Electronic Journal of Differential Equations, Vol. 2021 (2021), No. 34, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

DIRICHLET (p,q)-EQUATIONS WITH GRADIENT DEPENDENT AND LOCALLY DEFINED REACTION

ZHENHAI LIU, NIKOLAOS S. PAPAGEORGIOU

Abstract. We consider a Dirichlet (p,q)-equation, with a gradient dependent reaction which is only locally defined. Using truncations, theory of nonlinear operators of monotone type, and fixed point theory (the Leray-Schauder Alternative Theorem), we show the existence of a positive smooth solution.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this article we study the (p,q)-equation with gradient dependence (convection)

$$\begin{gathered} -\Delta_p u(z) - \Delta_q u(z) = f(z, u(z), Du(z)) \quad \text{in}\Omega, \\ u|_{\partial\Omega} = 0, \quad u > 0, \quad 1 < q < p. \end{gathered} \tag{1.1}$$

Given $r \in (1, +\infty)$ by Δ_r we denote the r-Laplace differential operator by

$$\Delta_r u = \operatorname{div}(|Du|^{r-2} Du) \quad \text{for all } u \in W_0^{1,r}(\Omega).$$

In problem (1.1) we have the sum of two such operators ((p,q)-equation). So the differential operator (left hand side) of the problem is not homogeneous. The reaction term (right hand side) of (1.1), depends also one the gradient of u (convection). This classifies the problem as non-variational and for this reason our method of proof is topological and uses the fixed point theory (in particular, the Leray-Schauder Alternative Principle). Our aim is to obtain positive solutions.

Recently such problems were studied by Faraci-Motreanu-Puglisi [5], Gasiński-Papageorgiou [7], Hu-Papageorgiou [9], Liu-Motreanu-Zeng [12], Papageorgiou-Vetro-Vetro [17], Papageorgiou-Zhang [18] (problems with Laplacian or p-Laplacian), Bai [2], Bai-Gasinski-Papageorgiou [4], Gasinski-Winkert [8], Liu-Papageorgiou [13] (nonlinear nonhomogeneous problems), and Bai-Gasinski-Papageorgiou [3], Papageorgiou-Radulescu-Repovs [15], Papageorgiou-Zhang [19], (problems with singular and convection terms). In all the aforementioned works, it is required that the reaction is nonnegative and/or it satisfies a restrictive growth condition involving the principal eigenvalue of the Dirichlet p-Laplacian (see, for example, [5, 7, 8]). In contrast here the reaction term is sign-changing and exhibits an oscillatory behavior near zero (namely the reaction function starts positive and at a certain point becomes strictly negative). Moreover, $f(z, \cdot, y)$ is only locally defined (near zero).

2010 *Mathematics Subject Classification.* 35J20, 35J60,35J92.
Key words and phrases. (p,q)-differential operator; convection; fixed point; nonlinear; regularity; positive solution; Leray-Schauder alternative theorem.

Submitted October 17, 2020. Published April 30, 2021.

Our approach differs from the above works which employed the so-called "frozen variable method" (see Liu-Papageorgiou [13]). Here instead, we use the theory of nonlinear operators of monotone type.

2. Mathematical background-hypotheses

Let X be a Banach space and $g : X \to X$ a map. We say that $g(\cdot)$ is compact, if it is continuous and maps bounded sets to relatively compact sets. We will use the Leray-Schauder Alternative Principle that asserts the following.

Theorem 2.1. *If X is a Banach space, $g : X \to X$ is a compact map and $D = \{x \in X : x = tg(x) \text{ for some } 0 < t < 1\}$, then one of the following statements holds*

(a) *D is unbounded, or*
(b) *g admits a fixed point.*

We consider the nonlinear eigenvalue problem

$$-\Delta_q u(z) = \hat{\lambda}|u(z)|^{q-2}u(z) \quad \text{in } \Omega, u|_{\partial\Omega} = 0. \tag{2.1}$$

An "eigenvalue" of (2.1), is a number $\hat{\lambda} \in \mathbb{R}$ such that problem (2.1) admits a nontrivial solution $\hat{u} \in W_0^{1,q}(\Omega)$, called an "eigenfunction" corresponding to the eigenvalue $\hat{\lambda}$. Nonlinear regularity theory (see, for example, Gasinski-Papageorgiou [6, Section 6.2]) implies that $\hat{u} \in C^1(\bar{\Omega})$. We know that problem (2.1) admits a smallest eigenvalue $\hat{\lambda}_1(q) > 0$ such that

- $\hat{\lambda}_1(q)$ is isolated (that is, we can find $\varepsilon > 0$ such that $(\hat{\lambda}_1(q), \hat{\lambda}_1(q) + \varepsilon)$ contains no eigenvalue);
- $\hat{\lambda}_1(q)$ is simple (that is, if $\hat{u}, \hat{v} \in C_0^1(\bar{\Omega})$ are eigenfunctions corresponding to $\hat{\lambda}_1(q)$, then $\hat{u} = \theta\hat{v}$ for some $\theta \in \mathbb{R} \setminus \{0\}$);
- $$\hat{\lambda}_1(q) = \inf[\frac{\|Du\|_q^q}{\|u\|_q^q} : u \in W_0^{1,q}(\Omega), u \neq 0]. \tag{2.2}$$

In (2.2) the infimum is realized on the corresponding one dimensional eigenspace. From the above properties it follows that the elements of this eigenspace do not change sign. For every other eigenvalue $\hat{\lambda} \neq \hat{\lambda}_1(q)$ the corresponding eigenfunctions are nodal (sign-changing).

We will also need the following weighted version of the eigenvalue problem (2.1)

$$-\Delta_q u(z) = \tilde{\lambda} m(z)|u(z)|^{q-2}u(z) \quad \text{in } \Omega, u|_{\partial\Omega} = 0. \tag{2.3}$$

Here $m \in L^\infty(\Omega), m(z) \geq 0$ for a.a. $z \in \Omega, m \not\equiv 0$. Then (2.3) has a smallest eigenvalue $\tilde{\lambda}_1(m,q) > 0$, which is isolated, simple and admits the variational characterization

$$\tilde{\lambda}_1(m,q) = \inf\big[\frac{\|Du\|_q^q}{\int_\Omega m(z)|u|^q dz} : u \in W_0^{1,q}(\Omega), u \neq 0\big]. \tag{2.4}$$

Again the infimum in (2.4) is realized on the corresponding one dimensional eigenspace, the elements of which have fixed sign and belong in $C_0^1(\bar{\Omega})$. Let $C_+ = \{u \in C_0^1(\bar{\Omega}) : u(z) \geq 0 \text{ for all } z \in \bar{\Omega}\}$ (the positive (order) cone of $C_0^1(\bar{\Omega})$). This cone has a nonempty interior given by

$$\inf C_+ = \big\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}\big|_{\partial\Omega} < 0\big\}$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The nonlinear maximum principle (see Gasiński-Papageorgiou [6],p.738), implies that the eigenfunctions corresponding to $\tilde\lambda_1(m,q) > 0$ are in $\operatorname{int} C_+$ or in $-\operatorname{int} C_+$.

Using all these properties, we infer the following strict monotonicity property for the map $m \to \tilde\lambda_1(m,q)$.

Proposition 2.2. *If $m, m' \in L^\infty(\Omega)$, $0 \le m(z) \le m'(z)$ for a.a. $z \in \Omega$, $m \ne 0$, and $m \ne m'$, then $\tilde\lambda_1(m',q) < \tilde\lambda_1(m,q)$.*

Our conditions on the reaction term $f(z,x,y)$ are the following:

(H1) $f : \Omega \times \mathbb{R} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0,0) = 0$ for a.a. $z \in \Omega$ and
  (i) if $|f(z,x,y)| \le a(z)[1 + x^{p-1} + |y|^{p-1}]$ for a.a. $z \in \Omega$, all $x \ge 0$, all $y \in \mathbb{R}^N$ with $a \in L^\infty(\Omega)$;
  (ii) there exist $M > 0$ and $\delta > 0$ such that
$$f(z,M,y) < 0 \text{ for a.a. } z \in \Omega, \text{ all } |y| \le \delta;$$
  (iii) there exist $\delta_0 > 0$ and $\eta \in L^\infty(\Omega)$ such that
$$\hat\lambda_1(q) \le \eta(z) \quad \text{for a.a. } z \in \Omega,\ \eta \ne \hat\lambda_1(q),$$
$$\eta(z)x^{q-1} \le f(z,x,y) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \le x \le \delta_0, \text{ all } y \in \mathbb{R}^N,$$
$$\limsup_{x\to 0^+} \frac{f(z,x,y)}{x^{q-1}} \le \hat c_\theta \quad \text{uniformly for a.a. } z \in \Omega, \text{ all } |y| \le \theta.$$

Evidently hypotheses (H1)(ii) is satisfied if $f(z,M,0) \le -\hat c < 0$ for a.a. $z \in \Omega$. Hypotheses (H1)(ii) and (H1)(iii) imply the oscillatory behavior of $f(z,\cdot,y)$ near zero, mentioned in the Introduction.

As examples of functions satisfy (H1) we have following, (For the sake of simplicity we drop the z-dependence).
$$f(x,y) = c_0[x^{q-1} - x^{p-1}] + c_1|y|^{p-1}$$
for all $x \ge 0$, all $y \in \mathbb{R}^{\mathbb{N}}$, with $c_0 > \hat\lambda_1(q), c_1 > 0$; and
$$f(x,y) = c_2 x^{q-1}[1 - x^{\tau-q}\ln x] + x|y|^{p-1}$$
for all $x \ge 0$, all $y \in \mathbb{R}^{\mathbb{N}}$, with $c_2 > \hat\lambda_1(q)$, $\tau \ge q$.

In what follows, $p_M : \mathbb{R} \to \mathbb{R}$ denotes the truncation function at level M, that is,
$$p_M(x) = \begin{cases} x & \text{if } x \le M, \\ M & \text{if } M < x. \end{cases}$$
Evidently $p_M(\cdot)$ is Lipschitz continuous.

Also for $x \in \mathbb{R}$, we denote $x^\pm = \max\{\pm x, 0\}$. For $u \in W_0^{1,p}(\Omega)$ we define $u^\pm(z) = u(z)^\pm$ for all $z \in \Omega$. Then $u^\pm \in W_0^{1,p}(\Omega)$, $u = u^+ - u^-$, $|u| = u^+ + u^-$. Finally for $r \in (1,\infty)$, by $A_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega) = W_0^{1,r}(\Omega)^*$ (with $\frac{1}{r} + \frac{1}{r'} = 1$), we denote the nonlinear map
$$\langle A_r(u), h\rangle = \int_\Omega |Du|^{r-2}(Du, Dh)_{\mathbb{R}^N}\,dz \quad \text{for all } u, h \in W_0^{1,p}(\Omega).$$
This map is bounded (maps bounded sets to bounded ones), continuous, strictly monotone (hence maximal monotone) and of type $(S)_+$ (see [16, p. 157]).

3. Positive solutions

In this section using the theory of nonlinear operators of monotone type and fixed point arguments based on Theorem 2.1, we show the existence of a positive smooth solution for problem (1.1).

Let $V : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ (with $\frac{1}{p} + \frac{1}{p'} = 1$) be defined by

$$V(u) = A_p(u) + A_q(u) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Proposition 3.1. *$V^{-1} : W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega)$ exists and is bounded and continuous.*

Proof. The map $V(\cdot)$ is continuous, strictly monotone (hence maximal monotone too) and coercive (since $\langle V(u), u\rangle = \|Du\|_p^p + \|Du\|_q^q$). If follows that $V(\cdot)$ is surjective (see Papageorgiou-Rădulescu-Repovš [16, Corollary 2.8.7, p. 135]. Therefore $V^{-1} : W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega)$ is well-defined and on account of the coercivity of $V(\cdot), V^{-1}(\cdot)$ is bounded (maps bounded sets to bounded ones). We examine the continuity of $V(\cdot)$. So, let $u_n^* \to u^*$ in $W^{-1,p'}(\Omega)$ and set $u_n = V^{-1}(u_n^*) \in W_0^{1,p}(\Omega)$ for all $n \in \mathbb{N}$. Then $u_n^* = V(u_n)$ for all $n \in \mathbb{N}$ which implies $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded, using the coercivity of $V(\cdot)$).

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to \infty.$$

We have that $\langle V(u_n), u_n - u\rangle = \langle u_n^*, u_n - u\rangle \to 0$, which implies $\|Du_n\|_p \to \|Du\|_p$, which in turn implies $u_n \to u$ in $W_0^{1,p}(\Omega)$, by the Kadec-Klee property of $W_0^{1,p}(\Omega)$); this implies $V(u) = u^*$ which in turn implies $u = V^{-1}(u^*)$ and so $V^{-1}(\cdot)$ is continuous. □

For $\varepsilon > 0$, let $\hat{f}_M^\varepsilon : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be the Carathéodory function defined by

$$\hat{f}_M^\varepsilon(z, x, y) = f(z, p_M(x) + \varepsilon, y).$$

Let $N_{\hat{f}_M^\varepsilon} : W_0^{1,p}(\Omega) \to L^{p'}(\Omega)$ be the corresponding Nemytskii (superposition) operator, defined by

$$N_{\hat{f}_M^\varepsilon}(u)(\cdot) = f(\cdot, p_M(u(\cdot)) + \varepsilon, Du(\cdot)) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

On account of hypothesis (H1)(i) and using Krasnoselskii's theorem (see, for example, Gasiński-Papageorgiou [6, Theorem 3.4.4, p. 407]), we have that

$$N_{\hat{f}_M^\varepsilon} : W_0^{1,p}(\Omega) \to L^{p'}(\Omega) \text{ is continuous .} \tag{3.1}$$

Also, let $i_+ : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ be defined by

$$i_+(u) = u^+ \quad \text{for all } u \in W_0^{1,p}(\Omega). \tag{3.2}$$

We introduce the map $\hat{N}_\varepsilon : W_0^{1,p}(\Omega) \to L^{p'}(\Omega)$ defined by

$$\hat{N}_\varepsilon(u) = (N_{\hat{f}_M^\varepsilon} \circ i_+)(u) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (3.1) and (3.2) we see that

$$\hat{N}_\varepsilon(\cdot) \text{ is bounded and continuous.} \tag{3.3}$$

We set $K_\varepsilon = V^{-1} \circ \hat{N}_\varepsilon$.

Proposition 3.2. *If* (H1) *holds, then $K_\varepsilon : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ is compact.*

Proof. From Proposition 3.1 and (3.3), we infer that $K_\varepsilon(\cdot)$ is continuous. Let $B \subseteq W_0^{1,p}(\Omega)$ be bounded. From (3.3) we have that

$$\hat{N}_\varepsilon(B) \subseteq L^{p'}(\Omega) \text{ is bounded }. \tag{3.4}$$

From the Sobolev embedding theorem, we know that $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ compactly and densely.

Invoking [6, Lemma 2.2.27, p. 141] and Schauder's Theorem (see Gasinski-Papageorgiou [6, Theorem 3.1.22, p. 275]) we have that

$$L^{p'}(\Omega) = L^p(\Omega)^* \hookrightarrow W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \text{ compactly and densely.}$$

Then from (3.4) it follows that $\hat{N}_\varepsilon(B) \subseteq W^{-1,p'}(\Omega)$ is relatively compact, Therefore, $\hat{N}_\varepsilon(\cdot)$ is a compact map. $\square$

Let $0 < \varepsilon \le \delta_0$ and define

$$D_\varepsilon = \{u \in W_0^{1,p}(\Omega) : u = tK_\varepsilon(u) \text{ for some } 0 < t < 1\}.$$

Proposition 3.3. *If* (H1) *holds, and* $0 < \varepsilon \le \delta_0$*, then* $D_\varepsilon \subseteq W_0^{1,p}(\Omega)$ *is bounded.*

Proof. Let $u \in D_\varepsilon$. We have

$$\frac{1}{t}u = K_\varepsilon(u) = (V^{-1} \circ \hat{N}_\varepsilon)(u),$$

which implies $V(\frac{1}{t}u) = \hat{N}_\varepsilon(u)$; therefore,

$$-\frac{1}{t^{p-1}}\Delta_p(u) - \frac{1}{t^{q-1}}\Delta_q(u) = f(z, p_M(u^+) + \varepsilon, Du^+) \quad \text{in } \Omega. \tag{3.5}$$

On account of hypothesis (H1)(iii) and since $0 < \varepsilon \le \delta_0$, from (3.5) we see that $u \neq 0$. On (3.5) we act with $-u^- \in W_0^{1,p}(\Omega)$ and obtain

$$\frac{1}{t^{p-1}}\|Du^-\|_p^p + \frac{1}{t^{q-1}}\|Du^-\|_q^q = \int_\Omega f(z,\varepsilon,0)(-u^-)dz \le 0$$

(see hypothesis (H1)(iii)). This implies $u \ge 0$, $u \neq 0$.

From (3.5) and Ladyzhenskaya-Uraltseva [10, Theorem 7.1, p. 286], we have that $u \in L^\infty(\Omega)$. Then the regularity theory of Lieberman [11] implies that $u \in C_+ \setminus \{0\}$. In fact on account of hypotheses (H1)(i) and (H1)(iii), given $r \in (p, p^*)$, we can find $c_3 = c_3(r) > 0$ such that

$$f(z,x,y) \ge \eta(z)x^{q-1} - c_3x^{r-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge 0, \text{ all } y \in \mathbb{R}^N.$$

Then from (3.5) we have

$$\Delta_p u + t^{p-q}\Delta_q u \le c_3(M+\delta_0)^{r-p}u^{r-p} \text{ in } \Omega;$$

therefore, $u \in \operatorname{int} C_+$, see Pucci-Serrin [20, pp. 111,120].

Claim: $0 \le u(z) \le M$ for all $z \in \bar{\Omega}$. Arguing by contradiction, suppose that the assertion of the Claim is not true. Then we can find $z_0 \in \Omega$ such that $u(z_0) = \max_{\bar{\Omega}} u > M$. Then we can find an open neighborhood Ω_0 of z_0, with Lipschitz boundary and $\bar{\Omega}_0 \subseteq \Omega$ such that

$$Du(z_0) = 0, \quad \frac{\partial u}{\partial n}\Big|_{\partial\Omega_0} < 0, \quad f(z, M+\varepsilon, Du(z)) \le 0, \quad \text{for a.a. } z \in \Omega_0, \tag{3.6}$$

see hypothesis (H1)(iii).

Recall that by (3.5),

$$-\Delta_p u(z) - t^{p-q}\Delta_q u(z) = t^{p-1} f(z, M+\varepsilon, Du(z)) \quad \text{for a.a. } z \in \Omega_0$$

Acting with u and using the nonlinear Green's identity (see Papageorgiou-Rădulescu-Repovš [16, p.35]), by (3.6) we have

$$\begin{aligned} 0 &\le \int_{\Omega_0} |Du|^p dz + \int_{\Omega_0} |Du|^q dz \\ &= t^{p-1} \int_{\Omega_0} f(z, M+\varepsilon, Du) dz + t^{p-1} \int_{\partial\Omega_0} \frac{\partial u}{\partial n}[|Du|^{p-2} + |Du|^{q-2}] u d\sigma < 0\,. \end{aligned}$$

This contradiction contradiction proves the Claim.

From (3.5), the Claim, and $0 < t < 1$, we have

$$\|Du\|_p^p \le M \int_\Omega |f(z, u+\varepsilon, Du)| dz,$$

which implies

$$\|Du\|_p^p \le c_4[1 + \|Du\|_p^{p-1}]$$

for some $c_4 > 0$, see hypothesis (H1)(i)). Therefore, $D_\varepsilon \subseteq W_0^{1,p}(\Omega)$ is bounded. $\square$

Propositions 3.2 and 3.3 permit the use of Theorem 2.1 (the Leray-Schauder Alternative Principle). So, for $0 < \varepsilon \le \delta_0$, we can find $u_\varepsilon \in W_0^{1,p}(\Omega)$ such that $u_\varepsilon = K_\varepsilon(u_\varepsilon)$. Therefore,

$$-\Delta_p u_\varepsilon - \Delta_q u_\varepsilon = f(z, p_M(u_\varepsilon^+) + \varepsilon, Du_\varepsilon^+) \quad \text{in } \Omega, u_\varepsilon|_{\partial\Omega} = 0.$$

From the proof of Proposition 3.3, we have

$$u_\varepsilon \in \operatorname{int} C_+ \quad \text{and} \quad 0 \le u_\varepsilon(z) \le M \text{ for all } z \in \bar\Omega.$$

Then it follows that

$$-\Delta_p u_\varepsilon(z) - \Delta_q u_\varepsilon(z) = f(z, u_\varepsilon(z) + \varepsilon, Du_\varepsilon(z)) \text{ in } \Omega, \tag{3.7}$$

which implies that

$$\{u_\varepsilon\}_{0<\varepsilon\le\delta_0} \subseteq W_0^{1,p}(\Omega) \quad \text{is bounded.} \tag{3.8}$$

We let $\varepsilon \to 0^+$ to obtain a positive solution for problem (1.1).

Theorem 3.4. *If* (H1) *holds, then* (1.1) *admits a positive solution* $\bar u \in \operatorname{int} C_+$.

Proof. Let $\varepsilon_n = 1/n$ for $n \in \mathbb{N}$ and let $u_n = u_{\varepsilon_n} \in \operatorname{int} C_+$ from (3.7). From (3.8) and the nonlinear regularity theory of Lieberman [11], we know that there exists $\alpha \in (0,1)$ such that $\{u_n\}_{n\in\mathbb{N}} \subseteq C_0^{1,\alpha}(\bar\Omega)$ is bounded.

Since $C_0^{1,\alpha}(\bar\Omega) \hookrightarrow C_0^1(\bar\Omega)$ compactly, we may assume that

$$u_n \to \bar u \quad \text{in } C_0^1(\bar\Omega). \tag{3.9}$$

From (3.7) and (3.9), if follows that

$$-\Delta_p \bar u(z) - \Delta_q \bar u(z) = f(z, \bar u(z), D\bar u(z)) \quad \text{in } \Omega, \bar u|_{\partial\Omega} = 0.$$

So, if we can show that $\bar u \ne 0$, then this will be the desired positive solution of (1.1). We argue indirectly. So, suppose that $\bar u = 0$. We set $v_n = u_n/\|u_n\|$, $n \in \mathbb{N}$. Then we have that $\|v_n\| = 1, v_n \ge 0$ for all $n \in \mathbb{N}$ and so we may assume that

$$v_n \xrightarrow{w} v \quad \text{in } W_0^{1,p}(\Omega). \tag{3.10}$$

From (3.7) we have

$$\|u_n\|^{p-q}\langle A_p(v_n), h\rangle + \langle A_q(v_n), h\rangle = \int_\Omega \frac{f(z, u_n + \frac{1}{n}, Dv_n)}{\|u_n\|^{p-1}} h dz \tag{3.11}$$

for all $h \in W_0^{1,p}(\Omega)$.

Let $\theta = \sup_{n\in\mathbb{N}} \|u_n\|_{C_0^1(\bar\Omega)} < \infty$ (see (3.9)). On account of hypotheses (H1)(i) and (H1)(iii), we have

$$|f(z, x, y)| \le c_5[x^{q-1} + x^{p-1}]$$

for a.a. $z \in \Omega$, all $x \ge 0$, all $y| \le \theta$ and some $c_5 > 0$. This implies

$$\{\frac{f(\cdot, u_n(\cdot) + \frac{1}{n}, Du_n(\cdot))}{\|u_n\|^{p-1}}\}_{n\in\mathbb{N}} \subseteq L^{p'}(\Omega) \quad \text{is bounded.} \tag{3.12}$$

From (3.11) and of Papageorgiou-Rădulescu [14, Proposition 2.10], we can find $c_6 > 0$ such that $\|v_n\|_\infty \le c_6$ for all $n \in \mathbb{N}$. Then the nonlinear regularity theory of Lieberman [11, p. 320] implies the existence of $\alpha \in (0, 1)$ and $c_7 > 0$ such that $v_n \in C_0^{1,\alpha}(\bar\Omega) = C^{1,\alpha}(\bar\Omega) \cap C_0^1(\bar\Omega)$ and $\|v_n\|_{C_0^{1,\alpha}(\bar\Omega)} \le c_7$ for all $n \in \mathbb{N}$. The compact embedding of $C_0^{1,\alpha}(\bar\Omega)$ into $C_0^1(\bar\Omega)$, implies that we may assume that $v_n \to v$ in $C_0^1(\bar\Omega)$, hence $\|v\| = 1$, $v \ge 0$. Also from (3.12) and hypothesis (H1)(iii), we have

$$\frac{f(\cdot, u_n(\cdot) + \frac{1}{n}, Du_n(\cdot))}{\|u_n\|^{p-1}} \xrightarrow{w} \eta_0(\cdot)v(\cdot)^{q-1} \quad \text{in } L^{p'}(\Omega). \tag{3.13}$$

With $\eta_0 \in L^\infty(\Omega), \eta(z) \le \eta_0(z)$ for a.a. $z \in \Omega$ (see Aizicovici-Papageorgiou-Staicu [1, Proposition 16]. If in (3.11) we pass to the limit as $n \to \infty$ and use (3.13), we obtain

$$\langle A_q(v), h\rangle = \int_\Omega \eta_0(z)v^{q-1} h dz \quad \text{for all } h \in W_0^{1,p}(\Omega),$$

which implies

$$-\Delta_q v(z) = \eta_0(z)v(z)^{q-1} \quad \text{in } \Omega, v|_{\partial\Omega} = 0, \quad v \ge 0. \tag{3.14}$$

Using Proposition 2.2, we have

$$\tilde\lambda_1(\eta_0, q) < \tilde\lambda_1(\hat\lambda_1(q), q) = 1$$

So, from (3.14) if follows that $v = 0$ or v is nodal, both cases leading to a contradiction. Therefore $\bar u \ne 0$ and as before $\bar u \in \operatorname{int} C_+$. This is the smooth positive solution of (1.1). □

Acknowledgments. This research was partially supported by the NNSF of China Grant No. 12071413, and by the NSF of Guangxi Grant No. 2018GXNSFDA138002.

References

[1] S. Aizicovici, N. S. Papageorgiou, V. Staicu; Degree theory for operators of monotone and nonlinear elliptic equations with inequality constraints, *Memoirs AMS 196*, No. 915, 2008.

[2] Y. Bai; Existence of solutions to nonhomogeneous Dirichlet problems with dependence on the gradient, *Electr. Jour. Diff. Equ.*, **2018** (2018), 101.

[3] Y. Bai, L. Gasiński, N. S. Papageorgiou; Nonlinear problems with the combined effects of singular and convection terms, *Electr. Jour. Diff. Equ.*, **2019** (2019), 57.

[4] Y. Bai, L. Gasiński, N. S. Papageorgiou; Nonlinear nonhomogeneous Robin problems with dependence on the gradient, *Bound. Value Probl.*, **2018** (2018), 17.

[5] F. Faraci, D. Motreanu, D. Puglisi; Positive solutions of quasilinear elliptic equations with dependence on the gradient, *Calc. Var.*, **54** (2015), 525-538.

[6] L. Gasiński, N. S. Papageorgiou; Nonlinear Analysis, *Series in Mathematical Analysis and Applications, 9.* Chapman & Hall/CRC, Boca Raton, FL, 2006. xii+971 pp
[7] L. Gasiński, N. S. Papageorgiou; Positive solutions for nonlinear elliptic problems with dependence on the gradient, *J. Differential Equ.*, **263** (2017), 1451-1476.
[8] L. Gasiński, P. Winkert; Existence and uniqueness results for double phase problems with convection term, *J. Differential Equ.*, **268** (2020), 4183-4193.
[9] S. Hu, N. S. papageorgiou; Positive solutions for nonliner Direchlet problems with convection, *Appl. Math. Optim.*, **82** (2020), 451-470.
[10] O. A. Ladyzhenskaya, N. N. Uraltseva; Linear and Quasilinear Elliptic Equations. Translated from the Russian by Scripta Technica, *Inc. Translation editor: Leon Ehrenpreis Academic Press*, New York-London 1968 xviii+495 pp.
[11] G. Lieberman; The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations. *Comm. Partial Differential Equations*, **16** (1991), 311-361.
[12] Z. H. Liu, D. Motreanu, S. D. Zeng; Positive solitions for nonlinear singular elliptic equations of p-Laplacian type with dependence on the gradient, *Calc. Var.*, **58** (2019) No. 28.
[13] Z. H. Liu, N. S. Papageorgiou; Positive solutions for resonant (p,q)-equations with convection, *Adv. Nonlinear Anal.*, **10** (2021), 217-232.
[14] N. S. Papageorgiou, V. D. Rădulescu; Nonlinear nonhomogeneous Robin problems with superlinear reaction term, *Adv. Nonlinear Studies*, **16** (2016), 737-764.
[15] N. S. Papageorgiou, V. D. Rădulescu, D. Repovš; Positive solutions for nonlinear Neumann problems with singular terms and convection, *J. Math. Pures Appl.*, 136 (2020), 1-21.
[16] N. S. Papageorgiou, V. D. Rădulescu, D. Repovš; Nonlinear Analysis-Theory and Methods, *Springer Nature*, Swizerland AG, 2019.
[17] N. S. Papageorgiou, C. Vetro, F. Vetro; Multiiple solutions with sign information for semilinear Neumann problems with convection, *Revista. Mat. Compl.* **33** (2020), 19-38.
[18] N. S. Papageorgiou, C. Zhang; Existence of positive solutions for nonlinear Robin problems with gradient dependence, Ann. Acad.Scient. *Fennicae. Math.*, **44** (2019), 739-753.
[19] N. S. Papageorgiou, Y. Zhang; Nonlinear nonhomogeneous Dirichlet problems with singular and convection terms, *Bound. Value Probl.*, **2020** (2020), 153.
[20] P. Pucci, J. Serrin; The Maximum Principle, *Birkhäuser*, Basel,2007.

Zhenhai Liu (corresponding author)
Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing, Yulin Normal University, Yulin 537000, China.
Guangxi Key Laboratory of Hybrid Computation and IC Design Analysis, Guangxi University for Nationalities, Nanning, Guangxi, 530006, China
Email address: zhhliu@hotmail.com

Nikolaos S. Papageorgiou
Department of Mathematics, National Technical University, Zografou Campus, 15780 Athens, Greece
Email address: npapg@math.ntua.gr