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ENERGY-DEPENDENT HAMILTONIAN IN A NUCLEAR OPTICAL MODEL

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Abstract. We study the spectral properties of a 1D model of optical potential introduced by Morillon and Romain [21] in the context of nuclear reactions. We can localize the discrete spectrum and estimate the number of eigenvalues. We also study the continuous spectrum with possibly embedded spectral singularities and give an expansion formula for an arbitrary function on a set of generalized eigenfunctions. We briefly study the resonances of a related model.

1. Introduction

The optical model in [11, 12, 31] is an approximation scheme introduced in nuclear reactions [5] theory for describing the interaction between a nucleus and an incident nucleon. It aims at replacing the complicated initial N-body problem by a single Schrödinger equation, provided that a suitable potential is exhibited. It relies on the assumption that the main characteristics of the interaction can be described with good accuracy by a single particle potential called optical potential. The price to pay is that this phenomenological potential is generally non-local, complex and energy-dependent.

Romain and Morillon [21] proposed a global spherical and dispersive optical potential for neutrons able to reproduce scattering data for a large mass range of nuclei. This phenomenogical potential is the sum of various contributions: a term of volume, a term of surface and the so-called spin-orbit term

$$U(r,E) = U_V(r,E) + U_S(r,E) + U_{SO}(r,E),$$

where the potentials are given by the formulae

$$U_V(r,E) = -[V_V(E) + iW_V(E)]f(r,R_V,a_V),$$
$$U_S(r,E) = -[V_S(E) + iW_S(E)]g(r,R_S,a_S),$$

and

$$U_{SO}(r,E) = -[V_{SO}(E) + iW_{SO}(E)]\Big(\frac{h^2}{m_\pi c}\Big)^2 \frac{1}{r}\ g(r,R_{SO},a_{SO})\vec{\ell}\cdot\vec{s}, \tag{1.1}$$

where $\vec{\ell}$ is the orbital momentum, $\vec{s}$ the spin, j the angular momentum and potentials $V_{V,S,SO}$, $W_{V,S,SO}$ (volume, surface and spin-orbit) are real functions and h, M_π, c are positive constants of quantum origin.

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The function f is the so-called Woods-Saxon form factor and g is the "derivative of Woods-Saxon" form factor. Both are given by

$$
\begin{gathered}
f(r,R,a)=\frac{1}{1+e^{\frac{r-R}{a}}},\\
g(r,R,a)=-4a\,\frac{d}{dr}f(r,R,a).
\end{gathered}
$$

The potential $U(r,E)$ is complex valued and explicitely depends on energy E through its real and imaginary parts $V_{V,S,SO}$ et $W_{V,S,SO}$ related according [21] by subtracted (renormalized) dispersion relations.

So the corresponding Schrödinger equation in the spherically symmetric case reads

$$
\begin{aligned}
&-\frac{d^2}{dr^2}v+\frac{\ell(\ell+1)}{r^2}\,v-\big\{[V_V(E)+iW_V(E)]f(r,R_V,a_V)+[V_S(E)\\
&+iW_S(E)]g(r,R_S,a_S)+\lambda_{j,\ell,s}[V_{SO}(E)\\
&+iW_{SO}(E)]\Big(\frac{h^2}{m_\pi c}\Big)^2\frac{1}{r}\,g(r,R_{SO},a_{SO})\big\}v=Ev,
\end{aligned}
\tag{1.2}
$$

for $r>0$, where $\lambda_{j,\ell,s}=j(j+1)-\ell(\ell+1)-s(s+1)$, with $j=\ell\pm 1/2$ if $\ell\neq 0$ and $j=1/2$ if $\ell=0$ and $s=1/2$. So we have $j(j+1)-\ell(\ell+1)-s(s+1)=j(j+1)-\ell(\ell+1)-3/4$ which is zero for S-waves corresponding to $j=1/2,\ell=0$.

In the following we consider a simplified version of (1.2) restricted to S-waves ($\ell=0$) leading to the 1D problem

$$
\begin{gathered}
-\frac{d^2}{dr^2}v+U(r,E)v=Ev\quad E\in\mathbb{C},\ r>0,\\
u(0,E)=0,
\end{gathered}
\tag{1.3}
$$

where the potential U has the form

$$
U(r,E)=\sum_{j=1}^{N}V_j(r)W_j(E),
\tag{1.4}
$$

and N is a given integer.

To comply with the model (1.1) we assume that

$$
V_j\in L^2(0,\infty;\mathbb{R})\cap L^1(0,\infty;\mathbb{R}),
\tag{1.5}
$$

and to take advantage of Jost theory [23] we assume the following analytic properties of $W_j(z)$ for $z\in\mathbb{C}$.

(1) The functions $z\to W_j(z)$ for $j=1,\dots N$ are meromorphic in $\mathbb{C}$ but holomorphic in the physical half-plane $\{\Im m(z)>0\}$, with single poles defined by $z_{j,m}=E_F-iB_{j,m}$ for $m=1,\dots,M$ with $B_{j,m}>0$.
(2) $|W_j(z)|\le\overline{W}_j$ when $|z|\to\infty$,

and we denote by Γ_W the common analyticity domain of the family $\{W_j(z)\}_{j\le N}$

$$
\Gamma_W=\mathbb{C}\backslash\big(\cup_{j=1}^N\cup_{m=1}^M z_{j,m}\big).
\tag{1.6}
$$

Note that because the two previous potentials W_j are complex, the hamiltonian $H:=\frac{d^2}{dr^2}+U$ is not self-adjoint and the standard spectral theory [8] has to be adapted.

The 1D selfadjoint Schrödinger equation is a classical topic, and is the matter of a number of books (among them [4, 8, 23]). The 1D non-selfadjoint case goes back to the pioneer work of Naimark [22] (and a large part of our study relies on this work). Among his followers, let us quote Lyantse [19, 20], Pavlov [24, 25, 26], Krall [17] and more recently Folland [13], Hruščev [16] and Brown [6].

Concerning energy-dependent potentials, let us mention studies by Friedman and Mishoe [14] and Bairamov and coauthors [1, 2, 3, 18], for specific types of energy dependence.

This article is organized as follows: In Section 2 we consider special solutions of (1.3). In Section 3 we study the discrete spectrum of H. In Section 4 we analyze the continuous spectrum of H together with its scattering properties. Finally in Section 5 we study resonances of the model.

2. Some special solutions of (1.3)

In analogy with classical studies of 1D selfadjoint operators [8, 23], we introduce several particular solutions of (1.3).

2.1. **The solution $\phi(r,\rho)$.** Taking benefit of the monodimensional geometry, for $E=\rho^2$ we classically [4] introduce the problems

$$\begin{gathered} H\phi = \rho^2\phi \quad \rho\in\mathbb{C},\ r>0,\\ \phi(0,\rho)=0,\\ \phi'(0,\rho)=1, \end{gathered} \tag{2.1}$$

and

$$\begin{gathered} H\psi = \rho^2\psi \quad \rho\in\mathbb{C},\ r>0,\\ \psi(0,\rho)=1,\\ \psi'(0,\rho)=0, \end{gathered} \tag{2.2}$$

where $Hv=-v''+Uv$. It is well known that both these both of these two problem have a unique solution $\phi(r,E)$ and $\psi(r,E)$, holomorphic in the positive complex half-plane $\{\Im m(E)>0\}$ for any $r\geq 0$.

Lemma 2.1. *For any $r\geq 0$ and $\rho\neq 0$ with $\Im m(\rho)\geq 0$ one has*

$$|\rho\phi(r,\rho^2)| \leq \exp\Big(r\Im m(\rho)+\sum_{j=1}^N \overline{W}_j\int_0^r |V_j(r')|\,dr'\Big). \tag{2.3}$$

Proof. Putting $E=\rho^2$ for $E>0$, ϕ satisfies the equation

$$\phi(r,\rho^2)=\frac{\sin\rho r}{\rho}+\int_0^r \frac{\sin\rho(r-r')}{\rho}U(r',\rho^2)\phi(r',\rho^2)\,dr'. \tag{2.4}$$

Defining $z(r,\rho^2):=\rho e^{i\rho r}\phi(r,\rho^2)$, z solves

$$z(r,\rho)=e^{i\rho r}\sin\rho r+\int_0^r \sin(\rho(r-r'))e^{i\rho(r-r')}U(r',\rho^2)z(r',\rho)\,dr'. \tag{2.5}$$

Supposing that $\Im m\rho\geq 0$, the first term in the right-hand side is less that 1; then $|z(r,\rho)|\leq 1+\int_0^r |U(r',\rho^2)z(r',\rho)|\,dr'$.

Multiplying (2.5) by $|U(r,\rho^2)|\,\big(1+\int_0^r |U(r',\rho^2)z(r',\rho)|\,dr'\big)^{-1}$ we find the bound:

$$\log\Big(1+\int_0^r |U(r,\rho^2)z(r',\rho)|\,dr'\Big)\leq \int_0^r |U(r',\rho^2)|\,dr' \leq \sum_{j=1}^N \overline{W}_j\int_0^r |V_j(r')|\,dr',$$

which gives

$$|z(r,\rho)| \le \exp\Big(\sum_{j=1}^{N} \overline{W}_j \int_0^r |V_j(r')|\,dr'\Big).$$

Then estimate (2.3) follows. □

2.2. **The solution** $e(r,\rho)$**.** Denoting by σ_j the well-defined functions

$$\sigma_j(r) = \int_r^\infty |V_j(r')|\,dr' \quad \text{for } j = 1,\dots,N, \tag{2.6}$$

we have the following result.

Proposition 2.2. *(1) The equation*

$$-y'' + U(r,\rho^2)y = \rho^2 y, \tag{2.7}$$

admits a solution $y = e(r,\rho)$ *satisfying the integral equation*

$$e(r,\rho) = e^{i\rho r} - \int_r^\infty \frac{\sin(\rho(r-r'))}{\rho} U(r',\rho^2)e(r',\rho)\,dr', \tag{2.8}$$

for $\rho \ne 0$ *and* $\Im m(\rho) \ge 0$.

(2) For any $\delta > 0$ *and for* $r \to \infty$

$$\begin{gathered} e(r,\rho) = e^{i\rho r}(1 + o(1)),\\ \partial_r e(r,\rho) = e^{i\rho r}(i\rho + o(1)), \end{gathered} \tag{2.9}$$

uniformly with respect to ρ *in the domain* $\{\Im m(\rho) \ge 0,\ |\rho| > \delta\}$.

Moreover for $\Im m(\rho) \ge 0$ *and* $|\rho| \to \infty$,

$$\begin{gathered} e(r,\rho) = e^{i\rho r}(1 + o(\rho^{-1})),\\ \partial_r e(r,\rho) = ie^{i\rho r}(1 + o(\rho^{-1})), \end{gathered} \tag{2.10}$$

uniformly with respect to r *in the domain* $\{r \ge 0\}$.

(3) For any $r \ge 0$ *the function* $\rho \to e(r,\rho)$ *is holomorphic in the open half-plane* $\{\Im m(\rho) > 0\}$.

Proof. (1) It is clear that $e(r,\rho)$ given by (2.8) satisfies (2.7). Now noting $\varepsilon(r,\rho) = e^{-i\rho r}e(r,\rho)$, we see that ε satisfies

$$\varepsilon(r,\rho) = 1 + \frac{1}{2i\rho}\int_r^\infty [e^{2i\rho(r'-r)} - 1]U(r',\rho^2)\varepsilon(r',\rho)\,dr'. \tag{2.11}$$

We look for a solution of (2.11) in the form of the expansion

$$\varepsilon(r,\rho) = \sum_{n=0}^\infty \varepsilon_n(r,\rho), \tag{2.12}$$

for a sequence $\{\varepsilon_n\}_{n\ge 0}$ given iteratively by

$$\varepsilon_{n+1}(r,\rho) = \frac{1}{2i\rho}\int_r^\infty [e^{2i\rho(r'-r)} - 1]U(r',\rho^2)\varepsilon_n(r',\rho)\,dr' \quad \text{for } n \ge 0,$$

and $\varepsilon_0(r,\rho) = 1$.

Observing that $|e^{2i\rho(r'-r)} - 1| \le 2$ when $r' \ge r$ and $\Im m(\rho) \ge 0$, we deduce that

$$|\varepsilon_{n+1}(r,\rho)| \le \frac{1}{|\rho|}\int_r^\infty |U(r',\rho^2)|\,|\varepsilon_n(r',\rho)|\,dr'.$$

Supposing by induction that

$$|\varepsilon_n(r',\rho)| \le \frac{1}{n!|\rho|^n} q^n(r),$$

where

$$q(r) := \frac{1}{|\rho|}\sum_{j=1}^N \overline{W}_j \sigma_j(r),$$

we obtain

$$|\varepsilon_{n+1}(r,\rho)| \le \frac{1}{|\rho|^{n+1}}\int_r^\infty \frac{dq(r')}{dr'} q^n(r')\,dr' \le \frac{1}{(n+1)!|\rho|^{n+1}} q^{n+1}(r).$$

Then for $\rho \neq 0$, series (2.12) is uniformly convergent and its sum $\varepsilon(r,\rho)$ is uniformly bounded on any domain $\{\Im m(\rho) \ge 0, |\rho| > \delta\}$ by $e^{q(r)/|\rho|}$.

(2) Going back to e with

$$e(r,\rho) = e^{i\rho r}\Big(1 + \sum_{n=1}^\infty \varepsilon_n(r,\rho)\Big),$$

and using estimate on ε_n in (1), asymptotic statements (2.9) and (2.10) follow.

(3) The result holds after applying analytic Fredholm theory to the integral equation (2.11) depending analytically on the parameter ρ in the open set $\{\Im m\rho > 0\}$ (see [28, 30]). $\square$

Proposition 2.3. *If the integrals*

$$\sigma_j'(r) = \int_r^\infty r'|V_j(r')|\,dr' \quad \textit{for } j = 1,\dots,N, \tag{2.13}$$

are convergent, then the solution e of (2.7) is well defined for $\rho = 0$ and

$$e(r,\rho) = e^{i\rho r} + \int_r^\infty K(r,r')e^{i\rho r'}\,dr'. \tag{2.14}$$

where the kernel K is continuous with respect to r and r' and satisfies the bounds

$$\begin{gathered}
|K(r,r')| \le \frac{1}{2}\exp\Big(\sum_{j=1}^N \overline{W_j}\sigma_j'(r)\Big)\sum_{j=1}^N \overline{W_j}\sigma_j\Big(\frac{r+r'}{2}\Big),\\
|\partial_r K(r,r')|, |\partial_{r'}K(r,r')| \le \frac{1}{4}\sum_{j=1}^N \overline{W_j}\Big|V_j\Big(\frac{r+r'}{2}\Big)\Big| + \exp\Big(\sum_{j=1}^N \overline{W_j}\sigma_j(r)\Big), \\
\times \sum_{j=1}^N \overline{W_j}\sigma_j\Big(\frac{r+r'}{2}\Big)\sum_{j=1}^N \overline{W_j}\sigma_j(r)),
\end{gathered} \tag{2.15}$$

Proof. Comparing (2.8) and (2.14) one has

$$\int_r^\infty K(r,r')e^{i\rho r'}\,dr' = -\int_r^\infty \frac{\sin(\rho(r-r'))}{\rho} U(r',\rho^2)e(r',\rho)\,dr'.$$

In the right-hand side, replacing $e(r',\rho)$ by its expression in (2.14) we obtain

$$\begin{aligned}
&-\int_r^\infty K(r,r')e^{i\rho r'}\,dr'\\
&= \int_r^\infty \frac{\sin(\rho(r-r'))}{\rho} U(r',\rho^2)\Big[e^{i\rho r'} + \int_{r'}^\infty K(r',r'')e^{i\rho r''}\,dr''\Big]\,dr'
\end{aligned}$$

$$
\begin{aligned}
&= \int_r^\infty \frac{\sin(\rho(r-r'))}{\rho} U(r',\rho^2) e^{i\rho r'} dr' \\
&\quad + \int_r^\infty \frac{\sin(\rho(r-r'))}{\rho} U(r',\rho^2) \int_{r'}^\infty K(r',r'') e^{i\rho r''} dr''\, dr' \\
&=: J_1 + J_2.
\end{aligned}
$$

Using the identity $\frac{\sin(\rho(r-r'))}{\rho} e^{i\rho r'} = \frac{1}{2}\int_r^{2r'-r} e^{i\rho r''} dr''$ and applying Fubini's theorem ($\sigma_j'(r) < \infty$), we obtain

$$
J_1 = \int_r^\infty e^{i\rho r''} dr'' \frac{1}{2} \int_{\frac{r+r''}{2}} V_j(r') dr'.
$$

In the same stroke, using identity $\frac{\sin(\rho(r-r'))}{\rho} e^{i\rho r''} = \frac{1}{2}\int_{r-r'+r''}^{r'-r+r''} e^{i\rho\tau} d\tau$ we obtain

$$
\begin{aligned}
J_2 = \int_r^\infty & e^{i\rho r''} \Big\{ \frac{1}{2} \int_r^{\frac{r+r''}{2}} U(\xi,\rho^2) \int_{r''+\xi-r}^{r''+r-\xi} K(\xi,\eta)\, d\eta\, d\xi \\
&+ \frac{1}{2} \int_{\frac{r+r''}{2}}^\infty U(\xi,\rho^2) \int_\xi^{r''+\xi-r} K(\xi,\eta)\, d\eta\, d\xi \Big\} dr''.
\end{aligned}
$$

Then K satisfies the integral equation

$$
\begin{aligned}
K(r,r') = \frac{1}{2} \int_{\frac{r+r'}{2}}^\infty U(\xi,\rho^2) d\xi + \frac{1}{2} \int_r^{\frac{r+r'}{2}} U(\xi,\rho^2) d\xi \int_{r'+r-\xi}^{r'+\xi-r} K(\xi,\eta)\, d\eta \\
+ \frac{1}{2} \int_{\frac{r+r'}{2}}^\infty U(\xi,\rho^2) d\xi \int_\xi^{r'+\xi-r} K(\xi,\eta)\, d\eta,
\end{aligned}
\tag{2.16}
$$

for any $0 \le r \le r'$. Similarly as in Proposition 2.2, we look for a solution in the form of the expansion

$$
K(r,r') = \sum_{n=0}^\infty K_n(r,r'), \tag{2.17}
$$

for a sequence $\{K_n\}_{n\ge 0}$ given iteratively by

$$
\begin{aligned}
K_{n+1}(r,r') = \frac{1}{2} \int_r^{\frac{r+r'}{2}} U(\xi,\rho^2) \int_{r'+r-\xi}^{r'+\xi-r} K_n(\xi,\eta)\, d\eta\, d\xi \\
+ \frac{1}{2} \int_{\frac{r+r'}{2}}^\infty U(\xi,\rho^2) \int_\xi^{r'+\xi-r} K_n(\xi,\eta)\, d\eta\, d\xi,
\end{aligned}
$$

for $n \ge 0$, and

$$
K_0(r,r') = \frac{1}{2} \int_{\frac{r+r'}{2}}^\infty U(\xi,\rho^2)\, d\xi.
$$

We conclude by induction that

$$
|K_n(r,r')| \le \frac{1}{2} \sum_{j=1}^N \overline{W_j} \sigma_j\Big(\frac{r+r'}{2}\Big) \frac{1}{n!} \Big[\sum_{j=1}^N \overline{W_j} \sigma_j'(r)\Big]^n \quad \text{for } n \ge 0,
$$

which gives the first bound. Bounds on derivatives are obtained in the same way by derivating equation (2.16) and using the first bound (2.15). $\square$

2.3. **Solution** $\widetilde{e}(r,\rho)$**.** By Proposition 2.2 we know that $r \to e(r,\rho)$ is asymptotic to the free solution $e^{i\rho r}$. Let us show that a similar asymptotic result holds for another solution $\widetilde{e}(r,\rho)$ which will be asymptotic to $e^{-i\rho r}$.

Proposition 2.4. *For any $\delta > 0$ there exists a positive number $a = a_\delta$ large enough such that equation*

$$-y'' + U(r,\rho^2)y = \rho^2 y, \tag{2.18}$$

admits a solution $y = \widetilde{e}(r,\rho)$ satisfying the integral equation

$$\begin{aligned}\widetilde{e}(r,\rho) = e^{-i\rho r} &+ \frac{1}{2i\rho}\int_a^r e^{i\rho(r-r'))}U(r',\rho^2)\widetilde{e}(r',\rho)\,dr' \\ &+ \frac{1}{2i\rho}\int_r^\infty e^{-i\rho(r-r'))}U(r',\rho^2)\widetilde{e}(r',\rho)\,dr',\end{aligned} \tag{2.19}$$

for $0 < a \le r < \infty$, in the domain $\{\rho \ge \delta;\ \Im m(\rho) \ge 0\}$.

There exists C_δ large enough such that in this domain

$$|\widetilde{e}(r,\rho)| \le C_\delta e^{r\Im m(\rho)}.$$

Moreover, for any $\alpha > 0$ and for $r \to \infty$,

$$\begin{aligned}\widetilde{e}(r,\rho) &= e^{-i\rho r}(1+o(1)),\\ \partial_r\widetilde{e}(r,\rho) &= e^{-i\rho r}(-i\rho + o(1)),\end{aligned} \tag{2.20}$$

uniformly with respect to ρ in the domain $\{\Im m(\rho) \ge \alpha,\ |\rho| \ge \delta\}$.

Moreover $\rho \to \widetilde{e}(r,\rho)$ is holomorphic in the domain $\{\Im m(\rho) \ge 0 : |\rho| \ge \delta\}$ and for $|\rho| \to \infty$,

$$\begin{aligned}\widetilde{e}(r,\rho) &= e^{i\rho r}(1+o(\rho^{-1})),\\ \partial_r\widetilde{e}(r,\rho) &= ie^{i\rho r}(1+o(\rho^{-1})),\end{aligned} \tag{2.21}$$

uniformly with respect to r in the domain $\{r \ge a\}$.

Proof. As the proof is similar to that of Proposition 2.2, we only sketch it, leaving the details to the reader. It is clear that $\widetilde{e}(r,\rho)$ given by (2.19) satisfies (2.18). Now noting $\widetilde{\varepsilon}(r,\rho) = e^{i\rho r}\widetilde{e}(r,\rho)$, we see that ε satisfies

$$\widetilde{\varepsilon}(r,\rho) = 1 + \frac{1}{2i\rho}\int_a^r e^{2i\rho(r-r'))}U(r',\rho^2)\widetilde{\varepsilon}(r',\rho)\,dr' + \frac{1}{2i\rho}\int_r^\infty U(r',\rho^2)\widetilde{\varepsilon}(r',\rho)\,dr',$$

Looking for a solution of (2.11) in the form

$$\widetilde{\varepsilon}(r,\rho) = \sum_{n=0}^\infty \widetilde{\varepsilon}_n(r,\rho), \tag{2.22}$$

for a sequence $\{\widetilde{\varepsilon}_n\}_{n\ge0}$ given iteratively by

$$\begin{aligned}\widetilde{\varepsilon}_{n+1}(r,\rho) = &\frac{1}{2i\rho}\int_a^r e^{2i\rho(r-r'))}U(r',\rho^2)\widetilde{\varepsilon}_n(r',\rho)\,dr' \\ &+ \frac{1}{2i\rho}\int_r^\infty U(r',\rho^2)\widetilde{\varepsilon}_n(r',\rho)\,dr' \quad \text{for } n \ge 1.\end{aligned}$$

Then for $\Im m(\rho) \ge 0$ we deduce that

$$|\widetilde{\varepsilon}_n(r,\rho)| \le \Big[\frac{1}{|\rho|}\sum_{j=1}^N \overline{W_j}\sigma_j(a)\Big]^n \quad \text{for } n \ge 1.$$

Then as $\sigma_j(r) \to 0$ when $r \to \infty$, for any $\delta > 0$ there is a number a large enough such that

$$\sum_{j=1}^{N} \overline{W_j}\sigma_j(a) < \delta,$$

so series (2.22) is uniformly convergent and its sum $\widetilde{\varepsilon}(r,\rho)$ is uniformly bounded on any domain

$$\{\Im m(\rho) \geq 0 : |\rho| > \delta,\ r \geq a\}.$$

$\square$

Finally we have the following consequence.

Proposition 2.5. *If the integrals $\sigma_j'(r) = \int_r^\infty r'|V_j(r')|\,dr'$ are convergent, then the solution $\widetilde{e}(r) := \widetilde{e}(r,0)$ of (2.7) is well defined as the solution of the integral equation*

$$\widetilde{e}(r) = r - r\int_r^\infty U(r',0)\widetilde{e}(r')\,dr' - \int_a^r r'U(r',0)\widetilde{e}(r')\,dr', \tag{2.23}$$

and $\widetilde{e}$ rewrites

$$\widetilde{e}(r) = r\widetilde{\varepsilon}(r),$$

where $\widetilde{\varepsilon}$ is bounded as $r \to \infty$.

Moreover if for some $\gamma > 0$, the integrals $\int_r^\infty \xi^{1+\gamma}|V_j(\xi)|\,d\xi$ are convergent, then as $r \to \infty$, one has

$$\begin{aligned} \widetilde{e}(r,0) &= r(1 + o(r^{-\gamma})), \\ \partial_r\widetilde{e}(r,\rho) &= 1 + o(r^{-\gamma}). \end{aligned} \tag{2.24}$$

We omit the proof of the above proposition.

2.4. **The Wronskian** $W\big(e(\rho,r),\widetilde{e}(\rho,r)\big)$**.** From the asymptotic behavior of solutions e and $\widetilde{e}$ for $r \to \infty$ we observe that their Wronskian

$$W(e,\widetilde{e}) := e'\widetilde{e} - e,\widetilde{e}',$$

has the asymptotic behavior

$$W(e,\widetilde{e}) = -2i\rho + o(1). \tag{2.25}$$

As first order derivative are absent from equation (2.7), $W(r)$ does not depend on r and in fact

$$W(e,\widetilde{e}) = -2i\rho \quad \text{in the domain } \{\Im m(\rho) \geq 0 : |\rho| > \delta\}. \tag{2.26}$$

3. Discrete spectrum $\sigma_d(H)$

For $\Im m(\rho) \leq 0$, we put

$$E(r,\rho) = e(r,-\rho),$$

where e is the solution studied in Proposition 2.2.

Using the previous information about solutions of (2.7) we have the following result.

Theorem 3.1. *(1) H has no positive eigenvalue.*

(2) If for a $\gamma > 0$
$$\int_r^\infty \xi^{1+\gamma} |V_j(\xi)|\, d\xi < \infty,$$
for $j = 1, \dots, N$, then 0 is not an eigenvalue of H.

(3) A necessary and sufficient condition for $\lambda \neq 0$ to be an eigenvalue of H is
$$\begin{gathered} \lambda = \rho^2, \\ \Im m(\rho) > 0, \\ e(0, \rho) = 0. \end{gathered}$$

(4) The set of eigenvalues of H is at most countable and its limit points belong to the half line $\{\lambda \geq 0\}$.

Proof. (1) The general solution y of (2.7) reads
$$y(r, \rho) = C_1 e(r, \rho) + C_2 E(r, \rho) \quad \text{with } \rho = \sqrt{\lambda}.$$
By Proposition 2.2, we have
$$y(r, \rho) = C_1 e^{i\rho r} + C_2 e^{-i\rho r} + o(1).$$
As the leading part is periodic, it cannot belong to $L^2(0, \infty)$.

(2) In the same stroke, after Propositions 2.3 and 2.5, the general solution Y of $-Y'' + U(r, 0)Y = 0$ satisfies
$$Y(r) = C_1 + C_2 r\big[1 + o\big(\frac{1}{r^\gamma}\big)\big],$$
which cannot belong to $L^2(0, \infty)$.

(3) We consider now solutions of (2.7) when ρ is non-positive or complex. As the general solution of (1.3) is given by
$$y(r) = C\phi(r, \rho),$$
it is clear that λ is an eigenvalue of H if and only if $\phi(\cdot, \lambda) \in L^2(0, \infty)$.

From (2.26) and the definition (2.1) of ϕ,
$$\phi(r, \rho^2) = \frac{\widetilde{e}(\rho) e(r, \rho) - e(\rho) \widetilde{e}(r, \rho)}{2i\rho}, \tag{3.1}$$
where $\widetilde{e}(\rho)) = \widetilde{e}(0, \rho)$ and $e(\rho) = e(0, \rho)$, for $\rho \in \Gamma_W$, $\Im m(\rho) > 0$ and $|\rho| \geq \delta$, the result follows.

Indeed, from Proposition 2.2 $e(\cdot, \rho) \in L^2(0, \infty)$ but by Proposition 2.4, $\widetilde{e}(\cdot, \rho)) \notin L^2(0, \infty)$ so $s(\cdot, \rho) \in L^2(0, \infty)$ if and only if $e(\rho) = 0$.

(4) We know (by Proposition 2.2) that for $\Im m(\rho) \geq 0$ and $|\rho| \to \infty$
$$e(\rho) = 1 + o\left(\rho^{-1}\right).$$
Therefore $\rho \to e(\rho)$ is bounded and holomorphic in the half-plane $\Im m(\rho) > 0$, therefore the set of zeros of $e(\rho) = 0$ is at most countable and can have limit points only on the axis $\square$

Now we are in position to give an approximate localization of the discrete spectrum, following Davies [7].

Proposition 3.2. *Suppose that $V_j \in L^1(0,\infty) \cap L^2(0,\infty)$ then every eigenvalue λ of H satisfies*

$$|\lambda| \le \frac{1}{2}\Big(\sum_{j=1}^{N} \overline{W}_j \|V_j\|_1\Big)^2. \tag{3.2}$$

Proof. Let $\lambda = -z^2$ be an eigenvalue of H with $\Re e(z) > 0$ and let ψ be the corresponding eigenfunction. Then $\psi \in H^2(\mathbb{R}_+) \subset C_0(\mathbb{R}_+)$. We have

$$(-\partial_r^2 + z^2)\psi = -U\psi,$$

so

$$-\psi = (-\partial_r^2 + z^2)^{-1} U\psi.$$

Noting $X := |U|^{1/2}$, $W := U/X$ and $\chi := W\psi \in L^2(0,\infty)$, we obtain

$$-\chi = W(-\partial_r^2 + z^2)^{-1} X\chi,$$

so

$$-1 \in \operatorname{Spec}\big(W(-\partial_r^2 + z^2)^{-1} X\big)\,.$$

We estimate the Hilbert-Schmidt norm of the operator $W(-\partial_r^2+z^2)^{-1}X$ with kernel

$$W(r)\Big[\frac{e^{-z(r-r')}}{2z} - \frac{e^{-z(r+r')}}{2z}\Big]\ X(r').$$

We have

$$\begin{aligned}
1 &\le \|W(-\partial_r^2 + z^2)^{-1} X\|_2^2 \\
&= \frac{1}{4|z|^2} \int_{\mathbb{R}_+\times\mathbb{R}_+} |W(r)|^2 \big[e^{-2\Re e(z)|r-r'|} + e^{-2\Re e(z)(r+r')}\big] |X(r')|^2 dr'\, dr \\
&\le \frac{1}{2|z|^2} \|U\|_1^2,
\end{aligned}$$

which gives the bound (3.2). □

Now following Naimark [22] and Stepin [29] we can show that the number of eigenvalues of H is finite

Proposition 3.3. *Suppose that*

$$\int_0^\infty e^{\alpha r} |V_j(r)|\, dr < \infty, \tag{3.3}$$

for an exponent $\alpha > 0$. Then the following properties hold

1. *The number of eigenvalues of H is finite.*
2. *Denoting by $\overline{E}$ the modulus of the largest eigenvalue E_N of H, suppose that A an arbitrary positive number such that*

$$A > \frac{\overline{E}^2}{\alpha} - \frac{\alpha}{4}. \tag{3.4}$$

Then the number $N(H)$ of eigenvalues of H in the disk $D(0,\Lambda)$, with

$$\Lambda = \frac{1}{2}\ \Big(\sum_{j=1}^{N} \overline{W}_j \|V_j\|_1\Big)^2,$$

satisfies the bound

$$
\begin{aligned}
N(H) \le \Big(\log \frac{A+\frac{\alpha}{2}}{\sqrt{A^2+\Lambda^2}}\Big)^{-1} \Big\{ &\sum_{j=1}^{N} \overline{W}_j \|(1+e^{\alpha r'})V_j\|_1 \\
&- \log\Big(2-\exp\Big(\frac{1}{A\log 2}\sqrt{\frac{\Lambda}{2}}\Big)\Big)\Big\}.
\end{aligned} \tag{3.5}
$$

Proof. (1) By (3.3) the discrete spectrum $\sigma_d(H)$ of H which we know not to have any accumulation point on the open positive semi-axis cannot include 0. So in fact $\sigma_d(H) \subset D(0,\Lambda)$ has no accumulation point, and therefore is finite.

(2) As $\overline{E}$ is the modulus of the largest eigenvalue E_N of H, we have $\overline{E} \le \Lambda$. Supposing A satisfy (3.4), pick $\overline{A}$ arbitrary in the interval $\big(\sqrt{A^2+\overline{E}^2}, A+\frac{\alpha}{2}\big)$ and consider the translated function $\Phi(z) := e(z+iA)$.

The number $N(H)$ is at most the number of zeroes of Φ in $D(0,\sqrt{A^2+\overline{E}^2})$. Applying Jensen's Theorem to the analytic function $\Phi(z)$ we obtain

$$
\begin{aligned}
N(H)\log\frac{\overline{A}}{\sqrt{A^2+\overline{E}^2}} &\le \sum_{|z_k|\le\sqrt{A^2+\overline{E}^2}} \log\frac{\overline{A}}{|z_k|} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log|\Phi(\overline{A}e^{i\theta})|\,d\theta - \log|\Phi(0)|.
\end{aligned}
$$

We observe now that

$$
\begin{aligned}
|\Phi(\overline{A}e^{i\theta})| &\le \exp\Big(\int_r^\infty \big(1+e^{-2(A+\overline{A}\sin\theta)r'}\big)|U(r',\rho^2)|\,dr'\Big) \\
&\le \exp\Big(\int_0^\infty \big(1+e^{2(\overline{A}-A)r'}\big)|U(r',\rho^2)|\,dr'\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
|\Phi(0)\| &\ge 2-\exp\Big(\frac{1}{2A}\int_r^\infty \big(1+e^{-2Ar'}\big)|U(r',\rho^2)|\,dr'\Big) \\
&\ge 2-\exp\Big(\frac{1}{2A}\int_0^\infty |U(r',\rho^2)|\,dr'\Big) \\
&\ge 2-\exp\Big(\frac{1}{2A}\sum_{j=1}^{N}\overline{W}_j\|V_j\|_1\Big).
\end{aligned}
$$

Then for any $\overline{A} \in \Big(\sqrt{A^2+\overline{E}^2}, A+\frac{\alpha}{2}\Big)$ we obtain

$$
\begin{aligned}
&N(H)\log\frac{\overline{A}}{\sqrt{A^2+\overline{E}^2}} \\
&\le \int_0^\infty \big(1+e^{2(\overline{A}-A)r'}\big)|U(r',\rho^2)|\,dr' - \log\Big(2-\exp\Big(\frac{1}{A\log 2}\sqrt{\frac{\Lambda}{2}}\Big)\Big),
\end{aligned}
$$

which gives (3.5) if we choose $\overline{A} = A+\frac{\alpha}{2}$. $\square$

Remark 3.4. Using the bound given by Franck, Laptev, Lieb and Seiringer [15] we can give another estimate for $N(H)$. Let $\kappa > 0$. Then for the eigenvalues outside the cone $\{|\Im m(z)| < \kappa \Re e(z)\}$, one has

$$\sum_{\{|\Im m(z)|<\kappa\Re e(z)\}} |\lambda_j|^{1/2} \le 2^{3/2}(1+\frac{2}{\kappa})\frac{1}{4}\sum_{k=1}^{N}\overline{W}_j \int_0^\infty |V_j(r)|\,dr.$$

Then supposing one has an estimate for $m := \inf\{|\lambda_j|\ j = 1,\dots N\}$ (the eigenvalue with least module), we infer that

$$N(H) \le m^{-1/2}2^{3/2}(1+\frac{2}{\kappa})\frac{1}{4}\sum_{k=1}^{N}\overline{W}_j \int_0^\infty |V_j(r)|\,dr.$$

4. Continuous spectrum $\sigma_c(H)$

Our goal is to show that, as in the self-adjoint case, $\sigma_{ess}(H)$ the essential spectrum of H is the positive half-line $\{\Im m(E) > 0\}$, that $\sigma_{ess}(H) = \sigma_c(H)$ (no positive eigenvalues) and moreover that $\sigma_{ac}(H) = (0, +\infty)\backslash\Omega_s$ where Ω_s is finite.

4.1. Continuous spectrum and resolvent of H.

Proposition 4.1. *(1) Each real number $\lambda = \rho^2$ such that $e(\rho) \neq 0$ and $\Im m(\rho) > 0$ belongs to the resolvent set $\varrho(H)$ of H.*

(2) The resolvent $R(\lambda) := (H-\lambda)^{-1}$ of H is given by the formula

$$R(\lambda)f(r) = \int_0^\infty R(r,r';\lambda)f(r')\,dr',$$

where

$$R(r,r';\lambda) = \begin{cases} \frac{e(r,\rho)\phi(r',\lambda)}{e(\rho)} & \text{if } 0 < r' < r,\\ \frac{\phi(r,\lambda)e(r',\rho)}{e(\rho)} & \text{if } 0 < r < r'. \end{cases} \tag{4.1}$$

(3) For any $\delta > 0$ there exists a constant $C_\delta > 0$ such that

$$\|R(\lambda)\| \le \frac{C_\delta}{|e(\rho)|\Im m(\rho)}.$$

Proof. Rewriting $F(r) := R(\lambda)f(r) = (R_1 + R_2)f(r)$ with

$$R_1f(r) = \frac{e(r,\rho)}{e(\rho)}\int_0^r \phi(r',\rho^2)\,dr' \quad\text{and}\quad R_2f(r) = \frac{\phi(r,\lambda)}{e(\rho)}\int_0^r e(r',\rho)\,dr',$$

one checks that F solves equation $HF = \rho^2F + f$ and the boundary condition $F(0) = 0$.

From Lemma 2.1 and Proposition 2.2 we obtain the estimates

$$|\phi(r,\rho^2)| \le C_\delta e^{\tau r}, \quad |e(r,\rho)| \le C_\delta e^{-\tau r},$$

for $\Im m\rho = \tau > 0$ and $|\rho| \ge \delta$. We deduce that

$$\|R_1\| \le \frac{C_\delta}{|e(\rho)|\tau} \quad\text{and}\quad \|R_2\| \le \frac{C_\delta}{|e(\rho)|\tau},$$

and the proof is complete. □

Proposition 4.2. *For any $R > 0$, there exists $C_\delta > 0$ such that*

$$\|R(\rho^2)\| \geq \frac{C_\delta}{|e(\rho)\sqrt{\Im m(\rho)}},$$

for any $\rho \in \{\Im m(\rho) > 0 : |\rho| \geq R\}$.

In particular $\|R(\lambda)\| \to \infty$ as long as $\lambda \in Res(H) \to \lambda_0 \in \mathbb{R}_+$. In other words $\mathbb{R}_+ \subset \sigma_c(H)$.

Proof. Consider for a $R > 0$ the truncated function $\Phi_R(r)$

$$\Phi_R(r) = \begin{cases} \overline{\phi(r,\lambda)} & \text{if } 0 < r < R, \\ 0 & \text{if } R < r. \end{cases}$$

From Proposition 4.1, $\Phi_R \in L^2$ and

$$R(\lambda)\Phi_R(r) = \|\Phi_R\|_2^2 \frac{e(r,\rho)}{e(\rho)} \quad \text{for } r > R.$$

Then

$$\|R(\lambda)\Phi_R\|^2 \geq \int_R^\infty |R(\lambda)\Phi_R(r)|^2 dr = \frac{\|\Phi_R\|_2^4}{|e(\rho)|^2} \int_R^\infty |e(r,\rho)|^2 dr.$$

Choosing $R = R_\delta$ so large that for $r > R$, $\Im m(\rho) \geq 0$ and $|\rho| \geq \delta$ one has

$$|e(r,\rho)| > \frac{1}{2}\, e^{-\tau r}, \quad \tau = \Im m(\rho),$$

we obtain

$$\int_R^\infty |e(r,\rho)|^2 dr \geq \frac{e^{-\tau R}}{8\tau},$$

$$\|R(\lambda)\Phi_R\|^2 \geq \frac{\|\Phi_R\|^2 e^{-\tau R}}{|2e(\rho)|\sqrt{2\tau}},$$

which completes the proof. $\square$

We are now in a position to characterize the continuous spectrum of H.

Theorem 4.3. *Every number $\lambda > 0$ belongs to the continuous spectrum of H. Moreover, under Condition 2 in Theorem 3.1,*

$$\sigma_c(H) = \mathbb{R}_+.$$

Proof. It is sufficient to check that for any $\lambda > 0$, the range $\mathcal{R}_\lambda$ of $H - \lambda$ is dense in $L^2(\mathbb{R}_+)$ or equivalently that its orthogonal $\mathcal{R}_\lambda^\perp$ is reduced to 0. As $\mathcal{R}_\lambda^\perp$ is the set of solutions of $H^* f = \lambda f$, where H^* is the adjoint of H defined by

$$-\frac{d^2}{dr^2}\, v + \overline{U(r,E)} v, \;\; v(0) = 0,$$

one sees from Theorem 3.1 that λ cannot be eigenvalue of H^*. $\square$

4.2. **Spectral expansion.** We show that the set of generalized eigenfunctions of H is complete (see below) and we give a convergent expansion in term of these eigenfunction for any L^2 function.

In the following we assume an additional constraint on the potential U,

$$\int_0^\infty e^{\epsilon r}|U(r,E)\,dr < \infty, \tag{4.2}$$

which is clearly satisfied by (1.1). This condition implies that

$$\int_r^\infty |U(r,E)\,dr < C_\epsilon e^{-\epsilon r} \quad \text{and} \quad \int_r^\infty r|U(r,E)\,dr < C_{\epsilon'} e^{-\epsilon' r},$$

for $0 < \epsilon' < \epsilon$, then estimates (2.15) can be improved as follows

$$\begin{gathered} |K(r,r')| \le Ce^{-\epsilon\frac{r+r'}{2}}, \\ |\partial_r K(r,r')|, |\partial_{r'} K(r,r')| \le \frac{1}{4}\sum_{j=1}^N \overline{W_j}\big|V_j\big(\frac{r+r'}{2}\big)\big| + Ce^{-\epsilon(\frac{3}{2}r+r')}, \end{gathered} \tag{4.3}$$

for $0 \le r \le r'$ and $C > 0$.

Proposition 4.4. *Under Condition 4.2 and for any $r \ge 0$, the function*

$$e(r,\rho) = e^{i\rho r} + \int_r^\infty K(r,r')e^{-i\rho r'}\,dr' \tag{4.4}$$

is holomorphic in the half-plane $\{\Im m(\rho) > -\frac{\epsilon}{2}\}$. The same holds for $e(\rho) = e(0,\rho)$.

Proof. As $\Im m(\rho) > -\epsilon/2$, using (4.3), the integral at the right hand side of (4.4) converges, and so do all the derivatives with respect to ρ □

We have the following simple Corollary of Proposition 4.4

Proposition 4.5. *On the strip $\{|\Im m(\rho)| < \frac{\epsilon}{2}\}$, the second-order ode $H\psi = \rho^2\psi$ admits the fundamental system of solutions*

$$e_1 = e(r,\rho), \quad e_2 = \tilde{e}(r,\rho) = e(r,-\rho),$$

with Wronskian

$$W(e_1,e_2) = -2i\rho, \quad |\Im m(\rho)| < \frac{\epsilon}{2}.$$

Proof. By analytic continuation, $e(r,\rho)$ satisfies the integral equation (2.8) on the half-plane $\{\Im m(\rho)| > -\epsilon/2\}$ then it is also solution of $He = \rho^2 e$ in the same region. □

Observe that from (3.1) the relation

$$\phi(r,\rho^2) = \frac{\widetilde{e}(\rho)e(r,\rho) - e(\rho)\widetilde{e}(r,\rho)}{2i\rho} = \frac{e(-\rho)e(r,\rho) - e(\rho)e(r,-\rho)}{2i\rho}, \tag{4.5}$$

holds on the strip $\{|\Im m(\rho)| < \epsilon/2\}$.

As Naimark, we call *singular value* of H any root ρ_k of the equation $e(\rho) = 0$ such that $\rho_k \neq 0$ and $\Im m(\rho_k) \ge 0$.

Proposition 4.6. *Under condition (4.2), the set Ω_s of singular values of H is finite.*

Proof. This result is a direct consequence of analytic properties of e in a neighborhood of the real given in Proposition 4.4 and asymptotic properties of e given in Proposition 2.2: as e is analytic and bounded on the positive half-line, Ω_s is actually finite. $\square$

The non-real singular values of H are denoted by $\rho_1, \rho_2, \dots \rho_K$, and its real singular values by $\rho_{K+1}, \rho_{K+2}, \dots \rho_L$, with multiplicities m_k. Then $\Im m\rho_k > 0$ for $k = 1, \dots, K$ and $\Im m\rho_k = 0$ for $k = K+1, \dots, L$.

From assertion 3 in Theorem 3.1 we know that the $\lambda_k = \rho_k^2$ for $k = 1, \dots, K$ are the eigenvalues of H and from Schwartz [27] we call *spectral singularities* of H the remaining $\lambda_k = \rho_k^2$ for $k = K+1, \dots, L$.

From Theorem 4.3 we see that spectral singularities of H are embedded into its continuous spectrum.

4.3. **Principal functions.** Under Condition (4.2), for any integer $m \in \mathbb{N}$,

$$e^{(m)}(\cdot, \rho) := \partial_\rho^m e(\cdot, \rho) \in L^2(\mathbb{R}_+) \quad \text{for } \Im m(\rho) > 0,$$

and from (4.5), taking into account that

$$e^{(m)}(\rho_k) = 0 \quad \text{for } k = 1, \dots, K,\ m = 0, \dots, m_k - 1,$$

we see that

$$\phi^{(m)}(\cdot, \rho_k) := \partial_\rho^m \phi(\cdot, \rho_k) \in L^2(\mathbb{R}_+) \quad \text{for } k = 1, \dots, K,\ m = 0, \dots, m_k - 1,$$

where m_k is the multiplicity of the singular value ρ_k and $\lambda_k = \rho_k^2$.

Functions $\phi^{(m)}(\cdot, \rho_k)$, $k = 1, \dots, K$, $m = 0, \dots, m_k - 1$ are called *principal functions of the point spectrum.* Functions $\phi^{(m)}(\cdot, \lambda)$, $\lambda > 0$ are called *principal functions of the continuous spectrum* (note that $\phi^{(m)}(\cdot, \lambda) \notin L^2$). In the same stroke: functions $\phi^{(m)}(\cdot, \rho_k)$, $k = K+1, \dots, L$, $m = 0, \dots, m_k - 1$ are called *principal functions for the spectral singularities.*

Lemma 4.7. *The following estimates hold*

$$\sup_{r \geq 0} \frac{|\partial_\rho^m e(r, \rho)|}{(1+r)^m} < \infty \quad \text{for } \Im m(\rho) > 0,\ m = 0, 1, \dots,$$

$$\sup_{r \geq 0} \frac{|\partial_\lambda^m \phi(r, \lambda)|}{(1+r)^m} < \infty \quad \text{for } \lambda > 0,\ m = 0, 1, \dots .$$

The above lemma is a consequence of (4.3) and the formula in formula (4.4) in Proposition 4.4.

4.4. **Eigenfunction expansion of the resolvent in term of principal functions.**

Theorem 4.8. *The integral kernel $R(r, r'; \lambda)$ of the resolvent of H has the expansion*

$$\begin{aligned} R(r, r'; z) = \frac{1}{\pi}\ Pf \int_0^\infty \frac{\phi(r, \lambda)\phi(r', \lambda)}{\lambda - z} \frac{\sqrt{\lambda}}{e(\sqrt{\lambda})e(-\sqrt{\lambda})}\, d\lambda \\ + \sum_{k=1}^L \partial_\lambda^{m_k - 1} \mathcal{M}(\lambda) \frac{\phi(r, \lambda)\phi(r', \lambda)}{\lambda - z}\Big|_{\lambda = \lambda_k}, \end{aligned}$$

where ϕ is defined by (2.1) and Pf is the finite part (in the Hadamard's sense) of the corresponding singular integral in formula (4.7) below.

Proof. Denoting the two disks $D(0,R)=\{\lambda\in\mathbb{C}:|\lambda|<R\}$ and $D(0,\eta)=\{\lambda\in\mathbb{C}:|\lambda|<\eta\}$, where $R>>1$ and $\eta<<1$, and the half-strip $S_\eta=\{\lambda\in\mathbb{C}:|\Im m(\lambda)<\eta,\ \Re e(\lambda>0\}$, we define the domain

$$\Omega_{R,\eta}:=D(0,R)\backslash(S_\eta\cup D(0,\eta)),$$

and we define the closed contour $\Gamma_{R,\eta}$ as the boundary of $\Omega_{R,\eta}$,

$$\Gamma_{R,\eta}=\partial\Omega_{R,\eta},$$

where the direction on $C(0,R)$ is clockwise.

Let $z\in\varrho(H)$, choose R large enough and η small enough to ensure that $\{z,\lambda_1,\dots,\lambda_K\}\in\Omega_{R,\eta}$ and consider the Cauchy integral

$$\mathcal{J}_{R,\eta}=\int_{\Gamma_{R,\eta}}\frac{R(r,r';\lambda)}{\lambda-z}\,d\lambda,$$

where $R(r,r';\lambda)$ is the integral kernel of the resolvent of H.

From the previous results $\lambda\to R(r,r';\lambda)$ is meromorphic in $\Omega_{R,\eta}$ with poles $\lambda_1,\dots,\lambda_K$ with multiplicities $m_1,\dots,m_K$. So, applying Cauchy's residue theorem, we obtain

$$\mathcal{J}_{R,\eta}=R(r,r';z)+\sum_{k=1}^{K}\mathrm{Res}\big|_{\lambda=\lambda_k}\{\frac{R(r,r';\lambda)}{\lambda-z}\}. \tag{4.6}$$

From (2.1) and (2.2) we see that

$$e(r,\rho)=\partial_r e(0,\rho)\phi(r,\lambda)+e(0,\rho)\psi(r,\lambda).$$

As $\lambda\to\psi(r,\rho)$ is an entire function in λ, with $\rho=\sqrt{\lambda}$, the resolvent kernel reads

$$R(r,r';\lambda)=\frac{\partial_r e(0,\rho)}{e(0,\rho)}\phi(r,\lambda)\phi(r',\lambda)+\Psi(r,r';\lambda),$$

for $\Im m(\rho)>0$, where $\lambda\to\Psi(r,r';\lambda)$ is entire. Plugging this into (4.6) we obtain

$$R(r,r';z)=\mathcal{J}_{R,\eta}+\sum_{k=1}^{K}\partial_\lambda^{m_k-1}\mathcal{M}(\lambda)\frac{\phi(r,\lambda)\phi(r',\lambda)}{\lambda-z}\big|_{\lambda=\lambda_k},$$

where

$$\mathcal{M}(\lambda)=\frac{(\lambda-\lambda_k)^{m_k}}{(m_k-1)!}\frac{\partial_r e(0,\rho)}{e(0,\rho)},$$

for $k=1,\dots,K$.

Observing that for $|\rho|$ large, $|R(r,r';\lambda)|<\frac{C}{\rho}$ and checking that the multiplicity of 0, as a root of $e(\rho)$, cannot be greater than unity, we conclude that we can pass to the limit $R\to\infty,\ \eta\to 0$ in the contributions on the circles $C(0,R)$ and $C(0,\eta)$ of $\mathcal{J}_{R,\eta})$.

To evaluate the contribution of spectral singularities on the real positive axis, we consider the indented domains

$$\omega_{+,\epsilon}:=\{\Im\lambda>0\}\backslash\cup_{k=K+1}^{L}D(\lambda_k,\epsilon),$$
$$\omega_{-,\epsilon}:=\{\Im\lambda<0\}\backslash\cup_{k=K+1}^{L}D(\lambda_k,\epsilon),$$

for $\epsilon>0$ small enough, and their boundaries $\gamma_{+,\epsilon}=\partial\omega_{+,\epsilon}$ and $\gamma_{-,\epsilon}=\partial\omega_{-,\epsilon}$.

Now, after the previous steps, we can clearly replace the limit $\lim_{R\to\infty,\eta\to0}\mathcal{J}_{R,\eta}$ by $\lim_{\epsilon\to0}\int_{\gamma_{+,\epsilon}\cup\gamma_{+,\epsilon}}$ with suitable orientations.

Applying once more Cauchy's residue theorem and plugging into (4.6), we obtain finally

$$\begin{aligned}\lim_{R\to\infty,\eta\to 0} \mathcal{J}_{R,\eta} = \frac{1}{\pi} Pf \int_0^\infty \frac{\phi(r,\lambda)\phi(r',\lambda)}{\lambda - z} \frac{\sqrt{\lambda}}{e(\sqrt{\lambda})e(-\sqrt{\lambda})}\, d\lambda \\ + \sum_{k=1}^{K} \partial_\lambda^{m_k-1} \mathcal{M}(\lambda) \frac{\phi(r,\lambda)\phi(r',\lambda)}{\lambda - z}\big|_{\lambda=\lambda_k},\end{aligned} \tag{4.7}$$

where Pf is the ("Hadamard's) finite part of the corresponding singular integral, which completes the proof. ☐

Note that in the case where real spectral singularities are simple poles we can simply erase the Pf in front of the integral. Denoting

$$\phi(f,\lambda) := \int_0^\infty \phi(r,\lambda) f(r)\, dr,$$

we have the following Fourier-type property.

Lemma 4.9. *(1) For any function $f \in L^2(\mathbb{R}_+)$ there exists $C > 0$ such that*

$$\int_0^\infty |\phi(f,\lambda)|^2 \sqrt{\lambda}\, d\lambda \le C \|f\|^2_{L^2(\mathbb{R}_+)}. \tag{4.8}$$

(2) For any $f \in L^2(\mathbb{R}_+)$ one has

$$\int_0^\infty \Big|\phi(f,\lambda) - \int_0^r f(r')\phi(r,\lambda)\, dr'\Big|^2 \sqrt{\lambda}\, d\lambda \to 0,$$

when $r \to \infty$.

Proof. (1) Supposing first that f is compactly supported and defining the operator K by

$$Kf(r) = \int_0^\infty K(r,r') f(r')\, dr',$$

where the kernel $K(r,r')$ satisfies (4.5) one checks

$$|Kf(r)| \le C e^{-\frac{\epsilon}{2} r} \int_0^r e^{-\frac{\epsilon}{2} r'} |f(r')|\, dr', \tag{4.9}$$

then $I + K$ is bounded in $L^2(\mathbb{R}_+)$. From (4.4) we see that

$$\phi(f,\lambda) = \int_0^\infty \phi(r,\lambda) f(r)\, dr = \int_0^\infty (I+K)f \; e^{i\rho r}\, dr, \tag{4.10}$$

and from (4.5) we have

$$2i\rho\phi(f,\lambda) = e(-\rho)\phi(f,\rho) - e(\rho)\phi(f,-\rho);$$

so using (4.10) and Parseval's Theorem we obtain

$$\int_0^\infty |\phi(f,\lambda)|^2 \sqrt{\lambda}\, d\lambda \le C \int_0^\infty |(I+K)f(r)|^2\, dr,$$

which is (4.8). The general case holds for any $f \in L^2(\mathbb{R}_+)$ by denseness.

Statement (2) follows from (1). ☐

At this point, one can show a spectral expansion for an arbitrary L^2 function. The following expansion holds

Theorem 4.10. *One has the following expansion for an arbitrary function $f \in L^2(\mathbb{R}_+)$ in the spirit of* [10],

$$\begin{aligned} f(r) = \frac{1}{\pi} \int_0^\infty \phi(f,\lambda)\phi(r,\lambda) \frac{\sqrt{\lambda}}{e(\sqrt{\lambda})e(-\sqrt{\lambda})}\, d\lambda \\ + \sum_{k=1}^{L} \partial_\lambda^{m_k - 1} \mathcal{M}(\lambda)\phi(f,\lambda)\phi(r,\lambda)\big|_{\lambda=\lambda_k}. \end{aligned} \tag{4.11}$$

Proof. (1) We first assume that H has no singular value. Supposing that $f \in C_0^2(\mathbb{R}_+)$, we define

$$g(r) := -f''(r) + U(r,z)f(r). \tag{4.12}$$

From (4.1) (see Proposition 4.1), the kernel $R(r,r';z)$ satisfies the equation

$$-R''(r,r';z) + U(r,z)R(r,r';z) = zR(r,r';z) + \delta(r-r'). \tag{4.13}$$

Multiplying (4.12) by $R(r,r';z)$ and (4.13) by $f(r)$, subtracting and integrating by parts, we obtain

$$\int_0^\infty R(r,r';z)f(r')\,dr' = -\frac{1}{z}\ f(r) + \frac{1}{z}\int_0^\infty R(r,r';z)f(r')\,dr'.$$

Integrating this relation on the circle $\gamma_R = \{z \in \mathbb{C} :\ |z| = R\}$ and using the asymptotic properties of $z \to R(r,r';z)$, we obtain the identity

$$f(r) = -\lim_{R\to\infty} \int_{\gamma_R} \Big[\int_0^\infty R(r,r';z)f(r')\,dr'\Big]\,dz.$$

Plugging (4.1) into this formula and deforming γ_R exactly as in the proof of Theorem 3.1, we obtain (4.11) for f smooth.

Using the denseness of $C_0^2(\mathbb{R}_+)$ in $L^2(\mathbb{R}_+)$, the general case follows.

(2) If H has spectral singularities, formula (4.11) still holds provided that the function f satisfy the conditions

$$\frac{d^{j_k}}{d\lambda^{j_k}}\phi(f,\lambda_k) = 0, \quad \lambda_k = \rho_k^2,\ k = K+1,\dots,L.$$

As one checks easily that this set of functions is dense in L^2, formula (4.11) holds for any $f \in L^2(\mathbb{R}_+)$. □

5. Resonances for a simplified model

We are interested now in the resonances of the (simplified) truncated Hamiltonian

$$H^b = -\frac{d^2}{dr^2} + U_V^b(r,E),$$

where

$$U_V^b(r,E) = -[V_V(E) + iW_V(E)]f^b(r,R_V,a_V),$$

and f^b is the truncated Woods-Saxon form factor

$$f^b(r,R,a) = \begin{cases} \frac{1}{1+e^{\frac{r-R}{a}}} & \text{if } r < b,\\ 0 & \text{if } r > b, \end{cases}$$

where $b > R$ and we discarded surface and spin-orbit contributions.

Following the 1D presentation of Dyatlov and Zworski in [9], resonance is a $\lambda \in \mathbb{C}\backslash\{0\}$ for which there exists a pure outgoing state: a solution ϕ to

$$(H^b - \lambda^2)\phi \equiv \big[-\frac{d^2}{dr^2} + U_V^b(r, \lambda^2) - \lambda^2 \big]\phi = 0,$$

satisfying

$$\phi(r) = e^{i\lambda r} \quad \text{for } r > b.$$

Let g a non decreasing C^1 function with $g(0) = 0$ and supported outside the interval $(0, b)$, we consider curves $\Gamma \subset \mathbb{C}$ which are the graphs of such g.

Let $\gamma(t)$ a parametrization of Γ and let $f \in C^1(\Gamma)$ in the sense: $f \circ \gamma \in C^1(\mathbb{R})$. We define

$$\partial_z^\Gamma f(z_0) = \gamma'(t_0)^{-1}\partial_t(f \circ \gamma)(t_0),$$

where $z = x + iy$ and $\gamma(t_0) = z_0$ and put $D_z = -i\partial_z^\Gamma$. By the chain rule, if f is differentiable near Γ, this is independent of the parametrization and if $\gamma(t) := \gamma_r(t) + i\gamma_i(t)$ one gets

$$\gamma'(t_0)^{-1}\partial_t(f \circ \gamma)(t_0) = \gamma'(t_0)^{-1} \left(\partial_x f(z_0)\gamma_r'(t_0) + i\partial_y f(z_0)\gamma_i'(t_0)\right).$$

If f is holomorphic near Γ one checks that $\partial_z^\Gamma f(z_0)$ reduces to the holomorphic differential ∂_z.

The space $L^2(\Gamma)$ is

$$L^2(\Gamma) = \big\{ f : \Gamma \to \mathbb{C} : \int_\gamma |f|^2 dz := \int |f \circ \gamma)(t)|^2 \gamma'(t)|\, dt < \infty \big\}.$$

As $U_V^b(z, \lambda^2)$ is well defined, so is

$$H_\Gamma := \left(D_z^\Gamma\right)^2 + U_V^b(z, \lambda^2).$$

Our point is that some resonances of H_V coincide with eigenvalues of H_Γ considered as an operator in $L^2(\Gamma)$.

Let us introduce two fixed angles $0 \le \theta_1 \le \theta_2 \le \pi/2$ and $\epsilon > 0$ such that for any $z \in \Gamma$ outside a given compact set K, one has $\theta_1 + \epsilon \le \arg z \le \theta_2 - \epsilon$.

Theorem 5.1. *Each λ such that $-\theta_1 \le \arg \lambda \le \pi - \theta_2$ is a resonance of H_V of multiplicity m if and only if it is an eigenvalue of H_Γ of multiplicity m.*

Proof. Suppose that λ is a resonance of multiplicity m of H_V. Then there is a function $\phi : \mathbb{R} \to \mathbb{C}$ such that

$$(H^b - \lambda^2)^{m-1}\phi(r) = Ae^{i\lambda r} \quad \text{if } r \ge b.$$

This means $\phi(r) = P(r)e^{i\lambda r}$, for $r \ge b$, for a suitable polynomial $P(r)$. We defining now $\Phi : \mathbb{R} \to \mathbb{C}$ such that

$$\Phi(r) = \begin{cases} P(r)e^{i\lambda r} & \text{if } \Re e(r) \ge b, \\ \phi(r) & \text{if } 0 \le \Re e(r) \le b, \end{cases}$$

we observe that $(H^b - \lambda^2)^m\Phi = 0$ and $(H^b - \lambda^2)^{m-1}\Phi \ne 0$, also that H_Γ has λ as eigenvalue of multiplicity m provided that $\Phi \in L^2(\Gamma)$ and the converse is proved in the same way. $\square$

Then the proof of Theorem 5.1 is complete provided we show the following lemma.

Lemma 5.2. *Let $P(z)$ be a polynomial. Any continuous function $f : \Gamma \to \mathbb{C}$ such that*
$$f(z) = P(z)e^{i\lambda z} \quad \textit{if } \Re e(z) \geq b,$$
belongs to $L^2(\Gamma)$.

Proof. Clearly it is sufficient to prove that f is square integrable on the exterior part of Γ with $\theta_1 \leq \arg z \leq \theta_2$. Namely suppose that this inequality holds for $\Re e(z) \geq R_0$ with $R_0 \geq b$ and denote by $\Gamma_{R_0} = \Gamma \cap \{\Re e(z) \geq R_0\}$. One has
$$\begin{aligned}
\int_{\Gamma_{R_0}} |f(z)|^2 |\Gamma'(t)|\, dt &= \int_{\Gamma_{R_0}} |P(z)|^2 \left|e^{i\lambda z}\right|^2 |\Gamma'(t)|\, dt \\
&= \int_{\Gamma_{R_0}} |P(z)|^2 e^{-2(\Im\lambda\Re z + \Re\lambda\Im z)} |\Gamma'(t)|\, dt \\
&= \int_{\Gamma_{R_0}} |P(z)|^2 e^{-2(|z||\lambda|(\sin(\arg\lambda + \arg z)}|\Gamma'(t)|\, dt \\
&\leq \int_{\Gamma_{R_0}} |P(z)|^2 e^{-2(|z||\lambda|(\sin\epsilon)}|\Gamma'(t)|\, dt \\
&\leq \int_{R_0}^{\infty} |P(t+ig(t))|^2 e^{-C\sqrt{t^2+g(t)^2}}\sqrt{1+g'(t)^2}\, dt =: J.
\end{aligned}$$
Now for $|z|$ large enough, we have $|P(z)| \leq C'|z|^n$ and $C'|z|^n e^{-C|z|} \leq e^{-C''|z|}$ for a suitable positive constant C''. Then our last integral J can be estimated as follows
$$J \leq \int_{R_0}^{\infty} e^{-C''\sqrt{t^2+g(t)^2}}\sqrt{1+g'(t)^2}\, dt.$$
Now splitting $[R_0, \infty) = A \cup B$ where $A = \{t \geq R_0 :\ g'(t) \leq 1\}$ and $B = \{t \geq R_0 : g'(t) \geq 1\}$, we see that
$$\int_A e^{-C''\sqrt{t^2+g(t)^2}}\sqrt{1+g'(t)^2}\, dt \leq \sqrt{2}\int_A e^{-C''t} dt < \infty,$$
and as g is increasing we obtain finally
$$\begin{aligned}
\int_B e^{-C''\sqrt{t^2+g(t)^2}}\sqrt{1+g'(t)^2}\, dt &\leq \sqrt{2}\int_B e^{-C''g(t)} g'(t)\, dt \\
&= \sqrt{2}\int_{g(R_0,\infty)} e^{-C''G} dG < \infty,
\end{aligned}$$
which completes the proof. □

Remark 5.3. An unsolved problem is to find sufficient conditions on U to avoid the presence of spectral singularities, and then to obtain the absolute continuity of the continuous spectrum $\sigma_c(H)$. As these singularities correspond to "zero-width" resonances, such a result would result from the existence of a small neighborhood of $\mathbb{R}_+$ free of resonances.

References

[1] E. Bairamov, O. Çakar; *Quadratic pencil of Schrödinger operators with spectral singularities,* J. of Math. Anal. and App., **216** (1997), 302–320.

[2] E. Bairamov, O. Çakar, A. M. Krall; *An eigenfunction expansion for a quadratic pencil of Schrödinger operators with spectral singularities,* J. of Diff. Equ., **151** (1999), 268–289.

[3] G. Başcanbaz Tunca, E. Bairamov; *Discrete spectrum and principal functions of non-selfadjoint differential operators,* Czech. Math. Journal, **49** (1999), 689–700.
[4] F. A. Berezin, M. A. Shubin; *The Schrödinger equation,* Kluwer Academic Publishers, 1991.
[5] C. Bloch; *Cours sur la théorie des réactions nucléaires,* Commissariat à l'Energie Atomique, Centre d'Etudes Nucléaires de Saclay, 1955-1956.
[6] B. M. Brown, M. S. P. Eastham, D. K. R. McCormack; *Resonances and analytic continuation for exponentially decaying Sturm-Liouville potentials,* Jour. of Comp. and Appl. Math., **116** (2000), 181–193.
[7] E. B. Davies; *Linear operators and their spectra,* Cambridge University Press, 2007.
[8] V. de Alfaro, T. Regge; *Potential scattering,* North-Holland, 1965.
[9] S. Dyatlov, M. Zworski; *Mathematical theory of scattering resonances,* Graduate Texts in Mathematics, AMS., 2021.
[10] L. D. Fadeev; *The inverse problem in the quantum theory of scattering,* J. Math. Phys., **1** (1963), 72-104.
[11] H. Feschbach, C. Porter, V. Weisskopf; *Model for nuclear reactions with neutrons,* Phys. Rev., **96** (1954), 448–464.
[12] H. Feschbach; *The optical model and its justification,* Ann. Rev. of Nucl. Sci., **8** (1958), 49–104.
[13] G. B. Folland; *Spectral analysis of a singular nonself-adjoint boundary value problem,* J. of Diff. Equ., **37** (1980), 206–224.
[14] B. Friedman, L. I. Mishoe; *Eigenfunction expansions associated with a non-self-adjoint differential equation,* Pacific J. of Math., **6** (1956), 249–270.
[15] R. L. Franck, A. Laptev, E. H. Lieb, R. Seiringer; *Lieb-Thirring inequalities for Schrödinger operators with complex-valued potentials,* Lett. in Math. Phys., **77** (2006), 309-316.
[16] S. V. Hruščev; *Spectral singularities of dissipative Schrödinger operators with rapidly decreasing potential,* Indiana Univ. Math. Journal, **33** (1984), 613-638.
[17] A. M. Krall. *A nonhomogeneous eigenfunction expansion,* Trans. AMS, **117** (1965), 352–361.
[18] A. M. Krall, E. Bairamov, O. Çakar; *Spectrum and spectral singularities of a quadratic pencil of Schrödinger operators with a general boundary condition,* J. of Diff. Equ., **151** (1999), 252–267.
[19] V. E. Lyantse; *On a differential operator with spectral singularities. I*, (in Russian) Math. Sbornik, **64** (1964), 521–561.
[20] V. E. Lyantse; *On a differential operator with spectral singularities. II, (in Russian)* Math. Sbornik **65** (1964), 47–103.
[21] B. Morillon, P. Romain; *Dispersive and global spherical optical model with a local energy approximation for the scattering of neutrons by nuclei from $1KeV$ to $300MeV$,* Phys. Rev. C, **70** (2004), 014601.
[22] M. A. Naimark; *Investigations on the spectrum and the expansion in eigenfunctions of a non-selfadjoint differential operator of the second order on a semi-axis,* Trud. Mosk. Mat. Obch., AMS Translations, **16** (1960), 103–193.
[23] R. Newton; *Scattering theory of waves and particles,* Springer-Verlag, 1982.
[24] B. S. Pavlov; *The non-selfadjoint Schrödinger operator I,* in: Topics in Mathematical Physics, Consultant Bureau, New York, 1967.
[25] B. S. Pavlov; *The non-selfadjoint Schrödinger operator II,* in: Topics in Mathematical Physics, Consultant Bureau, New York (1968).
[26] B.S. Pavlov. *The non-selfadjoint Schrödinger operator III,* in: Topics in Mathematical Physics, Consultant Bureau, New York, 1969.
[27] J. T. Schwartz; *Some non self-adjoint operators,* Comm. Pure and Appl. Math., **13** (1960), 609-639.
[28] R. T. Seeley; *Integral equations depending analytically on a parameter,* Indag. Math., **24** (1962), 434-442.
[29] S. A. Stepin; *Scattering and bound states for nonselfadjoint Schrödinger operator,* arXiv: 1306.6457v1 [math.CA] 27 Jun 2013.
[30] J. D. Tamarkin; *On Fredholm integral equations whose kernels are analytic in a parameter,* Ann. of Math., **28** (1926), 127-152.
[31] I. Úlehla, L. Gomolčák, Pluhař; *Optical model of the atomic nucleus,* Academic Press, 1964.

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