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POSITIVE STEPANOV-LIKE ALMOST AUTOMORPHIC SOLUTIONS FOR SYSTEMS OF NONLINEAR DELAY INTEGRAL EQUATIONS

ABDELLATIF SADRATI, ABDERRAHIM ZERTITI

Abstract. In this article we show the existence of positive Stepanov-like almost automorphic solutions for systems of nonlinear delay integral equations. To do this, we apply the well-known Guo-Krasnosel'skii fixed point theorem for cone expansion and compression.

1. Introduction

Almost automorphic functions, as a generalization of the classical periodic and almost periodic functions, was introduced by Bochner in the earlier sixties [2, 3, 4] to avoid some assumptions of uniform convergence that arise when using almost periodic functions. From that time, the theory of almost automorphic functions has been generalized and developed extensively because of its applications in mathematical biology, physics, control theory, and other fields. For more details on these functions we refer the reader to [5, 6, 7, 14, 20] and the references therein.

The study of almost automorphic solutions of various types of integral equations and systems of integral equations is new and is an attractive area of research. A comprehensive theory of almost automorphy and the applications can be found in [12, 21, 22, 24] and references therein.

However, the concept of Stepanov-like almost automorphic functions, which was introduced by N'Guérékata and Pankov [23], is more general than that of almost automorphic functions. Such a notion was then utilized to study the existence of weak Stepanov-like almost automorphic solutions to some parabolic evolution equations. Since then, these functions have generated lot of developements and applications.

In this work we consider a system of nonlinear delay integral equations where the delays are specified functions. Models of this form play a fundamental role in many biological systems and thus occur in many applications such as the evolution in time of populations, the spread of infectious disease, etc. Our gool in this paper is to study the existence of positive Stepanov-like almost automorphic solutions to

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the type

$$x(s) = \int_0^{\tau(s)} f(s,\sigma,x(s-\sigma-l))\,d\sigma, \tag{1.1}$$

where $x = (x_1,\dots,x_n) : \mathbb{R} \to \mathbb{R}_+^n$, $\tau = (\tau_1,\dots,\tau_n) : \mathbb{R} \to \mathbb{R}_+^n$ and $f = (f_1,\dots,f_n) : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^n \to \mathbb{R}_+^n$ are appropriate functions specified later. Hence system (1.1) means that for each $i \in \{1,2,\dots,n\}$,

$$x_i(s) = \int_0^{\tau_i(s)} f_i(s,\sigma,x_1(s-\sigma-l),\dots,x_n(s-\sigma-l))\,d\sigma.$$

As an application, problem (1.1) models the evolution in time, of n species $x_1,\dots,x_n$ $(n \geq 2)$ with interaction. In the case $n = 2$, this equation generalizes the one studied in [27], if one does the change of variable $s - \sigma = u$ and sets $l = 0$,

$$\begin{aligned} x(s) &= \int_{s-\tau_1(s)}^{s} \tilde f(\sigma,x(\sigma),y(\sigma))\,d\sigma,\\ y(s) &= \int_{s-\tau_2(s)}^{s} \tilde g(\sigma,x(\sigma),y(\sigma))\,d\sigma. \end{aligned} \tag{1.2}$$

In deed, this system also generalizes the system proposed by Cooke and Kaplan [11] when $\tau_1(t) = \tau_1$, $\tau_2(t) = \tau_2$, and $l = 0$. There have been many papers concerning the existence of positive periodic, positive almost periodic, positive weighted pseudo almost automorphic solutions, etc, for various system. We refer the reader to [8, 9, 25, 26, 27, 28]. However, the existence of Stepanov-like almost automorphic solution to (1.1) is an untreated topic and this is the main motivation of the present work. This article is organized as follows. In section 2, we recall some basic facts about the notions of almost automorphy and Stepanov-like almost automorphy. In section 3, we prove our results for the existence of positive Stepanov-like almost automorphic solutions.

2. Preliminaries

This section includes notation, definitions, lemmas and preliminary facts which will be used later.

Throughout this paper, $p \in [1,+\infty)$. We denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}_+$ the set of nonnegative real numbers and for $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, $\|x\| = \sum_{i=1}^n |x_i|$. Let $\operatorname{meas} E$ be the Lebesgue measure for a subset $E \subset \mathbb{R}$. $L^p_{\rm loc}(\mathbb{R},\mathbb{R}^n)$ denotes the space of all equivalence classes of measurable functions $f : \mathbb{R} \to \mathbb{R}^n$ such that the restriction of f to every bounded subinterval of $\mathbb{R}$ is in $L^p(\mathbb{R},\mathbb{R}^n)$. $L^{p,1}_{\rm loc}(\mathbb{R}\times\mathbb{R}_+,\mathbb{R}^n)$ denotes the space of all equivalence classes of measurable functions $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^n$, $(s,\sigma) \mapsto f(s,\sigma)$, such that the restriction of f to every bounded subset of $\mathbb{R} \times \mathbb{R}_+$ is in $L^{p,1}(\mathbb{R} \times \mathbb{R}_+,\mathbb{R}^n) = L^p(\mathbb{R}, L^1(\mathbb{R}_+,\mathbb{R}^n))$.

Definition 2.1 ([1])**.** A continuous function $f : \mathbb{R} \to \mathbb{R}^n$ is called almost automorphic if for every sequence of real numbers $(s'_n)_n$ there exists a subsequence $(s_n)_n$ such that

$$\lim_{m\to+\infty}\lim_{n\to+\infty} f(t+s_n-s_m) = f(t), \quad \forall t \in \mathbb{R}.$$

This limit means that

$$f^*(t) = \lim_{n\to+\infty} f(t+s_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$f(t) = \lim_{n \to +\infty} f^*(t - s_n), \quad \forall t \in \mathbb{R}.$$

The collection of all such functions will be denoted by $AA(\mathbb{R}, \mathbb{R}^n)$.

Note that some fundamental properties of almost periodic functions are not satisfied by the almost automorphic functions, as example the property of uniform continuity. A classical example of almost automorphic function which is not almost periodic, as it is not uniformly continuous, is the function

$$f(t) = \sin big(\frac{1}{2 + \cos t + \cos \sqrt{2}t}), \quad t \in \mathbb{R}.$$

Remark 2.2. The function f^* obtained in Definition 2.1 is measurable but not necessarily continuous. Moreover, if f^* is continuous, then f is uniformly continuous [22, Theorem 2.6]. If the convergence in Definition 2.1 is uniform in $t \in \mathbb{R}$, then f is almost periodic [15].

Lemma 2.3 ([21])**.** *Assume that $f, g \in AA(\mathbb{R}, \mathbb{R}^n)$ and λ is a scalar. Then the following statements hold:*

(i) *$f + g$, λf, $f_\tau(t) = f(t + \tau)$, $\tilde{f}(t) = f(-t)$ are almost automorphic.*
(ii) *The range $R_f = \{f(t) : t \in \mathbb{R}\}$ is precompact in $\mathbb{R}^n$, and so f is bounded in norm.*
(iii) *If $\{f_n\}$ is a sequence of almost automorphic functions and $f_n \to f$ uniformly on $\mathbb{R}$, then f is almost automorphic.*
(iv) *$AA(\mathbb{R}, \mathbb{R}^n)$ is a Banach space with the supremum norm*

$$\|f\|_\infty = \sup_{t \in \mathbb{R}} \|f(t)\|.$$

Definition 2.4 ([1])**.** A continuous function $f : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^n \to \mathbb{R}^n$ is said to be almost automorphic if $f(s, \sigma, u)$ is almost automorphic in $s \in \mathbb{R}$ uniformly for all $(\sigma, u) \in K$, where K is any bounded subset of $\mathbb{R}_+ \times \mathbb{R}_+^n$. The collection of all such functions will be denoted by $AA(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^n, \mathbb{R}^n)$.

Among others things, almost automorphic functions satisfy the following property:

Let $K = K_1 \times K_2$ where, $K_1 \subset \mathbb{R}_+$, $K_2 \subset \mathbb{R}_+^n$ are compact subsets, and $\Omega \subset \mathbb{R}$. We denote by $C_K(\Omega \times \mathbb{R}_+ \times \mathbb{R}_+^n, \mathbb{R}^n)$ the set of all functions $f : \Omega \times \mathbb{R}_+ \times \mathbb{R}_+^n \to \mathbb{R}^n$ such that $f(s, \cdot, \cdot)$ is uniformly continuous on K uniformly for $s \in \Omega$. If we consider $x \in AA(\mathbb{R}, \mathbb{R}^n)$, K_1 is a compact subset of $\mathbb{R}_+$, $K_2 = \overline{\{x(s) : s \in \mathbb{R}\}} \subset \mathbb{R}_+^n$ and $f \in AA(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^n, \mathbb{R}^n) \cap C_K(\Omega \times \mathbb{R}_+ \times \mathbb{R}_+^n, \mathbb{R}^n)$, then the function $s \mapsto f(s, \sigma, x(s - \sigma - l)$ belong to $AA(\mathbb{R}, \mathbb{R}^n)$, for a fixed constant $l \in \mathbb{R}_+$ and all $\sigma \in K_1$.

Definition 2.5 ([13])**.** For a function $f : \mathbb{R} \to \mathbb{R}^n$, with $t \in \mathbb{R}$, and $s \in [0, 1]$, The Bochner transform is defined as

$$f^b(t, s) := f(t + s).$$

Remark 2.6 ([13])**.** Note that a function $\varphi(t, s)$, is the Bochner transform of a certain function $f(t)$, $\varphi(t, s) = f^b(t, s)$, if, and only if $\varphi(t + \tau, s - \tau) = \varphi(s, t)$ for all $t \in \mathbb{R}, s \in [0, 1]$ and $\tau \in [s - 1, s]$.

Definition 2.7 ([13])**.** For a function $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, with $t \in \mathbb{R}$, $s \in [0,1]$, $(\sigma, u) \in \mathbb{R} \times \mathbb{R}^n$, The Bochner transform is defined as

$$f^b(t, s, \sigma, u) := f(t + s, \sigma, u).$$

Definition 2.8 ([23])**.** Let $p \in [1, +\infty)$.

(i) The space $BS^p(\mathbb{R}, \mathbb{R}^n)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions f on $\mathbb{R}$ with values in $\mathbb{R}^n$ such that $f^b \in L^\infty(\mathbb{R}, L^p([0,1], \mathbb{R}^n))$. This is a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R}, L^p)} = \sup_{t \in \mathbb{R}} \Big(\int_t^{t+1} \|f(s)\|^p ds \Big)^{1/p}.$$

(ii) The space $BS^p(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^n, \mathbb{R}^n)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^n \to \mathbb{R}^n$ such that

$$f^b(\cdot, \cdot, \sigma, u) \in L^\infty(\mathbb{R}, L^p([0,1], \mathbb{R}^n)), \quad t \mapsto f^b(t, ., \sigma, u) \in L^p([0,1], \mathbb{R}^n),$$

for each $t \in \mathbb{R}$ and each $(\sigma, u) \in \mathbb{R}_+ \times \mathbb{R}_+^n$.

One can see that the Bochner transform f^b is a continuous function on $\mathbb{R}$ with values in $L^p([0,1], \mathbb{R}^n)$, and thus $f^b \in BC(\mathbb{R}, L^p([0,1], \mathbb{R}^n))$. In fact, for $p \geq 1$ we have that $(BC(\mathbb{R}, \mathbb{R}^n), \| \cdot \|_{BC})$ is continuously embeded in $(BS^p(\mathbb{R}, \mathbb{R}^n), \| \cdot \|_{S^p})$.

Definition 2.9 ([23])**.** The space $AS^p(\mathbb{R}, \mathbb{R}^n)$ of Stepanov-like almost automorphic functions (or S^p-almost automorphic) consists of all $f \in BS^p(\mathbb{R}, \mathbb{R}^n)$ such that $f^b \in AA(\mathbb{R}, L^p([0,1], \mathbb{R}^n))$.

In other words, a function $f \in L^p_{\rm loc}(\mathbb{R}, \mathbb{R}^n)$ is said to be S^p-almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p([0,1], \mathbb{R}^n)$ is almost automorphic in the sense that for every sequence of real numbers $(s'_n)_n$, there exist a subsequence $(s_n)_n$ and a function $f^* \in L^p_{\rm loc}(\mathbb{R}, \mathbb{R}^n)$ such that

$$\begin{gathered} \Big(\int_t^{t+1} \|f(s + s_n) - f^*(s)\|^p ds \Big)^{1/p} \to 0, \\ \Big(\int_t^{t+1} \|f^*(s - s_n) - f(s)\|^p ds \Big)^{1/p} \to 0 \end{gathered} \tag{2.1}$$

as $n \to +\infty$ pointwise on $\mathbb{R}$.

Lemma 2.10 ([23])**.** (i) *$(AS^p(\mathbb{R}, \mathbb{R}^n), \|.\|_{S^p})$ is a Banach space.*

(ii) *$AA(\mathbb{R}, \mathbb{R}^n)$ is continuously embeded in $AS^p(\mathbb{R}, \mathbb{R}^n)$.*

Remark 2.11. (1) The operator $J : AS^p(\mathbb{R}, \mathbb{R}^n) \to AS^p(\mathbb{R}, \mathbb{R}^n)$ such that $(Jx)(s) := x(-s)$ is well defined and linear. Moreover it is an isometry and $J^2 = I$.

(2) the operator T_a defined by $(T_a x)(s) := x(s + a)$ for a fixed $a \in \mathbb{R}$ leaves $AS^p(\mathbb{R}, \mathbb{R}^n)$ invariant.

Definition 2.12 ([23])**.** A function $f : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^n \to \mathbb{R}^n$, $(s, \sigma, u) \to f(s, \sigma, u)$ with $f(., \sigma, u) \in L^p_{\rm loc}(\mathbb{R}, \mathbb{R}^n)$ for each $(\sigma, u) \in \mathbb{R}_+ \times \mathbb{R}_+^n$ is said to be Stepanov-like almost automorphic in $s \in \mathbb{R}$ uniformly for $(\sigma, u) \in \mathbb{R}_+ \times \mathbb{R}_+^n$, if $s \to f(s, \sigma, u)$ is Stepanov-like almost automorphic for each $(\sigma, u) \in \mathbb{R}_+ \times \mathbb{R}_+^n$. That is, for every sequence of real numbers $(s'_n)_n$, there exist a subsequence $(s_n)_n$ and a function $f^* : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n$ with $f^*(\cdot, \sigma, u) \in L^p_{\rm loc}(\mathbb{R}, \mathbb{R}^n)$ such that

$$\begin{gathered} \Big(\int_t^{t+1} \|f(s+s_n, \sigma, u) - f^*(s, \sigma, u)\|^p ds\Big)^{1/p} \to 0, \\ \Big(\int_t^{t+1} \|f^*(s-s_n, \sigma, u) - f(s, \sigma, u)\|^p ds\Big)^{1/p} \to 0 \end{gathered} \tag{2.2}$$

as $n \to +\infty$ for all $t \in \mathbb{R}$ and $(\sigma, u) \in \mathbb{R}_+ \times \mathbb{R}^n_+$. We denote by $AS^p(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n_+, \mathbb{R}^n)$ the set of all such functions.

Definition 2.13. Let E be a real Banach space. A closed convex set $\mathbb{P}$ in E is called a convex cone if the following conditions are satisfied:

(1) if $x \in \mathbb{P}$, then $\lambda x \in \mathbb{P}$ for any $\lambda \in \mathbb{R}^+$;
(2) if $x \in \mathbb{P}$ and $-x \in \mathbb{P}$, then $x = 0$.

A cone $\mathbb{P}$ induces a partial ordering $\leq$ in E by $x \leq y$ if and only if $y - x \in \mathbb{P}$. A cone $\mathbb{P}$ is called normal if there exists a constant $N > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq N\|y\|$, where $\|.\|$ is the norm on E. We denote by $\overset{\circ}{\mathbb{P}}$ the interior set of $\mathbb{P}$. A cone $\mathbb{P}$ is called a solid cone if $\overset{\circ}{\mathbb{P}} \neq \emptyset$.

We shall prove the existence of a positive solution of (1.1) by using the well-known Guo-Krasnosel'skii fixed point theorem of cone expansion and compression.

Theorem 2.14 ([16])**.** *Let E be a Banach space and $\mathbb{P} \subset E$ be a cone. Suppose Ω_1 and Ω_2 are two bounded open sets in Banach space E such that $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and suppose that the operator $T : \mathbb{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathbb{P}$ is completely continuous such that*

(1) $\|Tx\| \leq \|x\|, \forall x \in \mathbb{P} \cap \partial\Omega_1$ *and* $\|Tx\| \geq \|x\|, \forall x \in \mathbb{P} \cap \partial\Omega_2$ *or*
(2) $\|Tx\| \geq \|x\|, \forall x \in \mathbb{P} \cap \partial\Omega_1$ *and* $\|Tx\| \leq \|x\|, \forall x \in \mathbb{P} \cap \partial\Omega_2$.

Then T has a fixed point in $\mathbb{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Existence of positive Stepanov-Like almost automorphic solutions

In this section, we study the existence of positive Stepanov-like almost automorphic solution to the system (1.1) . For that, we need firstly to prove a composition theorem.

Consider the set of all bounded functions $BAS^p(\mathbb{R}, \mathbb{R}^n) \subset AS^p(\mathbb{R}, \mathbb{R}^n)$, that is, for each $x \in BAS^p(\mathbb{R}, \mathbb{R}^n)$ we have $\|x\|_\infty = \sup_{s\in\mathbb{R}} \|x(s)\| < \infty$. It is clear that $(BAS^p(\mathbb{R}, \mathbb{R}^n), \|.\|_{S^p})$ is a Banach space. Let $AS^{p,1}(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n_+, \mathbb{R}^n)$ be the subset of $AS^p(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n_+, \mathbb{R}^n)$ consists of all functions f such that $f(\cdot, \cdot, u) \in L^{p,1}_{\rm loc}(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}^n)$ for all $u \in \mathbb{R}^n_+$.

In the rest of this paper, we assume that the following holds

(H1) For each compact subset $K \subset \mathbb{R}^n_+$, there exist constants $L_K, M_K > 0$ such that
 (i) for all $u, v \in K$, all $\sigma_1, \sigma_2 \in \mathbb{R}_+$ and all $s \in \mathbb{R}$, it holds
$$\|f(s, \sigma_1, u) - f(s, \sigma_2, v)\| \leq L_K(|\sigma_1 - \sigma_2| + \|u - v\|)\,.$$
 (ii) for all $(s, \sigma) \in \mathbb{R} \times \mathbb{R}_+$ and all $u \in K$, it holds
$$\|f(s, \sigma, x)\| \leq M_K\|u\|,$$

Theorem 3.1. *Assume that* $\tau, x \in BAS^p(\mathbb{R}, \mathbb{R}^n)$ *and* $f \in AS^{p,1}(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n_+, \mathbb{R}^n)$ *such that* (H1) *holds. Then, the function* $Tx : \mathbb{R} \to \mathbb{R}^n$ *defined by*

$$Tx(s) = \int_0^{\tau(s)} f(s, \sigma, x(s - \sigma - l))\, d\sigma$$

is in $BAS^p(\mathbb{R}, \mathbb{R}^n)$.

As in [15], to prove the above theorem we introduce some lemmas.

Lemma 3.2. *Assume that* $f \in AS^{p,1}(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n_+, \mathbb{R}^n)$*,* K *is a compact subset of* $\mathbb{R}^n_+$*,* $\tau \in \mathbb{R}_+$ *and (H1)(i) holds. Then, for each* $t \in \mathbb{R}$ *and each sequence of real numbers* (s_n)*, there exist a subsequence* (s_m) *and a set* $E \subset [0, 1]$ *with* $\operatorname{meas} E = 0$ *such that* $\lim_{m\to+\infty} f(t + s + s_m, \sigma, u)$ *exists for each* $\sigma \in [0, \tau]$*,* $u \in K$ *and* $s \in [0, 1] \setminus E$.

The proof of the above lemma is similar to that of [15, Lemma 2.1], we omit it here.

Lemma 3.3. *Assume that* $f \in AS^{p,1}(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n_+, \mathbb{R}^n)$*,* K *is a compact subset of* $\mathbb{R}^n_+$*,* $\tau \in \mathbb{R}_+$ *and (H1) holds. Then, for each sequence of real numbers* (s'_n)*, there exist a subsequence* (s_n)*, a function* $f^* : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n$ *with* $f^*(\cdot, \cdot, u) \in L^{p,1}_{\rm loc}(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}^n)$ *and a set* $E \subset \mathbb{R}$ *with* $\operatorname{meas} E = 0$ *such that for all* $\sigma \in [0, \tau]$*, all* $u, v \in K$ *and* $s \in \mathbb{R} \setminus E$ *we have*

$$\|f^*(s, \sigma, u) - f^*(s, \sigma, v)\| \le L_K \|u - v\|,$$
$$\|f^*(s, \sigma, u)\| \le M_K \|u\|.$$

Moreover (2.2) *holds.*

The proof of the above lemma is similar to that of [15, Lemma 2.2], we omit it here.

Lemma 3.4. *Assume that* $f \in AS^{p,1}(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n_+, \mathbb{R}^n)$*,* K_1*,* K_2 *are compact subsets of* $\mathbb{R}^n_+$ *and* (H1) *holds. Then, for every sequence of real numbers* (s'_n) *there exist a subsequence* (s_n) *and a function* $f^* : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n$ *with* $f^*(., ., x) \in L^{p,1}_{\rm loc}(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}^n)$ *such that*

$$\lim_{n\to\infty} \Big[\int_t^{t+1} \big(\sup_{(w,u)\in K} \| \int_0^w (f(s + s_n, \sigma, u) - f^*(s, \sigma, u))\, d\sigma \|\big)^p ds \Big]^{1/p} = 0,$$
$$\lim_{n\to\infty} \Big[\int_t^{t+1} \big(\sup_{(w,u)\in K} \| \int_0^w (f^*(s - s_n, \sigma, u) - f(s, \sigma, u))\, d\sigma \|\big)^p ds \Big]^{1/p} = 0,$$

for each $t \in \mathbb{R}$*, where* $K = K_1 \times K_2$.

Proof. Let f^* be as in Lemma 3.3. Then for every sequence of real numbers (s'_n) there exist a subsequence (s_n) such that

$$\lim_{n\to\infty} \Big(\int_t^{t+1} \|f(s + s_n, \sigma, u) - f^*(s, \sigma, u)\|^p ds \Big)^{1/p} = 0,$$
$$\lim_{n\to\infty} \Big(\int_t^{t+1} \|f^*(s - s_n, \sigma, u) - f(s, \sigma, u)\|^p ds \Big)^{1/p} = 0,$$

for each $t \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$ and $u \in \mathbb{R}^n_+$. In addition, there exists a set $E \subset \mathbb{R}$ with $\operatorname{meas} E = 0$ such that for all $\sigma \in [0, \|K_1\|]$, where $\|K_1\| = \sup_{w\in K_1} \|w\|$, for all $u, v \in K_2$ and $s \in \mathbb{R} \setminus E$, we have

$$\|f^*(s,\sigma,u) - f^*(s,\sigma,v)\| \le L_{K_2}\|u - v\|. \tag{3.1}$$

Fix $t \in \mathbb{R}$. For any $\varepsilon > 0$, there exist $(\tau_1, x_1), \dots, (\tau_m, x_m) \in K_1 \times K_2 = K$ such that $K \subset \cup_{i=1}^m B((\tau_i, x_i), \frac{\varepsilon}{\|K\|})$, where $\|K\| = \sup_{(w,u)\in K}\{\|w\| + \|u\|\}$.

For the above ε, there exists a $n_0 \in \mathbb{N}$ such that

$$\Big(\int_t^{t+1} \|f(s+s_n,\sigma,x_i) - f^*(s,\sigma,x_i)\|^p ds\Big)^{1/p} < \frac{\varepsilon}{m}, \tag{3.2}$$

for $n > n_0$ and $i \in \{1, 2, \dots, m\}$.

For $(w, u) \in K$ there exists $i_0 \in \{1, 2, \dots, m\}$ such that $(w, u) \in B((\tau_{i_0}, x_{i_0}), \frac{\varepsilon}{\|K\|})$; that is, $\|w-\tau_{i_0}\| < \frac{\varepsilon}{\|K\|}$ and $\|u-x_{i_0}\| < \frac{\varepsilon}{\|K\|}$. From (H1) and (3.1), for each $n > n_0$ and $s \in [0, 1]$ with $t + s \notin E$, we have

$$\begin{aligned}
&\|\int_0^w [f(t+s+s_n,\sigma,u) - f^*(t+s,\sigma,u)]\, d\sigma\| \\
&\le \|\int_0^w f(t+s+s_n,\sigma,u)\, d\sigma - \int_0^{\tau_{i_0}} f(t+s+s_n,\sigma,x_{i_0})\, d\sigma\| \\
&\quad + \|\int_0^{\tau_{i_0}} [f(t+s+s_n,\sigma,x_{i_0}) - f^*(t+s,\sigma,x_{i_0})]\, d\sigma\| \\
&\quad + \|\int_0^{\tau_{i_0}} f^*(t+s,\sigma,x_{i_0})\, d\sigma - \int_0^w f^*(t+s,\sigma,u)\, d\sigma\| \\
&\le \|\int_0^{\tau_{i_0}} [f(t+s+s_n,\sigma,u) - f(t+s+s_n,\sigma,x_{i_0}]\, d\sigma\| \\
&\quad + \|\int_{\tau_{i_0}}^w f(t+s+s_n,\sigma,u)\, d\sigma\| \\
&\quad + \int_0^{\|\tau_{i_0}\|} \|f(t+s+s_n,\sigma,x_{i_0}) - f^*(t+s,\sigma,x_{i_0})\|\, d\sigma \\
&\quad + \|\int_0^w [f^*(t+s,\sigma,x_{i_0}) - f^*(t+s,\sigma,u)]\, d\sigma\| + \|\int_w^{\tau_{i_0}} f^*(t+s,\sigma,x_{i_0})\, d\sigma\|.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|\int_0^w [f(t+s+s_n,\sigma,u) - f^*(t+s,\sigma,u)]\, d\sigma\| \\
&\le L_{K_2}\|\tau_{i_0}\|\|u - x_{i_0}\| + M_{K_2}\|w - \tau_{i_0}\|\|u\| \\
&\quad + \int_0^{\|\tau_{i_0}\|} \|f(t+s+s_n,\sigma,x_{i_0}) - f^*(t+s,\sigma,x_{i_0})\|\, d\sigma \\
&\quad + L_{K_2}\|w\|\|x_{i_0} - u\| + M_{K_2}\|w - \tau_{i_0}\|\|x_{i_0}\| \\
&\le \int_0^{\|\tau_{i_0}\|} \|f(t+s+s_n,\sigma,x_{i_0}) - f^*(t+s,\sigma,x_{i_0})\|\, d\sigma + 2(L_{K_2} + M_{K_2})\varepsilon.
\end{aligned}$$

Now, by Minkowski's inequality, the Hölder's inequality, and (3.2), for each $n > n_0$ and $s \in [0,1]$ with $t+s \notin E$ we have

$$\begin{aligned}
&\Big[\int_t^{t+1}\Big(\sup_{(w,u)\in K}\|\int_0^w (f(s+s_n,\sigma,u)-f^*(s,\sigma,u))\,d\sigma\|\Big)^p ds\Big]^{1/p}\\
&=\Big[\int_0^1\Big(\sup_{(w,u)\in K}\|\int_0^w (f(t+s+s_n,\sigma,u)-f^*(t+s,\sigma,u))\,d\sigma\|\Big)^p ds\Big]^{1/p}\\
&\le \sum_{i=1}^m \|\tau_i\|^{\frac{p-1}{p}}\Big[\int_0^{\|\tau_i\|}\int_0^1 \|f(t+s+s_n,\sigma,x_i)-f^*(t+s,\sigma,x_i)\|^p ds\,d\sigma\Big]^{1/p}\\
&\quad + 2(L_{K_2}+M_{K_2})\varepsilon\\
&< \sum_{i=1}^m \|\tau_i\|^{\frac{p-1}{p}}\|\tau_i\|^{\frac{1}{p}}\frac{\varepsilon}{m} + 2(L_{K_2}+M_{K_2})\varepsilon\\
&\le [\|K_1\| + 2(L_{K_2}+M_{K_2})]\varepsilon.
\end{aligned}$$

Hence, we obtain

$$\lim_{n\to\infty}\Big[\int_t^{t+1}\Big(\sup_{(w,u)\in K}\|\int_0^w (f(s+s_n,\sigma,u)-f^*(s,\sigma,u))\,d\sigma\|\Big)^p ds\Big]^{1/p} = 0,$$

for each $t \in \mathbb{R}$. Analogously, one can show that

$$\lim_{n\to\infty}\Big[\int_t^{t+1}\Big(\sup_{(w,u)\in K}\|\int_0^w (f^*(s-s_n,\sigma,u)-f(s,\sigma,u))\,d\sigma\|\Big)^p ds\Big]^{1/p} = 0,$$

for each $t \in \mathbb{R}$. The proof is complete. $\square$

Proof of Theorem 3.1. Since $\tau, x \in BAS^p(\mathbb{R},\mathbb{R}^n)$, $f \in AS^{p,1}(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}^n_+,\mathbb{R}^n)$, and (H1)(ii) holds, it is easy to show that Tx is bounded and $Tx(\cdot) \in L^p_{\rm loc}(\mathbb{R},\mathbb{R}^n_+)$. In addition, there exist x^* and $\tau^* \in L^p_{loc}(\mathbb{R},\mathbb{R}^n)$ such that (2.1) holds, and $f^* : \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}^n_+ \to \mathbb{R}^n$ (as defined in Lemma 3.4) satisfies (2.2). Let

$$T^*x^*(s) = \int_0^{\tau^*(s)} f^*(s,\sigma,x^*(s-\sigma-l))ds.$$

Then we have

$$\begin{aligned}
&\Big[\int_t^{t+1}\|Tx(s+s_n)-T^*x^*(s)\|^p ds\Big]^{1/p}\\
&=\Big[\int_t^{t+1}\Big\|\int_0^{\tau(s+s_n)} f(s+s_n,\sigma,x(s+s_n-\sigma-l))\,d\sigma\\
&\quad -\int_0^{\tau^*(s)} f^*(s,\sigma,x^*(s-\sigma-l))\,d\sigma\Big\|^p ds\Big]^{1/p}\\
&\le \Big[\int_t^{t+1}\Big\|\int_0^{\tau(s+s_n)}[f(s+s_n,\sigma,x(s+s_n-\sigma-l))\\
&\quad - f^*(s,\sigma,x(s+s_n-\sigma-l))]\,d\sigma\Big\|^p ds\Big]^{1/p}\\
&\quad +\Big[\int_t^{t+1}\Big\|\int_0^{\tau^*(s)}[f^*(s,\sigma,x(s+s_n-\sigma-l))
\end{aligned}$$

$$
\begin{aligned}
&- f^*(s,\sigma,x^*(s-\sigma-l))]\,d\sigma\big\|^p ds\Big]^{1/p}\\
&+ \Big[\int_t^{t+1} \|\int_{\tau^*(s)}^{\tau(s+s_n)} f^*(s,\sigma,x(s+s_n-\sigma-l))\,d\sigma\|^p ds\Big]^{1/p}\\
\le{}& \Big[\int_t^{t+1} \big(\sup_{(w,u)\in K} \|\int_0^w (f(s+s_n,\sigma,u)-f^*(s,\sigma,u))\,d\sigma\|\big)^p ds\Big]^{1/p}\\
&+ \|\tau^*\|_\infty^{\frac{p-1}{p}} \Big[\int_t^{t+1}\int_0^{\|\tau^*\|_\infty} \big\|f^*(s,\sigma,x(s+s_n-\sigma-l))\\
&- f^*(s,\sigma,x^*(s-\sigma-l))\big\|^p\,d\sigma ds\Big]^{1/p}\\
&+ M_{K_2}\|x\|_\infty \Big[\int_t^{t+1} \|\tau(s+s_n)-\tau^*(s)\|^p ds\Big]^{1/p},
\end{aligned}
$$

where $K_1=\overline{\{\tau(s):s\in\mathbb{R}\}}$, $K_2=\overline{\{x(s):s\in\mathbb{R}\}}$ and $K=K_1\times K_2$.

Using Lemma 3.4, (2.2) and (2.1) we obtain

$$
\lim_{n\to+\infty}\Big[\int_t^{t+1}\|Tx(s+s_n)-T^*x^*(s)\|^p ds\Big]^{1/p}=0.
$$

Analogously we prove that $\lim_{n\to+\infty}\big[\int_t^{t+1}\|T^*x^*(s-s_n)-Tx(s)\|^p ds\big]^{1/p}=0$. The proof is complete. □

Now, we are ready to present our main results. In the sequel, we will consider that the functions f and τ are defined as in system (1.1).

Theorem 3.5. *Let $f\in AS^{p,1}(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+^n,\mathbb{R}_+^n)$ be a function satisfying* (H1) *and let $\tau\in BAS^p(\mathbb{R},\mathbb{R}_+^n)$. Assume that the following hypotheses hold:*

- (H2) *There exist numbers $r_1,r_2\in\mathbb{R}$ with $r_2-r_1\ge 1$ and $\gamma>0$ such that for each compact subset $K\subset\mathbb{R}_+^n\times\mathbb{R}_+^n$,*
$$
\inf_{r\in[r_1,r_2],\,(w,u)\in K}\|\int_0^w f(r,\sigma,u)\,d\sigma\|\ge\gamma\|\int_0^\tau f(s,\sigma,x)\,d\sigma\|,
$$
for all $(\tau,x)\in K$ and $s\in\mathbb{R}$.
- (H3) *There exists a function $a:\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}^n$ such that*
$$
\limsup_{\|u\|\to 0}\frac{f(s,\sigma,u)}{\|u\|}=a(s,\sigma)
$$
uniformly in $(s,\sigma)\in\mathbb{R}\times\mathbb{R}_+$, that is, for $i=1,\dots,n$, there exists a function $a_i:\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}$ such that
$$
\limsup_{\|u\|\to 0}\frac{f_i(s,\sigma,u)}{\|u\|}=a_i(s,\sigma)
$$
uniformly in $(s,\sigma)\in\mathbb{R}\times\mathbb{R}_+$, where $(a_1,a_2,\dots,a_n)=a$.
- (H4) *There exist a function $b:\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}^n$ such that*
$$
\liminf_{\|u\|\to+\infty}\frac{f(s,\sigma,u)}{\|u\|}=b(s,\sigma)
$$

uniformly in $(s,\sigma)\in\mathbb{R}\times\mathbb{R}_+$, *that is, for* $i=1,\dots,n$, *there exists a function* $b_i:\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}$ *such that*

$$\liminf_{\|u\|\to+\infty}\frac{f_i(s,\sigma,u)}{\|u\|}=b_i(s,\sigma)$$

uniformly in $(s,\sigma)\in\mathbb{R}\times\mathbb{R}_+$, *where* $(b_1,b_2,\dots,b_n)=b$.

If

$$\sup_{r\in[r_1,r_2]}\|\int_0^{\tau(r)}a(r,\sigma)\,d\sigma\|<\gamma\quad\text{and}\quad\inf_{r\in[r_1,r_2]}\|\int_0^{\tau(r)}b(r,\sigma)\,d\sigma\|>\frac{1}{\gamma^2},$$

then system (1.1) *has a nonzero positive solution* x *in* $BAS^p(\mathbb{R},\mathbb{R}^n_+)$. *That is,* x *not identically equal to zero and* $x_i(t)\geq 0$, *for all* $t\in\mathbb{R}$ *and all* $i=1,\dots,n$.

Proof. Let $T:BAS^p(\mathbb{R},\mathbb{R}^n_+)\to BAS^p(\mathbb{R},\mathbb{R}^n_+)$ be the integral operator defined by

$$Tx(s)=\int_0^{\tau(s)}f(s,\sigma,x(s-\sigma-l))\,d\sigma,\quad x\in\mathbb{P}.$$

Consider the positive cone of $BAS^p(\mathbb{R},\mathbb{R}^n_+)$ defined by

$$\mathbb{P}=\{x\in BAS^p(\mathbb{R},\mathbb{R}^n_+):\inf_{r\in[r_1,r_2]}\|x(r)\|\geq\gamma\|x\|_{S^p}\}.$$

It is clear that $x\in BAS^p(\mathbb{R},\mathbb{R}^n_+)$ is a solution of system (1.1) if and only if x is a fixed point of the operator T in $BAS^p(\mathbb{R},\mathbb{R}^n_+)$. We will prove that all assumptions of Theorem 2.14 are satisfied. For the sake of convenience, we divide the proof into three steps.

Step 1. We show that $T(BAS^p(\mathbb{R},\mathbb{R}^n_+))\subset\mathbb{P}$. For $\tau,x\in BAS^p(\mathbb{R},\mathbb{R}^n_+)$, denote by $K_1=\overline{\tau(\mathbb{R})}$, $K_2=\overline{x(\mathbb{R})}$ and $K=K_1\times K_2$. Then, for all $r\in[r_1,r_2]$,

$$\begin{aligned}
\|Tx(r)\|&=\|\int_0^{\tau(r)}f(r,\sigma,x(r-\sigma-l))\,d\sigma\|\\
&\geq\inf_{r\in[r_1,r_2],\,(w,u)\in K}\|\int_0^{w}f(r,\sigma,u)\,d\sigma\|\\
&=\Big[\int_t^{t+1}\Big(\inf_{r\in[r_1,r_2],\,(w,u)\in K}\|\int_0^{w}f(r,\sigma,u)\,d\sigma\|\Big)^p ds\Big]^{1/p}\\
&\geq\gamma\Big[\int_t^{t+1}\|\int_0^{\tau(s)}f(s,\sigma,x(s-\sigma-l))\,d\sigma\|^p ds\Big]^{1/p},
\end{aligned}$$

for all $s\in\mathbb{R}$ and $t\in\mathbb{R}$. Hence, for all $r\in[r_1,r_2]$,

$$\|Tx(r)\|\geq\gamma\|Tx\|_{S^p}.$$

Thus, $\inf_{r\in[r_1,r_2]}\|Tx(r)\|\geq\gamma\|Tx\|_{S^p}$, which proves the assertion.

Step 2. We prove that $T:\mathbb{P}\to\mathbb{P}$ is completely continuous. Firstly, we claim that T is a compact operator. That is, for every sequence $\{x_k\}\subset\mathbb{P}$ with $\{x_k\}$ is bounded in $BAS^p(\mathbb{R},\mathbb{R}^n_+)$, the sequence $\{Tx_k\}$ has a convergent subsequence in $BAS^p(\mathbb{R},\mathbb{R}^n_+)$. Indeed, since $\{x_k\}$ is bounded in $BAS^p(\mathbb{R},\mathbb{R}^n_+)$, there exists a constant $\overline{M}>0$ such that $\|x_k\|_\infty\leq\overline{M}$, for all $k=0,1,2,\dots$. In this case, there exist a compact subset $\mathbf{K}\subset\mathbb{R}^n_+$ such that $\overline{x_k(\mathbb{R})}\subset\mathbf{K}$ for all k. By using (H1)(ii), for $k=0,1,2,\dots$ we have

$$\|Tx_k\|_\infty=\sup_{s\in\mathbb{R}}\|Tx_k(s)\|$$

$$
\begin{aligned}
&= \sup_{s\in\mathbb{R}} \| \int_0^{\tau(s)} f(s,\sigma,x_k(s-\sigma-l))\,d\sigma\| \\
&\le \sup_{s\in\mathbb{R}} \int_0^{\|\tau\|_\infty} \|f(s,\sigma,x_k(s-\sigma-l))\|\,d\sigma \\
&\le \sup_{s\in\mathbb{R}} \int_0^{\|\tau\|_\infty} M_{\mathbf{K}}\|x_k(s-\sigma-l)\|\,d\sigma \\
&\le \|\tau\|_\infty M_{\mathbf{K}}\overline{M} < +\infty.
\end{aligned}
$$

Therefore, $\{Tx_k\}$ is uniformly bounded. Moreover, it is well known that if $h \in L^p(\mathbb{R}^n)$, $1 \le p < +\infty$, then $\lim_{\alpha\to 0}\int_{\mathbb{R}^n} \|h(s+\alpha)-h(s)\|^p ds = 0$. Hence, since $Tx_k \in BAS^p(\mathbb{R},\mathbb{R}^n_+)$ for all k, we obtain

$$
\lim_{\alpha\to 0}\int_t^{t+1} \|Tx_k(s+\alpha)-Tx_k(s)\|^p ds = 0, \quad \forall t\in\mathbb{R}.
$$

Thus, for all $k = 0,1,2,\dots$ and all $s\in\mathbb{R}$,

$$
\lim_{\alpha\to 0}\|Tx_k(s+\alpha)-Tx_k(s)\| \le \lim_{\alpha\to 0}\sup_{t\in\Lambda}\Big[\int_t^{t+1} \|Tx_k(\theta+\alpha)-Tx_k(\theta)\|^p d\theta\Big]^{1/p} = 0,
$$

where Λ is any compact subset in $\mathbb{R}$, which implies that the sequence $\{Tx_k\}$ of continuous functions is equicontinuous.

Note that $\{Tx_k\}$ is uniformly bounded and equicontinuous, this together with the Ascoli's theorem [19, p. 233-234] ensures that $\{Tx_k\}$ is relatively compact in uniform convergence on compacta. Thus $\{Tx_k\}$ has a convergent subsequence with respect to the topology of uniform convergence on compacta, say, $\{Tx_{k'}\}$. Hence, there exists a continuous function $x:\mathbb{R}\to\mathbb{R}^n$ such that for each compact subset $\Lambda\subset\mathbb{R}$ we have

$$
\lim_{k'\to\infty}\sup_{s\in\Lambda}\|Tx_{k'}(s)-x(s)\| = 0.
$$

It follows that

$$
\lim_{k'\to\infty}\Big[\int_t^{t+1}\|Tx_{k'}(\theta)-x(\theta)\|^p d\theta\Big]^{1/p} \le \lim_{k'\to\infty}\sup_{s\in[t,t+1]}\|Tx_{k'}(s)-x(s)\| = 0.
$$

That means $\lim_{k'\to\infty}\Big[\int_t^{t+1}\|Tx_{k'}(\theta)-x(\theta)\|^p d\theta\Big]^{1/p} = 0$, for each $t\in\mathbb{R}$. Then, using [17, Lemma 2.7], one deduce that $\{Tx_{k'}\}$ is a convergent subsequence of $\{Tx_k\}$ in $BAS^p(\mathbb{R},\mathbb{R}^n_+)$.

On the other hand, let $\{x_k\}\subset\mathbb{P}$, $x\in\mathbb{P}$ such that $\lim_{k\to+\infty}\|x_k-x\|_{S^p} = 0$ and let Λ be a compact subset of $\mathbb{R}^n_+$ such that for all k, $\overline{x_k(\mathbb{R})}, \overline{x(\mathbb{R})}\subset\Lambda$. By using (H1)(i), we have

$$
\begin{aligned}
&\|Tx_k - Tx\|_{S^p} \\
&= \sup_{t\in\mathbb{R}}\Big[\int_t^{t+1}\|\int_0^{\tau(s)}[f(s,\sigma,x_k(s-\sigma-l)) - f(s,\sigma,x(s-\sigma-l))]\,d\sigma\|^p ds\Big]^{1/p} \\
&\le \sup_{t\in\mathbb{R}}\Big[\int_t^{t+1}\Big(\int_0^{\|\tau\|_\infty}\|f(s,\sigma,x_k(s-\sigma-l)) - f(s,\sigma,x(s-\sigma-l))\|\,d\sigma\Big)^p ds\Big]^{1/p} \\
&\le \|\tau\|_\infty^{\frac{p-1}{p}} L_\Lambda \sup_{t\in\mathbb{R}}\Big[\int_t^{t+1}\int_0^{\|\tau\|_\infty}\|x_k(s-\sigma-l)-x(s-\sigma-l)\|^p\,d\sigma ds\Big]^{1/p}
\end{aligned}
$$

$$\leq \|\tau\|_\infty L_\Lambda \|x_k - x\|_{S^p}.$$

This means that T is continuous. The proof of the assertion is complete.

Step 3. In this step, we show that (1) of Theorem 2.14 is satisfied. By (H3), for every $\varepsilon > 0$ verifying $\sup_{r\in[r_1,r_2]} \|\int_0^{\tau(r)}(a(r,\sigma)+\overline{\varepsilon})\,d\sigma\| \leq \gamma$, where $\overline{\varepsilon} = (\varepsilon, \varepsilon, \dots, \varepsilon) \in \mathbb{R}^n$, there exists $\delta > 0$ such that

$$f_i(s,\sigma,u) \leq (a_i(s,\sigma)+\varepsilon)\|u\|, \quad i = 1,2,\dots,n,$$

for all $(s,\sigma) \in \mathbb{R}\times\mathbb{R}_+$ and all $u \in \mathbb{R}_+^n$ with $\|u\| \leq \delta$.

We set $\Omega_1 = \{x \in \mathbb{P} : \|x\|_{S^p} < \delta\}$. Then, for $x \in \partial\Omega_1$ and $t \in \mathbb{R}$ we have

$$\begin{aligned}
\sup_{t\in\mathbb{R}}\Big[\int_t^{t+1}\|Tx(s)\|^p ds\Big]^{1/p} &= \sup_{t\in\mathbb{R}}\Big[\int_t^{t+1}\|\int_0^{\tau(s)} f(s,\sigma,x(s-\sigma-l))\,d\sigma\|^p ds\Big]^{1/p} \\
&\leq \frac{1}{\gamma}\sup_{t\in\mathbb{R}}\Big[\int_t^{t+1}\inf_{\substack{r\in[r_1,r_2]\\(w,u)\in K}}\|\int_0^w f(r,\sigma,u)\,d\sigma\|^p ds\Big]^{1/p} \\
&= \frac{1}{\gamma}\inf_{\substack{r\in[r_1,r_2]\\(w,u)\in K}}\|\int_0^w f(r,\sigma,u)\,d\sigma\|.
\end{aligned}$$

Since $\|x\|_{S^p} = \delta$, there exist $r_0 \in [r_1,r_2]$ such that $\|x(r_0)\| \leq \delta$. Therefore

$$\begin{aligned}
\|Tx\|_{S^p} &\leq \frac{1}{\gamma}\|\int_0^{\tau(r_0)} f(r_0,\sigma,x(r_0))\,d\sigma\| \\
&\leq \frac{1}{\gamma}\|x(r_0)\|\|\int_0^{\tau(r_0)}(a(r_0,\sigma)+\overline{\varepsilon})\,d\sigma\| \\
&\leq \frac{1}{\gamma}\|x(r_0)\|\sup_{r\in[r_1,r_2]}\|\int_0^{\tau(r)}(a(r,\sigma)+\overline{\varepsilon})\,d\sigma\| \\
&\leq \|x(r_0)\| \leq \delta = \|x\|_{S^p}.
\end{aligned}$$

Inversely, by (H4), for every $\varepsilon > 0$ satisfying $\inf_{r\in[r_1,r_2]}\|\int_0^{\tau(r)} b(r,\sigma)-\overline{\varepsilon}\|\,d\sigma \geq \frac{1}{\gamma^2}$, there exists $M_0 > 2\delta$ such that

$$f_i(s,\sigma,u) \geq (b_i(s,\sigma)-\varepsilon)\|u\|, \quad i = 1,2,\dots,n,$$

for all $(s,\sigma) \in \mathbb{R}\times\mathbb{R}_+$ and all $u \in \mathbb{R}_+^n$ with $\|u\| \geq M_0$.

Let $M = \max\{2\delta, M_0/\gamma\}$, and set $\Omega_2 = \{x \in \mathbb{P} : \|x\|_{S^p} < M\}$. Then, for $x \in \partial\Omega_2$ and $r \in [r_1,r_2]$ we have

$$\|x(r)\| \geq \inf_{r\in[r_1,r_2]}\|x(r)\| \geq \gamma\|x\|_{S^p} \geq M_0.$$

It follows that for $r \in [r_1,r_2]$,

$$\begin{aligned}
\|\int_0^{\tau(r)} f(r,\sigma,x(r-\sigma-l))\,d\sigma\| &\geq \inf_{r\in[r_1,r_2]\,,(w,u)\in K}\|\int_0^w f(r,\sigma,u)\,d\sigma\| \\
&\geq \gamma\|\int_0^{\tau(r_0)} f(r_0,\sigma,x(r_0))\,d\sigma\| \\
&\geq \gamma\|\int_0^{\tau(r_0)}(b(r_0,\sigma)-\overline{\varepsilon})\,d\sigma\|\|x(r_0)\|,
\end{aligned}$$

for all $r_0 \in [r_1, r_2]$. Hence

$$\begin{aligned}
\|T(x)\|_{S^p} &= \sup_{t\in\mathbb{R}} \Big[\int_t^{t+1} \Big\| \int_0^{\tau(s)} f(s,\sigma,x(s-\sigma-l))\, d\sigma \Big\|^p ds \Big]^{1/p} \\
&\geq \gamma \inf_{r\in[r_1,r_2]} \Big\| \int_0^{\tau(r)} (b(r,\sigma)-\overline{\varepsilon})\, d\sigma \Big\| \inf_{r\in[r_1,r_2]} \|x(r)\| \\
&\geq \gamma^2 \inf_{r\in[r_1,r_2]} \Big\| \int_0^{\tau(r)} (b(r,\sigma)-\overline{\varepsilon})\, d\sigma \Big\| \|x\|_{S^p} \\
&\geq \|x\|_{S^p}.
\end{aligned}$$

The proof is complete. $\square$

Corollary 3.6. *Let $f \in AS^{p,1}(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+^n, \mathbb{R}_+^n)$ be a function satisfying* (H1) *and let $\tau \in BAS^p(\mathbb{R},\mathbb{R}_+^n)$. Assume that* (H2)–(H4) *hold. In addition assume*

- (H5) *The function $\tau = (\tau_1,\dots,\tau_n)$ is such that $\tau_i(t) > 0$, for all $t \in \mathbb{R}$ and all $i \in \{1,\dots,n\}$.*
- (H6) *$f(s,\sigma,0) = 0$ for all $(s,\sigma) \in \mathbb{R}\times\mathbb{R}_+$, and for $x = (x_1,\dots,x_n) \in \mathbb{R}_+^n$, if there is $j \in \{1,\dots,n\}$ such that $x_j > 0$ then $f_i(s,\sigma,x) > 0$, for all $i \neq j$ and $(s,\sigma) \in \mathbb{R}\times\mathbb{R}_+$.*

Then system (1.1) *has a strictly positive solution $x = (x_1,\dots,x_n)$ in $BAS^p(\mathbb{R},\mathbb{R}_+^n)$. That is, for each $i \in \{1,\dots,n\}$, $x_i(s) \geq 0$ for all $s \in \mathbb{R}$ and $x_i \neq 0$.*

Proof. We prove that if x is a solution as in the above theorem, then $x_i \neq 0$ for all $i \in \{1,\dots,n\}$. In fact, suppose that $x_{j_0} \neq 0$ for a $j_0 \in \{1,\dots,n\}$, then there exists $s_0 \in \mathbb{R}$ such that $x_{j_0}(s_0) > 0$ and consequently, for each $i \neq j_0$, there exist $s_i \in \mathbb{R}$ and $\sigma_i \in [0, \tau_i(s_i)]$ such that $x_{j_0}(s_i - \sigma_i - l) > 0$. Then by (H6) we have $f_i(s_i, \sigma_i, x(s_i - \sigma_i - l)) > 0$. Thus

$$\int_0^{\tau_i(s_i)} f_i(s_i,\sigma,x(s_i-\sigma-l))\, d\sigma > 0.$$

This implies that $x_i(s_i) > 0$ and therefore $x_i \neq 0$ for all $i \in \{1,\dots,n\}$. $\square$

Remark 3.7. System (1.1) with hypotheses (H5) and (H6) can be interpreted as an epidemic model combining with population ecology. More precisely, it is a model of n species $x_1,\dots,x_n$ with Stepanov-like almost automorphic interaction and infectious disease (assuming the n species are uniformly distributed in a given geographical area). In this context, $x_i(t)$ is the population at time t of infectious individuals in the species x_i, function f_i present the instantaneous rate of infection in the species x_i and $\tau_i(t)$ is the duration of infectivity in x_i.

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Abdellatif Sadrati
MSISI Laboratory, AM2CSI Group, Department of Mathematics, FST, Erracidia, University Moulay Ismaïl of Meknes, BP 509, Boutalamine, 52000, Errachidia, Morocco
Email address: abdo2sadrati@gmail.com

Abderrahim Zertiti
Department of Mathematics, Faculty of Sciences, University Abdelmalek Essaâdi, BP 2121, Tetouan, Morocco
Email address: abdzertiti@hotmail.fr