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COMPLETE CLASSIFICATION OF BIFURCATION CURVES FOR A MULTIPARAMETER DIFFUSIVE LOGISTIC PROBLEM WITH GENERALIZED HOLLING TYPE-IV FUNCTIONAL RESPONSE

JYUN-YUAN CIOU, TZUNG-SHIN YEH

Abstract. We study exact multiplicity and bifurcation curves of positive solutions for the diffusive logistic problem with generalized Holling type-IV functional response

$$
\begin{gathered}
u''(x) + \lambda\big[ru(1-\frac{u}{q}) - \frac{u}{1+mu+u^2}\big] = 0, \quad -1<x<1, \\
u(-1)=u(1)=0,
\end{gathered}
$$

where the quantity in brackets is the growth rate function and $\lambda > 0$ is a bifurcation parameter. On the $(\lambda, \|u\|_\infty)$-plane, we give a complete classification of two qualitatively different bifurcation curves: a C-shaped curve and a monotone increasing curve.

1. Introduction

We study the exact multiplicity and bifurcation curves of positive solutions for the diffusive logistic problem with generalized Holling type-IV functional response,

$$
\begin{gathered}
u''(x) + \lambda\big[ru(1-\frac{u}{q}) - \frac{u}{1+mu+u^2}\big] = 0, \quad -1<x<1, \\
u(-1)=u(1)=0,
\end{gathered}
\tag{1.1}
$$

where the growth rate function is $f(u) = ug(u)$ with

$$
g(u) = r(1-\frac{u}{q}) - \frac{1}{1+mu+u^2}\,, \tag{1.2}
$$

$m \geq 1$, q, r are two positive dimensionless parameters, and $\lambda > 0$ is a bifurcation parameter.

In population dynamics, a functional response of the predator to the prey density depends on the change in the density of prey susceptible to each predator per unit time. The simplest response function is

$$
p_1(u) = \begin{cases} au, & 0 \leq u < k/a, \\ k, & u \geq k/a, \end{cases}
$$

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where $k, a > 0$, which is called Holling type-I function in [6]. Michaelis and Menten proposed the response function

$$p_2(u) = \frac{cu}{a+u},$$

in the studying enzymatic reactions, where $a, c > 0$, which is called Holling type-II function in [6]. Another class of response function is

$$p_3(u) = \frac{cu^2}{a+u^2},$$

where $a, c > 0$, it is known as a Holling type-III function. Wang and Yeh [24, 26] studied a multiparameter diffusive logistic problem with Holling type-III functional response. Note that $p_1(u)$, $p_2(u)$, $p_3(u)$ are monotonic on $(0, \infty)$. Sokol and Howell [19] proposed a non-monotonic response function

$$p_4(u) = \frac{cu}{a+u^2},$$

which is called the simplified Holling type-IV function. This casse has been extensively studied by many authors, see Baek [1], Li and Xiao [9], Lian and Xu [10], Qolizadeh Amirabad et al. [16], Ruan and Xiao [17], and Yeh [25]. Another class of non-monotonic response functions is the generalized Holling type-IV function

$$\bar{p}_4(u) = \frac{cu}{a+bu+u^2},$$

where $a, c > 0$ and $b \geq 0$. The response function $\bar{p}_4(u)$ satisfies $\bar{p}_4'(u) > 0$ on $(0, \sqrt{a})$ and $\bar{p}_4'(u) < 0$ on $(\sqrt{a}, \infty)$. Collings [4] used the response function $\bar{p}_4(u)$ in a mite predator-prey interaction model for $b \geq \sqrt{a}$. The generalized Holling type-IV function has been studied by Huang and Xiao [7], Liu and Huang [11], and Upadhyay et al. [21].

The idea of using diffusion to study population dynamics was introduced by Skellam [18] in the early 1950s. Since then, reaction-diffusion equations have been widely used for the formation of spatial population patterns and the description of the effects of organisms' spatial dispersal in population dynamics; see Britton [2], Cantrell and Cosner [3], Fife [5], Murray[14], and Okubo [15]. Sounvoravong et al. [20] studied a reaction-diffusion system for a SIRS epidemic model. Problem (1.1) is motivated by the reaction-diffusion population model

$$\frac{\partial N}{\partial T} = D\frac{\partial^2 N}{\partial X^2} + r_N N\Big(1 - \frac{N}{K_N}\Big) - \frac{cN}{a+bN+N^2}, -\frac{L}{2}\sqrt{\frac{D}{r_N}} < X < \frac{L}{2}\sqrt{\frac{D}{r_N}}, \quad (1.3)$$

where $T > 0$, $D > 0$ is the diffusion constant, N is the prey population density, r_N is the intrinsic growth rate of the prey population, K_N is the carrying capacity, and $a, c > 0$, $b \geq 0$. The second term on the right-hand side of (1.3) is a logistic term. The third term on the right-hand side of (1.3) gives the rate of consumption of prey by predators, which is called the predation term; see Ludwig et al. [12, 13].

We consider the problem (1.3) with

$$w = \frac{N}{\sqrt{a}}, \quad \tilde{t} = r_N T, \quad \tilde{x} = \sqrt{\frac{r_N}{D}}X, \quad r = \frac{r_N a}{c}, \quad q = \frac{K_N}{\sqrt{a}}, \quad m = \frac{b}{\sqrt{a}}.$$

Then problem (1.3) takes the form

$$\frac{\partial w}{\partial \tilde{t}} = \frac{\partial^2 w}{\partial \tilde{x}^2} + w\big(1 - \frac{w}{q}\big) - \frac{1}{r}\frac{w}{1+mw+w^2}, \quad -\frac{L}{2} < \tilde{x} < \frac{L}{2}, \quad \tilde{t} > 0. \qquad (1.4)$$

Assume that a habitat $-L/2 \le \tilde{x} \le L/2$ is surrounded by a totally hostile, outer environment. That is, equation (1.4) holds in the strip $|\tilde{x}| < L/2$ and

$$w(-L/2, \tilde{t}) = w(L/2, \tilde{t}) = 0, \quad \tilde{t} > 0. \tag{1.5}$$

Let $v(x,t) = w(\tilde{x}, \tilde{t})$ with $x = 2\tilde{x}/L$, $t = 4\tilde{t}/L^2$, and let

$$\lambda = \frac{L^2}{4r}.$$

Then problem (1.4), (1.5) takes the form

$$\begin{gathered}\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \lambda\big[rv(1-\frac{v}{q}) - \frac{v}{1+mv+v^2}\big], \quad -1 < x < 1,\ t > 0,\\ v(-1,t) = v(1,t) = 0, \quad t > 0.\end{gathered} \tag{1.6}$$

Let $u(x)$ denote a positive steady-state population density of (1.6). Then $u(x)$ satisfies (1.1).

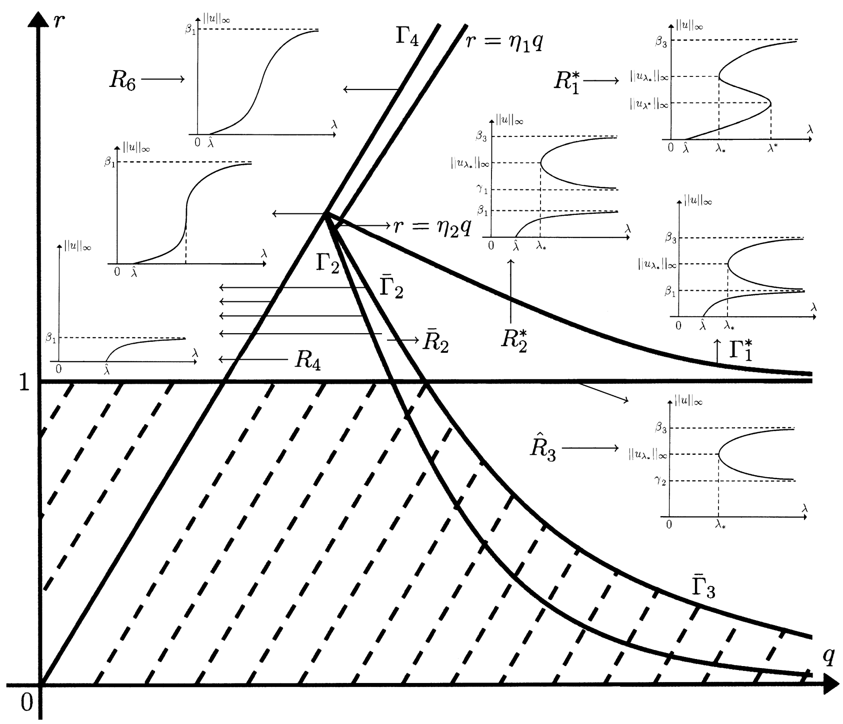

FIGURE 1. Classified bifurcation curves of (1.7), drawn on the first quadrant of (q, r)-parameter plane.

For $m = 0$, problem (1.1) takes the form

$$\begin{gathered}u''(x) + \lambda\big[ru(1-\frac{u}{q}) - \frac{u}{1+u^2}\big] = 0, \quad -1 < x < 1,\\ u(-1) = u(1) = 0.\end{gathered} \tag{1.7}$$

Applying the quadrature method (time-map method), Yeh [25] proved that either $r \le \eta_1 q$ and (q,r) lies above the curve

$$\Gamma = \Big\{(q,r) : q(a) = \frac{1+3a^2}{2a}, r(a) = \frac{1+3a^2}{(1+a^2)^2}, 0 < a < 1/\sqrt{3}\Big\}$$

or $r \le \eta_2 q$ for some constants $\eta_1 \approx 0.618$ and $\eta_2 \approx 0.601$. Then on the $(\lambda, \|u\|_\infty)$-plane, he gave a classification of four qualitatively different bifurcation curves: an S-shaped curve, a broken S-shaped curve, a C-shaped curve and a monotone increasing curve, see [25, Theorems 2.1–2.4] and Figure 1.

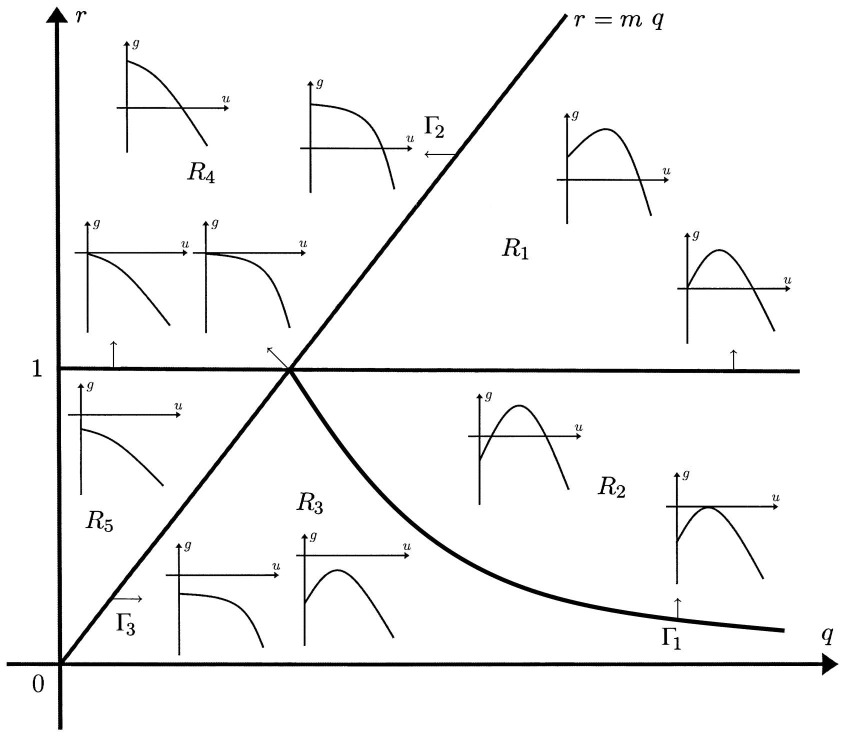

FIGURE 2. Classified graphs of growth rate per capita $g(u) = r(1-\frac{u}{q}) - \frac{1}{1+mu+u^2}$ with $m \ge 1$ on $(0,\infty)$, drawn on the first quadrant of (q,r)-parameter plane.

In this article we study exact multiplicity of positive solutions and shapes of bifurcation curves of (1.1) for parameters $m \ge 1$ and $q, r > 0$. We first find the number of positive zeros of growth rate per capita

$$g(u) = r(1-\frac{u}{q}) - \frac{1}{1+mu+u^2}.$$

Then we give a classification of $g(u)$ on the first quadrant of (q,r)-parameter plane according to the monotonicity of $g(u)$. We divide the first quadrant of (q,r)-parameter plane into the disjoint union of three curves Γ_1, Γ_2, Γ_3 and five regions

R_1, R_2, R_3, R_4, R_5 defined as follows:

$$\Gamma_1 = \Big\{(q,r) : q(a) = \frac{1+2ma+3a^2}{m+2a}, r(a) = \frac{1+2ma+3a^2}{(1+ma+a^2)^2}, 0 < a < \infty\Big\},$$
$$\Gamma_2 = \{(q,r) : r = mq > 1\},$$
$$\Gamma_3 = \big\{(q,r) : 0 < r = mq \le 1\big\},$$
$$R_1 = \{(q,r) : 0 < r < mq \text{ and } r \ge 1\},$$
$$R_2 = \big\{(q,r) : 0 < r < mq \text{ and } (q,r) \text{ lies between the curve } \Gamma_1 \text{ and the line } r = 1\big\},$$
$$R_3 = \Big\{(q,r) : 0 < r < mq \text{ and } (q,r) \text{ lies below the curves } \Gamma_1 \text{ and } \Gamma_3\Big\},$$
$$R_4 = \{(q,r) : r > mq > 0 \text{ and } r > 1\},$$
$$R_5 = \{(q,r) : r > mq > 0 \text{ and } r \le 1\}.$$

It is well known that the curve Γ_1 is continuous and strictly decreasing on the (q,r)-plane. Note that, for $(q(a), r(a)) \in \Gamma_1$, $\lim_{a\to 0^+} q(a) = 1/m$ and $\lim_{a\to 0^+} r(a) = 1$, $\lim_{a\to\infty} q(a) = \infty$ and $\lim_{a\to\infty} r(a) = 0$. Therefore, we write on curve Γ_1, the function $r_1(q)$ with $(q, r_1(q)) \in \Gamma_1$ for $q > 1/m$.

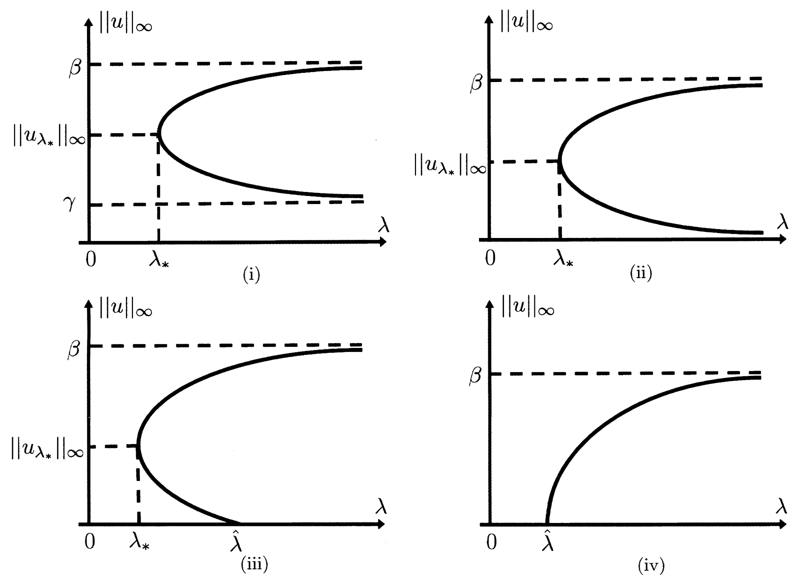

FIGURE 3. (i),(ii),(iii) C-shaped bifurcation curves $\bar{S}$ of (1.1). (iv) monotone increasing bifurcation curve $\bar{S}$ of (1.1).

We define the bifurcation curve of (1.1)

$$\bar{S} = \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.1)}\}.$$

By (1.2), $g(0) = r - 1$, $g'(0) = m - r/q$, and $\lim_{u\to\infty} g(u) = -\infty$. According to the monotonicity of $g(u)$, we give a classification of growth rate per capita $g(u)$ on the first quadrant of (q,r)-parameter plane, see Figure 2. We have:

(i) If $(q,r) \in \Gamma_1 \cup \Gamma_3 \cup R_3 \cup R_5$, then $g(u) \le 0$ for all $u \in [0,\infty)$. Hence problem (1.1) has no positive solution for all $\lambda > 0$.

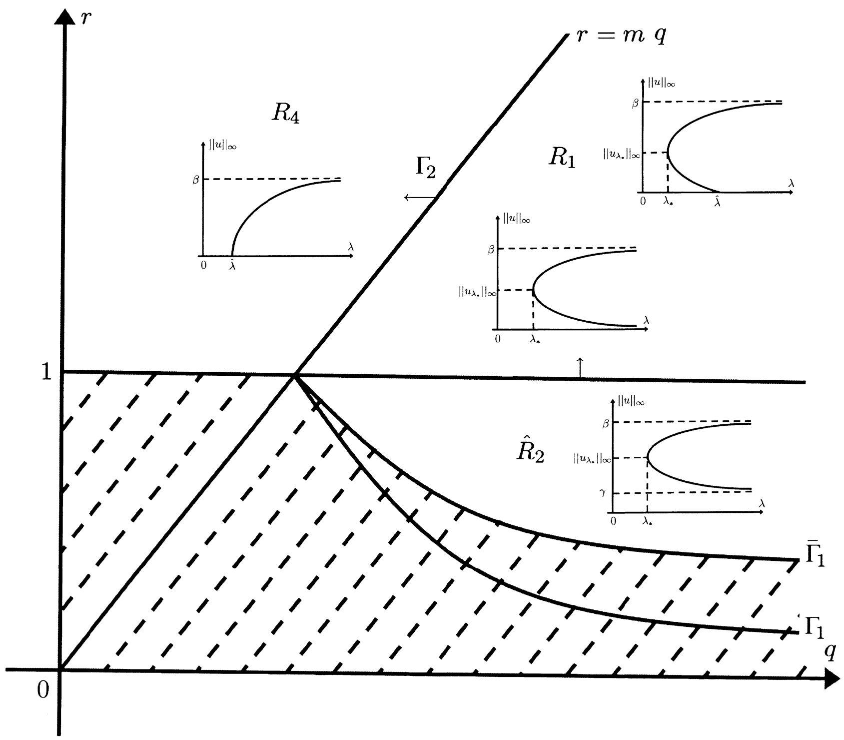

FIGURE 4. Classified bifurcation curves of (1.1), drawn on the first quadrant of (q, r)-parameter plane.

(ii) If $(q, r) \in \Gamma_2 \cup R_4$, $g(u)$ has exactly one positive zero at some β, $g'(u) < 0$ on $(0, \beta)$, and $g(u) \le 0$ on $[\beta, \infty)$. Thus, the bifurcation curve $\bar{S}$ of (1.1) is a monotone increasing curve since $f(u) - uf'(u) = -u^2 g'(u) > 0$ on $(0, \beta)$, see Figures 3(iv) and 4.

(iii) If $(q, r) \in R_1$, $g(u)$ has exactly one positive zero at some β such that $g(\beta) = 0$, $g(u) > 0$ on $(0, \beta)$, and $g(u) < 0$ on (β, ∞). In addition, $g(u)$ changes from increasing to decreasing on $(0, \beta)$. In Theorem 2.1 stated below, we prove that the bifurcation curve $\bar{S}$ of (1.1) has exactly one critical point, where the curve turns to the right on the $(\lambda, \|u\|_\infty)$-plane when $(q, r) \in R_1$, see Figures 3(ii)-(iii) and 4.

(iv) If $(q, r) \in R_2$, $g(u)$ has exactly two positive zeros at some $\beta_1 < \beta$ such that $g(\beta_1) = g(\beta) = 0$, $g(u) < 0$ on $(0, \beta_1) \cup (\beta, \infty)$ and $g(u) > 0$ on (β_1, β). In addition, $g(u)$ changes from increasing to decreasing on $(0, \beta)$. Thus, for each fixed $q > 1/m$, we have

$$\int_0^\beta f(u)du \begin{cases} < 0 & \text{for } r \text{ near } r_1(q)^+, \\ > 0 & \text{for } r \text{ near } 1^-. \end{cases}$$

In addition, for each fixed $q > 1/m$,

$$\frac{d}{dr} \int_0^\beta f(u)du = \frac{1}{6q}\beta^3 + \frac{1}{2r} \frac{\beta^2}{1 + m\beta + \beta^2} > 0$$

because $f(\beta)=0$. Hence for each fixed $q>1/m$, there exists $\bar{r}_1(q)\in(r_1(q),1)$ such that

$$\begin{gathered}\int_0^\beta f(u)du<0,\quad \text{for } r_1(q)<r<\bar{r}_1(q),\\ \int_0^\beta f(u)du=0,\quad \text{for } r=\bar{r}_1(q),\\ \int_0^\beta f(u)du>0,\quad \text{for } \bar{r}_1(q)<r<1.\end{gathered}$$

We define the curve

$$\bar{\Gamma}_1=\{(q,r):q>\frac{1}{m} \text{ and } r=\bar{r}_1(q)\}$$

and regions

$$\begin{gathered}\hat{R}_2=\big\{(q,r):0<r<mq \text{ and } (q,r) \text{ lies between the curve } \bar{\Gamma}_1 \text{ and the line } r=1\big\},\\ \bar{R}_2=\big\{(q,r):0<r<mq \text{ and } (q,r) \text{ lies between curves } \Gamma_1 \text{ and } \bar{\Gamma}_1\big\}.\end{gathered}$$

(So $R_2=\bar{\Gamma}_1\cup\bar{R}_2\cup\hat{R}_2$.) Notice that, for each fixed $(q,r)\in\hat{R}_2$, $\int_0^\beta f(u)du>0$. Thus, for each fixed $(q,r)\in\hat{R}_2$, there exists a positive number $\gamma\in(\beta_1,\beta)$ satisfying $\int_0^\gamma f(u)du=0$ since $f(u)<0$ on $(0,\beta_1)$ and $f(u)>0$ on (β_1,β). In Theorem 2.2 stated below, we prove that the bifurcation curve $\bar{S}$ of (1.1) is a C-shaped curve on the $(\lambda,\|u\|_\infty)$-plane when $(q,r)\in\hat{R}_2$, see Figures 3(i) and 4. In addition, we know that for $(q,r)\in\bar{\Gamma}_1\cup\bar{R}_2$, the problem (1.1) has no positive solution for all $\lambda>0$ since $\int_0^\beta f(u)du\le 0$.

2. Main results

Let u_λ be a positive solution of (1.1) with $\alpha\equiv\|u_\lambda\|_\infty>0$.

Theorem 2.1 (See Figures 3(ii)-(iii) and 4). *Consider* (1.1). *If* $(q,r)\in R_1$, *then*

$$\lim_{\alpha\to0^+}\lambda(\alpha)\equiv\hat{\lambda}=\frac{\pi^2}{4(r-1)}\in(0,\infty],\quad \lim_{\alpha\to\beta^-}\lambda(\alpha)=\infty, \tag{2.1}$$

and the bifurcation curve $\bar{S}$ of (1.1) *is a C-shaped curve on the $(\lambda,\|u\|_\infty)$-plane. More precisely, $\bar{S}$ consists of a continuous curve with exactly one turning point, $(\lambda_*,\|u_{\lambda_*}\|_\infty)$ such that $0<\lambda_*<\hat{\lambda}\le\infty$ and $0<\|u_{\lambda_*}\|_\infty<\beta$, where the curve turns to the right. Problem* (1.1) *has:*

(i) *exactly two positive solutions u_λ, v_λ with $u_\lambda<v_\lambda$ for $\lambda_*<\lambda<\hat{\lambda}$,*
(ii) *exactly one positive solution u_λ for $\lambda=\lambda_*$ and exactly one positive solution v_λ for $\lambda\ge\hat{\lambda}$ (if $r>1$),*
(iii) *no positive solution for $0<\lambda<\lambda_*$.*

Furthermore,

$$\begin{gathered}\lim_{\lambda\to\infty}\|u_\lambda\|_\infty=0,\quad \textit{if } r=1,\\ \lim_{\lambda\to(\hat{\lambda})^-}\|u_\lambda\|_\infty=0,\quad \textit{if } r>1,\end{gathered} \tag{2.2}$$

and

$$\lim_{\lambda\to\infty}\|v_\lambda\|_\infty=\beta. \tag{2.3}$$

Theorem 2.2 (See Figures 3(i) and 4). *Consider* (1.1). *If* $(q,r) \in \hat{R}_2$, *then*

$$\lim_{\alpha \to \gamma^+} \lambda(\alpha) = \lim_{\alpha \to \beta^-} \lambda(\alpha) = \infty, \tag{2.4}$$

and the bifurcation curve $\bar{S}$ *of* (1.1) *is a C-shaped curve on the* $(\lambda, \|u\|_\infty)$*-plane. More precisely,* $\bar{S}$ *consists of a continuous curve with exactly one turning point,* $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ *such that* $0 < \lambda_* < \infty$ *and* $\gamma < \|u_{\lambda_*}\|_\infty < \beta$, *where the curve turns to the right. Problem* (1.1) *has:*

(i) *exactly two positive solutions* u_λ, v_λ *with* $u_\lambda < v_\lambda$ *for* $\lambda_* < \lambda < \infty$,
(ii) *exactly one positive solution* u_λ *for* $\lambda = \lambda_*$,
(iii) *no positive solution for* $0 < \lambda < \lambda_*$.

Furthermore,

$$\lim_{\lambda \to \infty} \|u_\lambda\|_\infty = \gamma, \quad \lim_{\lambda \to \infty} \|v_\lambda\|_\infty = \beta. \tag{2.5}$$

3. Proofs of main results

For $f(u) = ug(u)$ from the analysis of $g(u)$ in Section 1, we obtain the following result.

Lemma 3.1. *Consider*

$$f(u) = ru(1 - \frac{u}{q}) - \frac{u}{1 + mu + u^2}$$

with $m \geq 1$, q, $r > 0$.

(i) *If* $(q,r) \in R_1$, *then there exists a positive number* β *such that* $f(0) = f(\beta) = 0$, $f(u) > 0$ *on* $(0, \beta)$, *and* $f(u) < 0$ *on* (β, ∞).
(ii) *If* $(q,r) \in \hat{R}_2$, *then there exist two positive numbers* $\beta_1 < \beta$ *such that* $f(0) = f(\beta_1) = f(\beta) = 0$, $f(u) < 0$ *on* $(0, \beta_1) \cup (\beta, \infty)$, *and* $f(u) > 0$ *on* (β_1, β). *Also, there exists a positive number* $\gamma \in (\beta_1, \beta)$ *satisfying* $\int_0^\gamma f(u)du = 0$.

Let $F(u) \equiv \int_0^u f(t)dt$, and u_λ be a positive solution of (1.1) with $\alpha \equiv \|u_\lambda\|_\infty > 0$. The time map formula for studying problem (1.1) takes the form:

(i) if $(q,r) \in R_1$, then the time map is

$$T(\alpha) \equiv \frac{1}{\sqrt{2}} \int_0^\alpha \frac{1}{[F(\alpha) - F(u)]^{1/2}} du = \sqrt{\lambda} \quad \text{for } 0 < \alpha < \beta; \tag{3.1}$$

(ii) if $(q,r) \in \hat{R}_2$, then the time map is

$$T(\alpha) \equiv \frac{1}{\sqrt{2}} \int_0^\alpha \frac{1}{[F(\alpha) - F(u)]^{1/2}} du = \sqrt{\lambda} \quad \text{for } \gamma < \alpha < \beta\,. \tag{3.2}$$

See Laetsch [8] for the derivation of the time map formula. So positive solutions u_λ of (1.1) correspond to $\|u_\lambda\|_\infty = \alpha$ and $T(\alpha) = \sqrt{\lambda}$. Thus, studying the exact number of positive solutions of (1.1) is equivalent to studying the number of roots of the equation $T(\alpha) = \sqrt{\lambda}$. We define

$$\theta(u) = 2F(u) - uf(u) = \frac{r}{3q}u^3 + \frac{u^2}{1 + mu + u^2} - 2\int_0^u \frac{t}{1 + mt + t^2} dt. \tag{3.3}$$

Lemma 3.2. *Consider*

$$f(u) = ru(1 - \frac{u}{q}) - \frac{u}{1 + mu + u^2}$$

with $m \geq 1$, $q, r > 0$. If $(q,r) \in R_1 \cup \hat{R}_2$, then there exists a positive number $B \in (0,\beta)$ such that

$$\theta''(u) = -uf''(u) \begin{cases} = 0 & \text{for } u = B, \\ < 0 & \text{on } (0,B), \\ > 0 & \text{on } (B,\beta). \end{cases}$$

Proof. For $m \geq 1$ and q, $r > 0$, by (3.3), we have

$$\theta'(u) = f(u) - uf'(u) = u^2\big[\frac{r}{q} - \frac{m+2u}{(1+mu+u^2)^2}\big],$$

$$\theta''(u) = -uf''(u) = 2u\big[\frac{r}{q} - \frac{m+3u-u^3}{(1+mu+u^2)^3}\big] = 2u\big[\frac{r}{q} - I(u)\big]\,,$$

where

$$I(u) \equiv \frac{m+3u-u^3}{(1+mu+u^2)^3}.$$

We compute that

$$I'(u) = \frac{3u^4 - 18u^2 - 12mu + 3(1-m^2)}{(1+mu+u^2)^4}.$$

Now we define

$$p(u) = 3u^4 - 18u^2 - 12mu + 3(1-m^2)$$

and obtain

$$p'(u) = 12(u^3 - 3u - m)\,, \quad p''(u) = 36(u^2-1).$$

Therefore,

$$\begin{gathered} p''(1) = 0, \\ p''(u) < 0 \quad \text{on } [0,1), \\ p''(u) > 0 \quad \text{on } (1,\infty). \end{gathered} \tag{3.4}$$

Since $p(0) = 3(1-m^2) \leq 0$, $p'(0) = -12m < 0$, $\lim_{u\to\infty} p(u) = \infty$, and by (3.4), there exists $C > 0$ such that

$$\begin{gathered} p(C) = 0, \\ p(u) < 0 \quad \text{on } (0,C), \\ p(u) > 0 \quad \text{on } (C,\infty). \end{gathered}$$

Then

$$\begin{gathered} I'(C) = 0, \\ I'(u) < 0 \quad \text{on } (0,C), \\ I'(u) > 0 \quad \text{on } (C,\infty). \end{gathered} \tag{3.5}$$

By (3.5), $I(0) = m \geq 1$ and $\lim_{u\to\infty} I(u) = 0$, there exists a D with $0 < D < C$ such that

$$\begin{gathered} I(D) = 0, \\ I(u) > 0 \quad \text{on } (0,D), \\ I(u) < 0 \quad \text{on } (D,\infty). \end{gathered}$$

It follows that $\max_{u\in[0,\infty)} I(u) = I(0) = m$ and $I(u)$ is strictly decreasing on $[0, D]$. So we obtain that for $0 < r/q \le m$, there exists $B > 0$ such that $I(B) = r/q$ and

$$\begin{gathered} \theta''(B) = 0, \\ \theta''(u) < 0 \quad \text{on } (0, B), \\ \theta''(u) > 0 \quad \text{on } (B, \infty). \end{gathered} \tag{3.6}$$

In addition, if $0 < r/q \le m$, we have

$$\theta'(u) = u^2 \big[\frac{r}{q} - \frac{m + 2u}{(1 + mu + u^2)^2}\big] > 0 \tag{3.7}$$

for u large enough. By (3.6), (3.7) and $\theta(0) = \theta'(0) = 0$, there exists $E > B$ such that

$$\begin{gathered} \theta(0) = \theta(E) = 0, \\ \theta(u) < 0 \quad \text{on } (0, E), \\ \theta(u) > 0 \quad \text{on } (E, \infty). \end{gathered} \tag{3.8}$$

(i) For $(q, r) \in R_1$, we know that $0 < r/q < m$ and $\int_0^\beta f(u)du > 0$ by $f(u) > 0$ on $(0, \beta)$. Thus,

$$\theta(\beta) = 2F(\beta) - \beta f(\beta) = 2\int_0^\beta f(u)du > 0$$

by $f(\beta) = 0$. It follows that $\beta > E > B$ by (3.8). So we obtain $B \in (0, \beta)$ and

$$\theta''(u) = -uf''(u) \begin{cases} = 0 & \text{for } u = B, \\ < 0 & \text{on } (0, B), \\ > 0 & \text{on } (B, \beta) \end{cases}$$

by (3.6).

(ii) For $(q, r) \in \hat{R}_2$, we know that $0 < r/q < m$ and $\int_0^\beta f(u)du > 0$. Thus,

$$\theta(\beta) = 2F(\beta) - \beta f(\beta) = 2\int_0^\beta f(u)du > 0$$

by $f(\beta) = 0$. It follows that $\beta > E > B$ by (3.8). So we obtain $B \in (0, \beta)$ and

$$\theta''(u) = -uf''(u) \begin{cases} = 0 & \text{for } u = B, \\ < 0 & \text{on } (0, B), \\ > 0 & \text{on } (B, \beta) \end{cases}$$

by (3.6). The proof of Lemma 3.2 is complete. □

By Lemmas 3.1 (i), 3.2, and a slight modification of the proof of [22, Theorem], we obtain the following Lemma.

Lemma 3.3. *Consider* (1.1) *with* $m \ge 1$, q, $r > 0$. *If* $(q, r) \in R_1$, *then*

$$\lim_{\alpha\to 0^+} T(\alpha) = \begin{cases} \frac{\pi}{2\sqrt{f'(0)}} \in (0, \infty), & \text{if } f'(0) > 0, \\ \infty, & \text{if } f'(0) = 0, \end{cases}$$
$$\lim_{\alpha\to\beta^-} T(\alpha) = \infty\,.$$

In addition, $T(\alpha)$ *has exactly one critical point, a minimum, on* $(0, \beta)$.

Proof of Theorem 2.1. By (3.1), $f'(0) = r - 1$ and Lemma 3.3, the results in (2.1) hold, and the bifurcation curve $\bar{S}$ of (1.1) is a C-shaped curve on the $(\lambda, \|u\|_\infty)$-plane. More precisely, $\bar{S}$ consists of a continuous curve with exactly one turning point, $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ such that $0 < \lambda_* < \hat{\lambda} \leq \infty$ and $0 < \|u_{\lambda_*}\|_\infty < \beta$, where the curve turns to the right. So we obtain immediately the exact multiplicity result and ordering results of the solutions in parts (i)–(iii). The proofs of results in (2.2) and (2.3) are easy but tedious; we omit them. □

By Lemmas 3.1 (ii), 3.2, and [23, Theorem 1, Notes 1, 2 and Remark 2], we obtain the following Lemma.

Lemma 3.4. *Consider* (1.1) *with* $m \geq 1$, q, $r > 0$. *If* $(q, r) \in \hat{R}_2$, *then*

$$\lim_{\alpha \to \gamma^+} T(\alpha) = \lim_{\alpha \to \beta^-} T(\alpha) = \infty.$$

In addition, $T(\alpha)$ has exactly one critical point, a minimum, on (γ, β).

Proof of Theorem 2.2. By (3.2) and Lemma 3.4, the results in (2.4) hold, and the bifurcation curve $\bar{S}$ of (1.1) is a C-shaped curve on the $(\lambda, \|u\|_\infty)$-plane. More precisely, $\bar{S}$ consists of a continuous curve with exactly one turning point, $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ such that $0 < \lambda_* < \infty$ and $\gamma < \|u_{\lambda_*}\|_\infty < \beta$, where the curve turns to the right. So we obtain immediately the exact multiplicity result and ordering results of the solutions in parts (i)–(iii). The proofs of results in (2.5) are easy but tedious; we omit them. □

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References

[1] H. Baek; A food chin system with Holling type IV functional response and impulsive perturbations, *Comput. Math. Appl.*, **60** (2010), 1152-1163.

[2] N. F. Britton; Reaction-diffusion equations and their applications to biology, *Academic Press, NewYork*, 1986.

[3] R. S. Cantrell, C. Cosner; The effects of spatial heterogeneity in population dynamics, *J. Math. Biol.*, **29** (1991), 315-338.

[4] J. B. Collings; The effects of the functional response on the bifurcation behavior of a mite predator-prey interaction model, *J. Math. Biol.*, **36** (1997), 149-168.

[5] P. C. Fife; Mathematical aspects of reacting and diffusing systems, *Lecture Notes in Biomathematics 28, Springer-Verlag, New-York*, 1979.

[6] C. S. Holling; The functional response of predators to prey density and its role in mimicry and population regulation, *Mem. Entomolog. Soc. Can.*, **45** (1965), 3-60.

[7] J. C. Huang, D. M. Xiao; Analyses of bifurcations and Stability in a predator-prey system with Holling Type-IV functional response, *Acta Math. Appl. Sin. Engl. Ser.*, **20** (2004), 167-178.

[8] T. Laetsch; The number of solutions of a nonlinear two point boundary value problem, *Indiana Univ. Math. J.*, **20** (1970), 1-13.

[9] Y. Li, D. Xiao; Bifurcations of a predator prey system of Holling and Leslie types, *Chaos Solitons Fractals*, **34** (2007), 606-620.

[10] F. Lian, Y. Xu; Hopf bifurcation analysis of a predator–prey system with Holling type IV functional response and time delay, *Applied Mathematics and Computation*, **215** (2009), 1484-1495.

[11] X. Liu, Q. Huang; The dynamics of a harvested predator-prey system with Holling type IV functional response, *BioSystems*, **169-170** (2018), 26-39.

[12] D. Ludwig, D. G. Aronson, H. F. Weinberger; Spatial patterning of the spruce budworm, *J. Math. Biol.*, **8** (1979), 217-258.

[13] D. Ludwig, D. D. Jones, C. S. Holling; Qualitative analysis of insect outbreak systems: the spruce budwork and the forest, *J. Anim. Ecol.*, **47** (1978), 315-332.
[14] J. D. Murray; Mathematical biology, 2nd. ed., *Biomathematics 19, Springer-Verlag, Berlin*, 1993.
[15] A. Okubo; Diffusion and ecological problems: mathematical models, *Biomathematics 10, Springer-Verlag, Berlin*, 1980.
[16] H. Qolizadeh Amirabad, O. RabieiMotlagh, H. M. MohammadiNejad; Permanency in predator-prey models of Leslie type with ratio-dependent simplified Holling type-IV functional response, *Mathematics and Computers in Simulation*, **157** (2019), 63-76.
[17] S. Ruan, D. Xiao; Global analysis in a predator-prey system with nonmonotonic functional response, *SIAM J. Appl. Math.*, **61** (2001), 1445-1472.
[18] J. G. Skellam; Random dispersal in theoretical populations, *Biometrika*, **38** (1951), 196-281.
[19] W. Sokol, J. A. Howell; Kinetics of phenol oxidation by washed cells, *Biotechnol. Bioeng.*, **23** (1981), 2039-2049.
[20] B. Sounvoravong, S. Guo, Y. Bai; Bifurcation and stability of a diffusive SIRS epidemic model with time delay, *Electron. J. Differential Equations*, **45** (2019), 1-16.
[21] R. K. Upadhyay, N. Kumaria, V. Raia; Wave of chaos and pattern formation in spatial predator–prey systems with Holling type IV predator response, *Math. Model. Nat. Phenom.*, **3** (2008), 71-95.
[22] S.-H. Wang, N. D. Kazarinoff; Bifurcation and stability of positive solutions of a two-point boundary value problem, *J. Austral. Math. Soc. Ser. A*, **52** (1992), 334-342.
[23] S.-H. Wang, N. D. Kazarinoff; Bifurcation of steady-state solutions of a scalar reaction-diffusion equation in one space variable, *J. Austral. Math. Soc. Ser. A*, **52** (1992), 343-355.
[24] S.-H. Wang, T.-S. Yeh; S-shaped and broken S-shaped bifurcation diagrams with hysteresis for a multiparameter spruce budworm population problem in one space dimension, *J. Differential Equations*, **255** (2013), 812-839.
[25] T.-S. Yeh; Classification of bifurcation diagrams for a multiparameter diffusive logistic problem with Holling type-IV functional response, *J. Math. Anal. Appl.*, **418** (2014), 283-304.
[26] T.-S. Yeh; S-shaped and broken S-shaped bifurcation curves for a multiparameter diffusive logistic problem with Holling type-III functional response, *Comm. Pure Appl. Anal.*, **16** (2017), 645-670.

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