Electronic Journal of Differential Equations, Vol. 2021 (2021), No. 14, pp. 1–31.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE AND CONCENTRATION RESULTS FOR FRACTIONAL SCHRÖDINGER-POISSON SYSTEM VIA PENALIZATION METHOD

ZHIPENG YANG, WEI ZHANG, FUKUN ZHAO

Abstract. This article concerns the positive solutions for the fractional Schrödinger-Poisson system

$$\begin{gathered}\varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = f(u) \quad \text{in } \mathbb{R}^3,\\ \varepsilon^{2t}(-\Delta)^t \phi = u^2 \quad \text{in } \mathbb{R}^3,\end{gathered}$$

where $\varepsilon > 0$ is a small parameter, $(-\Delta)^\alpha$ denotes the fractional Laplacian of orders $\alpha = s, t \in (3/4, 1)$, $V \in C(\mathbb{R}^3, \mathbb{R})$ is the potential function and $f : \mathbb{R} \to \mathbb{R}$ is continuous and subcritical. Under a local condition imposed on the potential function, we relate the number of positive solutions with the topology of the set where the potential attains its minimum values. Moreover, we considered some properties of these positive solutions, such as concentration behavior and decay estimate. In the proofs we apply variational methods, penalization techniques and Ljusternik-Schnirelmann theory.

1. Introduction and statement of main results

In recent years, a great deal of work has been done on the semiclassical standing waves for the fractional Schrödinger equation

$$i\hbar\frac{\partial \Psi}{\partial t} = \hbar^{2s}(-\Delta)^s\Psi + \widetilde{V}(x)\Psi - f(|\Psi|) \quad \text{in } \mathbb{R}^3 \times \mathbb{R}, \tag{1.1}$$

where i is the imaginary unit, $\hbar$ is the Planck constant, $\widetilde{V}(x) = V(x) + E$ is the potential function with the constant E and $f(exp(i\theta)\xi) = exp(i\theta)f(\xi)$ for $\theta, \xi \in \mathbb{R}$ is a nonlinear function.

Equations of type (1.1) were introduced by Laskin [21, 22], and come from an expansion of the Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths. It also appeared in several areas such as optimization, finance, phase transitions, stratified materials, crystal dislocation, flame propagation, conservation laws, materials science and water waves (see [8, 12]).

An interesting Schrödinger equation appears when it describes quantum (non-relativistic) particles interacting with the electromagnetic field generated by the

---

2010 *Mathematics Subject Classification.* 49J35, 58E05.
Key words and phrases. Penalization method; fractional Schrödinger-Poisson; Lusternik-Schnirelmann theory.

Submitted October 5, 2018. Published March 16, 2021.

motion, that is a fractional nonlinear Schrödinger-Poisson system (also called fractional Schrödinger-Maxwell system),

$$
\begin{gathered}
i\hbar\frac{\partial\Psi}{\partial t} = \hbar^{2s}(-\Delta)^s\Psi + \widetilde{V}(x)\Psi + \phi\Psi - f(|\Psi|) \quad \text{in } \mathbb{R}^3\times\mathbb{R}, \\
\hbar^{2t}(-\Delta)^t\phi = |\Psi|^2 \quad \text{in } \mathbb{R}^3.
\end{gathered} \tag{1.2}
$$

A solution of the form $e^{-iEt/\hbar}u(x)$ is called a standing wave and $(e^{-iEt/\hbar}u(x), \phi(x))$ is a solution of (1.2) if and only if $(u(x), \phi(x))$ satisfies

$$
\begin{gathered}
\varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = f(u) \quad \text{in } \mathbb{R}^3, \\
\varepsilon^{2t}(-\Delta)^t\phi = u^2 \quad \text{in } \mathbb{R}^3.
\end{gathered} \tag{1.3}
$$

For $s = t = 1$, Equation (1.3) gives back the classical Schrödinger-Poisson equation

$$
\begin{gathered}
-\varepsilon^2\Delta u + V(x)u + \phi u = f(u) \quad \text{in } \mathbb{R}^3, \\
-\varepsilon^2\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3,
\end{gathered} \tag{1.4}
$$

which was proposed by Benci and Fortunato [6] in 1998 for a bounded domain, and is related to the Hartree equation [23]. Recently, to better simulate the interaction effect among many particles in quantum mechanics, a nonlinear version of the Schrödinger equation coupled with a Poisson equation have begun to receive much attention, we refer the interested readers to [2, 3, 4, 9, 14, 17, 35, 41] and the references therein.

When $s, t \in (0, 1)$ to the best of our knowledge, there are just a few references on the existence for the fractional Schrödinger-Poisson systems, maybe because the standard techniques developed for the local Laplacian do not work immediately. Teng [34] considered the fractional Schrödinger-Poisson system (1.2) with subcritical and critical nonlinearity and he established the existence of ground state solutions. For other existence results we refer to [19, 24, 27, 38, 40] and the references therein.

Assuming $f(u) \sim |u|^{p-2}u(4 < p < 2_s^*)$ and the global condition

$$
0 < \inf_{x\in\mathbb{R}^3} V(x) < \liminf_{|x|\to\infty} V(x) = V_\infty,
$$

firstly introduced by Rabinowitz [29], the authors in [25] obtained the multiplicity and concentration of positive solutions for the system

$$
\begin{gathered}
\varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = f(u) + |u|^{2_s^*-2}u \quad \text{in } \mathbb{R}^3, \\
\varepsilon^{2t}(-\Delta)^t\phi = u^2 \quad \text{in } \mathbb{R}^3,
\end{gathered} \tag{1.5}
$$

via the standard Nehari manifold method. The concentration behavior of ground state solutions for subcritical and critical cases with two competing positive potentials was obtained in [39, 37].

Different from [25, 37, 39], in this paper, we establish the existence and concentration of positive solutions for the fractional Schrödinger-Poisson system (1.3) when the potential function $V \in C(\mathbb{R}^3, \mathbb{R})$ satisfies the following conditions:

(A1) There is constant $V_0 > 0$ such that $V_0 = \inf_{x\in\mathbb{R}^3} V(x)$,
(A2) there is a bounded domain Ω such that

$$
V_0 < \min_{\partial\Omega} V.
$$

Without loss of generality, we assume that $\mathcal{M} = \{x \in \Omega : V(x) = V_0\} \neq \emptyset$ and $0 \in \mathcal{M}$.

Moreover, we assume that the continuous function f vanishes on $(-\infty, 0)$ and satisfies

(A3) $f(u) = o(u^3)$ as $u \to 0$.

(A4) There is $p \in (4, 2_s^*)$ such that

$$\lim_{u\to\infty} \frac{f(u)}{u^{p-1}} = 0.$$

(A5) There is $\theta \in (4, 2_s^*)$ such that

$$0 < \theta F(u) = \theta \int_0^u f(\tau)d\tau \leq f(u)u, \quad \forall u > 0.$$

(A6) The function $f(u)/u^3$ is increasing on $(0, \infty)$.

Remark 1.1. In view of (A4) and (A5), we have $4 < 2_s^* = \frac{6}{3-2s}$, which implies that $s > \frac{3}{4}$. Moreover, if $s, t \in (3/4, 1)$, then we have that $2s + 2t > 3$.

We remark that the nonlinearity f is only continuous, we can not use standard arguments on the Nehari manifold as in [25]. To overcome the nondifferentiability of the Nehari manifold, we shall use some variants of critical point theorems from Szulkin and Weth [33]. Now we state our main results as follows.

Theorem 1.2. *Assume that* (A1)–(A6) *are satisfied with* $s, t \in (3/4, 1)$. *Then for each* $\delta > 0$ *such that*

$$\mathcal{M}_\delta = \{x \in \mathbb{R}^3 : \operatorname{dist} x, \mathcal{M}) \leq \delta\} \subset \Omega,$$

there exists $\varepsilon_\delta > 0$ *such that* (1.3) *has at least* $\operatorname{cat}_{\mathcal{M}_\delta}(\mathcal{M})$ *positive solutions for any* $\varepsilon \in (0, \varepsilon_\delta)$. *Moreover, if* u *denotes one of these positive solutions and* $\eta_\varepsilon \in \mathbb{R}^3$ *its global maximum, then*

$$\lim_{\varepsilon\to 0} V(\eta_\varepsilon) = V_0,$$

and there exists a constant $C > 0$ *(independent of* ε*) such that*

$$u(x) \leq \frac{C\varepsilon^{3+2s}}{\varepsilon^{3+2s} + |x - \eta_\varepsilon|^{3+2s}}, \quad \forall x \in \mathbb{R}^3.$$

This article is organized as follows. In section 2, besides describing the functional setting to stu dy problem (1.3), we give some preliminary Lemmas which will be used later. In section 3, influenced by the work [10] and [32], we introduce a modified functional and show it satisfies the Palais-Smale condition. In section 4, we stu dy the autonomous problem associated. This stu dy allows us to show that the modified problem has multiple solutions. Finally, we show the critical point of the modified functional which satisfies the original problem, and investigate its concentration and polynomial decay behavior, which completes the proof Theorem 1.2.

2. Variational setting and preliminary results

Throughout this paper, we denote $\|\cdot\|_{L^r}$ the usual norm of the space $L^r(\mathbb{R}^3)$, $1 \leq r < \infty$, $B_r(x)$ denotes the open ball with center at x and radius r, C or $C_i (i = 1, 2, \cdots)$ denote some positive constants may change from line to line. $\rightharpoonup$ and $\to$ mean the weak and strong convergence.

2.1. **Functional space setting.** Firstly, fractional Sobolev spaces are the convenient setting for our problem, so we sketch the fractional order Sobolev spaces, the complete introduction can be found in [11]. We recall that, for any $\alpha \in (0,1)$, the fractional Sobolev space $H^\alpha(\mathbb{R}^3) = W^{\alpha,2}(\mathbb{R}^3)$ is defined as follows:

$$H^\alpha(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} \big(|\xi|^{2\alpha}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2\big) d\xi < \infty\},$$

whose norm is defined as

$$\|u\|^2_{H^\alpha(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \big(|\xi|^{2\alpha}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2\big) d\xi,$$

where $\mathcal{F}$ denotes the Fourier transform. We also define the homogeneous fractional Sobolev space $\mathcal{D}^{\alpha,2}(\mathbb{R}^3)$ as the completion of $\mathcal{C}_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)} := \Big(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2\alpha}}\,dx\,dy\Big)^{1/2} = [u]_{H^\alpha(\mathbb{R}^3)}.$$

The fractional Laplacian, $(-\Delta)^\alpha u$, of a smooth function $u : \mathbb{R}^3 \to \mathbb{R}$, is defined by

$$\mathcal{F}((-\Delta)^\alpha u)(\xi) = |\xi|^{2\alpha}\mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^3.$$

Also $(-\Delta)^\alpha u$ can be equivalently represented [11] as

$$(-\Delta)^\alpha u(x) = -\frac{1}{2}C(\alpha)\int_{\mathbb{R}^3} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{3+2\alpha}}\,dy, \quad \forall x \in \mathbb{R}^3,$$

where

$$C(\alpha) = \Big(\int_{\mathbb{R}^3} \frac{(1-cos\xi_1)}{|\xi|^{3+2\alpha}} d\xi\Big)^{-1}, \quad \xi = (\xi_1,\xi_2,\xi_3).$$

Also, by the Plancherel formula in Fourier analysis, we have

$$[u]^2_{H^\alpha(\mathbb{R}^3)} = \frac{2}{C(\alpha)}\|(-\Delta)^{\alpha/2}u\|_2^2.$$

As a consequence, the following three norms are equivalent on $H^\alpha(\mathbb{R}^3)$:

$$\Big(\int_{\mathbb{R}^3} |u|^2\,dx + \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2\alpha}}\,dx\,dy\Big)^{1/2},$$
$$\Big(\int_{\mathbb{R}^3} (|\xi|^{2\alpha}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2) d\xi\Big)^{1/2},$$
$$\Big(\int_{\mathbb{R}^3} |u|^2\,dx + \|(-\Delta)^{\alpha/2}u\|_2^2\Big)^{1/2}.$$

Making the change of variable $x \mapsto \varepsilon x$, we can rewrite system (1.3) as the equivalent system

$$\begin{gathered} (-\Delta)^s u + V(\varepsilon x)u + \phi u = f(u) \quad \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2 \quad \text{in } \mathbb{R}^3. \end{gathered} \tag{2.1}$$

If u is a solution of (2.1), then $v(x) := u(x/\varepsilon)$ is a solution of (1.3). Thus, to study the system (1.3), it suffices to study the system (2.1). In view of the potential $V(x)$, we introduce the subspace

$$H_\varepsilon = \Big\{u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(\varepsilon x)u^2\,dx < +\infty\Big\},$$

which is a Hilbert space equipped with the inner product

$$(u,v)_{H_\varepsilon} = \int_{\mathbb{R}^3} (-\Delta)^{s/2} u (-\Delta)^{s/2} v \, dx + \int_{\mathbb{R}^3} V(\varepsilon x) uv \, dx,$$

and the norm

$$\|u\|^2_{H_\varepsilon} = (u,u) = \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 \, dx + \int_{\mathbb{R}^3} V(\varepsilon x) u^2 \, dx.$$

For convenience, we denote $\|\cdot\|_{H_\varepsilon}$ by $\|\cdot\|_\varepsilon$. For the reader's convenience, we review some useful result for this class of fractional Sobolev spaces.

Lemma 2.1 ([11])**.** *Let $0 < \alpha < 1$, then there exists a constant $C = C(\alpha) > 0$, such that*

$$\|u\|^2_{L^{2^*_\alpha}(\mathbb{R}^3)} \le C[u]^2_{H^\alpha(\mathbb{R}^3)}$$

for every $u \in H^\alpha(\mathbb{R}^3)$, where $2^*_\alpha = \frac{6}{3-2\alpha}$ is the fractional critical exponent. Moreover, the embedding $H^\alpha(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3)$ is continuous for any $r \in [2, 2^*_\alpha]$ and is locally compact whenever $r \in [2, 2^*_\alpha)$.

Lemma 2.2 ([30])**.** *If $\{u_n\}$ is bounded in $H^\alpha(\mathbb{R}^3)$ and for some $R > 0$,*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_n|^2 \, dx = 0,$$

then $u_n \to 0$ in $L^r(\mathbb{R}^3)$ for any $2 < r < 2^*_\alpha$.

Lemma 2.3 ([28])**.** *Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^3)$, $\varphi \in C_0^\infty(\mathbb{R}^3)$ and for each $r > 0$, $\varphi_r(x) = \varphi(\frac{x}{r})$. Then*

$$u\varphi_r \to 0 \text{ in } \mathcal{D}^{s,2}(\mathbb{R}^3) \quad \text{as } r \to 0.$$

If, in addition, $\varphi \equiv 1$ in a neighbourhood of the origin, then

$$u\varphi_r \to u \text{ in } \mathcal{D}^{s,2}(\mathbb{R}^3) \quad \text{as } r \to +\infty.$$

2.2. **Reduction method.** It is clear that system (2.1) is the Euler-Lagrange equations of the functional $J : H_\varepsilon \times \mathcal{D}^{t,2}(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$J(u,\phi) = \frac{1}{2}\|u\|^2_\varepsilon - \frac{1}{4}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}\phi|^2 \, dx + \frac{1}{2}\int_{\mathbb{R}^3} \phi u^2 \, dx - \int_{\mathbb{R}^3} F(u) \, dx\,. \tag{2.2}$$

It is easy to see that J exhibits a strong indefiniteness, namely it is unbounded both from below and from above on infinitely dimensional subspaces. This indefiniteness can be removed using the reduction method described in [6]. First of all, for each fixed $u \in H_\varepsilon$, there exists a unique $\phi^t_u \in \mathcal{D}^{t,2}(\mathbb{R}^3)$ which is the solution of

$$(-\Delta)^t \phi = u^2 \quad \text{in } \mathbb{R}^3.$$

We can write an integral expression for ϕ^t_u in the form

$$\phi^t_u(x) = C_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}} \, dy, \quad \forall x \in \mathbb{R}^3,$$

which is called t-Riesz potential (see [20]), where

$$C_t = \frac{1}{\pi^{3/2}} \frac{\Gamma(3-2t)}{2^{2t}\Gamma(s)}.$$

Then (2.1) can be reduced to the first equation with ϕ represented by the solution of the fractional Poisson equation. This is the basic strategy of solving (2.1). To be more precise about the solution ϕ of the fractional Poisson equation, we collect some useful Lemmas.

Lemma 2.4 ([34]). *For each $u \in H^s(\mathbb{R}^3)$ and $4s+2t \geq 3$, we have:*

(i) $\phi_u^t \geq 0$;
(ii) $\phi_u^t : H^s(\mathbb{R}^3) \to \mathcal{D}^{t,2}(\mathbb{R}^3)$ *is continuous and maps bounded sets into bounded sets;*
(iii) $\|\phi_u^t\|^2_{\mathcal{D}^{t,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \phi_u^t u^2\,dx \leq S_t^2 \|u\|^4_{\frac{12}{3+2t}}$;
(iv) *if* $u_n \rightharpoonup u$ *in* $H^s(\mathbb{R}^3)$, *then* $\phi_{u_n}^t \rightharpoonup \phi_u^t$ *in* $\mathcal{D}^{t,2}(\mathbb{R}^3)$;
(v) *if* $u_n \to u$ *in* $H^s(\mathbb{R}^3)$, *then* $\phi_{u_n}^t \to \phi_u^t$ *in* $\mathcal{D}^{t,2}(\mathbb{R}^3)$ *and* $\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2\,dx \to \int_{\mathbb{R}^3} \phi_u^t u^2\,dx$.

We define $N : H^s(\mathbb{R}^3) \to \mathbb{R}$ by

$$N(u) = \int_{\mathbb{R}^3} \phi_u^t u^2\,dx\,.$$

It is clearly that $N(u(\cdot + y)) = N(u)$ for any $y \in \mathbb{R}^3$, $u \in H^s(\mathbb{R}^3)$ and N is weakly lower semi-continuous in $H^s(\mathbb{R}^3)$. Moreover, similarly to the well-know Brezis-Lieb Lemma ([7]), we have the next Lemma.

Lemma 2.5 ([34]). *Let $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$ with $2s+2t > 3$. Then*

(i) $N(u_n - u) = N(u_n) - N(u) + o(1)$;
(ii) $N'(u_n - u) = N'(u_n) - N'(u) + o(1)$, *in* $(H^s(\mathbb{R}^3))^{-1}$.

Putting $\phi = \phi_u^t$ into the first equation of (2.1), we obtain a nonlocal semilinear elliptic equation

$$(-\Delta)^s u + V(\varepsilon x)u + \phi_u^t u = f(u) \text{ in } \mathbb{R}^3.$$

The corresponding functional $I : H_\varepsilon \to \mathbb{R}$ is defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x)u^2\,dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2\,dx - \int_{\mathbb{R}^3} F(u)\,dx.$$

Note that if $4s+2t \geq 3$, then $2 \leq \frac{12}{3+2t} \leq 2_s^*$, and thus $H^s(\mathbb{R}^3) \hookrightarrow L^{\frac{12}{3+2t}}(\mathbb{R}^3)$. Then by Hölder inequality and Sobolev inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} \phi_u^t u^2\,dx &\leq \Big(\int_{\mathbb{R}^3} |u|^{\frac{12}{3+2t}}\,dx\Big)^{\frac{3+2t}{6}} \Big(\int_{\mathbb{R}^3} |\phi_u^t|^{2_t^*}\,dx\Big)^{1/2_t^*} \\
&\leq \mathcal{S}_t^{-1/2}\Big(\int_{\mathbb{R}^3} |u|^{\frac{12}{3+2t}}\,dx\Big)^{\frac{3+2t}{6}} \|\phi_u^t\|_{\mathcal{D}^{t,2}} \\
&\leq C\|u\|_\varepsilon^2 \|\phi_u^t\|_{\mathcal{D}^{t,2}} < \infty.
\end{aligned}$$

Therefore, the functional I is well-defined for every $u \in H_\varepsilon$. Next, we consider critical points of I using a variational method.

3. Modified problem

In this section, we adapt for our case an argument explored by the penalization method introduction by del Pino and Felmer [10]. In fact, from (A3) and (A6) we have

$$\lim_{u\to 0} \frac{f(u)}{u} = 0$$

and the map $u \mapsto \frac{f(u)}{u}$ is increasing in $(0, \infty)$.

Now we fix some notation. Let $K > 2$, $a > 0$ be such that $\frac{f(a)}{a} = \frac{V_0}{K}$ where V_0 is given by (A1). We define

$$\tilde{f}(u) = \begin{cases} f(u) & \text{if } u \leq a, \\ V_0 u/K & \text{if } u > a, \end{cases}$$
$$g(x,u) = \chi_\Omega(x) f(u) + \big(1 - \chi_\Omega(x)\big)\tilde{f}(u),$$

where χ is characteristic function of a set Ω. From hypotheses (A3)–(A6) we have that g is a Carathéodory function and satisfies the following properties:

$$\lim_{u \to 0} \frac{g(x,u)}{u^3} = 0 \quad \text{uniformly in } x \in \mathbb{R}^3; \tag{3.1}$$

$$\lim_{u \to \infty} \frac{g(x,u)}{u^{p-1}} = 0 \quad \text{uniformly in } x \in \mathbb{R}^3; \tag{3.2}$$

$$\begin{gathered} 0 \leq \theta G(x,u) := \theta \int_0^u g(x,\tau) d\tau < g(x,u)u, \quad \forall x \in \Omega, \ \forall u > 0, \\ 0 \leq 2G(x,u) \leq g(x,u)u \leq \frac{V_0}{K} t^2, \quad \forall x \in \mathbb{R}^3 \backslash \Omega, \ \forall u > 0. \end{gathered} \tag{3.3}$$

$$u \mapsto \frac{g(x,u)}{u^3} \text{ is increasing for } u > 0. \tag{3.4}$$

Moreover, from definition of g, we have

$$\begin{gathered} g(x,u) \leq f(u), \quad \forall u \in (0, +\infty), \ \forall x \in \mathbb{R}^3, \\ g(x,u) = 0, \quad \forall u \in (-\infty, 0), \ \forall x \in \mathbb{R}^3. \end{gathered}$$

Now, we stu dy the modified equation

$$\begin{gathered} (-\Delta)^s u + V(\varepsilon x)u + \phi u = g(\varepsilon x, t) \quad \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2 \quad \text{in } \mathbb{R}^3. \end{gathered} \tag{3.5}$$

Note that positive solutions of (3.5) with $u(x) \leq a$ for each $x \in \mathbb{R}^3 \backslash \Omega_\varepsilon$ are also positive solution of (1.3), where $\Omega_\varepsilon = \{x \in \mathbb{R}^3 : \varepsilon x \in \Omega\}$. The energy functional associated with (3.5) is

$$I_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^3} \big(|(-\Delta)^{s/2} u|^2 + V(\varepsilon x)u^2\big)\, dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u^t u^2 \, dx - \int_{\mathbb{R}^3} G(\varepsilon x, u)\, dx$$

which is of C^1 class and whose derivative is given by

$$\langle I'_\varepsilon(u), v\rangle = \int_{\mathbb{R}^3} \big((-\Delta)^{s/2} u (-\Delta)^{s/2} v + V(\varepsilon x)uv\big)\, dx + \int_{\mathbb{R}^3} \phi_u^t uv \, dx - \int_{\mathbb{R}^3} g(\varepsilon x, u)v \, dx.$$

for all $v \in H_\varepsilon$ and associated norm $\| \cdot \|_\varepsilon$. Hence the critical points of I_ε in H_ε are weak solutions of problem (3.5).

Now, we denote the Nehari manifold associated to I_ε by

$$\mathcal{N}_\varepsilon = \{u \in H_\varepsilon \backslash \{0\} : \langle I'_\varepsilon(u), u\rangle = 0\}.$$

Obviously, $\mathcal{N}_\varepsilon$ contains all nontrivial critical points of I_ε. But we do not know whether $\mathcal{N}_\varepsilon$ is of class C^1 under our assumptions and therefore we cannot use minimax theorems directly on $\mathcal{N}_\varepsilon$. To overcome this difficulty, we will adopt a technique developed in [32, 33] to show that $\mathcal{N}_\varepsilon$ is still a topological manifold, naturally homeomorphic to the unit sphere of H_ε, and then we can consider a new minimax characterization of the corresponding critical value for I_ε.

For this we denote by H_ε^+ the subset of H_ε given by

$$H_\varepsilon^+ = \{u \in H_\varepsilon : |\operatorname{supp}(u^+) \cap \Omega_\varepsilon| > 0\}$$

and $S_\varepsilon^+ = S_\varepsilon \cap H_\varepsilon^+$, where S_ε is the unit sphere of H_ε.

Lemma 3.1. *The set H_ε^+ is open in H_ε.*

Proof. Suppose by contradiction there are a sequence $\{u_n\} \subset H_\varepsilon \backslash H_\varepsilon^+$ and $u \in H_\varepsilon^+$ such that $u_n \to u$ in H_ε. Hence $|\operatorname{supp}(u_n^+) \cap \Omega_\varepsilon| = 0$ for all $n \in \mathbb{N}$ and $u_n^+(x) \to u^+(x)$ a.e. in $x \in \Omega_\varepsilon$. So,

$$u^+(x) = \lim_{n\to\infty} u_n^+(x) = 0, \text{ a.e. in } x \in \Omega_\varepsilon.$$

But, this contradicts the fact that $u \in H_\varepsilon^+$. Therefore H_ε^+ is open. $\square$

From definition of S_ε^+ and Lemma 3.1 it follows that S_ε^+ is a incomplete $C^{1,1}$-manifold of codimension 1, modeled on H_ε and contained in the open H_ε^+. Hence, $H_\varepsilon = T_u S_\varepsilon^+ \oplus \mathbb{R}u$ for each $u \in S_\varepsilon^+$, where $T_u S_\varepsilon^+ = \{v \in H_\varepsilon : (u, v)_\varepsilon = 0\}$.

In the rest of this section, we show some Lemmas related to the function I_ε and the set H_ε^+. First, we show the functional I_ε satisfying the mountain pass geometry.

Lemma 3.2. *The functional I_ε satisfies the following conditions:*

(i) *There exist $\alpha, \rho > 0$ such that $I_\varepsilon(u) \geq \alpha$ with $\|u\|_\varepsilon = \rho$;*

(ii) *there exists $e \in H_\varepsilon$ satisfying $\|e\|_\varepsilon > \rho$ such that $I_\varepsilon(e) < 0$.*

Proof. (i) For any $u \in H_\varepsilon \backslash \{0\}$, it follows from (3.1) and (3.2) that there exists $C > 0$ such that

$$\begin{aligned} |g(\varepsilon x, u)| &\leq |u|^3 + C|u|^{p-1}, \quad \text{for all } x \in \mathbb{R}^3, u \in \mathbb{R}, \\ |G(\varepsilon x, u)| &\leq \frac{1}{4}|u|^4 + C|u|^p, \quad \text{for all } x \in \mathbb{R}^3, u \in \mathbb{R}. \end{aligned}$$

By the Sobolev embedding $H_\varepsilon \hookrightarrow L^r(\mathbb{R}^3)$ for $r \in [2, 2_s^*]$, we have

$$\begin{aligned} I_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^3} \big(|(-\Delta)^{s/2}u|^2 + V(\varepsilon x)u^2\big)\,dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2\,dx - \int_{\mathbb{R}^3} G(\varepsilon x, u)\,dx \\ &\geq \frac{1}{2}\|u\|_\varepsilon^2 - C_1\|u\|_\varepsilon^4 - C_2\|u\|_\varepsilon^p \\ &= \frac{1}{2}\|u\|_\varepsilon^2\big(1 - C_1\|u\|^2 - \varepsilon - C_2\|u\|_\varepsilon^2\big) \end{aligned}$$

Therefore, we can choose positive constants α, ρ such that

$$I_\varepsilon(u) \geq \alpha \quad \text{with } \|u\|_\varepsilon = \rho.$$

(ii) For each $u \in H_\varepsilon^+$ and $\tau > 0$ we have

$$\begin{aligned} I_\varepsilon(\tau u) &= \frac{\tau^2}{2}\int_{\mathbb{R}^3} \big(|(-\Delta)^{s/2}u|^2 + V(\varepsilon x)u^2\big)\,dx + \frac{\tau^4}{4}\int_{\mathbb{R}^3} \phi_u^t u^2\,dx - \int_{\mathbb{R}^3} G(\varepsilon x, \tau u)\,dx \\ &\leq \frac{\tau^2}{2}\|u\|_\varepsilon^2 + \frac{\tau^4}{4}\int_{\mathbb{R}^3} \phi_u^t u^2\,dx - C_1\tau^\theta \int_{\Omega_\varepsilon} |u^+|^\theta\,dx + C_2|\operatorname{supp}(u^+) \cap \Omega_\varepsilon|. \end{aligned}$$

Since $\theta \in (4, 2_s^*)$, conclusion (ii) follows. $\square$

Since f is only continuous, the next two results are very important because they allow us to overcome the non-differentiability of $\mathcal{N}_\varepsilon$ and the incompleteness of S_ε^+.

Lemma 3.3. *Assume that* (A1)–(A6) *are satisfied. Then the following properties hold:*

(1) *For each $u \in H_\varepsilon^+$, let $h_u : \mathbb{R}^+ \to \mathbb{R}$ be given by $h_u(\tau) = I_\varepsilon(\tau u)$. Then there exists a unique $\tau_u > 0$ such that $h_u'(\tau) > 0$ in $(0, \tau_u)$ and $h_u'(\tau) < 0$ in (τ_u, ∞);*

(2) *there is a $\sigma > 0$ independent on u such that $\tau_u > \sigma$ for all $u \in S_\varepsilon^+$. Moreover, for each compact set $\mathcal{W} \subset S_\varepsilon^+$ there is $C_{\mathcal{W}} > 0$ such that $\tau_u \le C_{\mathcal{W}}$ for all $u \in \mathcal{W}$;*

(3) *The map $\hat{m}_\varepsilon : H_\varepsilon^+ \to \mathcal{N}_\varepsilon$ given by $\hat{m}_\varepsilon(u) = \tau_u u$ is continuous and $m_\varepsilon := \hat{m}_\varepsilon\big|_{S_\varepsilon^+}$ is a homeomorphism between S_ε^+ and $\mathcal{N}_\varepsilon$. Moreover, $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon}$.*

Proof. (1) From Lemma 3.2, it is sufficient to note that, $h_u(0) = 0$, $h_u(\tau) > 0$ when $\tau > 0$ is small and $h_u(\tau) < 0$ when $\tau > 0$ is large. Since $h_u \in C^1(\mathbb{R}^+, \mathbb{R})$, there is $\tau_u > 0$ global maximum point of h_u and $h_u'(\tau_u) = 0$. Thus, $I_\varepsilon'(\tau_u u)(\tau_u u) = 0$ and $\tau_u u \in \mathcal{N}_\varepsilon$. We see that $\tau_u > 0$ is the unique positive number such that $h_u'(\tau_u) = 0$. Indeed, suppose by contradiction that there are $\tau_1 > \tau_2 > 0$ with $h_u'(\tau_1) = h_u'(\tau_2) = 0$. Then, for $i = 1, 2$ we have that

$$\tau_i^2 \|u\|_\varepsilon^2 + \tau_i^4 \int_{\mathbb{R}^3} \phi_u^t u^2 \, dx = \int_{\mathbb{R}^3} g(\varepsilon x, \tau_i u) \tau_i u \, dx.$$

Therefore,

$$\frac{\|u\|_\varepsilon^2}{\tau_i^2} + \int_{\mathbb{R}^3} \phi_u^t u^2 \, dx = \int_{\mathbb{R}^3} \frac{g(\varepsilon x, \tau_i u)}{(\tau_i u)^3} u^4 \, dx$$

which implies that

$$\big(\frac{1}{\tau_1^2} - \frac{1}{\tau_2^2}\big) \|u\|_\varepsilon^2 = \int_{\mathbb{R}^3} \big(\frac{g(\varepsilon x, \tau_1 u)}{(\tau_1 u)^3} - \frac{g(\varepsilon x, \tau_2 u)}{(\tau_2 u)^3}\big) u^4 \, dx,$$

contrary to (3.4). Thus, (1) is proved.

(2) Suppose $u \in S_\varepsilon^+$, then from (3.1) and (3.2) and Sobolev embeddings we obtain

$$\tau_u \le \int_{\mathbb{R}^3} g(\varepsilon x, \tau_u u) u \, dx \le \epsilon C_1 \tau_u^4 + C_\epsilon C_2 \tau_u^p.$$

From previous inequality we obtain $\sigma > 0$ independent on u, such that $\tau_u > \sigma$.

Finally, if $\mathcal{W} \subset S_\varepsilon^+$ is compact, and suppose by contradiction that there is $\{u_n\} \subset \mathcal{W}$ such that $\tau_n := \tau_{u_n} \to \infty$. Since $\mathcal{W}$ is compact, there is $u \in \mathcal{W}$ such that $u_n \to u$ in H_ε. It follows from the arguments used in the proof of Lemma 3.2 that

$$I_\varepsilon(\tau_n u_n) \to -\infty.$$

On the other hand, by $v_n := \tau_n u_n \in \mathcal{N}_\varepsilon$ and (3.3) and (3.4) we deduce that

$$\begin{aligned}
&I_\varepsilon(v_n) \\
&= I_\varepsilon(v_n) - \frac{1}{\theta} I_\varepsilon'(v_n) v_n \\
&\ge \frac{\theta - 2}{2\theta} \|v_n\|_\varepsilon^2 + \big(\frac{1}{4} - \frac{1}{\theta}\big) \int_{\mathbb{R}^3} \phi_u^t u^2 \, dx + \frac{1}{\theta} \int_{\mathbb{R}^3 \setminus \Omega_\varepsilon} \big(g(\varepsilon x, v_n) v_n - \theta G(\varepsilon x, v_n)\big) \, dx \\
&\ge \frac{\theta - 2}{2\theta} \|v_n\|_\varepsilon^2 - \frac{\theta - 2}{2\theta} \frac{\|v_n\|_\varepsilon^2}{K} \\
&= \big(1 - \frac{1}{K}\big) \frac{\theta - 2}{2\theta} \|v_n\|_\varepsilon^2,
\end{aligned}$$

which yields a contradiction. Therefore (2) is proved.

(3) First of all we observe that $\hat{m}_\varepsilon$, m_ε and m_ε^{-1} are well defined. In fact, by (1), for each $u \in H_\varepsilon^+$, there exists a unique $\tau_u > 0$ such that $\tau_u u \in \mathcal{N}_\varepsilon$, hence there is a unique $\hat{m}_\varepsilon(u) = \tau_u u \in \mathcal{N}_\varepsilon$. On the other hand, if $u \in \mathcal{N}_\varepsilon$ then $u \in H_\varepsilon^+$. Otherwise, we have $|\operatorname{supp}(u^+) \cap \Omega_\varepsilon| = 0$ and by Lemma 2.3 and (3.3) we have

$$0 < \|u\|_\varepsilon^2 \le \int_{\mathbb{R}^3} g(\varepsilon x, u)u\,dx \le \frac{1}{K}\int_{\mathbb{R}^3\setminus\Omega_\varepsilon} V(\varepsilon x)u^2\,dx \le \frac{\|u\|_\varepsilon^2}{K}$$

which is impossible since $K > 2$ and $u \neq 0$. Therefore, $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon} \in S_\varepsilon^+$, is well defined and it is a continuous function. Since

$$m_\varepsilon^{-1}(m_\varepsilon(u)) = m_\varepsilon^{-1}(t_u u) = \frac{\tau_u u}{\tau_u\|u\|_\varepsilon} = u,\ \forall u \in S_\varepsilon^+,$$

we conclude that m_ε is a bijection.

To prove $\hat{m}_\varepsilon : H_\varepsilon^+ \to \mathcal{N}_\varepsilon$ is continuous, let $\{u_n\} \subset H_\varepsilon^+$ and $u \in H_\varepsilon^+$ be such that $u_n \to u$ in H_ε. By (a_2), there is a $\tau_0 > 0$ up to a subsequence such that $\tau_n := \tau_{u_n} \to \tau_0$. Since $\tau_n u_n \in \mathcal{N}_\varepsilon$ we obtain

$$\tau_n^2\|u_n\|_\varepsilon^2 + \tau_n^4\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2\,dx = \int_{\mathbb{R}^3} g(\varepsilon x, \tau_n u_n)\tau_n u_n\,dx, \quad \forall n \in \mathbb{N}.$$

By Lemma 2.4 and passing to the limit as $n \to \infty$, it follows that

$$\tau_0^2\|u\|_\varepsilon^2 + \tau_0^4\int_{\mathbb{R}^3}\phi_u^t u^2\,dx = \int_{\mathbb{R}^3} g(\varepsilon x, \tau_0 u)\tau_0 u\,dx,$$

which means that $\tau_0 u \in \mathcal{N}_\varepsilon$ and $\tau_u = \tau_0$. This proves $\hat{m}_\varepsilon(u_n) \to \hat{m}_\varepsilon(u)$ in H_ε^+. So, $\hat{m}_\varepsilon$, m_ε are continuous functions and (3) is proved. □

Now we define the functions $\hat{\Psi}_\varepsilon : H_\varepsilon^+ \to \mathbb{R}$ and $\Psi_\varepsilon : S_\varepsilon^+ \to \mathbb{R}$, by

$$\hat{\Psi}_\varepsilon(u) = I_\varepsilon(\hat{m}_\varepsilon(u)), \quad \Psi_\varepsilon := \hat{\Psi}_\varepsilon|_{S_\varepsilon^+}.$$

The next result is a direct consequence of Lemma 3.3. The details can be seen in [33]. For the convenience of the reader, here we do a sketch of the proof.

Lemma 3.4. *Assume that* (A1)–(A6) *are satisfied. Then*

1. $\hat{\Psi}_\varepsilon \in C^1(H_\varepsilon^+, \mathbb{R})$ *and*
$$\hat{\Psi}_\varepsilon'(u)v = \frac{\|\hat{m}_\varepsilon(u)\|_\varepsilon}{\|u\|_\varepsilon} I_\varepsilon'(\hat{m}_\varepsilon(u))v, \quad \forall u \in H_\varepsilon^+,\ \forall v \in H_\varepsilon.$$
2. $\Psi_\varepsilon \in C^1(S_\varepsilon^+, \mathbb{R})$ *and*
$$\Psi_\varepsilon'(u)v = \|m_\varepsilon(u)\|_\varepsilon I_\varepsilon'(m_\varepsilon(u))v, \quad \forall v \in T_u S_\varepsilon^+.$$
3. *If* $\{u_n\}$ *is a* $(PS)_c$ *sequence of* Ψ_ε, *then* $\{m_\varepsilon(u_n)\}$ *is a* $(PS)_c$ *sequence of* I_ε. *If* $\{u_n\} \subset \mathcal{N}_\varepsilon$ *is a bounded* $(PS)_c$ *sequence for* I_ε, *then* $\{m_\varepsilon^{-1}(u_n)\}$ *is a* $(PS)_c$ *sequence of* Ψ_ε.
4. u *is a critical point of* Ψ_ε *if and only if,* $m_\varepsilon(u)$ *is a critical point of* I_ε. *Moreover, corresponding critical values coincide and*
$$\inf_{S_\varepsilon^+}\Psi_\varepsilon = \inf_{\mathcal{N}_\varepsilon} I_\varepsilon.$$

Proof. (1) Let $u \in H_\varepsilon^+$ and $v \in H_\varepsilon$. From definition of $\hat{\Psi}_\varepsilon$ and t_u and the mean value theorem, we obtain

$$\begin{aligned}\hat{\Psi}_\varepsilon(u+hv) - \hat{\Psi}_\varepsilon(u) &= I_\varepsilon\big(\tau_{u+hv}(u+hv)\big) - I_\varepsilon(\tau_u u)\\ &\le I_\varepsilon\big(\tau_{u+hv}(u+hv)\big) - I_\varepsilon(\tau_{u+hv}u)\\ &= I_\varepsilon'\big(\tau_{u+hv}(u+\theta hv)\big)\tau_{u+hv}hv,\end{aligned}$$

where $|h|$ is small enough and $\theta \in (0,1)$. Similarly,

$$\hat{\Psi}_\varepsilon(u+hv) - \hat{\Psi}_\varepsilon(u) \ge I_\varepsilon(\tau_u(u+hv)) - I_\varepsilon(\tau_u u) = I_\varepsilon'(\tau_u(u+\varsigma hv))\tau_u hv,$$

where $\varsigma \in (0,1)$. Since the mapping $u \mapsto \tau_u$ is continuous according to Lemma 3.3, we see combining these two inequalities that

$$\lim_{h\to 0} \frac{\hat{\Psi}_\varepsilon(u+hv) - \hat{\Psi}_\varepsilon(u)}{h} = \tau_u I_\varepsilon'(\tau_u u)v = \frac{\|\hat{m}_\varepsilon(u)\|_\varepsilon}{\|u\|_\varepsilon} I_\varepsilon'(\hat{m}_\varepsilon(u))v.$$

Since $I_\varepsilon \in C^1$, it follows that the Gâteaux derivative of $\hat{\Psi}_\varepsilon$ is bounded linear in v and continuous on u. From [36] we know that $\hat{\Psi}_\varepsilon \in C^1(H_\varepsilon^+, \mathbb{R})$ and

$$\hat{\Psi}_\varepsilon'(u)v = \frac{\|\hat{m}_\varepsilon(u)\|_\varepsilon}{\|u\|_\varepsilon} I_\varepsilon'(\hat{m}_\varepsilon(u))v, \quad \forall u \in H_\varepsilon^+,\ \forall v \in H_\varepsilon.$$

The item (1) is proved.

(2) This item is a direct consequence of the item (1).

(3) We first note that $H_\varepsilon = T_u S_\varepsilon^+ \oplus \mathbb{R}u$ for every $u \in S_\varepsilon^+$ and the linear projection $P : H_\varepsilon \to T_u S_\varepsilon^+$ defined by $P(v+\tau u) = v$ is continuous, namely, there is $C > 0$ such that

$$\|v\|_\varepsilon \le C\|v+\tau u\|_\varepsilon, \quad \forall u \in S_\varepsilon^+,\ v \in T_u S_\varepsilon^+,\ \tau \in \mathbb{R}. \tag{3.6}$$

Moreover, by (1) we have

$$\|\Psi_\varepsilon'\| = \sup_{v\in T_u S_\varepsilon^+, \|v\|_\varepsilon = 1} \Psi_\varepsilon'(u)v = \|w\|_\varepsilon \sup_{v\in T_u S_\varepsilon^+, \|v\|_\varepsilon = 1} I_\varepsilon'(w)v, \tag{3.7}$$

where $w = m_\varepsilon(u)$. Since $w \in \mathcal{N}_\varepsilon$, we conclude that

$$I_\varepsilon'(w)u = I_\varepsilon'(w)\frac{w}{\|w\|_\varepsilon} = 0. \tag{3.8}$$

Hence, from (3.6) and (3.8) we have

$$\|\Psi_\varepsilon'(u)\| \le \|w\|_\varepsilon \|I_\varepsilon'(w)\| \le C\|w\|_\varepsilon \sup_{v\in T_u S_\varepsilon^+\setminus\{0\}} \frac{I_\varepsilon'(w)v}{\|v\|_\varepsilon} = C\|\Psi_\varepsilon'(u)\|,$$

which shows that

$$\|\Psi_\varepsilon'(u)\| \le \|w\|_\varepsilon \|I_\varepsilon'(w)\| \le C\|\Psi_\varepsilon'(u)\|,\ \forall u \in S_\varepsilon^+. \tag{3.9}$$

Since $w \in \mathcal{N}_\varepsilon$, we have $\|w\| \ge \gamma > 0$. Therefore, the inequality in (3.9) together with $I_\varepsilon(w) = \Psi_\varepsilon(u)$ imply the item (3).

(4) It follow from (3.9) that $\Psi_\varepsilon'(u) = 0$ if and only if $I_\varepsilon'(w) = 0$. The remainder follows from definition of Ψ_ε. □

As in [33], we have the following variational characterization of the infimum of I_ε over $\mathcal{N}_\varepsilon$:

$$c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} I_\varepsilon(u) = \inf_{u\in H_\varepsilon^+} \max_{\tau>0} I_\varepsilon(\tau u) = \inf_{u\in S_\varepsilon^+} \max_{\tau>0} I_\varepsilon(\tau u) > 0.$$

The main feature of the modified functional is that it satisfies the Palais-Smale condition, as we can see from the next results.

Lemma 3.5. *Let $\{u_n\}$ be a $(PS)_c$ sequence for I_ε with $c>0$, then $\{u_n\}$ is bounded in H_ε.*

Proof. Since $\{u_n\}$ be a $(PS)_c$ sequence for I_ε, then there is $C>0$ such that

$$C+\|u_n\|_\varepsilon \geq I_\varepsilon(u_n)-\frac{1}{\theta}I'_\varepsilon(u_n)u_n.$$

From (3.3) and (3.4) we have

$$C+\|u_n\|_\varepsilon \geq (1-\frac{1}{K})\frac{\theta-2}{2\theta}\|u_n\|_\varepsilon^2.$$

Therefore, $\{u_n\}$ is bounded in H_ε by the fact that $K,\theta>2$. $\square$

Lemma 3.6 ([16])**.** *Let $\{u_n\}$ be a $(PS)_c$ sequence for I_ε, then for each $\xi>0$, there is a number $R=R(\xi)>0$ such that*

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3\setminus B_R}\big(|(-\triangle)^{s/2}u_n|^2+V(\varepsilon x)u_n^2\big)\,dx<\xi.$$

Lemma 3.7. *The functional I_ε verifies the $(PS)_c$ condition with $c>0$ in H_ε.*

Proof. Let $\{u_n\}\subset H_\varepsilon$ be a $(PS)_c$ sequence for I_ε. From Lemma 3.5 we know that $\{u_n\}$ is bounded in H_ε. Passing to a subsequence, we may assume that

$$\begin{gathered}u_n \rightharpoonup u \quad \text{in } H^s(\mathbb{R}^3),\\ u_n\to u \quad \text{in } L^r_{\rm loc}(\mathbb{R}^3), 2\leq r<2_s^*,\\ u_n(x)\to u(x) \quad \text{a.e. in } \mathbb{R}^3.\end{gathered} \tag{3.10}$$

By Lemma 3.6 we have for any $\xi>0$, there exists an $R=R(\xi)>0$ such that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3\setminus B_R}\big(|(-\triangle)^{s/2}u_n|^2+V(\varepsilon x)u_n^2\big)\,dx<\xi. \tag{3.11}$$

and so

$$\int_{\mathbb{R}^3\setminus B_R}\big(|(-\triangle)^{s/2}u|^2+V(\varepsilon x)u^2\big)\,dx<\xi. \tag{3.12}$$

By (3.10)-(3.12), and the Sobolev's Imbedding Theorem, we have that for any $r\in[2,2_s^*)$ and any $\xi>0$, there exists a $C>0$ such that

$$\begin{aligned}\int_{\mathbb{R}^3}|u_n-u|^r\,dx &= \int_{B_R}|u_n-u|^r\,dx+\int_{\mathbb{R}^3\setminus B_R}|u_n-u|^r\,dx\\ &\leq o_n(1)+C(\|u_n\|_{H_\varepsilon(\mathbb{R}^3\setminus B_R)}+\|u\|_{H_\varepsilon(\mathbb{R}^3\setminus B_R)})\leq C\xi.\end{aligned}$$

Hence. we have proved that

$$u_n\to u \quad \text{in } L^r(\mathbb{R}^3), 2\leq r<2_s^*. \tag{3.13}$$

Observe that

$$\begin{aligned}\|u_n-u\|_\varepsilon^2 &= \langle I'_\varepsilon(u_n)-I'_\varepsilon(u),u_n-u\rangle+\int_{\mathbb{R}^3}\big(g(\varepsilon x,u_n)-g(\varepsilon x,u)\big)(u_n-u)\,dx\\ &\quad -\int_{\mathbb{R}^3}(\phi^t_{u_n}u_n-\phi^t_u u)(u_n-u)\,dx.\end{aligned}$$

It is clear that
$$\langle I'_\varepsilon(u_n) - I'_\varepsilon(u), u_n - u\rangle \to 0 \quad as\ n \to +\infty.$$
According to assumptions (3.1) and (3.2) and the Hölder inequality, we obtain
$$\begin{aligned}
&\int_{\mathbb{R}^3} \big(g(\varepsilon x, u_n) - g(\varepsilon x, u)\big)(u_n - u)\,dx \\
&\le \int_{\mathbb{R}^3} C(|u_n| + |u| + |u_n|^{p-1} + |u|^{p-1})|u_n - u|\,dx \\
&\le C(\|u_n\|_{L^2} + \|u\|_{L^2})\|u_n - u\|_{L^2} + C(\|u_n\|_{L^p}^{p-1} + \|u\|_{L^p}^{p-1})\|u_n - u\|_{L^p}.
\end{aligned}$$
Since $u_n \to u$ in $L^r(\mathbb{R}^3)$ for all $r \in [2, 2_s^*)$, we have that
$$\int_{\mathbb{R}^3} \big(g(\varepsilon x, u_n) - g(\varepsilon x, u)\big)(u_n - u)\,dx \to 0 \quad \text{as } n \to +\infty.$$
By Hölder inequality, the Sobolev inequality and Lemma 2.4 we have
$$\begin{aligned}
|\int_{\mathbb{R}^3} \phi_{u_n}^t u_n(u_n - u)\,dx| &\le \|\phi_{u_n}^t\|_{L^{2_t^*}} \|u_n\|_{L^{\frac{3}{t}}} \|u_n - u\|_{L^2} \\
&\le C\|\phi_{u_n}^t\|_{\mathcal{D}^{t,2}} \|u_n\|_{L^{\frac{3}{t}}} \|u_n - u\|_{L^2} \\
&\le C\|u_n\|^2_{L^{\frac{12}{3+2t}}} \|u_n\|_{L^{\frac{3}{t}}} \|u_n - u\|_{L^2},
\end{aligned}$$
where $C > 0$ is a constant and we have used the fact that $2s + 2t \ge 3$. Again using $u_n \to u$ in $L^r(\mathbb{R}^3)$ for any $r \in [2, 2_s^*)$, we have
$$\int_{\mathbb{R}^3} \phi_{u_n}^t u_n(u_n - u)\,dx \to 0 \quad \text{as } n \to \infty,$$
Similarly, we obtain
$$\int_{\mathbb{R}^3} \phi_u^t u(u_n - u)\,dx \to 0 \quad \text{as } n \to \infty.$$
Thus
$$\int_{\mathbb{R}^3} (\phi_{u_n}^t u_n - \phi_u^t u)(u_n - u)\,dx \to 0 \quad \text{as } n \to \infty,$$
so that $\|u_n - u\|_\varepsilon \to 0$ and consequently $u_n \to u$ in H_ε. □

Lemma 3.8. *The functional Ψ_ε satisfies the Palais-Smale condition in S_ε^+.*

Proof. Let $\{u_n\} \subset S_\varepsilon^+$ be a $(PS)_c$ sequence for Ψ_ε. Thus, $\Psi_\varepsilon(u_n) \to c$ and $\|\Psi'_\varepsilon\|_* \to 0$, where $\|\cdot\|_*$ is the norm in the dual space $(T_{u_n} S_\varepsilon^+)'$. It follows from Lemma 3.4(3) that $\{m_\varepsilon(u_n)\}$ is a $(PS)_c$ sequence for I_ε in H_ε. From Lemma 3.7 we see that there is a $u \in S_\varepsilon^+$ such that $m_\varepsilon(u_n) \to m_\varepsilon(u)$ in H_ε. From Lemma 3.3(3), it follows that $u_n \to u$ in S_ε^+. □

4. Multiplicity of solutions for the modified problem

4.1. **Autonomous problem.** Since we are interesting in giving a multiplicity result for the modified problem, we start by considering the limit problem associated to (3.5), namely, the problem
$$\begin{gathered}
(-\Delta)^s u + V_0 u + \phi u = f(u) \quad \text{in } \mathbb{R}^3, \\
(-\Delta)^t \phi = u^2 \quad \text{in } \mathbb{R}^3,
\end{gathered} \tag{4.1}$$

which has the associated functional

$$I_0(u)=\frac{1}{2}\int_{\mathbb{R}^3}(|(-\Delta)^{s/2}u|^2+V_0u^2)\,dx+\frac{1}{4}\int_{\mathbb{R}^3}\phi_u^t u^2\,dx-\int_{\mathbb{R}^3}F(u)\,dx.$$

The functional is well defined on the Hilbert space $H_0=H^s(\mathbb{R}^3)$ with the inner product

$$(u,v)_0=\int_{\mathbb{R}^3}(-\Delta)^{s/2}u(-\Delta)^{s/2}v\,dx+\int_{\mathbb{R}^3}V_0uv\,dx,$$

and the norm

$$\|u\|_0^2=\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2\,dx+\int_{\mathbb{R}^3}V_0u^2\,dx.$$

We denote the Nehari manifold associated to I_0 by

$$\mathcal{N}_0=\{u\in H_0\backslash\{0\}:\langle I_0'(u),u\rangle=0\},$$

and the open subset

$$H_0^+=\{u\in H_0:|\operatorname{supp}(u^+)|>0\},$$

and $S_0^+=S_0\cap H_0^+$, where S_0 is the unit sphere of H_0.

As in section 3, S_0^+ is an incomplete $C^{1,1}$-manifold of codimension 1, modeled on H_0 and contained in the open H_0^+. Thus, $H_0=T_uS_0^+\oplus\mathbb{R}u$ for each $u\in S_0^+$, where $T_uS_0^+=\{v\in H_0:(u,v)_0=0\}$.

Next we have the following Lemmas, their proofs follow from a similar argument the one used in Lemmas 3.3 and 3.4.

Lemma 4.1. *Let V_0 be given in* (A1) *and* (A3)–(A6) *be satisfied. Then the following properties hold:*

1. *For each $u\in H_0^+$, let $g_u:\mathbb{R}^+\to\mathbb{R}$ be given by $g_u(\tau)=I_0(\tau u)$. Then there exists a unique $\tau_u>0$ such that $g_u'(\tau)>0$ in $(0,\tau_u)$ and $g_u'(\tau)<0$ in (τ_u,∞).*
2. *There is a $\sigma>0$ independent on u such that $\tau_u>\sigma$ for all $u\in S_0^+$. Moreover, for each compact set $\mathcal{W}\subset S_0^+$ there is $C_{\mathcal{W}}>0$ such that $\tau_u\le C_{\mathcal{W}}$ for all $u\in\mathcal{W}$.*
3. *The map $\hat{m}:H_0^+\to\mathcal{N}_0$ given by $\hat{m}(u)=\tau_uu$ is continuous and $m:=\hat{m}|_{S_0^+}$ is a homeomorphism between S_0^+ and $\mathcal{N}_0$. Moreover, $m^{-1}(u)=\frac{u}{\|u\|_0}$.*

We define the applications $\hat{\Psi}_0:H_0^+\to\mathbb{R}$ by $\hat{\Psi}_0(u)=I_0(\hat{m}(u))$, and $\Psi_0:S_0^+\to\mathbb{R}$ by $\Psi_0:=\hat{\Psi}_0|_{S_0^+}$.

Lemma 4.2. *Let V_0 be given in* (A1) *and* (A3)–(A6) *be satisfied. Then:*

1. $\hat{\Psi}_0\in C^1(H_0^+,\mathbb{R})$ *and*
$$\hat{\Psi}_0'(u)v=\frac{\|\hat{m}(u)\|_0}{\|u\|_0}I_0'(\hat{m}(u))v,\quad\forall u\in H_0^+\ \forall v\in H_0.$$
2. $\Psi_0\in C^1(S_0^+,\mathbb{R})$ *and*
$$\Psi_0'(u)v=\|m(u)\|_0I_0'(m(u))v,\ \forall v\in T_uS_0^+.$$
3. *If $\{u_n\}$ is a $(PS)_c$ sequence of Ψ_0, then $\{m(u_n)\}$ is a $(PS)_c$ sequence of I_0. If $\{u_n\}\subset\mathcal{N}_0$ is a bounded $(PS)_c$ sequence for I_0, then $\{m^{-1}(u_n)\}$ is a $(PS)_c$ sequence of Ψ_0.*

(4) *u is a critical point of Ψ_0 if and only if, $m(u)$ is a critical point of I_0. Moreover, corresponding critical values coincide and*

$$\inf_{S_0^+} \Psi_0 = \inf_{\mathcal{N}_0} I_0.$$

As in the previous section, we have the following variational characterization of the infimum of I_0 over $\mathcal{N}_0$:

$$c_{V_0} = \inf_{u\in\mathcal{N}_0} I_0(u) = \inf_{u\in H_0^+} \max_{\tau>0} I_0(\tau u) = \inf_{u\in S_0^+} \max_{\tau>0} I_0(\tau u) > 0.$$

The next Lemma allows us to assume that the weak limit of a $(PS)_c$ sequence is non-trivial.

Lemma 4.3. *Let $\{u_n\} \subset H_0$ be a $(PS)_c$ sequence with $c > 0$ for I_0 with $u_n \rightharpoonup 0$. Then, only one of the following statements holds.*

(i) *$u_n \to 0$ in H_0, or*
(ii) *there exist a sequence $\{y_n\} \subset \mathbb{R}^3$ and constants $R, \beta > 0$ such that*

$$\liminf_{n\to+\infty} \int_{B_R(y_n)} u_n^2\,dx \ge \beta > 0.$$

Proof. Suppose (ii) does not occur. Then, for any $R > 0$, we have

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} u_n^2\,dx = 0.$$

Since $\{u_n\}$ is bounded in H_0, by Lemma 2.2, we have

$$u_n \to 0 \text{ in } L^r(\mathbb{R}^3) \text{ for } r \in (2, 2_s^*).$$

Thus,

$$\int_{\mathbb{R}^3} f(u_n)u_n\,dx = \int_{\mathbb{R}^3} F(u_n)\,dx \to 0 \quad \text{as } n \to +\infty.$$

Recalling that $I_0'(u_n)u_n \to 0$ and Lemma 2.4 we obtain

$$\|u_n\|_0^2 = o_n(1).$$

Therefore the conclusion follows. □

From Lemma 4.3 we can see that, if u is the weak limit of a $(PS)_{c_{V_0}}$ sequence $\{u_n\}$ for the functional I_0, then we can assume $u \neq 0$. Otherwise we would have $u_n \rightharpoonup 0$ and once it doesn't occur $u_n \to 0$, we conclude from Lemma 4.3 that there exist $\{y_n\} \subset \mathbb{R}^3$ and $R, \beta > 0$ such that

$$\liminf_{n\to+\infty} \int_{B_R(y_n)} u_n^2\,dx \ge \beta > 0.$$

Then set $v_n(x) = u_n(x + y_n)$, making a change of variable, we can prove that $\{v_n\}$ ia also a $(PS)_{c_{V_0}}$ sequence for the functional I_0, it is bounded in H_0 and there is $v \in H_0$ such that $v_n \to v$ in H_0 with $v \neq 0$.

In the next Lemma we obtain a positive ground state solution for the autonomous problem (4.1).

Theorem 4.4. *Problem* (4.1) *has a positive ground state solution.*

Proof. Let $\{u_n\} \subset H_0$ be a $(PS)_{c_{V_0}}$ sequence for I_0. Arguing as Lemma 3.5, we have that $\{u_n\}$ is bounded in H_0. Then, up to a subsequence, we have

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } H^s(\mathbb{R}^3), \\ u_n \to u \quad \text{in } L^r_{\rm loc}(\mathbb{R}^3), 2 \le r < 2_s^*, \\ u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^3. \end{gathered}$$

Similar to the proof in Lemmas 3.2 and 3.7, we know that the functional I_0 has the mountain pass geometry and satisfies the $(PS)_{c_{V_0}}$ condition. So problem (4.1) has ground state solution from [36].

Next we prove that the solution u is positive, which is a ground state solution of the equation

$$(-\Delta)^s u + V_0 u + \phi_u^t u = f(u) \text{ in } \mathbb{R}^3. \tag{4.2}$$

Using $u^- = \max -u, 0$ as a test function in (4.2) we obtain

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} u^- \,dx + \int_{\mathbb{R}^3} V_0 |u^-|^2 \,dx + \int_{\mathbb{R}^3} \phi_u^t (u^-)^2 \,dx = 0. \tag{4.3}$$

On the other hand,

$$\begin{aligned} \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} u^- \,dx &= \frac{1}{2} C(s) \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{3+2s}} \,dx\,dy \\ &\ge \frac{1}{2} C(s) \int_{\{u>0\}\times\{u<0\}} \frac{(u(x)-u(y))(-u^-(y))}{|x-y|^{3+2s}} \,dx\,dy \\ &\quad + \frac{1}{2} C(s) \int_{\{u<0\}\times\{u<0\}} \frac{(u^-(x)-u^-(y))^2}{|x-y|^{3+2s}} \,dx\,dy \\ &\quad + \frac{1}{2} C(s) \int_{\{u<0\}\times\{u>0\}} \frac{(u(x)-u(y))u^-(x)}{|x-y|^{3+2s}} \,dx\,dy \ge 0. \end{aligned}$$

Thus, from (4.3) and Lemma 2.4 (i), it follows that $u^- = 0$ and $u \ge 0$. Moreover, if $u(x_0) = 0$ for some $x_0 \in \mathbb{R}^3$ and the regularity of solutions [18] we have that $(-\Delta)^s u(x_0) = 0$ and by [11, Lemma 3.2], we have

$$(-\Delta)^s u(x_0) = -\frac{C(s)}{2} \int_{\mathbb{R}^3} \frac{u(x_0+y)+u(x_0-y)-2u(x_0)}{|y|^{3+2s}} \,dy;$$

therefore,

$$\int_{\mathbb{R}^3} \frac{u(x_0+y)+u(x_0-y)}{|y|^{3+2s}} \,dy = 0,$$

yielding $u \equiv 0$, a contradiction. Therefore, u is a positive solution of the system (4.1) and the proof is complete. □

The next result is a compactness result on autonomous problem which we will use later.

Lemma 4.5. *Let $\{u_n\} \subset \mathcal{N}_0$ be a sequence such that $I_0(u_n) \to c_{V_0}$. Then $\{u_n\}$ has a convergent subsequence in H_0.*

Proof. Since $\{u_n\} \subset \mathcal{N}_0$, it follows from Lemma 4.1(3), Lemma 4.2(4) and the definition of c_{V_0} that

$$v_n = m^{-1}(u_n) = \frac{u_n}{\|u_n\|_0} \in S_0^+, \quad \forall n \in \mathbb{N},$$

$$\Psi_0(v_n) = I_0(u_n) \to c_{V_0} = \inf_{S_0^+} \Psi_0.$$

Although S_0^+ is incomplete, due to Lemma 4.1, we can still apply the Ekeland's variational principle [13] to the functional $\Theta_0 : H \to \mathbb{R} \cup \{\infty\}$, defined by $\Theta_0(u) = \Psi_0(u)$ if $u \in S_0^+$ and $\Theta_0(u) = \infty$ if $u \in \partial S_0^+$, where $H = \overline{S_0^+}$ is the complete metric space equipped with the metric $d(u, v) := \|u - v\|_0$. In fact, take $\varepsilon = \frac{1}{k^2}$ in Theorem 1.1 of [13], we have a subsequence $\{v_{n_k}\} \subset \{v_n\}$ such that

$$c_{V_0} \le \Psi(v_{n_k}) \le c_{V_0} + \frac{1}{k^2}.$$

From [13, Theorem 1.1], for $\lambda = 1/k$, there exist a sequence $\{\tilde{v}_k\} \subset S_0^+$ such that

$$\Theta_0(\tilde{v}_k) \le \Theta_0(v_{n_k}) < c_{V_0} + \frac{1}{k^2} \quad \text{and} \quad \|v_{n_k} - \tilde{v}_k\|_0 \le \frac{1}{k}.$$

In particular, for any $u \in S_0^+$ we have

$$\Psi_0(u) > \Psi_0(\tilde{v}_k) - \frac{1}{k}\|u - \tilde{v}_k\|_0.$$

Hence, similar the proof for [13, Theorem 3.1], we have that there exists $\lambda_k \in \mathbb{R}$ such that

$$\|\hat{\Psi}_0'(\tilde{v}_k) - \lambda_k g_0'(\tilde{v}_k)\|_0 \le \frac{1}{k},$$

where $g_0(u) = \|u\|_0^2 - 1$. Which means that

$$\lambda_k = \frac{1}{\|g_0'(\tilde{v}_k)\|_0^2} \langle \hat{\Psi}_0'(\tilde{v}_k), g_0'(\tilde{v}_k)\rangle + o_k(1), \quad g_0'(\tilde{v}_k) = \tilde{v}_k.$$

From Lemma 4.2(1),

$$\lambda_k = \langle \hat{\Psi}_0'(\tilde{v}_k), \tilde{v}_k\rangle + o_k(1) = \tau_{\tilde{v}_k}\langle I_0'(t_{\tilde{v}_k}\tilde{v}_k), \tilde{v}_k\rangle + o_k(1) = o_k(1).$$

Therefore, we can conclude there is a sequence $\{\tilde{v}_n\} \subset S_0^+$ such that $\{\tilde{v}_n\}$ is a $(PS)_{c_{V_0}}$ sequence for Ψ_0 on S_0^+ and

$$\|u_n - \tilde{v}_n\|_0 = o_n(1).$$

The remainder of the proof follows from Lemma 4.2, Theorem 4.4 and arguing as in the proof of Lemma 3.8. □

4.2. **Technical results.** In this section we will relate the number of positive solutions of (3.5) to the topology of the set $\mathcal{M}$. For this, we consider $\delta > 0$ such that $\mathcal{M}_\delta \subset \Omega$ and by Theorem 4.4, we can choose $w \in \mathcal{N}_0$ with $I_0(w) = c_{V_0}$. Let η be a smooth nonincreasing cut-off function defined in $[0, +\infty)$ such that $\eta(t) = 1$ if $0 \le t \le \frac{\delta}{2}$ and $\eta(t) = 0$ if $t \ge \delta$. For each $y \in \mathcal{M}$, let

$$\Psi_{\varepsilon,y}(x) = \eta(|\varepsilon x - y|) w(\frac{\varepsilon x - y}{\varepsilon}).$$

Then for small $\varepsilon > 0$, one has $\Psi_{\varepsilon,y} \in H_\varepsilon \backslash \{0\}$ for all $y \in \mathcal{M}$. In fact, using the change of variable $z = x - \frac{y}{\varepsilon}$, one has

$$\begin{aligned} \int_{\mathbb{R}^3} V(\varepsilon x)\Psi_{\varepsilon,y}^2(x)\,dx &= \int_{\mathbb{R}^3} V(\varepsilon x)\eta^2(|\varepsilon x - y|)w^2(\frac{\varepsilon x - y}{\varepsilon})\,dx \\ &= \int_{\mathbb{R}^3} V(\varepsilon z + y)\eta^2(|\varepsilon z|)w^2(z)dz \end{aligned}$$

$$\leq C\int_{\mathbb{R}^3} w^2(z)dz < +\infty.$$

Moreover, using the change of variable $x' = x - \frac{y}{\varepsilon}$, $z' = z - \frac{y}{\varepsilon}$, we have

$$\begin{aligned}
&\|(-\Delta)^{s/2}\Psi_{\varepsilon,y}\|_2^2 \\
&= \frac{1}{2}C(s)\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\left|\eta(|\varepsilon x - y|)w(\frac{\varepsilon x-y}{\varepsilon}) - \eta(|\varepsilon z - y|)w(\frac{\varepsilon z-y}{\varepsilon})\right|^2}{|x-z|^{3+2s}}\,dxdz \\
&= \frac{1}{2}C(s)\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\left|\eta(|\varepsilon x'|)w(x') - \eta(|\varepsilon z'|)w(z')\right|^2}{|x'-z'|^{3+2s}}\,dx'dz' \\
&= \|(-\Delta)^{s/2}\eta(|\varepsilon x|)w(x)\|_2^2 = \|(-\Delta)^{s/2}\eta_\varepsilon w\|_2^2,
\end{aligned}$$

where $\eta_\varepsilon(x) = \eta(|\varepsilon x|)$. By Lemma 2.3, we see that $\eta_\varepsilon w \in \mathcal{D}^{s,2}(\mathbb{R}^3)$ as $\varepsilon \to 0$, and hence $\Psi_{\varepsilon,y} \in \mathcal{D}^{s,2}(\mathbb{R}^3)$ for $\varepsilon > 0$ small. Hence $\Psi_{\varepsilon,y} \in H_\varepsilon$. Now we proof $\Psi_{\varepsilon,y} \neq 0$. In fact,

$$\begin{aligned}
\int_{\mathbb{R}^3} \Psi_{\varepsilon,y}^2(x)\,dx &= \int_{\mathbb{R}^3} \eta^2(|\varepsilon x - y|)w^2(\frac{\varepsilon x - y}{\varepsilon})\,dx \\
&= \int_{|\varepsilon x - y|<\delta} \eta^2(|\varepsilon x - y|)w^2(\frac{\varepsilon x - y}{\varepsilon})\,dx \\
&\geq \int_{|z|\leq\delta/2\varepsilon} \eta^2(|\varepsilon z|)w^2(z)dz \\
&\geq \int_{B_0(\delta/2\varepsilon)} w^2(z)dz \\
&\to \int_{\mathbb{R}^3} w^2(z)dz > 0
\end{aligned}$$

as $\varepsilon \to 0$. Then $\Psi_{\varepsilon,y} \neq 0$ for small $\varepsilon > 0$. Therefore, there exists unique $\tau_\varepsilon > 0$ such that

$$\max_{\tau\geq 0} I_\varepsilon(\tau\Psi_{\varepsilon,y}) = I_\varepsilon(\tau_\varepsilon\Psi_{\varepsilon,y}) \quad\text{and}\quad \tau_\varepsilon\Psi_{\varepsilon,y} \in \mathcal{N}_\varepsilon.$$

We introduce the map $\Phi_\varepsilon : \mathcal{M} \to \mathcal{N}_\varepsilon$ by setting $\Phi_\varepsilon(y) = \tau_\varepsilon\Psi_{\varepsilon,y}$. By construction, $\Phi_\varepsilon(y)$ has a compact support for any $y \in \mathcal{M}$ and Φ_ε is a continuous map.

Lemma 4.6. *The functional $\Phi_\varepsilon(y)$ satisfies*

$$\lim_{\varepsilon\to 0} I_\varepsilon(\Phi_\varepsilon(y)) = c_{V_0} \quad \textit{uniformly in } y \in \mathcal{M}.$$

Proof. Suppose that the result is false. Then, there exist some $\delta_0 > 0$, $\{y_n\} \subset \mathcal{M}$ and $\varepsilon_n \to 0$ such that

$$|I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - c_{V_0}| \geq \delta_0. \tag{4.4}$$

From the definition of τ_{ε_n} we have

$$\begin{aligned}
0 &< \tau_{\varepsilon_n}^2 \int_{\mathbb{R}^3} |(-\Delta)^{s/2}\Psi_{\varepsilon_n,y_n}|^2\,dx + \tau_{\varepsilon_n}^2 \int_{\mathbb{R}^3} V(\varepsilon_n x)\Psi_{\varepsilon_n,y_n}^2\,dx \\
&\leq \tau_{\varepsilon_n}^2 \int_{\mathbb{R}^3} |(-\Delta)^{s/2}\Psi_{\varepsilon_n,y_n}|^2\,dx + \tau_{\varepsilon_n}^2 \int_{\mathbb{R}^3} V(\varepsilon_n x)\Psi_{\varepsilon_n,y_n}^2\,dx \\
&\quad + \tau_{\varepsilon_n}^4 \int_{\mathbb{R}^3} \phi_{\Psi_{\varepsilon_n,y_n}}^t \Psi_{\varepsilon_n,y_n}^2\,dx \\
&= \tau_{\varepsilon_n} \int_{\mathbb{R}^3} g(\varepsilon_n x, \tau_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})\Psi_{\varepsilon_n,y_n}\,dx.
\end{aligned} \tag{4.5}$$

It follows from (4.5) that $\tau_{\varepsilon_n} \not\to 0$; then $\tau_{\varepsilon_n} \geq \tau_0 > 0$ for some $\tau_0 > 0$. If $\tau_{\varepsilon_n} \to +\infty$, by $(A6)$ and the boundedness of Ψ_{ε_n,y_n}, we obtain

$$\begin{aligned}
&\frac{1}{\tau_{\varepsilon_n}^2}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}\Psi_{\varepsilon_n,y_n}|^2\,dx + \int_{\mathbb{R}^3} V(\varepsilon_n x)\Psi_{\varepsilon_n,y_n}^2\,dx\Big) + \int_{\mathbb{R}^3} \phi^t_{\Psi_{\varepsilon_n,y_n}}\Psi_{\varepsilon_n,y_n}^2\,dx \\
&= \int_{\mathbb{R}^3} \frac{g(\varepsilon_n x, \tau_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})}{(\tau_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})^3}\Psi_{\varepsilon_n,y_n}^4\,dx \to +\infty
\end{aligned}$$

as $n \to +\infty$. But the left side of the above inequality tends to $\int_{\mathbb{R}^3} \phi_w^t w^2\,dx$ since $\tau_{\varepsilon_n} \to +\infty$ as $\varepsilon_n \to 0$, which is impossible. Hence, $0 < \tau_0 \leq \tau_{\varepsilon_n} \leq C$. Without loss of generality, we may assume that $\tau_{\varepsilon_n} \to T > 0$.

Next we claim that $T = 1$. By Lemma 2.3 and Lebesgue's theorem we have

$$\begin{gathered}
\lim_{n\to+\infty} \|\Psi_{\varepsilon_n,y_n}\|_{\varepsilon_n}^2 = \|w\|_0^2, \\
\lim_{n\to+\infty} \int_{\mathbb{R}^3} \phi^t_{\Psi_{\varepsilon_n,y_n}} |\Psi_{\varepsilon_n,y_n}|^2\,dx = \int_{\mathbb{R}^3} \phi_w^t |w|^2\,dx, \\
\lim_{n\to+\infty} \int_{\mathbb{R}^3} f(\Psi_{\varepsilon_n,y_n})\Psi_{\varepsilon_n,y_n}\,dx = \int_{\mathbb{R}^3} f(w)w\,dx,
\end{gathered} \tag{4.6}$$

Therefore,

$$\frac{1}{T^2}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}w|^2\,dx + \frac{1}{T^2}\int_{\mathbb{R}^3} V_0 w^2\,dx + \int_{\mathbb{R}^3} \phi_w^t w^2\,dx = \int_{\mathbb{R}^3} \frac{f(Tw)}{(Tw)^3} w^4\,dx. \tag{4.7}$$

Since w is a ground state solution of (4.1), then

$$\int_{\mathbb{R}^3} |(-\Delta)^{s/2}w|^2\,dx + \int_{\mathbb{R}^3} V_0 w^2\,dx + \int_{\mathbb{R}^3} \phi_w^t w^2\,dx = \int_{\mathbb{R}^3} f(w)w\,dx. \tag{4.8}$$

Combining (4.7)-(4.8), we have

$$\begin{aligned}
&(1 - \frac{1}{T^2})\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}w|^2\,dx + \int_{\mathbb{R}^3} V_0 w^2\,dx\Big) \\
&= \int_{\mathbb{R}^3} (\frac{f(w)}{w^3} - \frac{f(Tw)}{(Tw)^3}) w^4\,dx.
\end{aligned} \tag{4.9}$$

By (3.4), we deduce that $T = 1$. It follows from (4.6), we have

$$\lim_{n\to+\infty} I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = I_{V_0}(w) = c_{V_0}, \tag{4.10}$$

which contradicts (4.4). This completes the proof. $\square$

Let $\rho = \rho(\delta) > 0$ be such that $\mathcal{M}_\delta \subset B_\rho(0)$. Consider $\chi : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as $\chi(x) = x$ for $|x| \leq \rho$ and $\chi(x) = \frac{\rho x}{|x|}$ for $|x| \geq \rho$. Finally, let us consider the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^3$ given by

$$\beta_\varepsilon(u) = \frac{\int_{\mathbb{R}^3} \chi(\varepsilon x)u^2(x)\,dx}{\int_{\mathbb{R}^3} u^2(x)\,dx} \in \mathbb{R}^3.$$

Lemma 4.7. *The functional β_ε satisfies $\lim_{\varepsilon\to 0} \beta_\varepsilon(\Phi_\varepsilon(y)) = y$ uniformly in $y \in \mathcal{M}$.*

Proof. Suppose by contradiction that there exist $\delta_0 > 0$, $\{y_n\} \subset \mathcal{M}$ and $\varepsilon_n \to 0$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \geq \delta_0. \tag{4.11}$$

Using the change of variables $z = \frac{\varepsilon_n x - y_n}{\varepsilon_n}$ and the definition of β_ε, we have

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^3}(\chi(\varepsilon_n z + y) - y_n)|\eta(|\varepsilon_n z|)w(z)|^2\,dx}{\int_{\mathbb{R}^3}|\eta(|\varepsilon_n z|)w(z)|^2\,dx}.$$

Since $\{y_n\} \subset \mathcal{M} \subset B_\rho(0)$ and $\chi\big|_{B_\rho} \equiv id$, we conclude that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| = o_n(1),$$

which contradicts (4.11) and the desired conclusion holds. $\square$

Lemma 4.8. *Let $\varepsilon_n \to 0$ and $\{u_n\} \subset \mathcal{N}_{\varepsilon_n}$ be such that $I_{\varepsilon_n}(u_n) \to c_{V_0}$. Then, there exists a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^3$ such that $v_n(x) = u_n(x + \tilde{y}_n)$ has a convergent subsequence in H_0. Moreover, passing to a subsequence, $y_n := \varepsilon_n \tilde{y}_n \to y_0 \in \mathcal{M}$.*

Proof. By Lemma 3.5, $\{u_n\}$ is bounded in H_0. Note that $c_{V_0} > 0$, and since $\|u_n\|_{\varepsilon_n} \to 0$ would imply $I_{\varepsilon_n}(u_n) \to 0$, we can argue as in the proof of Lemma 4.3 to obtain a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^3$ and constants $R, \beta > 0$ such that

$$\liminf_{n\to+\infty} \int_{B_R(\tilde{y}_n)} u_n^2\,dx \ge \beta > 0.$$

Define $v_n(x) := u_n(x + \tilde{y}_n)$, then $\{v_n\}$ is also bounded in H_0 and up to a subsequence, we have $v_n \rightharpoonup v \neq 0$ in H_0.

Let $\tau_n > 0$ be such that $\tilde{v}_n := \tau_n v_n \in \mathcal{N}_0$ and set $y_n = \varepsilon_n \tilde{y}_n$. Because $\{u_n\} \subset \mathcal{N}_{\varepsilon_n}$, we have

$$\begin{aligned}
c_{V_0} &\le I_0(\tilde{v}_n)\\
&\le \frac{1}{2}\int_{\mathbb{R}^3}|(-\Delta)^{s/2}\tilde{v}_n|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon_n x + y_n)\tilde{v}_n^2\,dx\\
&\quad + \frac{1}{4}\int_{\mathbb{R}^3}\phi^t_{\tilde{v}_n}\tilde{v}_n^2\,dx - \int_{\mathbb{R}^3} G(\varepsilon_n x + y_n, \tilde{v}_n)\,dx\\
&= \frac{\tau_n^2}{2}\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u_n|^2\,dx + \frac{\tau_n^2}{2}\int_{\mathbb{R}^3} V(\varepsilon_n x)u_n^2\,dx\\
&\quad + \frac{\tau_n^4}{4}\int_{\mathbb{R}^3}\phi^t_{u_n}u_n^2\,dx - \int_{\mathbb{R}^3} G(\varepsilon_n x, \tau_n u_n)\,dx\\
&= I_{\varepsilon_n}(\tau_n u_n)\\
&\le I_{\varepsilon_n}(u_n) = c_{V_0} + o_n(1).
\end{aligned}$$

This implies $\lim_{n\to+\infty} I_0(\tilde{v}_n) = c_{V_0}$. Because $\tilde{v}_n \in \mathcal{N}_0$, we obtain $\{\tilde{v}_n\}$ is bounded in H_0. It follows from the boundedness of $\{v_n\}$ in H_0 that $\{\tau_n\}$ is bounded, without loss of generality, we may assume that $\tau_n \to \tau_0 \ge 0$. If $\tau_0 = 0$, in view of the boundedness of $\{v_n\}$ in H_0, we have $\tilde{v}_n = \tau_n v_n \to 0$ in H_0. Hence $I_0(\tilde{v}_n) \to 0$, which contradicts $c_{V_0} > 0$. Thus, $\tau_0 > 0$ and the weak limit of $\{\tilde{v}_n\}$ is different from zero. Hence, up to a subsequence, we have $\tilde{v}_n \rightharpoonup \tau_0 v := \tilde{v} \neq 0$ in H_0 by the uniqueness of the weak limit. From Lemma 4.5, we know that $\tilde{v}_n \to \tilde{v}$ in H_0. Moreover, $\tilde{v} \in \mathcal{N}_0$.

Now, we show that $\{y_n\}$ is bounded in $\mathbb{R}^3$. Suppose that after passing to a subsequence, $|y_n| \to +\infty$. Then, by Fatou's Lemma and Lemma 2.4 we have

$$\begin{aligned}
c_{V_0} &= I_0(\tilde{v})\\
&< \liminf_{n\to\infty}\Big(\frac{1}{2}\int_{\mathbb{R}^3}|(-\Delta)^{s/2}\tilde{v}_n|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon_n x + y_n)\tilde{v}_n^2\,dx
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}\int_{\mathbb{R}^3}\phi^t_{\tilde{v}_n}\tilde{v}_n^2\,dx-\int_{\mathbb{R}^3}F(\tilde{v}_n)\,dx\Big)\\
&=\liminf_{n\to\infty}\Big(\frac{\tau_n^2}{2}\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u_n|^2\,dx+\frac{\tau_n^2}{2}\int_{\mathbb{R}^3}V(\varepsilon_n x)u_n^2\,dx\\
&+\frac{\tau_n^4}{4}\int_{\mathbb{R}^3}\phi^t_{u_n}u_n^2\,dx-\int_{\mathbb{R}^3}F(\tau_n u_n)\,dx\Big)\\
&\le\liminf_{n\to\infty}I_{\varepsilon_n}(\tau_n u_n)\\
&\le\liminf_{n\to\infty}I_{\varepsilon_n}(u_n)=c_{V_0},
\end{aligned}
$$

which yields a contradiction. Hence, $\{y_n\}$ is bounded and up to a subsequence, $y_n\to y_0$ in $\mathbb{R}^3$. If $y_0\notin\mathcal{M}$, then $V_0<V(y_0)$ and we obtain a contradiction by the same manner as above. So, $y_0\in\mathcal{M}$ and the proof is complete. $\square$

Let $h:\mathbb{R}^+\to\mathbb{R}^+$ be a positive function satisfying $h(\varepsilon)\to 0^+$ as $\varepsilon\to 0^+$. Define the set

$$
\tilde{\mathcal{N}}_\varepsilon=\{u\in\mathcal{N}_\varepsilon: I_\varepsilon(u)\le c_{V_0}+h(\varepsilon)\}.
$$

Given $y\in\mathcal{M}$, from Lemma 4.6 we conclude that $h(\varepsilon)=\sup_{y\in\mathcal{M}}|I_\varepsilon(\Phi_\varepsilon(y))-c_{V_0}|\to 0$ as $\varepsilon\to 0^+$. Thus, $\Phi_\varepsilon(y)\in\tilde{\mathcal{N}}_\varepsilon$ and $\tilde{\mathcal{N}}_\varepsilon\neq\emptyset$ for $\varepsilon>0$.

Lemma 4.9. *For any $\delta>0$, it holds that*

$$
\lim_{\varepsilon\to 0}\sup_{u\in\tilde{\mathcal{N}}_\varepsilon}\inf_{y\in\mathcal{M}_\delta}|\beta_\varepsilon(u)-y|=0.
$$

Proof. Let $\{\varepsilon_n\}\subset\mathbb{R}^+$ be such that $\varepsilon_n\to 0$. By definition, there exists $\{u_n\}\subset\tilde{\mathcal{N}}_{\varepsilon_n}$ such that

$$
\inf_{y\in\mathcal{M}_\delta}|\beta_{\varepsilon_n}(u_n)-y|=\sup_{u\in\tilde{\mathcal{N}}_{\varepsilon_n}}\inf_{y\in\mathcal{M}_\delta}|\beta_{\varepsilon_n}(u)-y|+o_n(1).
$$

So, it suffices to find a sequence $\{y_n\}\subset\mathcal{M}_\delta$ satisfying

$$
\lim_{n\to+\infty}|\beta_{\varepsilon_n}(u_n)-y_n|=0. \tag{4.12}
$$

Since $u_n\in\tilde{\mathcal{N}}_{\varepsilon_n}\subset\mathcal{N}_{\varepsilon_n}$, we obtain

$$
c_{V_0}\le c_{\varepsilon_n}\le I_{\varepsilon_n}(u_n)\le c_{V_0}+h(\varepsilon_n).
$$

It follows that $I_{\varepsilon_n}(u_n)\to c_{V_0}$. Thus, we can invoke Lemma 4.8 to obtain a sequence $\{\tilde{y}_n\}\subset\mathbb{R}^3$ such that $y_n=\varepsilon_n\tilde{y}_n\in\mathcal{M}_\delta$ for n large enough. Then

$$
\beta_{\varepsilon_n}(u_n)=y_n+\frac{\int_{\mathbb{R}^3}(\chi(\varepsilon_n x+y_n)-y_n)|u_n(x+\tilde{y}_n)|^2\,dx}{\int_{\mathbb{R}^3}|u_n(x+\tilde{y}_n)|^2\,dx}.
$$

For all $x\in\mathbb{R}^3$ fixed, since $\varepsilon_n x+y_n\to y\in\mathcal{M}_\delta$, we have that the sequence $\{y_n\}$ satisfies (4.12). This completes the proof. $\square$

4.3. **Multiplicity of solutions for** (3.5)**.** Next we prove our multiplicity result by presenting a relation between the topology of $\mathcal{M}$ the number of solutions of the modified problem (3.5), we will apply the Ljusternik-Schnirelmann abstract result in [31, 33].

Theorem 4.10. *Assume that conditions* (A1)–(A6) *hold. Then, given $\delta>0$ there is $\hat{\varepsilon}_\delta>0$ such that for any $\varepsilon\in(0,\hat{\varepsilon}_\delta)$, problem* (3.5) *has at least* $\mathrm{cat}_{\mathcal{M}_\delta}(\mathcal{M})$ *positive solutions.*

Proof. For $y \in \mathcal{M}$, set $\gamma_\varepsilon(y) = m_\varepsilon^{-1}(\Phi_\varepsilon(y))$. It follows from Lemma 3.4 and Lemma 4.6 that

$$\lim_{\varepsilon\to 0} \Psi_\varepsilon(\gamma_\varepsilon(y)) = \lim_{\varepsilon\to 0} I_\varepsilon(\Phi_\varepsilon(y)) = c_{V_0}, \tag{4.13}$$

uniformly for in $y \in \mathcal{M}$. Let

$$\tilde{S}_\varepsilon^+ = \{w \in S_\varepsilon^+ : \Psi_\varepsilon(w) \le c_{V_0} + h(\varepsilon)\},$$

where h is given in the definition of $\tilde{\mathcal{N}}_\varepsilon$. From (4.13), we know that there is a number $\hat{\varepsilon}$ such that $\tilde{S}_\varepsilon^+ \neq \emptyset$ for $\varepsilon \in (0, \hat{\varepsilon})$.

For a fixed $\delta > 0$, by Lemmas 3.3, 4.6, 4.7 and 4.9, we know that there exists a $\hat{\varepsilon} = \hat{\varepsilon}_\delta > 0$ such that for any $\varepsilon \in (0, \hat{\varepsilon}_\delta)$, the diagram

$$\mathcal{M} \xrightarrow{\Phi_\varepsilon} \tilde{\mathcal{N}}_\varepsilon \xrightarrow{m_\varepsilon^{-1}} \tilde{S}_\varepsilon^+ \xrightarrow{m_\varepsilon} \tilde{\mathcal{N}}_\varepsilon \xrightarrow{\beta_\varepsilon} \mathcal{M}_\delta$$

is well defined. From Lemma 4.7, there is a function $\lambda(\varepsilon, y)$ with $|\lambda(\varepsilon, y)| < \frac{\delta}{2}$ uniformly in $y \in \mathcal{M}$, for all $\varepsilon \in (0, \hat{\varepsilon})$, such that $\beta_\varepsilon(\Phi_\varepsilon(y)) := y + \lambda(\varepsilon, y)$ for all $y \in \mathcal{M}$. Define $H(t, y) = y + (1 - t)\lambda(\varepsilon, y)$. Then, $H : [0, 1] \times \mathcal{M} \to \mathcal{M}_\delta$ is continuous. Obviously, $H(0, y) = \beta_\varepsilon(\Phi_\varepsilon(y)), H(1, y) = y$ for all $y \in \mathcal{M}$. That is, $H(t, y)$ is homotopy between $\beta_\varepsilon \circ \Phi_\varepsilon$ and the inclusion map $id : \mathcal{M} \to \mathcal{M}_\delta$. This fact and [5, Lemma 4.3] implies that

$$\operatorname{cat}_{\tilde{S}_\varepsilon^+} \gamma_\varepsilon(\mathcal{M}) \ge \operatorname{cat}_{\mathcal{M}_\delta}(\mathcal{M}).$$

On the other hand, using the definition of $\tilde{\mathcal{N}}_\varepsilon$ and choosing $\hat{\varepsilon}_\delta$ small if necessary, we see that I_ε satisfies the (PS) condition in $\tilde{\mathcal{N}}_\varepsilon$ recalling Lemma 3.7. By Lemma 3.4 and 3.8, we obtain that Ψ_ε satisfies the (PS) condition in $\tilde{S}_\varepsilon^+$. Therefore, the standard Ljusternik-Schnirelmann theory provides at least $\operatorname{cat}_{\tilde{S}_\varepsilon^+} \gamma_\varepsilon(\mathcal{M})$ critical points of Ψ_ε restricted to $\tilde{S}_\varepsilon^+$. Using Lemma 3.7 again, we infer that I_ε has at least $\operatorname{cat}_{\mathcal{M}_\delta}(\mathcal{M})$ critical points. Using the same arguments contained in the proof Theorem 4.4, we see that the system (3.5) has at least $\operatorname{cat}_{\mathcal{M}_\delta}(\mathcal{M})$ positive solutions. $\square$

5. Proof of Theorem 1.2

In this section we will prove our main result. The idea is to show that the solutions obtained in Theorem 4.10 satisfy the following estimate $u_\varepsilon(x) \le a$, for all $x \in \Omega_\varepsilon^c$ for ε small enough. This fact implies that these solutions are in fact solutions of the original problem (2.1). The key ingredient is the following result, whose proof uses an adaptation of the arguments found in [12], which are related to the Moser iteration method [26].

Lemma 5.1. *Let $\varepsilon_n \to 0^+$ and $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$ be a solution of* (3.5)*. Then up to a subsequence, $v_n = u_n(x + \tilde{y}_n)$ satisfies that $v_n \in L^\infty(\mathbb{R}^3)$ and there exists $C > 0$ such that*

$$\|v_n\|_{L^\infty(\mathbb{R}^3)} \le C, \ \forall n \in \mathbb{N},$$

where $\{\tilde{y}_n\}$ is given in Lemma 4.8.

Proof. We define

$$h(\varepsilon_n x, v_n) := g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, v_n) - V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)v_n - \phi_{v_n}^t v_n.$$

From Lemma 3.5, $\{v_n\}$ is bounded in H_ε, and hence in $L^r(\mathbb{R}^3)$ for any $r \in [2, 2_s^*]$. So there exists some $C > 0$ such that

$$\|v_n\|_q \le C, \ \forall\, n \in \mathbb{N}.$$

Since v_n is a solution of (3.5), it follows that

$$\begin{aligned}
\phi_{v_n}^t(x) &= \int_{\mathbb{R}^3} \frac{v_n^2(y)}{|x-y|^{3-2t}}\, dy \\
&= \int_{\{|x-y|\le 1\}} \frac{v_n^2(y)}{|x-y|^{3-2t}}\, dy + \int_{\{|x-y|>1\}} \frac{v_n^2(y)}{|x-y|^{3-2t}}\, dy \\
&\le \int_{\{|x-y|\le 1\}} \frac{v_n^2(y)}{|x-y|^{3-2t}}\, dy + \int_{\{|x-y|>1\}} v_n^2(y)\, dy \\
&\le \Big(\int_{\{|x-y|\le 1\}} \frac{1}{|x-y|^{(3-2t)q'}}\, dy \Big)^{1/q'} \Big(\int_{\{|x-y|\le 1\}} v_n^{2q}(y)\, dy \Big)^{1/q} + C \\
&\le C\,,
\end{aligned}$$

where $q'(3-2t) < 3,\ 2q \in [2, 2_s^*],\ \frac{1}{q} + \frac{1}{q'} = 1$ since $2s + 2t \ge 3$. Therefore,

$$|h(\varepsilon_n x, v_n)| \le C(|v_n| + |v_n|^{p-1}) \le C(1 + |v_n|^{2_s^*-1}) \tag{5.1}$$

for n large enough.

For $T > 0$, we define

$$H(t) = \begin{cases} 0, & \text{if } t \le 0, \\ t^\beta, & \text{if } 0 < t < T, \\ \beta T^{\beta-1}(t-T) + T^\beta, & \text{if } t \ge T, \end{cases}$$

with $\beta > 1$ to be determined later. Since H is Lipschitz with constant $L_0 = \beta T^{\beta-1}$, we have

$$\begin{aligned}
[H(v_n)]_{\mathcal{D}^{s,2}} &= \Big(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|H(v_n(x)) - H(v_n(y))|^2}{|x-y|^{3+2s}}\, dx\, dy \Big)^{1/2} \\
&\le \Big(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{L_0^2 |v_n(x) - v_n(y)|^2}{|x-y|^{3+2s}}\, dx\, dy \Big)^{1/2} \\
&= L_0 [v_n]_{\mathcal{D}^{s,2}}.
\end{aligned}$$

Therefore, $H(v_n) \in \mathcal{D}^{s,2}(\mathbb{R}^3)$. Moreover, by the definition of H, we know that H is a convex function; then

$$(-\Delta)^s H(v_n) \le H'(v_n)(-\Delta)^s v_n \tag{5.2}$$

in the weak sense. Thus, from $H(v_n) \in \mathcal{D}^{s,2}(\mathbb{R}^3)$ and (5.1) and (5.2), we have

$$\begin{aligned}
\|H(v_n)\|_{2_s^*}^2 &\le C \int_{\mathbb{R}^3} |(-\Delta)^{s/2} H(v_n)|^2\, dx \\
&= C \int_{\mathbb{R}^3} H(v_n)(-\Delta)^s H(v_n)\, dx \\
&\le C \int_{\mathbb{R}^3} H(v_n) H'(v_n)(-\Delta)^s v_n\, dx \\
&= C \int_{\mathbb{R}^3} H(v_n) H'(v_n) h(\varepsilon_n x, v_n)\, dx
\end{aligned}$$

$$\leq C\int_{\mathbb{R}^3} H(v_n)H'(v_n)\,dx + C\int_{\mathbb{R}^3} H(v_n)H'(v_n)v_n^{2_s^*-1}\,dx.$$

Using that $H(v_n)H'(v_n)\leq \beta v_n^{2\beta-1}$ and $v_nH'(v_n)\leq \beta H(v_n)$, we have

$$\Big(\int_{\mathbb{R}^3}\big(H(v_n)\big)^{2_s^*}\,dx\Big)^{2/2_s^*} = C\beta\Big(\int_{\mathbb{R}^3} v_n^{2\beta-1}\,dx + \int_{\mathbb{R}^3}\big(H(v_n)\big)^2 v_n^{2_s^*-2}\,dx\Big), \tag{5.3}$$

where C is a positive constant that does not depend on β. Note that the last integral is well defined for T in the definition of H. Indeed

$$\begin{aligned}\int_{\mathbb{R}^3}\big(H(v_n)\big)^2 v_n^{2_s^*-2}\,dx &= \int_{v_n\leq T}\big(H(v_n)\big)^2 v_n^{2_s^*-2}\,dx + \int_{v_n>T}\big(H(v_n)\big)^2 v_n^{2_s^*-2}\,dx\\ &\leq T^{2\beta-2}\int_{\mathbb{R}^3} v_n^{2_s^*}\,dx + C\int_{\mathbb{R}^3} v_n^{2_s^*}\,dx<\infty.\end{aligned}$$

We choose now β in (5.3) such that $2\beta-1=2_s^*$, and we name it β_1, that is

$$\beta_1 := \frac{2_s^*+1}{2}. \tag{5.4}$$

Let $\hat{R}>0$ to be fixed later. Using last integral in (5.3) and applying the Holder's inequality with exponents $\gamma := \frac{2_s^*}{2}$ and $\gamma' := \frac{2_s^*}{2_s^*-2}$, we have

$$\begin{aligned}&\int_{\mathbb{R}^3}\big(H(v_n)\big)^2 v_n^{2_s^*-2}\,dx\\ &= \int_{v_n\leq \hat{R}}\big(H(v_n)\big)^2 v_n^{2_s^*-2}\,dx + \int_{v_n>\hat{R}}\big(H(v_n)\big)^2 v_n^{2_s^*-2}\,dx\\ &\leq \int_{v_n\leq \hat{R}}\frac{\big(H(v_n)\big)^2}{v_n}\hat{R}^{2_s^*-1}\,dx + \Big(\int_{\mathbb{R}^3}\big(H(v_n)\big)^{2_s^*}\,dx\Big)^{2/2_s^*}\Big(\int_{v_n>\hat{R}} v_n^{2_s^*}\,dx\Big)^{\frac{2_s^*-2}{2_s^*}}.\end{aligned} \tag{5.5}$$

By Lemma 4.8, we know that $\{v_n\}$ has a convergent subsequence in H_0, therefore we can choose $\hat{R}$ large enough so that

$$\Big(\int_{v_n>\hat{R}} v_n^{2_s^*}\,dx\Big)^{\frac{2_s^*-2}{2_s^*}} \leq \frac{1}{2C\beta_1},$$

where C is the constant appearing in (5.3). Therefore, we can absorb the last term in (5.5) by the left hand side of (5.3) to obtain

$$\Big(\int_{\mathbb{R}^3}\big(H(v_n)\big)^{2_s^*}\,dx\Big)^{2/2_s^*} \leq 2C\beta_1\Big(\int_{\mathbb{R}^3} v_n^{2_s^*}\,dx + \hat{R}^{2_s^*-1}\int_{\mathbb{R}^3}\frac{\big(H(v_n)\big)^2}{v_n}\,dx\Big).$$

Now we use that $H(v_n)\leq v_n^{\beta_1}$ and we take $T\to\infty$, to obtain

$$\Big(\int_{\mathbb{R}^3} v_n^{2_s^*\beta_1}\,dx\Big)^{2/2_s^*} \leq 2C\beta_1\Big(\int_{\mathbb{R}^3} v_n^{2_s^*}\,dx + \hat{R}^{2_s^*-1}\int_{\mathbb{R}^3} v_n^{2_s^*}\,dx\Big),$$

and therefore

$$v_n\in L^{2_s^*\beta_1}(\mathbb{R}^3). \tag{5.6}$$

Let us suppose now $\beta>\beta_1$. Thus, using that $H(v_n)\leq v_n^{\beta}$ in the right-hand side of (5.3) and letting $T\to\infty$ we obtain

$$\Big(\int_{\mathbb{R}^3} v_n^{2_s^*\beta}\,dx\Big)^{2/2_s^*} \leq C\beta\Big(\int_{\mathbb{R}^3} v_n^{2\beta-1}\,dx + \hat{R}^{2_s^*-1}\int_{\mathbb{R}^3} v_n^{2\beta+2_s^*-2}\,dx\Big). \tag{5.7}$$

Set $c_0 := \frac{2_s^*(2_s^*-1)}{2(\beta-1)}$ and $c_1 := 2\beta - 1 - c_0$. Note that since $\beta > \beta_1$, we have $0 < c_0 < 2_s^*, c_1 > 0$. Hence, applying Young's inequality with exponents $\gamma := 2_s^*/c_0$ and $\gamma' := 2_s^*/2_s^* - c_0$, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} v_n^{2\beta-1}\,dx &\le \frac{c_0}{2_s^*}\int_{\mathbb{R}^3} v_n^{2_s^*}\,dx + \frac{2_s^*}{2_s^*-c_0}\int_{\mathbb{R}^3} v_n^{\frac{2_s^* c_1}{2_s^*-c_0}}\,dx \\
&\le \int_{\mathbb{R}^3} v_n^{2_s^*}\,dx + \int_{\mathbb{R}^3} v_n^{2\beta+2_s^*-2}\,dx \\
&\le C\big(1 + \int_{\mathbb{R}^3} v_n^{2\beta+2_s^*-2}\,dx\big),
\end{aligned}$$

with $C > 0$ independent of β. Plugging into (5.7),

$$\Big(\int_{\mathbb{R}^3} v_n^{2_s^*\beta}\,dx\Big)^{2/2_s^*} \le C\beta\big(1 + \int_{\mathbb{R}^3} v_n^{2\beta+2_s^*-2}\,dx\big),$$

with C changing from line to line, but remaining independent of β. Therefore,

$$\Big(1 + \int_{\mathbb{R}^3} v_n^{2_s^*\beta}\,dx\Big)^{\frac{1}{2_s^*(\beta-1)}} \le \big(C\beta\big)^{\frac{1}{2(\beta-1)}}\Big(1 + \int_{\mathbb{R}^3} v_n^{2\beta+2_s^*-2}\,dx\Big)^{\frac{1}{2(\beta-1)}}. \tag{5.8}$$

Repeating this argument we define a sequence $\beta_m, m \ge 1$ such that

$$2\beta_{m+1} + 2_s^* - 2 = 2_s^*\beta_m.$$

Thus,

$$\beta_{m+1} - 1 = \big(\frac{2_s^*}{2}\big)^m(\beta_1 - 1).$$

Replacing this in (5.8) one has

$$\Big(1 + \int_{\mathbb{R}^3} v_n^{2_s^*\beta_{m+1}}\,dx\Big)^{\frac{1}{2_s^*(\beta_{m+1}-1)}} \le \big(C\beta_{m+1}\big)^{\frac{1}{2(\beta_{m+1}-1)}}\Big(1 + \int_{\mathbb{R}^3} v_n^{2_s^*\beta_m}\,dx\Big)^{\frac{1}{2_s^*(\beta_m-1)}}.$$

Defining $C_{m+1} := C\beta_{m+1}$ and

$$A_m := \Big(1 + \int_{\mathbb{R}^3} v_n^{2_s^*\beta_m}\,dx\Big)^{\frac{1}{2_s^*(\beta_m-1)}},$$

we conclude that there exists a constant $C_0 > 0$ independent of m, such that

$$A_m \le \prod_{k=1}^m C_k^{\frac{1}{2(\beta_k-1)}} A_1 \le C_0 A_1.$$

Thus,

$$\|v_n\|_\infty \le C_0 A_1 < \infty, \tag{5.9}$$

uniformly in $n \in \mathbb{N}$, thanks to (5.6). This completes the proof. □

Lemma 5.2. *As $|x| \to \infty$, $u_n(x) \to 0$ uniformly in n.*

Proof. Note that u_n satisfies

$$(-\Delta)^s u_n + u_n = \Upsilon_n,\ x \in \mathbb{R}^3,$$

where

$$\Upsilon_n(x) = u_n(x) - V(\varepsilon_n(x+\tilde{y}_n))u_n(x) - \phi_{u_n}^t u_n(x) + g(\varepsilon_n(x+\tilde{y}_n), u_n), \quad x \in \mathbb{R}^3.$$

Putting $\Upsilon(x) = u(x) - V(y_0)u(x) - \phi_u^t u(x) + g(y_0, u)$, by Lemma 4.7, we see that

$$\Upsilon_n \to \Upsilon \text{ in } L^q(\mathbb{R}^3),\ \forall\, q \in [2, +\infty),$$

and there exists a $C > 0$ such that

$$\|\Upsilon_n\|_{L^\infty(\mathbb{R}^3)} \le C, \ \forall\, n \in \mathbb{N}.$$

From [15], we have that

$$u_n(x) = \mathcal{G} * \Upsilon_n = \int_{\mathbb{R}^3} \mathcal{G}(x-y)\Upsilon_n(y)\, dy,$$

where $\mathcal{G}$ is the Bessel Kernel

$$\mathcal{G}(x) = \mathcal{F}^{-1}\Big(\frac{1}{1+|\xi|^{2s}}\Big).$$

From [15, Theorem 3.3] it is known that $\mathcal{G}$ is positive, radially symmetric and smooth in $\mathbb{R}^3\backslash\{0\}$, there is $C > 0$ such that $\mathcal{G}(x) \le C/|x|^{3+2s}$, and $\mathcal{G} \in L^q(\mathbb{R}^3)$ for all $q \in [1, \frac{3}{3-2s})$. Now arguing as in the proof of [1, Lemma 2.6], we conclude that

$$u_n(x) \to 0 \text{ as } |x| \to \infty, \tag{5.10}$$

uniformly in $n \in \mathbb{N}$. □

We are now ready to prove the main result of this article.

Proof of Theorem 1.2. We fix a small $\delta > 0$ such that $\mathcal{M}_\delta \subset \Omega$. We first claim that there exists some $\tilde{\varepsilon}_\delta > 0$ such that for any $\varepsilon \in (0, \tilde{\varepsilon}_\delta)$ and any solution $u_\varepsilon \in \tilde{\mathcal{N}}_\varepsilon$ of the problem (3.5), it holds

$$\|u_\varepsilon\|_{L^\infty(\mathbb{R}^3\backslash\Omega_\varepsilon)} < a. \tag{5.11}$$

To prove the claim we argue by contradiction. Suppose that for some sequence $\varepsilon_n \to 0^+$ we can obtain $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$ such that $I'_{\varepsilon_n}(u_n) = 0$ and

$$\|u_n\|_{L^\infty(\mathbb{R}^3\backslash\Omega_\varepsilon)} \ge a. \tag{5.12}$$

As in the proof of Lemma 4.8, we have that $I_{\varepsilon_n}(u_n) \to c_{V_0}$ and we can obtain a sequence $\{\tilde{y}_n\} \in \mathbb{R}^3$ such that $\varepsilon_n \tilde{y}_n \to y_0 \in \mathcal{M}$.

If we take $r > 0$ such that $B_r(y_0) \subset B_{2r}(y_0) \subset \Omega$ we have

$$B_{\frac{r}{\varepsilon_n}}\Big(\frac{y_0}{\varepsilon_n}\Big) = \frac{1}{\varepsilon_n} B_r(y_0) \subset \Omega_{\varepsilon_n}.$$

Moreover, for any $z \in B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n)$, it holds

$$|z - \frac{y_0}{\varepsilon_n}| \le ||z - \tilde{y}_n| + |\tilde{y}_n - \frac{y_0}{\varepsilon_n}| < \frac{1}{\varepsilon_n}(r + o_n(1)) < \frac{2r}{\varepsilon_n},$$

for n large. For these values of n we have that $B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n) \subset \Omega_{\varepsilon_n}$; that is, $\mathbb{R}^3\backslash\Omega_{\varepsilon_n} \subset \mathbb{R}^3\backslash B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n)$. On the other hand, it follows from Lemma 5.2 that there is $R > 0$ such that $u_n(x) < a$ for $|x| \ge R$ and all $n \in \mathbb{N}$. Then it follows that

$$v_n(x - \tilde{y}_n) < a \quad \text{for } x \in B_R^c(\tilde{y}_n),\ n \in \mathbb{N}.$$

Thus, there exists $n_0 \in \mathbb{N}$ such that for any $n \ge n_0$ and $\frac{r}{\varepsilon_n} > R$, it holds

$$\mathbb{R}^3\backslash\Omega_{\varepsilon_n} \subset \mathbb{R}^3\backslash B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n) \subset \mathbb{R}^3\backslash B_R(\tilde{y}_n).$$

Then, it holds

$$u_n(x) < a \ \ \forall x \in \mathbb{R}^3\backslash\Omega_{\varepsilon_n},$$

which contradicts (5.12) and the claim holds.

Let $\hat{\varepsilon}_\delta$ be given by Theorem 4.10, and let $\varepsilon_\delta := \min\{\hat{\varepsilon}_\delta, \tilde{\varepsilon}_\delta\}$. We will prove the theorem for this choice of ε_δ. Let $\varepsilon \in (0, \varepsilon_\delta)$ be fixed. By using Theorem 4.10 we obtain $\operatorname{cat}_{\mathcal{M}_\delta}(\mathcal{M})$ nontrivial solutions of problem (3.5). If $u \in H_\varepsilon$ is one of these solutions, we have that $u \in \tilde{\mathcal{N}}_\varepsilon$, and we can use (5.11) and the definition of g to conclude that $g(\cdot, u) = f(u)$. Hence, u is also a solution of (2.1). An easy calculation shows that $\omega(x) = u(\frac{x}{\varepsilon})$ is a solution of the original problem (1.3). Then (1.3) has at least $\mathrm{cat}_{\mathcal{M}_\delta}(\mathcal{M})$ nontrivial solutions.

Now we consider $\varepsilon_n \to 0^+$ and take a sequence $u_n \in H_{\varepsilon_n}$ of solutions of (3.5) as above. To study the behavior of the maximum points of u_n, we first notice that, by the definition of g and (3.1), (3.2), there exists $0 < \gamma < a$ such that

$$g(\varepsilon_n x, u)u \le \frac{V_0}{K}u^2, \quad \text{for all } x \in \mathbb{R}^3,\ u \le \gamma. \tag{5.13}$$

Using a similar discussion as above, we obtain $R > 0$ and $\{\tilde{y}_n\} \subset \mathbb{R}^3$ such that

$$\|u_n\|_{L^\infty(B_R^c(\tilde{y}_n))} < \gamma. \tag{5.14}$$

Up to a subsequence, we may assume that

$$\|u_n\|_{L^\infty(B_R(\tilde{y}_n))} \ge \gamma. \tag{5.15}$$

Indeed, if this is not the case, we have $\|u_n\|_{L^\infty} < \gamma$, and therefore from $I'_{\varepsilon_n}(u_n) = 0$ and (5.13) it follows that

$$\|u_n\|_{\varepsilon_n}^2 \le \int_{\mathbb{R}^3} g(\varepsilon_n x, u_n)u_n\,dx \le \frac{V_0}{K}\int_{\mathbb{R}^3} u_n^2\,dx. \tag{5.16}$$

The above expression implies that $\|u_n\|_{\varepsilon_n} \to 0$ as $n \to \infty$, which leads to a contradiction. Thus, (5.15) holds.

By using (5.14) and (5.15) we conclude that the maximal points $p_n \in \mathbb{R}^3$ of u_n belong to $B_R(\tilde{y}_n)$. Hence, $p_n = \tilde{y}_n + q_n$ for some $q_n \in B_R(0)$. Recalling that the associated solution of (1.3) is of the form $\omega_n(x) = u_n(\frac{x}{\varepsilon})$, we conclude that the maximum point η_ε of v_n is $\eta_\varepsilon := \varepsilon_n\tilde{y}_n + \varepsilon_n q_n$. Since $\{q_n\} \subset B_R(0)$ is bounded and $\varepsilon_n\tilde{y}_n \to y_0 \in \mathcal{M}$, we obtain

$$\lim_{n\to\infty} V(\eta_\varepsilon) = V(y_0) = V_0.$$

Next we estimate the decay properties of ω_n. For this, we first claim that there exist $C > 0$ such that

$$u_n(x) \le \frac{C}{1 + |x|^{3+2s}}, \quad \forall x \in \mathbb{R}^3.$$

Indeed, we can use [15, Lemma 4.2], according to which there exists a continuous function $\bar{\omega}$ such that

$$0 < \bar{\omega}(x) \le \frac{C}{1 + |x|^{3+2s}}, \tag{5.17}$$

$$(-\Delta)^s\bar{\omega} + \frac{V_0}{2}\bar{\omega} = 0 \quad \text{in } \mathbb{R}^3\backslash B_{\bar{R}}(0) \tag{5.18}$$

for some suitable $\bar{R} > 0$. Thanks to (5.10) we have that $u_n \to 0$ as $|x| \to \infty$ uniformly in n. Therefore, for some large $R_1 > 0$, we obtain

$$\begin{aligned}
(-\Delta)^s u_n + \frac{V_0}{2} u_n &= (-\Delta)^s u_n + V(\varepsilon_n(x+\tilde{y}_n))u_n - \big(V(\varepsilon_n(x+\tilde{y}_n)) - \frac{V_0}{2}\big)u_n \\
&= -\phi^s_{v_n} u_n + g(\varepsilon_n(x+\tilde{y}_n), u_n) - \big(V(\varepsilon_n(x+y_n)) - \frac{V_0}{2}\big)u_n \\
&\le g(\varepsilon_n(x+\tilde{y}_n), u_n) - \frac{V_0}{2} u_n \\
&\le \frac{V_0}{K} u_n - \frac{V_0}{2} u_n \le 0,
\end{aligned} \tag{5.19}$$

for $x \in \mathbb{R}^3 \backslash B_{R_1}(0)$. Now we take $R_2 := \max\{\bar{R}, R_1\}$ and set

$$b := \inf_{B_{R_2}(0)} \bar{\omega} > 0,\ z_n := (m+1)\bar{\omega} - b v_n, \tag{5.20}$$

where $m := \sup_{n\in\mathbb{N}} \|u_n\|_{L^\infty} < \infty$. We next show that $z_n \ge 0$ in $\mathbb{R}^3$. For this we suppose by contradiction that there is a sequence $\{x_n^j\}$ such that

$$\inf_{x\in\mathbb{R}^3} z_n(x) = \lim_{j\to\infty} z_n(x_n^j) < 0, \tag{5.21}$$

Notice that

$$\lim_{|x|\to\infty} \bar{\omega}(x) = \lim_{|x|\to\infty} u_n(x) = 0,$$

and so

$$\lim_{|x|\to\infty} z_n(x) = 0,$$

uniformly in $n \in \mathbb{N}$. Consequently, the sequence $\{x_n^j\}$ is bounded and therefore, up to a subsequence, we may suppose that $x_n^j \to x_n^*$ as $j \to \infty$ for some $x_n^* \in \mathbb{R}^3$. Hence (5.21) becomes

$$\inf_{x\in\mathbb{R}^3} z_n(x) = z_n(x_n^*) < 0. \tag{5.22}$$

From the minimality property of x_n^*, we have

$$(-\Delta)^s z_n(x_n^*) = -\frac{C(s)}{2} \int_{\mathbb{R}^3} \frac{z_n(x_n^*+y) + z_n(x_n^*-y) - 2z_n(x_n^*)}{|y|^{3+2s}}\,dy \le 0. \tag{5.23}$$

By (5.20), we obtain

$$z_n(x) \ge mb + \bar{\omega} - mb > 0,\ \text{in } B_{R_2}(0).$$

Therefore, combining this with (5.22), we see that

$$x_n^* \in \mathbb{R}^3 \setminus B_{R_2}(0). \tag{5.24}$$

From (5.17)-(5.19), we conclude that

$$(-\Delta)^s z_n + \frac{V_0}{2} z_n \ge 0, \quad \text{in } \mathbb{R}^3 \setminus B_{R_2}(0). \tag{5.25}$$

Thinks to (5.24), we can evaluate (5.25) at the point x_n^*, and recalling (5.22), (5.23), we conclude that

$$0 \le (-\Delta)^s z_n(x_n^*) + \frac{V_0}{2} z_n(x_n^*) < 0,$$

this is a contradiction, so $z_n(x) \geq 0$ in $\mathbb{R}^3$. That is to say, $u_n \leq (m+1)b^{-1}\bar{\omega}$ with (5.17) imply that

$$u_n(x) \leq \frac{C}{1+|x|^{3+2s}}, \ \forall x \in \mathbb{R}^3.$$

Therefore,

$$\begin{aligned}
\omega_n(x) &= u_n(\frac{x}{\varepsilon_n} - \tilde{y}_n) \\
&\leq \frac{C}{1+|\frac{x}{\varepsilon_n} - \tilde{y}_n|^{3+2s}} \\
&= \frac{C\varepsilon_n^{3+2s}}{\varepsilon_n^{3+2s} + |x - \varepsilon_n \tilde{y}_n|^{3+2s}} \\
&= \frac{C\varepsilon_n^{3+2s}}{\varepsilon_n^{3+2s} + |x - \eta_{\varepsilon_n}|^{3+2s}}, \ \forall \ x \in \mathbb{R}^3.
\end{aligned}$$

Thus, the proof is complete. □

Acknowledgments. This research was supported by the National Natural Science Foundation of China (No. 11771385). We would like to thank the anonymous referee for his/her careful readings of our manuscript and the useful comments made for its improvement.

References

[1] C. Alves, O.H. Miyagaki; *Existence and concentration of solution for a class of fractional elliptic equation in $\mathbb{R}^N$ via penalization method,* Calc. Var. Partial Differential Equations, 55 (2016), 1-19.

[2] A. Ambrosetti; *On Schrödinger-Poisson systems,* Milan J. Math., 76 (2008), 257–274.

[3] A. Azzollini; *Concentration and compactness in nonlinear Schrödinger-Poisson system with a general nonlinearity,* J. Differential Equations, 249 (2010), 1746–1763.

[4] A. Azzollini, P. d'Avenia, A. Pomponio; *On the Schrödinger-Maxwell equations under the effect of a general nonlinear termk,* Ann. Inst. H. Poincaré Anal. Non Linéaire, 27 (2010), 779–791.

[5] V. Benci, G. Cerami; *Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology,* Calc. Var. Partial Differential Equations, 2 (1994), 29–48.

[6] V. Benci, D. Fortunato; *An eigenvalue problem for the Schrödinger-Maxwell equations,* Topol. Methods Nonlinear Anal., 11 (1998), 283–293.

[7] H. Brézis, E. Lieb; *A relation between pointwise convergence of functions and convergence of functionals,* Proc. Amer. Math. Soc., 88 (1983), 486–490.

[8] C. Bucur, E. Valdinoci; *Nonlocal diffusion and applications*, volume 20 of Lecture Notes of the Unione Matematica Italiana, Springer; Unione Matematica Italiana, Bologna, 2016.

[9] G. Cerami, G. Vaira; *Positive solutions for some non-autonomous Schrödinger-Poisson systems,* J. Differential Equations, 248 (2010). 521–543.

[10] M. del Pino, P. L. Felmer; *Local mountain passes for semilinear elliptic problems in unbounded domains,* Calc. Var. Partial Differential Equations, 4 (1996), 121–137.

[11] E. Di Nezza, G. Palatucci, E. Valdinoci; *Hitchhiker's guide to the fractional Sobolev spaces,* Bull. Sci. Math., 136 (2012), 521–573.

[12] S. Dipierro, M. Medina, E. Valdinoci; *Fractional elliptic problems with critical growth in the whole of $\mathbb{R}^n$*, volume 15 of Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)], Edizioni della Normale, Pisa, 2017.

[13] I. Ekeland; *On the variational principle,* J. Math. Anal. Appl., 47 (1974), 324–353.

[14] H. Fan; *Multiple positive solutions for Schrödinger-Poisson systems involving concave-convex nonlinearities,* Electron. J. Differential Equations, 2019 no. 86 (2019), 1-19.

[15] P. Felmer, A. Quaas, J. Tan; *Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian,* Proc. Roy. Soc. Edinburgh Sect. A, 142 (2012), 1237–1262.

[16] X. He, W. Zou; *Existence and concentration result for the fractional Schrödinger equations with critical nonlinearities,* Calc. Var. Partial Differential Equations, 55 (2016), 1–39.
[17] Y. Jiang, H. Zhou; *Schrödinger-Poisson system with steep potential well,* J. Differential Equations, 251 (2011), 582–608.
[18] T. Jin, Y. Li, J. Xiong; *On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions,* J. Eur. Math. Soc., 16 (2014), 1111–1171.
[19] T. Jin and Z. Yang; *The fractional Schrödinger-Poisson systems with infinitely many solutions,* J. Korean Math. Soc., 57 (2020), 489–506.
[20] N. Landkof; *Foundations of modern potential theory*, Springer-Verlag, New York-Heidelberg, 1972.
[21] N. Laskin; *Fractional quantum mechanics and Lévy path integrals,* Phys. Lett. A, 268 (2000), 298–305.
[22] N. Laskin; *Fractional Schrödinger equation,* Phys. Rev. E (3), 66 (2002), 056108.
[23] P.-L. Lions; *Solutions of Hartree-Fock equations for Coulomb systems,* Comm. Math. Phys., 109 (1987), 33–97.
[24] W. Liu; *Existence of multi-bump solutions for the fractional Schrödinger-Poisson system,* J. Math. Phys., 57 (2016),091502.
[25] Z. Liu, J. Zhang; *Multiplicity and concentration of positive solutions for the fractional Schrödinger-Poisson systems with critical growth.* ESAIM Control Optim. Calc. Var. , 23 (2017), 1515–1542.
[26] J. Moser; *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations,* Comm. Pure Appl. Math., 13 (1960), 457–468.
[27] E. G. Murcia, G. Siciliano; *Positive semiclassical states for a fractional Schrödinger-Poisson system,* Differential Integral Equations, 30 (2017), 231–258.
[28] G. Palatucci, A. Pisante; *Improved Sobolev embeddings, profile decomposition, and concentration-compactness for fractional Sobolev spaces,* Calc. Var. Partial Differential Equations, 50 (2014), 799–829.
[29] P. H. Rabinowitz; *On a class of nonlinear Schrödinger equations,* Z. Angew. Math. Phys., 43(1992), 270–291.
[30] S. Secchi; *Ground state solutions for nonlinear fractional Schrödinger equations in $\mathbb{R}^N$,* J. Math. Phys., 54 (2013), 1–17.
[31] M. Struwe; *Variational methods.* Springer-Verlag, Berlin, 1990.
[32] A. Szulkin, T. Weth; *Ground state solutions for some indefinite variational problems,* J. Funct. Anal., 257 (2009), 3802–3822.
[33] A. Szulkin, T. Weth; *The method of Nehari manifold.* Int. Press, Somerville, MA, 2010.
[34] K. Teng; *Existence of ground state solutions for the nonlinear fractional Schrödinger-Poisson system with critical Sobolev exponent,* J. Differential Equations, 261(2016), 3061–3106.
[35] W. Wang, Q. Li; *Existence and concentration of positive ground states for Schrödinger-Poisson equations with competing potential functions,* Electron. J. Differential Equations, 2020, no. 78 (2020), 1-19.
[36] M. Willem; *Minimax theorems*, Birkhäuser Boston, Inc., Boston, MA, 1996.
[37] Z. Yang, Y. Yu, F. Zhao; *The concentration behavior of ground state solutions for a critical fractional Schrödinger-Poisson system,* Math. Nachr., 292 (2019), 1837–1868.
[38] Z. Yang, Y. Yu, F. Zhao; *Concentration behavior of ground state solutions for a fractional Schrödinger-Poisson system involving critical exponent,* Commun. Contemp. Math., 21 (2019),1–46
[39] Y. Yu, F. Zhao, L. Zhao; *The concentration behavior of ground state solutions for a fractional Schrödinger-Poisson system,* Calc. Var. Partial Differential Equations, 56 (2017), 1-25.
[40] J. Zhang, J. M. do ó, M. Squassina; *Fractional Schrödinger-Poisson systems with a general subcritical or critical nonlinearity,* Adv. Nonlinear Stud., 16(2016), 15–30.
[41] L. Zhao, H. Liu, F. Zhao; *Existence and concentration of solutions for the Schrödinger-Poisson equations with steep well potential,* J. Differential Equations, 255 (2013), 1–23.

Zhipeng Yang
Department of Mathematics, Yunnan Normal University, Kunming 650500, China
Email address: yangzhipeng326@163.com

Wei Zhang
School of Statistics and Mathematics, Yunnan University of Finance and Economics, Kunming 650221, China
Email address: weizyn@163.com

Fukun Zhao (corresponding author)
Department of Mathematics, Yunnan Normal University, Kunming 650500, China
Email address: fukunzhao@163.com