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HÉNON EQUATION WITH NOLINEARITIES INVOLVING SOBOLEV CRITICAL GROWTH IN $H^1_{0,\mathrm{rad}}(B_1)$

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Abstract. In this article we study the Hénon equation

$$\begin{gathered} -\Delta u = \lambda |x|^{\mu} u + |x|^{\alpha} |u|^{2^*_\alpha - 2} u \quad \text{in } B_1, \\ u = 0 \quad \text{on } \partial B_1, \end{gathered}$$

where B_1 is the ball centered at the origin of $\mathbb{R}^N$ ($N \geq 3$) and $\mu \geq \alpha \geq 0$. Under appropriate hypotheses on the constant λ, we prove existence of at least one radial solution using variational methods.

1. Introduction

In this article we search for a non-trivial radially symmetric solution to the Hénon-type Dirichlet problem

$$\begin{gathered} -\Delta u = \lambda |x|^{\mu} u + |x|^{\alpha} |u|^{2^*_\alpha - 2} u \quad \text{in } B_1, \\ u = 0 \quad \text{on } \partial B_1, \end{gathered} \tag{1.1}$$

where $\lambda > 0$, $\mu \geq \alpha \geq 0$, B_1 is a unity ball centered at the origin of $\mathbb{R}^N$ ($N \geq 3$), and $2^*_\alpha = \frac{2(N+\alpha)}{N-2}$.

When $\alpha = \mu = 0$, the pioneering work is due to Brézis and Nirenberg in [9], where they obtained a λ_1 and positive solutions when $\lambda < \lambda_1$. We refer the reader to the book [39] for a survey about this subject. Devillanova and Solimini [24] proved multiplicity results for $N \geq 7$, for all $\lambda > 0$. Then in [25], they complemented the former result for $N \geq 4$, but for $\lambda \in (0, \lambda_1)$. Clapp and Weth [20] extended the above results for $N \geq 4$, for all $\lambda > 0$, getting lower estimates for the number of solutions. Chen, Shioji and Zou [18] obtained a ground state solution and multiplicity results, and improved results in [20]. The existence is proved in [15], for all $\lambda > 0$ and $N \geq 5$, and when $N = 4$ only for $\lambda \neq \lambda_k$, where λ_k is eigenvalue of $(-\Delta)$. In [17] some multiplicity results were obtained for λ near λ_k. These existence results were improved in [26]. For a version of these results in the quasilinear see [21, 1].

When $\alpha, \mu > 0$, these problems are called Hénon type problems. Actually, Hénon [28] introduced problem (1.1) with $\lambda = 0$, as a model of clusters of stars for the case $N = 1$. Since then, many authors have worked with this type of

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the equations from several points of view. The pioneering paper is due to Ni [32]; he established the compact embedding $H^1_{0,\mathrm{rad}}(B_1) \subset L^p(B_1, |x|^\alpha)$ for all $p \in [1, 2^*_\alpha)$, where $2^*_\alpha = \frac{2(N+\alpha)}{N-2}$. This was used for obtaining radial solutions. Here $H^1_{0,\mathrm{rad}}(B_1) = \{u \in H^1_0(B_1) : u \text{ is radial, that is, } u(x) = u(|x|), \forall x \in B_1\}$. This result was extended to more general quasilinear operators in [21]. In the case $\lambda = 0$, Badiale and Serra [2] obtained multiplicity results for non-radial solutions (see [16] for some extensions). For ground state profile (when the solutions that concentrate at a boundary point of B_1 as $\alpha \to \infty$) and when the growth approaches to the usual Sobolev critical exponent, see [10, 11, 13, 14, 30, 34, 38], and references therein. For Hénon problems involving the usual Sobolev exponents we cite [31, 29, 35, 36] and their references. Up to our knowledge, there are only a few works treating problem (1.1) with $\lambda \neq 0$ involving the Sobolev critical exponent given by Ni, 2^*_α. Nonhomogeneous perturbations are studied in [3], when $\lambda > 0$ and smaller than the first eigenvalue. While some concentration phenomena for linear perturbation is studied in [27] when λ is small enough. Long and Yang [31] established the existence of nontrivial solutions for (1.1) with $\mu = 0$, when $\lambda \neq \lambda_k$, for all k, and $N \geq 7$. Also, they proved that $(\lambda_k, 0)$ is a bifurcation point of problem (1.1), for all k. The aim of this article is to extend above results, for instance, treating all λ positive.

To establish our results, we need to know the spectrum of the problem

$$\begin{gathered} -\Delta u = \lambda |x|^\mu u \quad \text{in } B_1; \\ u = 0 \quad \text{on } \partial B_1. \end{gathered} \tag{1.2}$$

Note that $H^1_{0,\mathrm{rad}}(B_1)$ is a Hilbert space, which is compactly embedded in $L^p(B_1, |x|^\mu)$, for all $p \in (1, 2^*_\mu)$ (see [32]). Arguing as in [22, 4], we can show that there exists a sequence of eigenvalues for (1.2), with

$$\lambda^*_1 \leq \lambda^*_2 \leq \lambda^*_3 \leq \dots \leq \lambda^*_k \leq \dots, \quad \lambda^*_k \to +\infty, \quad \text{as } k \to \infty.$$

The eigenvalues are characterized by

$$\lambda^*_1 = \min_{u \in H^1_{0,\mathrm{rad}}(B_1)\setminus\{0\}} \frac{\int_{B_1} |\nabla u|^2 dx}{\int_{B_1} |x|^\mu |u|^2 dx}, \quad \lambda^*_{k+1} = \min_{u \in \mathbb{P}_{k+1}\setminus\{0\}} \frac{\int_{B_1} |\nabla u|^2 \,\mathrm{d}x}{\int_{B_1} |x|^\mu |u|^2 \,\mathrm{d}x}, \tag{1.3}$$

where

$$\mathbb{P}_{k+1} = \big\{u \in H^1_{0,\mathrm{rad}}(B_1) : \langle u, e_j \rangle = \int_{B_1} \nabla u \nabla e_j \,\mathrm{d}x = 0,\ j = 1, 2, \dots, k\big\}, \tag{1.4}$$

and e_k denotes the eigenfunction associated with the eigenvalue λ^*_k. Also from [22], we know that $e_1 > 0$, and that e_j for $j \neq 1$ changes sign.

The results below follow from the linear theory, which are obtained by adapting the ideas in [7] or [37, Appendix A]):

(1) each λ^*_k has finite multiplicity,
(2) $e_k \in C^{0,\sigma}(\overline{B_1})$ for some $\sigma \in (0, 1)$;
(3) the sequence $\{e_k\}$ is an orthonormal basis in $L^2(B_1, |x|^\mu)$ and orthogonal in $H^1_{0,\mathrm{rad}}(B_1)$.

For a fix $k \in \mathbb{N}$ we can assume $\lambda^*_k < \lambda^*_{k+1}$, otherwise we can assume that λ^*_k has multiplicity $p \in \mathbb{N}$; that is,

$$\lambda^*_{k-1} < \lambda^*_k = \lambda^*_{k+1} = \ldots = \lambda^*_{k+p-1} < \lambda^*_{k+p},$$

and we denote $\lambda^*_{k+p} = \lambda^*_{k+1}$.

The proofs of our results are based on variational methods. To ensure that the considered minimax levels lie in a suitable range, we use approximating functions that are constructed from Talenti functions (Hénon version). When we work with nonlinearities involving Sobolov critical growth, it is common to follow the Brézis-Nirenberg approach to estimate the minimax levels with the help of the Talenti functions,

$$U_\epsilon(x) = \Big[\frac{N(N-2)\epsilon}{\epsilon + |x|^2}\Big]^{(N-2)/4} \tag{1.5}$$

which are solutions of the problem

$$\begin{gathered} -\Delta u = |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N; \\ u(x) \to 0 \quad \text{as } |x| \to \infty. \end{gathered}$$

It is well-know that they yield the best Sobolev embedding constant constant for $H^1(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N)$, given by

$$S = \inf_{u \in H_0^1(B_1), u \neq 0} \frac{\|u\|^2}{\|u\|_{2^*}^2}.$$

Using U_ϵ one can prove that the minimax level of the functional associated with problems with critical growth belongs to the interval where the Palais-Smale compactness condition holds.

When searching for solutions to Hénon type equations in $H^1_{0,\mathrm{rad}}(B_1)$, we note that the weight $|x|^\alpha$ modifies the critical exponent, it becomes $2^*_\alpha \geq 2^*$ for $\alpha \geq 0$. Consequently, we need to invoke a different family of functions adapted for the radial context. More precisely, since we are searching for radial solutions for (1.1) with critical growth, we let S_α be the best constant for the Sobolev-Hardy embedding

$$H^1_{0,\mathrm{rad}}(\mathbb{R}^N) \to L^{2^*_\alpha}(\mathbb{R}^N, |x|^\alpha).$$

The constant is

$$S_\alpha = \inf_{u \in H^1_{0,\mathrm{rad}}(B_1),\, u \neq 0} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x}{\big(\int_{\mathbb{R}^N} |x|^\alpha |u|^{2^*_\alpha} \,\mathrm{d}x\big)^{2/2^*_\alpha}} \tag{1.6}$$

which is achieved by the family of functions

$$u_\epsilon(x) = \frac{[(N+\alpha)(N-2)\epsilon]^{(N-2)/2(2+\alpha)}}{(\epsilon + |x|^{2+\alpha})^{(N-2)/(2+\alpha)}} \tag{1.7}$$

defined for $\epsilon > 0$. Indeed, these functions are minimizers of S_α in the set of radial functions in the case $\alpha > -2$. Furthermore, the u_ϵs are the positive radial solutions of

$$\begin{gathered} -\Delta u = |x|^\alpha |u|^{2^*_\alpha - 2}u \quad \text{in } \mathbb{R}^N; \\ u(x) \to 0 \quad \text{as } |x| \to \infty. \end{gathered} \tag{1.8}$$

For details and more general results, see [3, 12, 19, 21, 32].

1.1. **Statement of main results.** We present our results in three theorems. The first theorem deals with the non-trivial solution of problem (1.1) when $\lambda > 0$ and $N > 4 + \mu$. The possibility of resonance is also considered in this case. The second theorem also concerns the non-trivial solution, when the working dimension is $4 + \mu$; in this case we need to consider $\lambda \neq \lambda_j^*$ for $j \in \mathbb{N} = \{1, 2, 3, \dots\}$. In the third theorem considers non-trivial solutions when $N < 4 + \mu$. To recover the compactness of the functional associated with problem (1.1), we need λ large, with $\lambda \neq \lambda_j^*$.

Theorem 1.1. *For $0 < \lambda < \lambda_1^*$ or $\lambda_k^* \leq \lambda < \lambda_{k+1}^*$, problem (1.1) possesses a non-trivial radial solution when*
$$N > \frac{\mu - \alpha}{2} + 2 + (2+\mu)\sqrt{2}. \tag{1.9}$$

Theorem 1.2. *For $0 < \lambda < \lambda_1^*$ or $\lambda_k^* < \lambda < \lambda_{k+1}^*$, problem (1.1) possesses a non-trivial radial solution when $N = 4 + \mu$.*

Theorem 1.3. *For $\lambda > 0$ sufficiently large and $\lambda \neq \lambda_j^*$, for $j \in \mathbb{N}$, problem (1.1) possesses a non-trivial radial solution when $N < 4 + \mu$.*

Remark 1.4. Observe that (1.9) implies $N > 4 + \mu$. In this sense, Theorem 1.1 provides a partial answer to the question about existence of nontrivial radial solutions when $N > 4 + \mu$.

In [3], it was proved that the non-trivial solution of (1.1) is positive when $0 < \lambda < \lambda_1^*$.

This article is organized as follows. In Section 2, we introduce the variational framework, prove the boundedness of Palais-Smale sequences of the functional associated with problem (1.1). Since we search for a radial solutions for a problem with critical Sobolev growth nonlinearity, we show the minimax levels are bounded by constants depending on N, α and S_α. In Section 3, we obtain the geometric conditions on the functional for proving the existence of solutions to (1.1). In Section 4, following [15], we obtain estimates for recovering the compactness of the functional associated with problem (1.1). In Section 5, we prove our main results.

2. Variational formulation

Given a real Banach space E and a functional Φ of class C^1 on E, by definition Φ satisfies Palais-Smale condition at level $c \in \mathbb{R}$ (denoted $(PS)_c$) if every sequence (u_j) in E such that
$$\Phi(u_j) \to c \quad \text{and} \quad \Phi'(u_j) \to 0 \quad \text{in } E^* \tag{2.1}$$
has a convergent subsequence. Such a sequence is called a (PS) sequence (at level c). We shall use the following version of a well-known critical-point theorem (see [5]).

Theorem 2.1. *Let H be a real Hilbert space and $f \in C^1(H, \mathbb{R})$ be a functional satisfying the following assumptions:*

1. *$f(u) = f(-u)$, $f(0) = 0$ for any $u \in H$;*
2. *there exists $\beta > 0$ such that f satisfies $(PS)_c$ for $c \in (0, \beta)$;*
3. *there exist two closed subspaces $V, W \subset H$ and positive constants ρ, δ with $\delta < \beta$ such that*
   - (i) *$f(u) < \beta$ for any $u \in W$;*
   - (ii) *$f(u) \geq \delta$ for any $u \in V$, $\|u\| = \rho$;*
   - (iii) *$\operatorname{codim} V < \infty$.*

Then there exist at least m pairs of critical points, where
$$m = \dim(V \cap W) - \operatorname{codim}(V + W).$$

We consider $H^1_{0,\mathrm{rad}}(B_1)$, with the norm

$$\|u\| = \Big(\int_{B_1} |\nabla u|^2 \,\mathrm{d}x\Big)^{1/2}.$$

The subspace of functions in B_1 with weight $|x|^\mu$ and $\mu \geq 0$ is denoted by $L^z(B_1, |x|^\mu)$, and it is endowed the norm

$$\|u\|_{z,|x|^\mu} = \Big(\int_{B_1} |x|^\mu |u|^z \,\mathrm{d}x\Big)^{1/z}.$$

For finding (weak) solutions of (1.1) we look for critical points of the functional $J_\lambda : H^1_{0,\mathrm{rad}}(B_1) \to \mathbb{R}$ defined as

$$J_\lambda(v) = \frac{1}{2}\int_{B_1} (|\nabla v|^2 - \lambda |x|^\mu v^2)\,\mathrm{d}x - \frac{1}{2^*_\alpha}\int_{B_1} |x|^\alpha |v|^{2^*_\alpha}\,\mathrm{d}x.$$

We do not apply the standard variational arguments because the embedding of $H^1_{0,\mathrm{rad}}(B_1)$ in $L^{2^*_\alpha}(B_1, |x|^\alpha)$ is not compact, and that the functional J_λ does not satisfy the Palais-Smale condition. We need to adapt an idea introduced by Brézis and Nirenberg [9] and Secchi [35]. This idea was used for the Talenti functions (1.5) for proving that a functional associated with a problem with critical Sobolev growth nonlinearity satisfies the PS-condition in the interval $(0, S^{N/2}/N)$.

Here, in the radial context for a Hénon type equation, we construct minimax levels for the functional J_λ which lie in the interval

$$\Big(0, \frac{2+\alpha}{2(N+\alpha)} S_\alpha^{(N+\alpha)/(2+\alpha)}\Big).$$

For this purpose, we use that positive solutions (1.7) of (1.8) yield the constant S_α in the embedding of $H^1_{0,\mathrm{rad}}(\mathbb{R}^N)$ in $L^{2^*_\alpha}(\mathbb{R}^N, |x|^\alpha)$.

2.1. **Palais-Smale sequences.** Recall that the proof of the Palais-Smale condition for the functional associated with Problem (1.1) follows traditional methods. So we present a brief proof for this condition.

Lemma 2.2. *Let $(u_m) \subset H^1_{0,\mathrm{rad}}(B_1)$ be a $(PS)_c$ sequence of J_λ. Then (u_m) is bounded in $H^1_{0,\mathrm{rad}}(B_1)$.*

Proof. Let $(u_m) \subset H^1_{0,\mathrm{rad}}(B_1)$ be a $(PS)_c$ sequence, that is

$$J_\lambda(u_m) = \frac{1}{2}\|u_m\|^2 - \frac{\lambda}{2}\|u_m\|^2_{2,|x|^\mu} - \frac{1}{2^*_\alpha}\int_{B_1} |x|^\alpha |u_m|^{2^*_\alpha}\,\mathrm{d}x = c + o(1) \tag{2.2}$$

and

$$\begin{aligned} \langle J_\lambda'(u_m), v\rangle &= \int_{B_1} \nabla u_m \nabla v\,\mathrm{d}x - \lambda\int_{B_1} |x|^\mu u_m v\,\mathrm{d}x - \int_{B_1} |x|^\alpha |u_m|^{2^*_\alpha - 2} u_m v\,\mathrm{d}x \\ &= o(1)\|v\| \end{aligned} \tag{2.3}$$

for all $v \in H^1_{0,\mathrm{rad}}(B_1)$. From (2.2) and (2.3), it follows that

$$\begin{aligned} J_\lambda(u_m) - \frac{1}{2}\langle J_\lambda'(u_m), u_m\rangle &= \frac{2^*_\alpha - 2}{2\cdot 2^*_\alpha}\int_{B_1} |x|^\alpha |u_m|^{2^*_\alpha}\,\mathrm{d}x \\ &= c + o(1) + o(1)\|u_m\|. \end{aligned} \tag{2.4}$$

Considering $0 < \lambda < \lambda_1^*$, by the variational characterization of λ_1^*, we have

$$\langle J'_\lambda(u_m), u_m\rangle \geq \Big(1 - \frac{\lambda}{\lambda_1^*}\Big)\|u_m\|^2 - \int_{B_1} |x|^\alpha |u_m|^{2^*_\alpha}\, \mathrm{d}x.$$

Hence by (2.4), we obtain

$$\|u_m\|^2 \leq C_1 + C_2\|u_m\|$$

and consequently (u_m) is a bounded sequence in $H^1_{0,\mathrm{rad}}(B_1)$.

Now we consider $\lambda_k^* < \lambda < \lambda_{k+1}^*$. It is convenient to decompose $H^1_{0,\mathrm{rad}}(B_1)$ into the following subspaces,

$$H^1_{0,\mathrm{rad}}(B_1) = H_k \oplus H_k^\perp, \tag{2.5}$$

where H_k is finite dimensional defined by

$$H_k = [e_1, \dots, e_k]. \tag{2.6}$$

For u in $H^1_{0,\mathrm{rad}}(B_1)$, let $u = u^k + u^\perp$, where $u^k \in H_k$ and $u^\perp \in (H_k)^\perp$. We note that

$$\int_{B_1} \nabla u \nabla u^k \,\mathrm{d}x - \lambda\int_{B_1} |x|^\mu u u^k\,\mathrm{d}x = \|u^k\|^2 - \lambda\|u^k\|^2_{2,|x|^\mu}, \tag{2.7}$$

$$\int_{B_1} \nabla u \nabla u^\perp \,\mathrm{d}x - \lambda\int_{B_1} |x|^\mu u u^\perp\,\mathrm{d}x = \|u^\perp\|^2 - \lambda\|u^\perp\|^2_{2,|x|^\mu}. \tag{2.8}$$

By (2.3) and (2.8), we can see that

$$\langle J_\lambda(u_m), u_m^\perp\rangle = \|u_m^\perp\|^2 - \lambda\|u_m^\perp\|^2_{2,|x|^\mu} - \int_{B_1} |x|^\alpha |u_m|^{2^*_\alpha - 2} u_m u_m^\perp\,\mathrm{d}x = o(1)\|u_m^\perp\|.$$

Then, from the variational characterization of λ_{k+1}^*, the Holder and Young inequalities, and (2.4), we obtain

$$\begin{aligned}
&\Big(1 - \frac{\lambda}{\lambda_{k+1}^*}\Big)\|u_m^\perp\|^2 \\
&\leq \int_{B_1} |x|^\alpha |u_m|^{2^*_\alpha-2} u_m u_m^\perp\,\mathrm{d}x + o(1)\|u_m^\perp\| \\
&\leq \Big(\int_{B_1} |x|^\alpha |u_m|^{2^*_\alpha}\,\mathrm{d}x\Big)^{\frac{2^*_\alpha - 1}{2^*_\alpha}} \Big(\int_{B_1} |x|^\alpha |u_m^\perp|^{2^*_\alpha}\,\mathrm{d}x\Big)^{\frac{1}{2^*_\alpha}} \\
&\leq \epsilon\Big(\int_{B_1} |x|^\alpha |u_m^\perp|^{2^*_\alpha}\,\mathrm{d}x\Big)^{2/2^*_\alpha} + c_\epsilon\Big(\int_{B_1} |x|^\alpha |u_m|^{2^*_\alpha}\,\mathrm{d}x\Big)^{\frac{2(2^*_\alpha-1)}{2^*_\alpha}} + o(1)\|u_m^\perp\| \\
&\leq \epsilon\|u_m^\perp\|^2 + c_\epsilon\Big(\int_{B_1} |x|^\alpha |u_m|^{2^*_\alpha}\,\mathrm{d}x\Big)^{\frac{2(2^*_\alpha-1)}{2^*_\alpha}} + c\|u_m^\perp\|.
\end{aligned}$$

By (2.4) and [32, Compactness Lemma] which guarantees the compact embedding of $H^1_{0,\mathrm{rad}}(B_1)$ in $L^z(B_1, |x|^\alpha)$ for $2 \leq z < 2^*_\alpha$, we have

$$\|u_m^\perp\|^2 \leq (c + c\|u_m\|)^{\frac{2(2^*_\alpha-1)}{2^*_\alpha}} + c\|u_m^\perp\|. \tag{2.9}$$

For $u_m^k \in H_k$, using the variational characterization of λ_k^*, similar to (2.9), we obtain

$$\|u_m^k\|^2 \leq (c + c\|u_m\|)^{\frac{2(2^*_\alpha-1)}{2^*_\alpha}} + c\|u_m^k\|. \tag{2.10}$$

By summing the inequalities in (2.9) and (2.10), we have

$$\|u_m\|^2 \leq (C + C\|u_m\|)^{\frac{2(2^*_\alpha-1)}{2^*_\alpha}} + C\|u_m\|,$$

which proves the boundedness of the sequence (u_m) in $H^1_{0,\mathrm{rad}}(B_1)$ as desired.

Lastly, we consider $\lambda = \lambda^*_k$ for some $k \in \mathbb{N}$. We use the decomposition

$$H^1_{0,\mathrm{rad}}(B_1) = H_{k-1} \oplus H_k^\perp \oplus E_{\lambda^*_k}, \tag{2.11}$$

where $E_{\lambda^*_k}$ is the eigenspace associated with eigenvalue λ^*_k. For the sequence (u_m) in $H^1_{0,\mathrm{rad}}(B_1)$, we have

$$u_m = u_m^{k-1} + u_m^\perp + w_m = v_m + w_m,$$

where $u_m^{k-1} \in H_{k-1}$, $u_m^\perp \in (H_k)^\perp$, $v_m = u_m^{k-1} + u_m^\perp$ and $w_m = \sum_{i=1}^l y_{i,m} e_{i,\lambda^*_k} \in E_{\lambda^*_k}$, where e_{i,λ^*_k} is an eigenfunction associated with λ^*_k for $1 \leq i \leq l$, l is the multiplicity of λ^*_k, and w_m can be consider different from 0 for all $m \in \mathbb{N}$. Note that $\|w_m\| \leq y_m$, where $y_m = l \max\{|y_{i,m}|; 1 \leq i \leq l\}$. Using arguments similar to those used in (2.9) and (2.10), we conclude that

$$\|v_m\|^2 \leq C(1 + \|u_m\|)^{\frac{2(2^*_\alpha-1)}{2^*_\alpha}} + C\|v_m\|. \tag{2.12}$$

We can assume $\|u_m\| \geq 1$ (if $\|u_m\| \leq 1$, the sequence (u_m) is bounded in $H^1_{0,\mathrm{rad}}(B_1)$) and, since $\|u_m\| \leq \|v_m\| + y_m$, by (2.12), we obtain

$$\|v_m\|^2 \leq C(\|v_m\| + y_m)^{\frac{2(2^*_\alpha-1)}{2^*_\alpha}} + C\|v_m\|. \tag{2.13}$$

If y_m is bounded, from (2.13), we have that (v_m) is bounded in $H^1_{0,\mathrm{rad}}(B_1)$ and, consequently, (u_m) is bounded in $H^1_{0,\mathrm{rad}}(B_1)$. Now let us assume $y_m \to +\infty$. Using (2.13), we have

$$\begin{aligned} \|\frac{v_m}{y_m}\|^2 &\leq C\Big[\frac{(\|v_m\| + y_m)^{\frac{(2^*_\alpha-1)}{2^*_\alpha}}}{y_m}\Big]^2 + \frac{C}{y_m}\|\frac{v_m}{y_m}\| \\ &\leq C\Big[\frac{1}{y_m^{1-\frac{(2^*_\alpha-1)}{2^*_\alpha}}}\|\frac{v_m}{y_m}\|^{\frac{(2^*_\alpha-1)}{2^*_\alpha}} + \frac{1}{y_m^{1-\frac{(2^*_\alpha-1)}{2^*_\alpha}}}\Big]^2 + \frac{C}{y_m}\|\frac{v_m}{y_m}\|. \end{aligned} \tag{2.14}$$

Thus, we obtain

$$\|\frac{v_m}{y_m}\|^2 \leq C\|\frac{v_m}{y_m}\|^{\frac{2(2^*_\alpha-1)}{2^*_\alpha}} + C\|\frac{v_m}{y_m}\| + C,$$

which implies the sequence $\{\frac{v_m}{y_m}\}$ being bounded because $\frac{(2^*_\alpha-1)}{2^*_\alpha} < 1$, and, by (2.14), $\|\frac{v_m}{y_m}\| \to 0$ as $m \to 0$.

Therefore, possibly up to a subsequence, $v_m/y_m \to 0$ a.e. in B_1 and strongly in $L^q(B_1, |x|^\alpha)$, $1 \leq q < 2^*_\alpha$. Notice that

$$\begin{aligned} \langle J'_\lambda(u_m), \frac{w_m}{y_m}\rangle &= \frac{1}{y_m^2}\Big(\int_{B_1} |\nabla w_m|^2 \,\mathrm{d}x - \lambda\int_{B_1} |x|^\mu w_m^2 \,\mathrm{d}x\Big) \\ &\quad - \int_{B_1} |x|^\alpha |u_m|^{2^*_\alpha-1}\frac{w_m}{y_m} \,\mathrm{d}x = o(1) \end{aligned} \tag{2.15}$$

and since $w_m \in E_{\lambda^*_k}$, we have

$$\langle J'_\lambda(u_m), \frac{w_m}{y_m}\rangle = -\int_{B_1} |x|^\alpha |u_m|^{2^*_\alpha-1}\frac{w_m}{y_m} \,\mathrm{d}x = o(1). \tag{2.16}$$

Thus, we have

$$\int_{B_1} |x|^\alpha \Big|\frac{u_m}{y_m}\Big|^{2^*_\alpha-2}\frac{u_m}{y_m} w_m \,\mathrm{d}x = \frac{1}{y_m^{2^*_\alpha-1}}\int_{B_1}|x|^\alpha|u_m|^{2^*_\alpha-2}u_m\frac{w_m}{y_m}\,\mathrm{d}x \to 0 \tag{2.17}$$

as $n\to\infty$. Note, since $u_m = v_m + w_m$, we have that $\frac{u_m}{y_m}\to w_0$ in $L^q(B_1,|x|^\alpha)$ for all $1\le q<2^*_\alpha$ and a.e. in B_1 with $w_0\in E_{\lambda_k^*}\setminus\{0\}$. So, by the Dominated Convergence Theorem and using (2.17), it follows that

$$\int_{B_1}|x|^\alpha\Big|\frac{u_m}{y_m}\Big|^{2^*_\alpha-2}\frac{u_m}{y_m}\frac{w_m}{y_m}\,\mathrm{d}x \to \int_{B_1}|x|^\alpha|w_0|^{2^*_\alpha}\,\mathrm{d}x = 0 \tag{2.18}$$

which is a contradiction. So y_m is bounded and, consequently, (u_m) is also bounded in $H^1_{0,\mathrm{rad}}(B_1)$. $\square$

We need to show that the minimax levels are below a suitable constant. For this purpose, we need an estimate that allows us to simplify some calculations needed ahead. Initially, we consider a Palais-Smale sequence (u_m); thus, by Lemma 2.2, we may assume that (eventually passing to a subsequence)

$$\begin{gathered} u_m \rightharpoonup u \in H^1_{0,\mathrm{rad}}(B_1),\\ u_m \to u \in L^p(B_1,|x|^\alpha) \quad \text{for any } p\in[1,2^*_\alpha[,\\ u_m \to u\in L^p(B_1,|x|^\mu) \quad \text{for any } p\in[1,2^*_\alpha[,\ \text{if } \mu\ge\alpha,\\ u_m\to u \quad \text{a.e. in } B_1. \end{gathered} \tag{2.19}$$

To check that u is a solution for (1.1), we need the following lemma.

Lemma 2.3. *Let (u_m) be a $(PS)_c$ sequence in $H^1_{0,\mathrm{rad}}(B_1)$, with*

$$c < \frac{2+\alpha}{2(N+\alpha)} S_\alpha^{(N+\alpha)/(2+\alpha)},$$

and let $v_m = u_m - u$. Then $v_m\to 0$ strongly in $H^1_{0,\mathrm{rad}}(B_1)$.

Proof. By Lemma 2.2, $\|u_m\|$ is bounded, so from (2.19), u is a weak solution of (1.1). Then, by (2.3) we have

$$\|u\|^2 - \lambda\|u\|^2_{2,|x|^\mu} - \int_{B_1}|x|^\alpha|u|^{2^*_\alpha}\,\mathrm{d}x = 0. \tag{2.20}$$

By the Brézis-Lieb Lemma [8], it follows that

$$\int_{B_1}|x|^\alpha|u_m|^{2^*_\alpha}\,\mathrm{d}x = \int_{B_1}|x|^\alpha|v_m|^{2^*_\alpha}\,\mathrm{d}x + \int_{B_1}|x|^\alpha|u|^{2^*_\alpha}\,\mathrm{d}x + o(1). \tag{2.21}$$

On the other hand, since $H^1_{0,\mathrm{rad}}(B_1)$ is a Hilbert Space, we obtain

$$\|u_m\|^2 = \|v_m\|^2 + \|u\|^2 + o(1). \tag{2.22}$$

By (2.2), (2.21), and (2.22), as $u_m\to u$ in $L^2(B_1,|x|^\mu)$, we obtain

$$\begin{aligned} c+o(1) &= J_\lambda(u_m)\\ &= J_\lambda(u) + \frac12\|v_m\|^2 - \frac{\lambda}{2}\|v_m\|^2_{2,|x|^\mu} - \frac{1}{2^*_\alpha}\int_{B_1}|x|^\alpha|v_m|^{2^*_\alpha}\,\mathrm{d}x + o(1)\\ &= J_\lambda(u) + \frac12\|v_m\|^2 - \frac{1}{2^*_\alpha}\int_{B_1}|x|^\alpha|v_m|^{2^*_\alpha}\,\mathrm{d}x + o(1). \end{aligned} \tag{2.23}$$

Since $J'_\lambda(u)=0$ and $\|v_m\|^2_{2,|x|^\mu}=o(1)$, we conclude that

$$\langle J'_\lambda(u_m),v_m\rangle=\|v_m\|^2-\int_{B_1}|x|^\alpha|v_m|^{2^*_\alpha}\,\mathrm{d}x+o(1).$$

Then

$$\|v_m\|^2=\int_{B_1}|x|^\alpha|v_m|^{2^*_\alpha}\,\mathrm{d}x+o(1). \tag{2.24}$$

Now, by (2.3) and taking u_m as test function, we note that

$$\int_{B_1}|x|^\alpha|u_m|^{2^*_\alpha}\,\mathrm{d}x=\|u_m\|^2-\lambda\|u_m\|^2_{2,\mu}+o(1).$$

So, as $u_m\to u$ in $L^2(B_1,|x|^\mu)$ and using (2.22), we obtain

$$\begin{aligned} J_\lambda(u_m)=&\frac{1}{2}(\|u_m\|^2-\lambda\|u_m\|^2_{2,|\mu|})-\frac{1}{2^*_\alpha}\int_{B_1}|x|^\alpha|u_m|^{2^*_\alpha}\,\mathrm{d}x\\ =&\frac{1}{2}(\|u_m\|^2-\lambda\|u_m\|^2_{2,|\mu|})-\frac{1}{2^*_\alpha}(\|u_m\|^2-\lambda\|u_m\|^2_{2,\mu}+o(1))\\ =&\frac{2+\alpha}{2(N+\alpha)}(\|u_m\|^2-\lambda\|u_m\|^2_{2,|x|^\mu})+o(1)\\ =&\frac{2+\alpha}{2(N+\alpha)}(\|u\|^2-\lambda\|u\|^2_{2,|x|^\mu}+\|v_m\|^2)+o(1). \end{aligned} \tag{2.25}$$

From (2.20), we conclude that

$$\|u\|^2-\lambda\|u\|^2_{2,|x|^\mu}\ge 0. \tag{2.26}$$

Thus, by (2.25) and (2.26), we have

$$\|v_m\|^2\le\frac{2(N+\alpha)}{2+\alpha}J_\lambda(u_m)+o(1).$$

By (2.2), since $c<\frac{2+\alpha}{2(N+\alpha)}S_\alpha^{(N+\alpha)/(2+\alpha)}$, for m sufficiently large we obtain

$$\|v_m\|^2\le c+o(1)<S_\alpha^{(N+\alpha)/(2+\alpha)}. \tag{2.27}$$

From (1.6) and (2.24), we obtain

$$\|v_m\|^2\le S_\alpha^{-2^*_\alpha/2}\|v_m\|^{2^*_\alpha}+o(1),$$

which implies

$$\|v_m\|^2(S^{2^*_\alpha/2}-\|v_m\|^{2^*_\alpha-2})\le o(1).$$

This and (2.27) imply that $v_m\to 0$ strongly in $H^1_{0,\mathrm{rad}}(B_1)$. □

3. Geometric conditions

Here we prove that J_λ satisfies the geometric condition of Theorem 2.1. Firstly, given $\lambda>0$, we define $\lambda^+=\min\{\lambda^*_j:\ \lambda<\lambda^*_j\}$ and set

$$H_1=\overline{\oplus[e_j]_{\lambda^*_j\ge\lambda^+}}^{H^1_{0,\mathrm{rad}}(B_1)}\quad H_2=[e_1,\dots,e_j]_{\lambda^*_j<\lambda^+}. \tag{3.1}$$

Lemma 3.1. *There exist $\delta,\rho>0$ such that, for $u\in H_1$,*

$$J_\lambda(u)\ge\delta\quad \text{if } \|u\|=\rho.$$

Proof. Let us take $u \in H_1$, by the variational characterization of λ^+ we obtain that

$$J_\lambda(u) \geq \frac{1}{2}\big(1 - \frac{\lambda}{\lambda^+}\big)\|u\|^2 - C\|u\|^{2^*_\alpha} \geq \delta > 0$$

when $\|u\| = \rho$ with $\rho > 0$ small enough. □

4. Estimates of minimax levels

In this section, we obtain some estimates to show that the minimax levels are below an appropriate constant in order to recover a similar compactness property for the functional J_λ.

First, let $r \in (0,1)$ and $B_r = \{x \in \mathbb{R}^N : |x| \leq r\}$. We take $\xi_r \in C_0^\infty(B_r, [0,1])$, a radial cut-off function such that $\xi_r = 1$ in $B_{r/2}$ and $|\nabla \xi_r| \leq 4/r$, and set $u_\epsilon^r(x) = \xi_r(x)u_\epsilon(x)$. In [3, Proof of Theorem 3.3] were obtained the following estimates of Brézis-Nirenberg type [9, Lemma 1.2], which also can be found in [3, 21].

Lemma 4.1. *Let K_1, K_2 and K_3 be positive constants. For fixed $r \in (0,1)$ and $\mu, \alpha \geq 0$ and $\epsilon > 0$ small enough, we have*

(a) $\|u_\epsilon^r\|^2 = S_\alpha^{(N+\alpha)/(2+\alpha)} + O\left(\epsilon^{(N-2)/(2+\alpha)}\right)$;

(b) $\|u_\epsilon^r\|_{2^*_\alpha,|x|^\alpha}^{2^*_\alpha} = S_\alpha^{(N+\alpha)/(2+\alpha)} + O\left(\epsilon^{(N+\alpha)/(2+\alpha)}\right)$;

(c)

$$\|u_\epsilon^r\|_{2,|x|^\mu}^2 = \begin{cases} K_1\epsilon^{(2+\mu)/(2+\alpha)} & \text{if } N > 4+\mu; \\ K_1\epsilon^{(2+\mu)/(2+\alpha)}|\log\epsilon| + O\big(\epsilon^{(2+\mu)/(2+\alpha)}\big) & \text{if } N = 4+\mu; \\ K_1\epsilon^{(N-2)/(2+\alpha)} & \text{if } N < 4+\mu; \end{cases}$$

(d) $\|u_\epsilon^r\|_{1,|x|^\mu} \leq K_2\epsilon^{(N-2)/[2(2+\alpha)]}$;

(e) $\|u_\epsilon^r\|_{2^*_\alpha-1,|x|^\alpha}^{2^*_\alpha-1} \leq K_3\epsilon^{(N-2)/[2(2+\alpha)]}$.

Now we shall prove some main technical lemmas. First of all, we define

$$W(\epsilon, r) = \{u \in H^1_{0,\mathrm{rad}}(B_1); u = u^- + tu_\epsilon^r, u^- \in H_2, t \in \mathbb{R}\}.$$

Remark 4.2. Since u_ϵ is solution for (1.8), $u_\epsilon^r \notin [e_1, e_2, \dots, e_k]$ for any $k \in \mathbb{N}$. Thus, $W(\epsilon, r) \neq H_2$.

Lemma 4.3. *If $u \in W(\epsilon, r)$, then for $\epsilon > 0$ sufficiently small*

$$\|u\|_{2^*_\alpha,|x|^\alpha}^{2^*_\alpha} \geq \|tu_\epsilon^r\|_{2^*_\alpha,|x|^\alpha}^{2^*_\alpha} - Ct^{2^*_\alpha}\epsilon^{(N-2)(N+\alpha)/[(N+2\alpha+2)(2+\alpha)]} \tag{4.1}$$

for any $t \in \mathbb{R}$.

Proof. Note that from

$$\|u\|_{2^*_\alpha,|x|^\alpha}^{2^*_\alpha} = 2^*_\alpha \int_{B_1} |x|^\alpha \,\mathrm{d}x \int_0^u |s|^{2^*_\alpha-2}s \,\mathrm{d}s, \tag{4.2}$$

and the Mean Value Theorem, we obtain

$$\begin{aligned} &\|u\|_{2^*_\alpha,|x|^\alpha}^{2^*_\alpha} - \|tu_\epsilon^r\|_{2^*_\alpha,|x|^\alpha}^{2^*_\alpha} - \|u^-\|_{2^*_\alpha,|x|^\alpha}^{2^*_\alpha} \\ &= 2^*_\alpha \int_0^1 \mathrm{d}s \int_{B_1} |x|^\alpha[|tu_\epsilon^r + su^-|^{2^*_\alpha-2}(tu_\epsilon^r + su^-) - |su^-|^{2^*_\alpha-2}su^-]u^- \,\mathrm{d}x \\ &= 2^*_\alpha(2^*_\alpha - 1)\int_0^1 \mathrm{d}s \int_{B_1} |x|^\alpha |tu_\epsilon^r + \tau su^-|^{2^*_\alpha-2}tu_\epsilon^r \cdot u^- \,\mathrm{d}x \end{aligned} \tag{4.3}$$

where $\tau = \tau(x)$ is a measurable function such that $0 < \tau(x) < 1$.

Using (4.3) and since $u^- \in H_2$, which is a finite-dimension subspace, we obtain

$$\begin{aligned}
&\big|\|u\|_{2^*_\alpha,|x|^\alpha}^{2^*_\alpha} - \|tu_\epsilon^r\|_{2^*_\alpha,|x|^\alpha}^{2^*_\alpha} - \|u^-\|_{2^*_\alpha,|x|^\alpha}^{2^*_\alpha}\big| \\
&\le C\int_0^1 \mathrm{d}s \int_{B_1} |x|^\alpha (|tu_\epsilon^r|^{2^*_\alpha-1}|u^-| + |u^-|^{2^*_\alpha-1}|tu_\epsilon^r|)\,\mathrm{d}x \\
&\le C\|tu_\epsilon^r\|_{2^*_\alpha-1,|x|^\alpha}^{2^*_\alpha-1}\|u^-\|_\infty + \|u^-\|_{\infty,|x|^\alpha}^{2^*_\alpha-1}\|tu_\epsilon^r\|_1 \\
&\le C\|tu_\epsilon^r\|_{2^*_\alpha-1,|x|^\alpha}^{2^*_\alpha-1}\|u^-\|_2 + \|u^-\|_{2^*_\alpha,|x|^\alpha}^{2^*_\alpha-1}\|tu_\epsilon^r\|_1,
\end{aligned} \tag{4.4}$$

where C is positive constant. From (4.4), the Young inequality and the items (d) and (e) of Lemma 4.1, we have that

$$\begin{aligned}
&\big|\|u\|_{2^*_\alpha,|x|^\alpha}^{2^*_\alpha} - \|tu_\epsilon^r\|_{2^*_\alpha,|x|^\alpha}^{2^*_\alpha} - \|u^-\|_{2^*_\alpha,|x|^\alpha}^{2^*_\alpha}\big| \\
&\le Ct^{2^*_\alpha-1}\epsilon^{(N-2)/(2(2+\alpha))}\|u^-\|_2 + \frac{N+2+2\alpha}{2(N+\alpha)}\|u^-\|_{2^*_\alpha,|x|^\alpha}^{2^*_\alpha} + Ct^{2^*_\alpha}\epsilon^{(N+\alpha)/(2+\alpha)}.
\end{aligned}$$

Finally, again by the Young inequality, we have

$$\begin{aligned}
&\big|\|u\|_{2^*_\alpha,|x|^\alpha}^{2^*_\alpha} - \|tu_\epsilon^r\|_{2^*_\alpha,|x|^\alpha}^{2^*_\alpha} - \|u^-\|_{2^*_\alpha,|x|^\alpha}^{2^*_\alpha}\big| \\
&\le Ct^{2^*_\alpha-1}\epsilon^{\frac{(N-2)}{(2(2+\alpha))}}\|u^-\|_{2^*_\alpha,|x|^\alpha} + \frac{N+2+2\alpha}{2(N+\alpha)}\|u^-\|_{2^*_\alpha,|x|^\alpha}^{2^*_\alpha} + Ct^{2^*_\alpha}\epsilon^{\frac{(N+\alpha)}{(2+\alpha)}} \\
&\le Ct^{2^*_\alpha}\epsilon^{\frac{(N-2)(N+\alpha)}{[(N+2\alpha+2)(2+a)]}} + \frac{1}{2^*_\alpha}\|u^-\|_{2^*_\alpha,|x|^\alpha}^{2^*_\alpha} + \frac{N+2+2\alpha}{2(N+\alpha)}\|u^-\|_{2^*_\alpha,|x|^\alpha}^{2^*_\alpha} + Ct^{2^*_\alpha}\epsilon^{\frac{(N+\alpha)}{(2+\alpha)}} \\
&= Ct^{2^*_\alpha}\epsilon^{\frac{(N-2)(N+\alpha)}{[(N+2\alpha+2)(2+\alpha)]}} + \|u^-\|_{2^*_\alpha,|x|^\alpha}^{2^*_\alpha} + Ct^{2^*_\alpha}\epsilon^{\frac{(N+\alpha)}{(2+\alpha)}} \\
&\le Ct^{2^*_\alpha}\epsilon^{\frac{(N-2)(N+\alpha)}{[(N+2\alpha+2)(2+\alpha)]}} + \|u^-\|_{2^*_\alpha,|x|^\alpha}^{2^*_\alpha}.
\end{aligned}$$

for $\epsilon > 0$ small enough. The proof is complete. □

Lemma 4.4. *For $\epsilon > 0$ sufficiently small, we have*

$$\begin{aligned}
&\frac{\|u_\epsilon^r\|^2 - \lambda\|u_\epsilon^r\|_{2,|x|^\mu}^2}{\|u_\epsilon^r\|_{2^*_\alpha,|x|^\alpha}^2} \\
&= \begin{cases} S_\alpha - C\epsilon^{(2+\mu)/(2+\alpha)} & \text{if } N > 4+\mu; \\ S_\alpha - C\epsilon^{(2+\mu)/(2+\alpha)}|\log(\epsilon)| + O(\epsilon^{(2+\mu)/(2+\alpha)}) & \text{if } N = 4+\mu; \\ S_\alpha + \epsilon^{(N-2)/(2+\alpha)}(O(1) - \lambda C) & \text{if } N < 4+\mu. \end{cases}
\end{aligned} \tag{4.5}$$

The statement of the lemma above is obtained from (a)–(c) in Lemma 4.1.

Now we separate our study into three cases: non-resonant case assuming (1.9), and consequently, $N > 4+\mu$, or $N = 4+\mu$; resonant case when (1.9) holds; and non-resonant case with $N < 4+\mu$. This separation occurs because to prove the $(PS)_c$ condition for c below an appropriate constant when $\lambda = \lambda_j$ for some $j \in \mathbb{N}$, we need to have $N > 4+\mu$. When $N < 4+\mu$, it is crucial to assume in addition that λ is sufficiently large to prove the $(PS)_c$ condition.

4.1. **Non-resonant case with $N \ge 4+\mu$.** Initially, we consider the non-resonant case and we obtain the following results.

Lemma 4.5. *Assume* (1.9)*, for ϵ sufficiently small and positive. If $\lambda \neq \lambda_j^*$, for every $j \in \mathbb{N}$, then*

$$\sup_{W(\epsilon,r)} J_\lambda(u) < \frac{(2+\alpha)}{2(N+\alpha)} S_\alpha^{(N+\alpha)/(2+\alpha)}. \tag{4.6}$$

Proof. Note that for fixed $u \in H^1_{0,\mathrm{rad}}(B_1)$ with $u \neq 0$, we obtain

$$\sup_t J_\lambda(tu) = \frac{(2+\alpha)}{2(N+\alpha)} \Big(\frac{\|u\|^2 - \lambda \|u\|^2_{2,|x|^\mu}}{\|u\|^2_{2^*_\alpha,|x|^\alpha}} \Big)^{(N+\alpha)/(2+\alpha)}. \tag{4.7}$$

Since

$$\begin{aligned}
&\sup\{J_\lambda(u) : u \in W(\epsilon) \setminus \{0\}\} \\
&= \sup \Big\{ J_\lambda(\|u\|_{2^*_\alpha,|x|^\alpha} \frac{u}{\|u\|_{2^*_\alpha,|x|^\alpha}}) : u \in W(\epsilon, r) \setminus \{0\} \Big\} \\
&\leq \sup\{J_\lambda(tu) : u \in W(\epsilon,r) \setminus \{0\} \text{ with } \|u\|_{2^*_{\alpha,|x|^\alpha}} = 1 \text{ and } t \in \mathbb{R}\},
\end{aligned}$$

to show that (4.6) is true, we need to estimate

$$\sup_{u \in W(\epsilon,r),\, \|u\|_{2^*_\alpha,|x|^\alpha}=1} \big\{ \|u\|^2 - \lambda \|u\|^2_{2,|x|^\mu} \big\}. \tag{4.8}$$

Let $u = u^- + tu^r_\epsilon \in W(\epsilon, r)$ with $\|u\|_{2^*_\alpha,|x|^\alpha} = 1$. By (4.1) and item (b) of Lemma 4.1, for ϵ small enough, we have

$$\begin{aligned}
1 &= \|u\|^{2^*_\alpha}_{2^*_\alpha,|x|^\alpha} \\
&\geq \|tu^r_\epsilon\|^{2^*_\alpha}_{2^*_\alpha,|x|^\alpha} - Ct^{2^*_\alpha} \epsilon^{(N-2)(N+\alpha)/(N+2\alpha+2)(2+\alpha)} \\
&= t^{2^*_\alpha} \Big(S_\alpha^{(N+\alpha)/(2+\alpha)} + O\big(\epsilon^{(N-2)/(2+\alpha)}\big) \Big) - Ct^{2^*_\alpha} \epsilon^{(N-2)(N+\alpha)/(N+2\alpha+2)(2+\alpha)} \\
&= t^{2^*_\alpha} \Big(S_\alpha^{(N+\alpha)/(2+\alpha)} + O\big(\epsilon^{(N-2)(N+\alpha)/(N+2\alpha+2)(2+\alpha)}\big) \Big).
\end{aligned}$$

Thus, we can conclude that t is bounded for small positive ϵ. From item (e) in Lemma 4.1, the variational characterization of λ_j^* and Green's Theorem, we obtain

$$\begin{aligned}
&\|u\|^2 - \lambda \|u\|^2_{2,|x|^\mu} \\
&\leq \|tu^r_\epsilon\|^2 - \lambda \|tu^r_\epsilon\|^2_{2,|x|^\mu} + \|u^-\|^2 - \lambda \|u^-\|^2_{2,|x|^\mu} \\
&\quad + 2 \int_{B_1} \{ |tu^r_\epsilon| \, |\Delta u^-| + \lambda |x|^\mu |u^-| |tu^r_\epsilon| \} \, \mathrm{d}x \\
&\leq \|tu^r_\epsilon\|^2 - \lambda \|tu^r_\epsilon\|^2_{2,|x|^\mu} + \|u^-\|^2 - \lambda \|u^-\|^2_{2,|x|^\mu} + C\{ \|tu^r_\epsilon\|_1 \, \|\Delta u^-\|_\infty \\
&\quad + \lambda \|u^-\|_\infty \|tu^r_\epsilon\|_{1,|x|^\mu} \} \\
&\leq \|tu^r_\epsilon\|^2 - \lambda \|tu^r_\epsilon\|^2_{2,|x|^\mu} + \|u^-\|^2 - \lambda \|u^-\|^2_{2,|x|^\mu} + C\|u^-\|_2 \epsilon^{(N-2)/[2(2+\alpha)]} \\
&\leq \frac{\|tu^r_\epsilon\|^2 - \lambda \|tu^r_\epsilon\|^2_{2,|x|^\mu}}{\|tu^r_\epsilon\|^2_{2^*_\alpha,|x|^\alpha}} \|tu^r_\epsilon\|^2_{2^*_\alpha,|x|^\alpha} + (\overline{\lambda} - \lambda) \|u^-\|^2_{2,|x|^\mu} \\
&\quad + C\|u^-\|_{2,|x|^\mu} \epsilon^{(N-2)/[2(2+\alpha)]},
\end{aligned} \tag{4.9}$$

where $\overline{\lambda} = \max\{\lambda_j^* : \lambda_j^* < \lambda\}$.

Now we define $A(u^-,\epsilon,c)=(\overline{\lambda}-\lambda)\|u^-\|^2_{2,|x|^\mu}+C\|u^-\|_{2,|x|^\mu}\epsilon^{(N-2)/[2(2+\alpha)]}$. Notice that

$$A(u^-,\epsilon,c)\le 0 \quad\text{or}\quad A(u^-,\epsilon,c)\le \frac{c^2}{\lambda-\overline{\lambda}}\epsilon^{(N-2)/(2+\alpha)}. \tag{4.10}$$

On the other hand by (4.1) and the boundedness of t, we obtain

$$\begin{aligned}\|tu^r_\epsilon\|^2_{2^*_\alpha,|x|^\alpha}&\le \big(1+C\epsilon^{(N-2)(N+\alpha)/[(N+2\alpha+2)(2+\alpha)]}\big)^{2/2^*_\alpha}\\&\le 1+C\epsilon^{(N-2)(N+\alpha)/[(N+2\alpha+2)(2+\alpha)]}.\end{aligned} \tag{4.11}$$

From (1.9), we obtain $N>4+\mu$, then using (4.5), (4.9), (4.10) and (4.11), we have

$$\begin{aligned}&\|u\|^2-\lambda\|u\|^2_{2,|x|^\mu}\\&\le \Big(S_\alpha-C\epsilon^{(2+\mu)/(2+\alpha)}\Big)\Big(1+C\epsilon^{[(N-2)(N+\alpha)]/[(N+2\alpha+2)(2+\alpha)]}\Big)+A(u^-,\epsilon,c).\end{aligned} \tag{4.12}$$

By (1.9), we also conclude that

$$\frac{(N-2)(N+\alpha)}{(N+2\alpha+2)(2+\alpha)}>\frac{2+\mu}{2+\alpha}.$$

Thus, $\|u\|^2-\lambda\|u\|^2_{2,|x|^\mu}<S_\alpha$ for ϵ positive and small enough. □

Lemma 4.6. *For $\epsilon>0$ sufficiently small and $N=4+\mu$, if $\lambda\ne\lambda^*_j$, for every $j\in\mathbb{N}$, then*

$$\sup_{W(\epsilon,r)} J_\lambda(u)<\frac{(2+\alpha)}{2(N+\alpha)}S_\alpha^{(N+\alpha)/(2+\alpha)}. \tag{4.13}$$

Proof. When $N=4+\mu$, as for (4.12), from (4.5), (4.9), (4.10) and (4.11), we obtain

$$\begin{aligned}\|u\|^2-\lambda\|u\|^2_{2,|x|^\mu}&\le\Big(S_\alpha-C\epsilon^{(2+\mu)/(2+\alpha)}|\log(\epsilon)|+O(\epsilon^{(2+\mu)/(2+\alpha)})\Big)\\&\quad\times\Big(1+C\epsilon^{[(2+\mu+\alpha)(4+\mu+\alpha)]/[(6+\mu+2\alpha)(2+\alpha)]}\Big)+A(u^-,\epsilon,c).\end{aligned}$$

Because of the behavior of $|\log(\epsilon)|$ near zero, for ϵ small enough we conclude the result. □

4.2. **Resonant case with $N>4+\mu$.** Now we consider, $\lambda=\lambda^*_j$ for some $j\in\mathbb{N}$. We will find estimates which will help us in obtaining a result similar to Lemma 4.5 for the resonant case when (1.9) is satisfied.

First, we denote by P_j the projector on the eigenspace corresponding to λ^*_j and set

$$\tilde{u^r_\epsilon}=u^r_\epsilon-P_ju^r_\epsilon. \tag{4.14}$$

Thus, by item (d) in Lemma 4.1, we have

$$\|P_ju^r_\epsilon\|^2_{2,|x|^\mu}=\sum_k\Big(\int_{B_1}|x|^\mu e_ku^r_\epsilon\,\mathrm{d}x\Big)^2\le C\|u^r_\epsilon\|^2_{1,|x|^\mu}\le C\epsilon^{(N-2)/(2+\alpha)}. \tag{4.15}$$

Consequently, as $P_ju^r_\epsilon$ is in a finite dimensional space, we obtain

$$\|P_ju^r_\epsilon\|_{\infty,|x|^\mu}\le C\epsilon^{(N-2)/2[(2+\alpha)]}. \tag{4.16}$$

Furthermore,

$$\big|\|\tilde{u^r_\epsilon}\|^{2^*_\alpha}_{2^*_\alpha,|x|^\alpha}-\|u^r_\epsilon\|^{2^*_\alpha}_{2^*_\alpha,|x|^\alpha}\big|$$

$$
\begin{aligned}
&= 2_\alpha^* \Big| \int_0^1 \mathrm{d}s \int_{B_1} |x|^\alpha |u_\epsilon^r - sP_j u_\epsilon^r|^{2_\alpha^*-2}(u_\epsilon^r - sP_j u_\epsilon^r) P_j u_\epsilon^r \, \mathrm{d}x \Big| \\
&\le 2_\alpha^* \cdot 2^{2_\alpha^*-1} \int_0^1 \mathrm{d}s \int_{B_1} |x|^\alpha \big\{ |u_\epsilon^r|^{2_\alpha^*-1} + s^{2_\alpha^*-1} |P_j u_\epsilon^r|^{2_\alpha^*-1} \big\} |P_j u_\epsilon^r| \, \mathrm{d}x \\
&\le C \big\{ \|u_\epsilon^r\|_{2_\alpha^*-1,|x|^\alpha}^{2_\alpha^*-1} \|P_j u_\epsilon^r\|_{\infty,|x|^\mu} + \|P_j u_\epsilon^r\|_{2,|x|^\alpha}^{2_\alpha^*} \big\}.
\end{aligned}
$$

Then from item (e) in Lemma 4.1, (4.15) and (4.16), we obtain

$$
\big| \|\tilde{u_\epsilon^r}\|_{2_\alpha^*,|x|^\alpha}^{2_\alpha^*} - \|u_\epsilon^r\|_{2_\alpha^*,|x|^\alpha}^{2_\alpha^*} \big| \le C\epsilon^{(N-2)/(2+\alpha)}. \tag{4.17}
$$

By item (e) in Lemma 4.1 and (4.16), we notice that

$$
\begin{aligned}
\|\tilde{u_\epsilon^r}\|_{2_\alpha^*-1,|x|^\alpha}^{2_\alpha^*-1} &= \|u_\epsilon^r - P_j u_\epsilon^r\|_{2_\alpha^*-1,|x|^\alpha}^{2_\alpha^*-1} \\
&\le C\{ \|u_\epsilon^r\|_{2_\alpha^*-1,|x|^\alpha}^{2_\alpha^*-1} + \|P_j u_\epsilon^r\|_{2_\alpha^*-1,|x|^\alpha}^{2_\alpha^*-1} \} \\
&\le C\epsilon^{(N-2)/[2(2+\alpha)]}.
\end{aligned} \tag{4.18}
$$

As for (4.18), using item (d) in Lemma 4.1 and (4.16), we obtain

$$
\|\tilde{u}_\epsilon^r\|_{1,|x|^\mu} \le C\epsilon^{(N-2)/[2(2+\alpha)]}. \tag{4.19}
$$

Based on these estimates, we can conclude the following lemma.

Lemma 4.7. *For ϵ sufficiently small and positive, we have*

$$
\frac{\|\tilde{u_\epsilon^r}\|^2 - \lambda \|\tilde{u_\epsilon^r}\|_{2,|x|^\mu}^2}{\|\tilde{u_\epsilon^r}\|_{2_\alpha^*,|x|^\alpha}^2} = S_\alpha - C\epsilon^{(2+\mu)/(2+\alpha)} \quad \text{if } N > 4 + \mu. \tag{4.20}
$$

The proof of the above lemma follows from (4.17), (4.18) and (4.19), and arguments similar to those in Lemma 4.4. Now, we define

$$
\widetilde{W}(\epsilon) = \{ u \in H_{0,\mathrm{rad}}^1(B_1) : u = u^- + t\tilde{u_\epsilon^r},\ u^- \in H_2,\ t \in \mathbb{R} \}.
$$

Arguments analogous to those used in the Lemma 4.5, guarantee the following result.

Lemma 4.8. *Suppose* (1.9) *and $\lambda = \lambda_j^*$, for some $j \in \mathbb{N}$. Then, for ϵ positive and sufficiently small,*

$$
\sup_{\widetilde{W}(\epsilon)} J_\lambda(u) < \frac{(2+\alpha)}{2(N+\alpha)} S_\alpha^{(N+\alpha)/(2+\alpha)}. \tag{4.21}
$$

4.3. **Non resonant case with $N < 4 + \mu$.** In this case, to conclude a similar result to Lemma 4.5, we need another condition on λ. More precisely, we should have λ sufficiently large to guarantee that the minimax levels are below a suitable constant.

Lemma 4.9. *Suppose $N < 4 + \mu$ and $\lambda \ne \lambda_j^*$, for some $j \in \mathbb{N}$. Then, for $\epsilon > 0$ sufficiently small and λ large enough,*

$$
\sup_{W(\epsilon,r)} J_\lambda(u) < \frac{(2+\alpha)}{2(N+\alpha)} S_\alpha^{(N+\alpha)/(2+\alpha)}. \tag{4.22}
$$

Proof. As in Lemma 4.5, we need to show that

$$
\|u_\epsilon^r\|^2 - \lambda \|u_\epsilon^r\|_{2,|x|^\mu}^2 < S_\alpha, \tag{4.23}
$$

when $\lambda \neq \lambda_j^*$ for all $j \in \mathbb{N}$. Thus, following the same steps as in Lemma 4.5, and using (4.5) we obtain

$$\begin{aligned}\|u\|^2 - \lambda\|u\|^2_{2,|x|^\mu} \leq &\Big(S_\alpha + \epsilon^{(N-2)/(2+\alpha)}(O(1) - \lambda C)\Big)\\ &\times \Big(1 + C\epsilon^{[(N-2)(N+\alpha)]/[(2+\alpha)(N+2\alpha+2]}\Big) + A(u^-, \epsilon, C).\end{aligned}$$

Therefore, for ϵ positive and small enough, and λ sufficiently large, we obtain (4.23). □

5. Proof of main results

It is clear that $J_\lambda \in C^1(H^1_{0,\mathrm{rad}}(B_1), \mathbb{R})$ and complies with condition (f_1) of Theorem 2.1. Then Lemma 2.3 ensures that (2) in Theorem 2.1 is satisfied with $\beta = \frac{(2+\alpha)}{2(N+\alpha)} S_\alpha^{(N+\alpha)/(2+\alpha))}$.

If $0 < \lambda \neq \lambda_j^*$ for all $j \in \mathbb{N}$, we set $V = H_1$ and $W = W(\epsilon, r)$ with ϵ small enough to satisfy Lemma 4.5 for $N > 4 + \mu$, when (1.9) is satisfied, or Lemma 4.6 for $N = 4 + \mu$. Then (3)(iii) in Theorem 2.1 holds in both cases. Thus, (3)(i)) and (3)(ii)) are satisfied by Lemmas 3.1, 4.5 and 4.6, respectively. Since $\dim(V \cap W) = 1$ and $V + W = H^1_{0,\mathrm{rad}}(B_1)$, from Theorem 2.1, it follows that (1.1) has at least one non trivial solution.

If $0 < \lambda = \lambda_j^*$ for some $j \in \mathbb{N}$ and $N > 4 + \mu$, when (1.9) is true, we conclude this result repeating the above arguments using $W = \widetilde{W}(\epsilon)$ and the Lemma 4.8 and 3.1.

For $N < 4+\mu$, following the same steps as in the two previous cases, Lemmas 4.9 and 3.1 with $H_1 = H^1_{0,\mathrm{rad}}(B_1)$, we obtain the conclusion by applying Ambrosetti-Rabinowitz Mountain Pass Theorem [39]. Recall that there is a function $e \in H_1$ such that $J_\lambda(e) \leq 0$. By standard arguments and the maximum principle, we can show the solution is positive. This completes the proof.

Remark 5.1. We know that

$$J'_\lambda(v)w = 0, \quad \forall w \in H^1_{0,\mathrm{rad}}(B_1), \tag{5.1}$$

and v is a critical point of the functional J_λ restricted to the space $H^1_{0,rad}(B_1)$. Now, we follow the ideas of [6, 23, 33]. Since $H^1_{0,\mathrm{rad}}(B_1)$ is a closed subspace of $H^1_0(B_1)$, we can write

$$H^1_0(B_1) = H^1_{0,\mathrm{rad}}(B_1) \oplus H^1_{0,\mathrm{rad}}(B_1)^\perp,$$

where $^\perp$ denotes the orthogonal complement of the space. Therefore, for each $w \in H^1_0(B_1)$, there exist $\vartheta \in H^1_{0,\mathrm{rad}}(B_1)$ and $\vartheta^\perp \in H^1_{0,\mathrm{rad}}(B_1)^\perp$ such that

$$w = \vartheta + \vartheta^\perp. \tag{5.2}$$

Since $H^1_{0,\mathrm{rad}}(B_1)$ is a Hilbert space and $J'_\lambda(v) \in H^1_{0,\mathrm{rad}}(B_1)^*$, from the Riesz Representation Theorem there exists $z \in H^1_{0,\mathrm{rad}}(B_1)$ such that

$$J'_\lambda(v)w = \int_{B_1} \nabla z \cdot \nabla w \,\mathrm{d}x, \quad \text{for all } w \in H^1_{0,\mathrm{rad}}(B_1).$$

Thus, $J'_\lambda(v) \approx z$, as $z \in H^1_{0,\mathrm{rad}}(B_1)$ and $\vartheta^\perp \in H^1_{0,\mathrm{rad}}(B_1)^\perp$, we have

$$J'_\lambda(v)\vartheta^\perp = 0. \tag{5.3}$$

From (5.1), (5.2) and (5.3), for each $w \in H_0^1(B_1)$, we obtain

$$J'_\lambda(v)w = J'_\lambda(v)\vartheta + J'_\lambda(v)\vartheta^\perp = 0.$$

This implies that v is a critical point of the functional J_λ in $H_0^1(B_1)$ and consequently v is a weak solution for problem (1.1).

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