}{2\sqrt{t}}\Big).$$

Since $t \le 1+t$ for all $t > 0$ and $g(t)$ is bounded, we can see that the last term in (3.13) is uniformly bounded by

$$\frac{AM}{2\sqrt{\pi}}\int_0^t \frac{e^{\frac{-x^2}{4\tau}}}{t\sqrt{\tau}}\,d\tau \le \frac{AM}{2\sqrt{\pi}}\int_0^t \frac{1}{t\,\sqrt{\tau}}\,d\tau = \frac{AM}{\sqrt{\pi}}\,\frac{1}{\sqrt{t}}.$$

Now, we consider the uniform convergence of $xR_x(x,t)$. We have

$$\begin{aligned} xR_x(x,t) &= \frac{x}{(1+t)}\frac{1}{4\sqrt{\pi}t^{3/2}}\int_0^{\infty}\theta_0(\xi)(\xi - x)e^{\frac{-(\xi-x)^2}{4t}} \\ &\quad + \frac{x}{(1+t)}\frac{1}{4\sqrt{\pi}t^{3/2}}\int_0^{\infty}\theta_0(\xi)(\xi + x)e^{\frac{-(\xi+x)^2}{4t}}\,d\xi \\ &\quad + \frac{x}{2\sqrt{\pi}(1+t)}\int_0^t \frac{g(t-\tau)(1+t-\tau)e^{\frac{-x^2}{4\tau}}}{\tau^{3/2}}[1 - \frac{x^2}{2\tau}]d\tau \\ &=: xJ_1 + xJ_2 + xJ_3. \end{aligned}$$

By changing the variable $\eta = \frac{\xi - x}{2\sqrt{t}}$ in xJ_1 and $\eta = \frac{\xi + x}{2\sqrt{t}}$ in xJ_2, we can observe that both the terms vanishes uniformly. The third term xJ_3 can be expanded as

$$\begin{aligned} xJ_3 &= \frac{x}{2\sqrt{\pi}} \int_0^t g(t-\tau)\Big(\frac{1+t-\tau}{1+t}\Big)\frac{e^{\frac{-x^2}{4\tau}}}{\tau^{3/2}} d\tau + \frac{-x^3}{4\sqrt{\pi}} \int_0^t \frac{g(t-\tau)e^{\frac{-x^2}{4\tau}}}{\tau^{\frac{5}{2}}} d\tau \\ &\quad + \frac{x^3}{4\sqrt{\pi}} \int_0^t \frac{g(t-\tau)}{(1+t)} \frac{e^{\frac{-x^2}{4\tau}}}{\tau^{3/2}} d\tau =: M_1 + M_2 + M_3. \end{aligned} \tag{3.14}$$

Then

$$|xJ_3| \le |M_1 - \frac{1}{3}| + |M_2 + M_3 + \frac{1}{3}| \le |R(x,t) - \frac{1}{3}| + |M_2 + \frac{1}{3}| + |M_3|. \tag{3.15}$$

Now by changing the variable for M_2, we obtain

$$\begin{aligned} &M_2 + \frac{1}{3} \\ &= \frac{-4}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{t}}}^{\infty} g\Big(t - \frac{x^2}{4\eta^2}\Big)\eta^2 e^{-\eta^2} d\eta + \frac{4}{\sqrt{\pi}} \int_0^\infty \frac{1}{3}\,\eta^2\, e^{-\eta^2} d\eta \\ &= \frac{-4}{\sqrt{\pi}} \int_0^\infty \Big[g\Big(t - \frac{x^2}{4\eta^2}\Big) - \frac{1}{3}\Big]\eta^2 e^{-\eta^2} d\eta + \frac{4}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{t}}} g\Big(t - \frac{x^2}{4\eta^2}\Big)\eta^2 e^{-\eta^2} d\eta. \end{aligned}$$

The last term on the right side of the above equation is uniformly bounded by

$$\frac{4M}{\sqrt{\pi}} \frac{A^2}{4t} \int_0^{\frac{x}{2\sqrt{t}}} e^{-\eta^2} d\eta \le 2M \frac{A^2}{4t} \operatorname{erf}\Big(\frac{A}{2\sqrt{t}}\Big),$$

which vanishes uniformly as $t \to \infty$. By similar calculations, one can obtain that third term in (3.15) vanishes uniformly as $t \to \infty$. Hence, we can conclude that $xR_x(x,t) \to 0$ uniformly as $t \to \infty$ when x is bounded. $\square$

Theorem 3.5. *The functions $xL(x,t)$ and $xL_x(x,t)$ are uniformly convergent on compact sets and their limits are*

$$\lim_{t\to\infty} xL(x,t) = \frac{x}{3}, \quad \lim_{t\to\infty} xL_x(x,t) = 0.$$

Proof. First, we prove that $L(x,t)$ converges to $1/3$ uniformly on compact sets. Let M_1 and M_2 be the bounds for $g(t)$ and $|g(t) - \frac{1}{3}|$ respectively for all $t > 0$. Let $\epsilon > 0$ be given and a positive number A such that $-A \le x \le 0$. Then there exist T_1, T_2 and T_3 such that

$$|g(t) - \frac{1}{3}| < \frac{\epsilon}{3}, \quad \forall t > T_1. \tag{3.16}$$

$$\operatorname{erf}\big(\frac{A}{2\sqrt{t}}\big) < \frac{\epsilon}{3M_1}, \quad \forall t > T_2. \tag{3.17}$$

$$\operatorname{erf}\big(\frac{A}{2\sqrt{t}}\big) < \frac{\epsilon}{3M_2} \quad \forall t > T_3. \tag{3.18}$$

Hence, $|g\big(t - \frac{x^2}{4\eta^2}\big) - \frac{1}{3}| < \frac{\epsilon}{3}$ whenever $\eta < \frac{-A}{2\sqrt{t-T_1}}$. Assume that $t \ge T_1 + T_2 + T_3$. Then

$$|L(x,t) - \frac{1}{3}| = \Big| - \frac{x}{2\sqrt{\pi}} \int_0^t g(t-\tau)\frac{e^{\frac{-x^2}{4\tau}}}{\tau^{3/2}} d\tau - \frac{2}{\sqrt{\pi}} \int_{-\infty}^0 \frac{1}{3}\, e^{-\eta^2} d\eta\Big|. \tag{3.19}$$

Substituting $\eta = \frac{x}{2\sqrt{\tau}}$ in the first term of right-hand side, (3.19) reduces to

$$\Big|\frac{2}{\sqrt{\pi}}\Big[\int_{-\infty}^{0} g\Big(t-\frac{x^2}{4\eta^2}\Big)e^{-\eta^2}d\eta - \int_{\frac{-|x|}{2\sqrt{t}}}^{0} g\Big(t-\frac{x^2}{4\eta^2}\Big)e^{-\eta^2}d\eta\Big] - \frac{2}{\sqrt{\pi}}\int_{-\infty}^{0}\frac{1}{3}\,e^{-\eta^2}d\eta\Big|,$$

which is bounded by

$$\begin{aligned}
&\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\frac{-A}{2\sqrt{t-T_1}}}\Big|g\Big(t-\frac{x^2}{4\eta^2}\Big)-\frac{1}{3}\Big|e^{-\eta^2}d\eta + \frac{2}{\sqrt{\pi}}\int_{\frac{-A}{2\sqrt{t-T_1}}}^{0}\Big|g\Big(t-\frac{x^2}{4\eta^2}\Big)-\frac{1}{3}\Big|e^{-\eta^2}d\eta\\
&+\frac{2}{\sqrt{\pi}}\int_{\frac{-A}{2\sqrt{t}}}^{0}\Big|g\Big(t-\frac{x^2}{4\eta^2}\Big)\Big|e^{-\eta^2}d\eta\\
&\le \frac{2}{\sqrt{\pi}}\,\frac{\epsilon}{3}\int_{-\infty}^{0}e^{-\eta^2}d\eta + \frac{2}{\sqrt{\pi}}M_2\int_0^{\frac{A}{2\sqrt{t-T_1}}}e^{-\eta^2}d\eta + \frac{2}{\sqrt{\pi}}M_1\int_0^{\frac{A}{2\sqrt{t}}}e^{-\eta^2}d\eta\\
&\le \frac{\epsilon}{3} + M_2\,\mathrm{erf}\Big(\frac{A}{2\sqrt{t-T_1}}\Big) + M_1\,\mathrm{erf}\Big(\frac{A}{2\sqrt{t}}\Big) \le \epsilon.
\end{aligned}$$

Next we consider

$$\begin{aligned}
L_x(x,t) &= \frac{-1}{4\sqrt{\pi}t^{3/2}}\int_0^{\infty}\theta_0(-\xi)(\xi-x)e^{\frac{-(\xi-x)^2}{4t}}d\xi\\
&\quad+\frac{-1}{4\sqrt{\pi}t^{3/2}}\int_0^{\infty}\theta_0(-\xi)(\xi+x)e^{\frac{-(\xi+x)^2}{4t}}d\xi\\
&\quad-\frac{1}{2\sqrt{\pi}}\int_0^t\frac{g(t-\tau)}{\tau^{3/2}}e^{\frac{-x^2}{4\tau}}\big[1-\frac{x^2}{2\tau}\big]d\tau =: P_1+P_2+P_3.
\end{aligned}$$

Observe that $|xP_1| \le AP_1$ vanishes uniformly as $t\to\infty$. Similarly, xP_2 vanishes uniformly as $t\to\infty$. Hence, it is enough to show that xP_3 converges to zero uniformly when x is bounded. We consider

$$\begin{aligned}
P_3 &= -\frac{1}{2\sqrt{\pi}}\int_0^t\frac{g(t-\tau)}{\tau^{3/2}}e^{\frac{-x^2}{4\tau}}\big[1-\frac{x^2}{2\tau}\big]d\tau\\
&= \frac{2}{x\sqrt{\pi}}\int_{-\infty}^{\frac{-|x|}{2\sqrt{t}}} g\big(t-\frac{x^2}{4\eta^2}\big)e^{-\eta^2}d\eta - \frac{4}{x\sqrt{\pi}}\int_{-\infty}^{\frac{-|x|}{2\sqrt{t}}} g\big(t-\frac{x^2}{4\eta^2}\big)\eta^2e^{-\eta^2}d\eta\\
&=: P_3' + P_3''.
\end{aligned}$$

Then

$$|xP_3| \le |xP_3' - \frac{1}{3}| + |xP_3'' + \frac{1}{3}|. \tag{3.20}$$

We show that the terms in the right of the above expression (3.20) admit uniform bounds which vanish uniformly as t tends to infinity. We consider

$$\begin{aligned}
|xP_3' - \frac{1}{3}| &= \Big|\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\frac{-|x|}{2\sqrt{t}}} g\Big(t-\frac{x^2}{4\eta^2}\Big)e^{-\eta^2}d\eta - \frac{1}{3}\Big|\\
&\le \frac{2}{\sqrt{\pi}}\int_{-\infty}^{0}\Big|g\Big(t-\frac{x^2}{4\eta^2}\Big)-\frac{1}{3}\Big|e^{-\eta^2}d\eta + \frac{2}{\sqrt{\pi}}\int_{\frac{-|x|}{2\sqrt{t}}}^{0}\Big|g\Big(t-\frac{x^2}{4\eta^2}\Big)\Big|e^{-\eta^2}d\eta.
\end{aligned}$$

Observe that the second term of the above expression satisfies

$$\frac{2}{\sqrt{\pi}}\int_{\frac{-|x|}{2\sqrt{t}}}^{0}\Big|g\Big(t-\frac{x^2}{4\eta^2}\Big)\Big|e^{-\eta^2}d\eta \le \frac{2}{\sqrt{\pi}}\int_{\frac{-A}{2\sqrt{t}}}^{0}\Big|g\Big(t-\frac{x^2}{4\eta^2}\Big)\Big|e^{-\eta^2}d\eta$$

$$
\begin{aligned}
&\le \frac{2M_1}{\sqrt{\pi}} \int_0^{\frac{A}{2\sqrt{t}}} e^{-\eta^2} d\eta \\
&= M_1 \operatorname{erf}\big(\frac{A}{2\sqrt{t}}\big),
\end{aligned}
$$

which vanishes uniformly. The bound for the second term in right side of (3.20) is obtained as follows:

$$
\begin{aligned}
&|xP_3'' + \frac{1}{3}| \\
&= \Big| \frac{4}{\sqrt{\pi}} \int_{-\infty}^{\frac{-|x|}{2\sqrt{t}}} g\Big(t - \frac{x^2}{4\eta^2}\Big)\eta^2 e^{-\eta^2} d\eta - \frac{4}{\sqrt{\pi}} \int_{-\infty}^{0} \frac{1}{3}\eta^2 e^{-\eta^2} d\eta \Big| \\
&\le \frac{4}{\sqrt{\pi}} \Big[\int_{-\infty}^{0} \Big| g\Big(t - \frac{x^2}{4\eta^2}\Big) - \frac{1}{3}\Big| \eta^2 e^{-\eta^2} d\eta + \int_{\frac{-|x|}{2\sqrt{t}}}^{0} \Big| g\Big(t - \frac{x^2}{4\eta^2}\Big)\Big| \eta^2 e^{-\eta^2} d\eta \Big].
\end{aligned}
$$

It is observed that the second term in the above expression vanishes uniformly and is bounded by

$$
\frac{4M_1}{\sqrt{\pi}} \int_{\frac{-A}{2\sqrt{t}}}^{0} \eta^2 e^{-\eta^2} d\eta \le \frac{4M_1}{\sqrt{\pi}} \frac{A^2}{4t} \int_0^{\frac{A}{2\sqrt{t}}} e^{-\eta^2} d\eta = \frac{M_1\, A^2}{2t} \operatorname{erf}\Big(\frac{A}{2\sqrt{t}}\Big).
$$

This completes the proof. □

Theorem 3.6. *The unique weak solution $u(x,t)$ of* (1.1)-(1.2) *converges uniformly to zero on compact sets.*

Proof. Using the Theorem 3.2, the inverse Hopf-Cole transformation is well defined and given by (3.4). Hence, for $x > 0$, we have

$$
u(x,t) = -2 \frac{x\, R_x(x,t)}{x\, R(x,t)} \to 0 \quad \text{as } t \to \infty.
$$

Similarly, for $x < 0$, we obtain

$$
u(x,t) = -2 \frac{x\, L_x(x,t)}{x\, L(x,t)} \to 0 \quad \text{as } t \to \infty.
$$

In view of the uniform convergence of R, R_x, L, L_x, the unique weak solution $u(x,t)$ converges to zero uniformly on compact sets. □

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