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MULTIPLE SOLUTIONS TO BOUNDARY VALUE PROBLEMS FOR SEMILINEAR ELLIPTIC EQUATIONS

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Abstract. In this article, we study the multiplicity of weak solutions to the boundary value problem

$$-\Delta u = f(x,u) + g(x,u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$ $(N > 2)$, $f(x,\xi)$ is odd in ξ and g is a perturbation term. Under some growth conditions on f and g, we show that there are infinitely many solutions. Here we do not require that f be continuous or satisfy the Ambrosetti-Rabinowitz (AR) condition. The conditions assumed here are not implied by the ones in [3, 15]. We use the perturbation method by Rabinowitz combined with estimating the asymptotic behavior of eigenvalues for Schrödinger's equations.

1. Introduction

In the previous decades, the boundary value problem for semilinear elliptic equation

$$-\Delta u = f(x,u) + g(x,u), \quad u \in H_0^1(\Omega) \tag{1.1}$$

has been studied by many authors, see for example [2, 14, 3] and the references therein. Here Ω is a bounded smooth domain of $\mathbb{R}^N$ $(N \geq 2)$, $f(x,\xi)$ is odd in ξ and $g(x,\xi)$ is a non-odd perturbation term. The following condition was introduced in [1, 10]

(AR) For some $\mu > 2$, and $R > 0$, we have

$$0 < \mu F(x,\xi) \leq f(x,\xi)\xi, \quad \forall x \in \Omega,\ \forall |\xi| \geq R,$$

where $F(x,\xi) = \int_0^\xi f(x,\tau)\,d\tau$.

This condition plays an important role in the study of elliptic equations. Let us sketch some the results from the past 40 years.

Bahri and Berestycki [2] proved that if $f(x,\xi) \equiv |\xi|^{p-2}\xi$, $g(x,\xi) \equiv g(x) \in L^2(\Omega)$, $p \in (1, P_N)$, where P_N is the largest root of the equation

$$(2N-2)P^2 - (N+2)P - N = 0, \quad N \geq 2,$$

then problem (1.1) has infinitely many solutions in $H_0^1(\Omega)$. This case was first studied by Bahri and Berestycki [2], and independently by Struwe [14].

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Rabinowitz [11, 12] studied problem (1.1), assuming that $N \geq 3$ and f satisfies (AR), (R1), (R2), $g(x,\xi) \equiv g(x) \in L^2(\Omega)$, and

$$\frac{2p}{N(p-2)} - 1 > \frac{\mu}{\mu - 1}, \tag{1.2}$$

where

- (R1) $f(x,\xi) \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $f(x,-\xi) = -f(x,\xi)$ for all $(x,\xi) \in \Omega \times \mathbb{R}$.
- (R2) There exist $2 < p < 2^* := \frac{2N}{N-2}$, $C > 0$ such that almost everywhere in Ω,

$$|f(x,\xi)| \leq C(1 + |\xi|^{p-1}).$$

He then proved that problem (1.1) has an unbounded sequence of solutions in $H_0^1(\Omega)$ (see [11, Theorem 1.5]). Assuming that f satisfies (AR), (R1), (R2), $g(x,\xi) \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and

$$|g(x,\xi)| \leq C_1 + C_2|\xi|^{\sigma}, \quad 0 \leq \sigma < \mu - 1, \quad \frac{2p}{N(p-2)} - 1 > \frac{\mu}{\mu - \sigma - 1},$$

where C_1, C_2 are nonnegative real numbers, then he confirmed that the problem (1.1) has an unbounded sequence of solutions in $H_0^1(\Omega)$ (see [11, Remark 1.71]).

Bahri and Lions [3] assumed that $f(x,\xi) \equiv |\xi|^{p-2}\xi$, $2 < p < 2^*$, ($p < \infty$, if $N = 2$) such that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$|g(x,\xi)| \leq g_1(x) + C_3|\xi|^{\frac{N+2}{N-2}} \quad \text{a.e. in } \Omega \text{ for some } C_3 \geq 0,$$
$$|G(x,\xi)| \leq g_2(x) + g_3(x)|\xi|^{\sigma_1} \quad \text{a.e. in } \Omega \text{ for some } 0 \leq \sigma_1 < 2,$$

where $G(x,\xi) := \int_0^{\xi} g(x,\tau)\,\mathrm{d}\tau$, $g_1(x) \in L_+^{2N/(N+2)}(\Omega)$, $g_2(x) \in L_+^1(\Omega)$, $N > 2$, $g_3(x) \in L_+^{\beta}(\Omega)$,with $\beta > 1$, $\beta' < 2N/(N-2)(1/\sigma_1)$, $1/\beta + 1/\beta' = 1$, $L_+^{\beta}(\Omega) := \{g : \Omega \to \mathbb{R} | g \in L^{\beta}(\Omega), g(x) \geq 0 \text{ a.e. in } \Omega\}$ and

$$2 < p < \frac{2N - 2\sigma_1}{N-2}. \tag{1.3}$$

Under the above assumptions, Bahri and Lions proved that problem (1.1) has infinitely many solutions in $H_0^1(\Omega)$. Obviously, the assumption on p in (1.3) is weaker than the one in (1.2).

Later Tanaka [15] obtained a similar existence result as in [3], assuming that $f(x,\xi) \equiv f(\xi)$ satisfies (AR), (R1), (R2), $g(x,\xi) \equiv g(x) \in L^{\frac{p}{p-1}}(\Omega)$, and

$$\frac{2p}{N(p-2)} > \frac{\mu}{\mu - 1}. \tag{1.4}$$

He then proved that problem (1.1) has an unbounded sequence of solutions in $H_0^1(\Omega)$ (see [15, Theorem 1]). The assumption on p in (1.4) is weaker than the one in (1.2).

Tehrani [16] considered the case of a sign-changing potential. Bolle, Ghoussoub and Tehrani [4] also obtained some existence results on the perturbed elliptic equation

$$-\Delta u = |u|^{p-2}u + g(x) \;\text{ in }\; \Omega, \quad u = u_0 \text{ on } \partial\Omega,$$

where $u_0 \in C^2(\overline{\Omega}, \mathbb{R})$ with $\Delta u_0 = 0, 2 < p < 2^*$. Long [8] considered a perturbed superquadratic second order Hamiltonian systems.

Hirano and Zou [7] studied the elliptic boundary value problem

$$-\Delta u = |u|^{p-2}u + \beta g(x,u), \quad u \in H_0^1(\Omega), \tag{1.5}$$

where $2 < p < 2^*$, $(N \geq 3)$ and $g(x,\xi) \in C(\Omega \times \mathbb{R}, \mathbb{R})$, $g(x,\xi)\xi \geq 0$ for all $x \in \Omega$, $\xi \in \mathbb{R}$, $\lim_{\xi \to 0} g(x,\xi)/\xi = 0$ uniformly in $x \in \Omega$. Then they proved that for any $m \in \mathbb{N}$, there is a $\beta_m > 0$ such that for each $\beta \in (0, \beta_m)$, problem (1.5) has at least m distinct sign-changing solutions.

Recently, Santos [13] using Leray-Schauder degree theory and the method of upper and lower solutions proved existence and multiplicity of solutions the problem

$$
\begin{gathered}
(\varphi(u'))' = f(t,u,u') \\
u(0) = u(T) = u'(0),
\end{gathered}
$$

where φ is an increasing homeomorphism such that $\varphi(0) = 0$, and f is a continuous function.

In this article, we study the multiplicity of solutions to problem (1.1), using the following assumptions: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

(A1) $f(x,-\xi) = -f(x,\xi)$ for all $(x,\xi) \in \Omega \times \mathbb{R}$.

(A2) There exist $2 < p < 2^*$, $C_1 > 0$ such that

$$
|f(x,\xi)| \leq C_1(1 + |\xi|^{p-1}) \quad \text{a.e. in } \Omega \times \mathbb{R}.
$$

(A3) There exists a positive constant r_0 such that

$$
\begin{gathered}
F(x,\xi) \geq 0, \quad (x,\xi) \in \Omega \times \mathbb{R} \text{ and } |\xi| \geq r_0, \\
\lim_{|\xi| \to \infty} \frac{F(x,\xi)}{\xi^2} = \infty \ \text{ a.e. in } \ \Omega.
\end{gathered}
$$

(A4) There exist constants $C_2 > 0$ and $\kappa > N/2$ such that

$$
|F(x,\xi)|^\kappa \leq C_2 |\xi|^{2\kappa} \widehat{F}(x,\xi), \quad (x,\xi) \in \Omega \times \mathbb{R} \text{ and } |\xi| \geq r_0,
$$

where $\widehat{F}(x,\xi) = 2^{-1}\xi f(x,\xi) - F(x,\xi)$.

(A5) There exist a positive constant $C_3 > 0$ and $\rho_1 \in [2, 2^*)$ such that

$$
\widehat{F}(x,\xi) \geq C_3(|\xi|^{\rho_1} - 1), \quad \text{for all } (x,\xi) \in \Omega \times \mathbb{R}.
$$

(A6) $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying: There exist $g_1(x) \in L^{p_1}(\Omega)$, $g_2(x) \in L^{p_2}(\Omega)$, $p_1/(p_1 - 1) \leq \rho_1$, $p_2 > 1$, $(\sigma_1 + 1)p_2/(p_2 - 1) < \rho_1$, $\sigma_1 \in [0, \rho_1 - 1)$, $p_1 > \max\{1, \frac{2^* p_2}{p_2 \sigma_1 + 2^*}\}$, such that

$$
|g(x,\xi)| \leq g_1(x) + g_2(x)|\xi|^{\sigma_1} \quad \text{a.e. in } \Omega \times \mathbb{R}.
$$

The main results of this paper are the following theorems.

Theorem 1.1. *Suppose that* (A1)–(A6) *are satisfied, and*

$$
\frac{2p}{N(p-2)} > \frac{\rho_1}{\rho_1 - \sigma_1 - 1}. \tag{1.6}
$$

Then problem (1.1) *has an unbounded sequence of solutions in* $H_0^1(\Omega)$.

Remark 1.2. The result in Theorem 1.1 is not covered by the ones in [15]. For example, when $N = 3$,

$$
f(x,\xi) = 2\xi\Big[\ln(1 + |\xi|^{1/3}) + \frac{|\xi|^{1/3}}{6(1 + |\xi|^{1/3})}\Big],
$$

and $g:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that there exist $g_1(x)\in L^{p_1}(\Omega)$, $g_2(x)\in L^{p_2}(\Omega)$, $(\sigma_1+1)p_2/(p_2-1)<2$, $\sigma_1\in[0,\frac{4}{7})$, $p_1\ge\frac{6p_2}{p_2\sigma_1+6}$, $p_2\ge\frac{6}{5-\sigma_1}$, such that

$$|g(x,\xi)|\le g_1(x)+g_2(x)|\xi|^{\sigma_1}\quad \text{a.e. in } \Omega\times\mathbb{R},$$

then f,g satisfies the conditions in Theorem 1.1, but f does not satisfy the conditions in [15, Theorem 1].

Theorem 1.3. *Suppose that* (AR), (A1), (A2) *are satisfied, and g satisfies*

(A6') *there exist $g_3(x)\in L^{p_3}(\Omega)$, $g_4(x)\in L^{p_4}(\Omega)$, $p_3/(p_3-1)\le\mu$, $p_4>1$, $(\sigma_2+1)p_4/(p_4-1)<\mu$, $\sigma_2\in[0,\mu-1)$, $p_3>\max\{1,\frac{2^*p_4}{p_4\sigma_2+2^*}\}$, such that*

$$|g(x,\xi)|\le g_3(x)+g_4(x)|\xi|^{\sigma_2}\quad \textit{a.e. in } \Omega\times\mathbb{R},$$

$$\frac{2p}{N(p-2)}>\frac{\mu}{\mu-\sigma_2-1}.$$

Then problem (1.1) *has an unbounded sequence of solutions in $H_0^1(\Omega)$.*

The proofs of Theorems 1.1 and 1.3 are quite long, but they contain several arguments similar to those in [9, 11]. Therefore sometimes, we will omit detailed discussions by referring to these papers.

Remark 1.4. Theorem 1.3 generalizes results in Rabinowitz [11, 12] and in Tanaka [15], and it is not covered by [3] and [9, 11, 12, 14]. For example, when $N=3$,

$$f(x,\xi)=\xi|\xi|^{1/4}-\xi|\xi|^{1/8},\quad g(x,\xi)=|\xi|^{\sigma_2},\quad 0\le\sigma_2<\frac{21}{24}$$

then on one hand f,g satisfy the conditions in Theorem 1.3, but f does not satisfy the conditions in [3, Theorem 1]. On the other hand, the function f satisfies the conditions in [9, Theorem 1.1], [11, 12, Theorem 1.5] and in [14, Theorem 3]. However, the function g may grow faster than the perturbation term in [9, 11, 12, 14].

2. Proofs of the main results

We define the Euler-Lagrange functional associated with problem (1.1) as follows

$$\Phi(u)=\frac{1}{2}\int_\Omega|\nabla u|^2\,\mathrm{d}x-\int_\Omega F(x,u)\,\mathrm{d}x-\int_\Omega G(x,u)\,\mathrm{d}x.$$

From [9, Proposition 2.2], (A2), and (A6), we have Φ is well defined on $H_0^1(\Omega)$ and $\Phi\in C^1(H_0^1(\Omega),\mathbb{R})$ with

$$\Phi'(u)(v)=\int_\Omega\nabla u\cdot\nabla v\,\mathrm{d}x-\int_\Omega f(x,u)v\,\mathrm{d}x-\int_\Omega g(x,u)v\,\mathrm{d}x$$

for all $v\in H_0^1(\Omega)$. One can also check that the critical points of Φ are solutions of the problem (1.1).

Lemma 2.1. *Suppose that* (A2), (A5), (A6) *are satisfied, and u is a critical point of Φ. Then there is a constant C_5 such that*

$$\int_\Omega|u(x)|^{\rho_1}\,\mathrm{d}x\le C_5(\Phi^2(u)+1)^{1/2}. \tag{2.1}$$

Proof. Since u is a critical point of Φ, by (A2), (A5), and (A6), applying Hölder's inequality, we obtain

$$\begin{aligned} \Phi(u) &= \Phi(u) - \frac{1}{2}\Phi'(u)(u) \\ &\geq \int_\Omega \widehat{F}(x,u)\,\mathrm{d}x - \int_\Omega (|2^{-1}g(x,u)u| + |G(x,u)|)\,\mathrm{d}x \\ &\geq C_4 \int_\Omega |u|^\rho\,\mathrm{d}x - C_6\Big(\int_\Omega |u|^{\frac{(\sigma+1)p_2}{p_2-1}}\,\mathrm{d}x\Big)^{\frac{p_2-1}{p_2}} - C_7. \end{aligned} \tag{2.2}$$

Then (2.1) follows from (2.2) and Young's inequality. The proof is complete. $\square$

Next, we define a modified functional $\overline{\Phi}(u)$. Let $\chi \in C^\infty(\mathbb{R},\mathbb{R})$ such that $\chi(t) = 1$ for $t \leq 1$, $\chi(t) = 0$ for $t > 2$ and $-2 < \chi' < 0$ for $t \in (1,2)$. For $u \in H_0^1(\Omega)$, we put

$$\begin{gathered} \kappa(u) = 2\Theta\Big((\Phi(u))^2 + 1\Big)^{1/2}, \quad \psi(u) = \chi\Big(\kappa(u)^{-1}\int_\Omega |u(x)|^{\rho_1}\,\mathrm{d}x\Big), \\ \overline{\Phi}(u) = \int_\Omega \Big(\frac{1}{2}|\nabla u|^2 - F(x,u) - \psi(u)G(x,u)\Big)\,\mathrm{d}x, \end{gathered} \tag{2.3}$$

where Θ is a large enough positive constant, which will be chosen later in Lemma 2.3. Then, we obtain

$$\begin{aligned} \overline{\Phi}'(u)(u) &= (1 + T_1(u))\Big(\int_\Omega |\nabla u|^2\,\mathrm{d}x - \int_\Omega f(x,u)u\,\mathrm{d}x\Big) \\ &\quad - T_2(u)\int_\Omega G(x,u)\,\mathrm{d}x - (\psi(u) + T_1(u))\int_\Omega g(x,u)u\,\mathrm{d}x, \end{aligned} \tag{2.4}$$

where

$$\begin{gathered} T_1(u) = \chi'\Big(\kappa(u)^{-1}\int_\Omega |u|^{\rho_1}\,\mathrm{d}x\Big)\kappa(u)^{-3}(2\Theta)^2\Phi(u)\int_\Omega |u|^{\rho_1}\,\mathrm{d}x\int_\Omega G(x,u)\,\mathrm{d}x, \\ T_2(u) = \rho_1\chi'\Big(\kappa(u)^{-1}\int_\Omega |u|^{\rho_1}\,\mathrm{d}x\Big)\kappa(u)^{-1}\int_\Omega |u|^{\rho_1}\,\mathrm{d}x. \end{gathered}$$

Let $\mathrm{supp}(\psi)$ denote the support of ψ.

Lemma 2.2. *Suppose that* (A1), (A2), (A5), (A6) *are satisfied.*

(i) *If $u \in \mathrm{supp}(\psi)$ then*
$$\Big|\int_\Omega G(x,u)\,\mathrm{d}x\Big| \leq C_8\Big(|\Phi(u)|^{\frac{\sigma_1+1}{\rho_1}} + |\Phi(u)|^{\frac{1}{\rho_1}} + 1\Big).$$

(ii) *There is a constant C_9, such that for any $u \in H_0^1(\Omega)$,*
$$|\overline{\Phi}(u) - \overline{\Phi}(-u)| \leq C_9\Big(|\overline{\Phi}(u)|^{\frac{1}{\rho_1}} + |\overline{\Phi}(u)|^{\frac{\sigma_1+1}{\rho_1}} + 1\Big).$$

(iii) *There are constants $M_0, C_{10} > 0$ such that whenever $M \geq M_0$, $\overline{\Phi}(u) \geq M$, $u \in \mathrm{supp}(\psi)$ then $\Phi(u) \geq C_{10}M$.*

(iv) *For every $\delta > 0$ small enough there exists $M > 0$ large enough such that for all $u \in H_0^1(\Omega), \overline{\Phi}(u) \geq M$ we have $|T_1(u)| \leq \delta$, $|T_2(u)| \leq 4\rho_1$.*

The proof of the above lemma is similar to the ones of [9, Lemmas 3.4, 3.5 , 3.6], so we omit it here. Now, we shall show that large critical values of $\overline{\Phi}$ are critical values of Φ.

Lemma 2.3. *Suppose that* (A1), (A2), (A5), (A6) *are satisfied, and* Θ *is large enough. Then there exists* $M_1 > 0$ *such that if* $u \in H_0^1(\Omega)$ *is a critical point of* $\overline{\Phi}$ *and* $\overline{\Phi}(u) \geq M_1$*, then* u *is a critical point of* Φ *and* $\overline{\Phi}(u) = \Phi(u)$.

Proof. Let $u \in H_0^1(\Omega)$ be such that $\overline{\Phi}'(u) = 0$. For M_1 sufficiently large such that $M_1 > M_0$ then T_1 is sufficiently small and T_2 is bounded, with (2.4), we have

$$\begin{aligned}\Phi(u) &= \Phi(u) - \frac{\overline{\Phi}'(u)(u)}{2(1+T_1(u))}\\ &\geq C_4 \int_\Omega |u|^\rho \,\mathrm{d}x - C_{11}\Big(\int_\Omega |u|^{\frac{(\sigma+1)p_2}{p_2-1}}\,\mathrm{d}x\Big)^{\frac{p_2-1}{p_2}} - C_{11}.\end{aligned}$$

Therefore, if we choose Θ large enough,

$$\kappa(u)^{-1} \int_\Omega |u|^\rho \,\mathrm{d}x \leq 1,$$

it follows that $\psi(u) = 1$ and $\psi'(u) = 0$. □

Definition 2.4. Let $(\mathbb{V}, \|\cdot\|_{\mathbb{V}})$ be a real Banach space with its dual space $\mathbb{V}^*$ and $J \in C^1(\mathbb{V}, \mathbb{R})$. For $c \in \mathbb{R}$ we say that J satisfies condition $(C)_c$ if for each sequence $\{x_m\}_{m=1}^\infty \subset \mathbb{V}$ with

$$J(x_m) \to c \quad \text{and} \quad (1 + \|x_m\|_{\mathbb{V}})\|J'(x_m)\|_{\mathbb{V}} \to 0 \quad \text{as } m \to \infty,$$

there exists a subsequence $\{x_{m_k}\}_{k=1}^\infty$ that converges strongly in $\mathbb{V}$. If J satisfies condition (C_c) for all $c > 0$, then we say that J satisfies the Cerami condition.

Lemma 2.5. *Suppose that* (A1)–(A6) *are satisfied. Then* $\overline{\Phi} \in C^1(H_0^1(\Omega), \mathbb{R})$ *and there is a constant* $M_2 > 0$ *such that* $\overline{\Phi}$ *satisfies the* $(C)_c$ *condition for all* $c > M_2$.

Proof. Since (A2), (A5), and (A6) are satisfied, and $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$, it follows that $\overline{\Phi} \in C^1(H_0^1(\Omega), \mathbb{R})$. Let M_0 be as in Lemma 2.2 and take $M_2 \geq M_0$, $c > M_2$. Let $\{u_m\}_{m=1}^\infty \subset H_0^1(\Omega)$ be a $(C)_c$ sequence, i.e.,

$$\overline{\Phi}(u_m) \to c \text{ as } m \to \infty, \quad \lim_{m\to\infty}\Big(1 + \|u_m\|_{H_0^1(\Omega)}\Big)\|\overline{\Phi}'(u_m)\|_{(H_0^1(\Omega))^*} = 0. \tag{2.5}$$

Then

$$\begin{gathered}\overline{\Phi}'(u_m)(u_m) \to 0,\\ \|u_m\|_{H_0^1(\Omega)}^2 - \int_\Omega 2F(x,u_m)\,\mathrm{d}x - \int_\Omega 2\psi(u_m)G(x,u_m)\,\mathrm{d}x \to 2c \quad \text{as } m \to \infty.\end{gathered} \tag{2.6}$$

We first show that $\{u_m\}_{m=1}^\infty$ is bounded in $H_0^1(\Omega)$ by a contradiction argument. Indeed, we can (by passing to a subsequence if necessary) suppose that for any m, $\|u_m\|_{H_0^1(\Omega)} > 1$ and

$$\|u_m\|_{H_0^1(\Omega)} \to \infty \quad \text{as } m \to \infty. \tag{2.7}$$

Setting

$$w_m = \frac{u_m}{\|u_m\|_{H_0^1(\Omega)}},$$

we have $\|w_m\|_{H_0^1(\Omega)} = 1$ and

$$\|w_m\|_{L^\nu(\Omega)} \leq \tau_\nu \|w_m\|_{H_0^1(\Omega)} = \tau_\nu, 1 \leq \nu < 2^*.$$

Passing to a subsequence, assume that $w_m \rightharpoonup w$ in $H_0^1(\Omega)$, then $w_m \to w$ in $L^\nu(\Omega)$, $1 \le \nu < 2^*$. For $0 \le a < b$, let

$$\Omega_m(a,b) = \{x \in \Omega : a \le |u_m(x)| < b\}. \tag{2.8}$$

In view of (A5) and (A6), for m large enough, we have

$$c+1 \ge \overline{\Phi}(u_m) - \frac{1}{2(1+T_1(u_m))}\overline{\Phi}'(u_m)(u_m) > \frac{1}{2}\int_{\Omega_m(r_0,\infty)} \widehat{F}(x,u_m)\,\mathrm{d}x - C_{12}, \tag{2.9}$$

where C_{12} is a positive constant independent of m.

From (A6) and the definition of the functional ψ, for m large enough, we have

$$\Big|\int_\Omega \psi(u_m)G(x,u_m)\,\mathrm{d}x\Big| \le C_{13}. \tag{2.10}$$

By (A6), (2.6), (2.7) and (2.10), we obtain

$$\lim_{m\to\infty}\int_\Omega \frac{2F(x,u_m)}{\|u_m\|^2_{H_0^1(\Omega)}}\,\mathrm{d}x = 1. \tag{2.11}$$

Now, we consider two possible cases: $w = 0$ and $w \ne 0$.

Case 1: $w = 0$. Then $w_m \to 0$ in $L^\nu(\Omega)$, $1 \le \nu < 2^*$, and $w_m \to 0$ a.e. in Ω. From (A2) and Hölder's inequality, we deduce that

$$\int_{\Omega_m(0,r_0)} \frac{F(x,u_m)}{\|u_m\|^2_{H_0^1(\Omega)}}\,\mathrm{d}x \le C_{14}\Big(\frac{1}{\|u_m\|_{H_0^1(\Omega)}}\|w_m\|_{L^1(\Omega)} + \|w_m\|^2_{L^2(\Omega)}\Big) \to 0$$

as $m \to \infty$, hence

$$\int_{\Omega_m(0,r_0)} \frac{2F(x,u_m)}{\|u_m\|^2_{H_0^1(\Omega)}}\,\mathrm{d}x \to 0 \quad \text{as } m \to \infty. \tag{2.12}$$

Set $q' = q/(q-1)$ and $q > N/2$. Then $2q' \in [2, 2^*)$. Therefore, from (A4) and (2.9), we have

$$\begin{aligned}
&\Big|\int_{\Omega_m(r_0,+\infty)} \frac{F(x,u_m)}{\|u_m\|^2_{H_0^1(\Omega)}}\,\mathrm{d}x\Big| \\
&\le \int_{\Omega_m(r_0,+\infty)} \frac{|F(x,u_m)|}{|u_m|^2}|w_m|^2\,\mathrm{d}x \\
&\le \Big[\int_{\Omega_m(r_0,+\infty)} \Big(\frac{|F(x,u_m)|}{|u_m|^2}\Big)^q\,\mathrm{d}x\Big]^{1/q}\Big[\int_{\Omega_m(r_0,+\infty)} |w_m|^{2q'}\,\mathrm{d}x\Big]^{1/q} \\
&\le C_{15}\Big[\int_{\Omega_m(r_0,+\infty)} \widehat{F}(x,u_m)\,\mathrm{d}x\Big]^{1/q}\Big[\int_{\Omega_m(r_0,+\infty)} |w_m|^{2q'}\,\mathrm{d}x\Big]^{1/q'} \\
&\le C_{16}\Big[\int_{\Omega_m(r_0,+\infty)} |w_m|^{2q'}\,\mathrm{d}x\Big]^{1/q'}\,\mathrm{d}x \to 0, \quad \text{as } m \to \infty.
\end{aligned}$$

Then

$$\int_{\Omega_m(r_0,+\infty)} \frac{2F(x,u_m)}{\|u_m\|^2_{H_0^1(\Omega)}}\,\mathrm{d}x \to 0 \quad \text{as } m \to \infty. \tag{2.13}$$

In combination with (2.12), we obtain

$$\int_\Omega \frac{2F(x,u_m)}{\|u_m\|^2_{H_0^1(\Omega)}}\,\mathrm{d}x \to 0 \quad \text{as } m \to \infty,$$

which contradicts (2.11).

Case 2: $w \neq 0$. Setting $\Omega_0 := \{x \in \Omega : w(x) \neq 0\}$, we have $\operatorname{meas}(\Omega_0) > 0$ and

$$\lim_{m\to\infty} u_m(x) = \lim_{m\to\infty} \|u_m\|^2_{H_0^1(\Omega)} w_m(x) = \infty, \quad \text{a.e. in } \Omega_0.$$

It follows from (A2), (A3), (A6), (2.5), (2.10) and Fatou's lemma that

$$\begin{aligned}
\frac{1}{2} &= \lim_{m\to\infty} \int_\Omega \frac{F(x,u_m)}{\|u_m\|^2_{H_0^1(\Omega)}}\,\mathrm{d}x \ge \liminf_{m\to\infty} \int_\Omega \frac{F(x,u_m)}{\|u_m\|^2_{H_0^1(\Omega)}}\,\mathrm{d}x \\
&\ge \liminf_{m\to\infty} \int_{\Omega_0} \frac{F(x,u_m)}{\|u_m\|^2_{H_0^1(\Omega)}}\,\mathrm{d}x \ge \int_{\Omega_0} \liminf_{m\to\infty} \frac{F(x,u_m)}{\|u_m\|^2_{H_0^1(\Omega)}}\,\mathrm{d}x \\
&= \int_{\Omega_0} \liminf_{m\to\infty} \frac{F(x,u_m)}{|u_m|^2} w_m^2\,\mathrm{d}x = +\infty,
\end{aligned} \tag{2.14}$$

which is a contradiction. Because of the above result, without loss of generality, we can assume that

$$\begin{gathered}
u_m \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega) \text{ as } m \to \infty, \\
u_m \to u \quad \text{a.e. in } \Omega \text{ as } m \to \infty, \\
u_m \to u \text{ strongly in } L^\nu(\Omega),\ 1 \le \nu < 2^* \text{ as } m \to \infty.
\end{gathered} \tag{2.15}$$

Thus by (A2), (A6) and (2.15), we have

$$\int_\Omega (f(x,u_m) - f(x,u))(u_m - u)\,\mathrm{d}x \to 0 \quad \text{as } m \to \infty, \tag{2.16}$$

$$\int_\Omega (g(x,u_m) - g(x,u))(u_m - u)\,\mathrm{d}x \to 0 \quad \text{as } m \to \infty. \tag{2.17}$$

If M_2 is large enough, it follows from $\lim_{m\to\infty} \overline{\Phi}'(u_m) = 0$ and (2.15) that

$$\big\langle (1+T_1(u))\overline{\Phi}'(u_m) - (1+T_1(u_m))\overline{\Phi}'(u), u_m - u\big\rangle \to 0 \quad \text{as } m \to \infty. \tag{2.18}$$

Moreover,

$$\begin{aligned}
&\big\langle (1+T_1(u))\overline{\Phi}'(u_m) - (1+T_1(u_m))\overline{\Phi}'(u), u_m - u\big\rangle \\
&= (1+T_1(u))(1+T_1(u_m))\Big[\int_\Omega \Big(|\nabla u_m - \nabla u|^2 - f(x,u_m)(u_m-u) \\
&\quad + f(x,u)(u_m-u)\Big)\,\mathrm{d}x\Big] \\
&\quad - (1+T_1(u))(\psi(u_m)+T_1(u_m)) \int_\Omega g(x,u_m)(u_m-u)\,\mathrm{d}x \\
&\quad + (1+T_1(u_m))(\psi(u)+T_1(u)) \int_\Omega g(x,u)(u_m-u)\,\mathrm{d}x \\
&\quad - (1+T_1(u))T_3(u_m) \int_\Omega |u_m|^{\rho-1}(u_m-u)\,\mathrm{d}x \\
&\quad + (1+T_1(u_m))T_3(u) \int_\Omega |u|^{\rho-1}(u_m-u)\,\mathrm{d}x,
\end{aligned} \tag{2.19}$$

where

$$T_3(u) = \rho\chi'\Big(\kappa(u)^{-1}\int_\Omega |u|^\rho\,\mathrm{d}x\Big)\kappa(u)^{-1}\int_\Omega G(x,u)\,\mathrm{d}x.$$

By (2.16), (2.17), (2.18) and (2.19) we obtain

$$\int_\Omega |\nabla u_m - \nabla u|^2\,\mathrm{d}x \to 0 \text{ as } m \to \infty.$$

Therefore, we conclude that $u_m \to u$ strongly in $H_0^1(\Omega)$. The proof is complete. $\square$

Lemma 2.6. *Suppose that* (A2), (A3), (A6) *are satisfied. Then for any finite dimensional subspace* $\widehat{\mathbb{X}} \subset H_0^1(\Omega)$, *there is* $R = R(\widehat{\mathbb{X}}) > 0$ *such that*

$$\overline{\Phi}(u) \le 0, \quad \forall u \in \widehat{\mathbb{X}},\ \|u\|_{H_0^1(\Omega)} \ge R.$$

Proof. Arguing by contradiction, suppose that for some sequence $\{u_m\}_{m=1}^{\infty} \subset \widehat{\mathbb{X}}$ with $\|u_m\|_{H_0^1(\Omega)} > 0$ for all $m \in \mathbb{N}$ and $\|u_m\|_{H_0^1(\Omega)} \to \infty$ as $m \to \infty$, there is $M > 0$ such that $\overline{\Phi}(u_m) \ge -M$ for all $m \in \mathbb{N}$. Setting

$$w_m = \frac{u_m}{\|u_m\|_{H_0^1(\Omega)}},$$

then $\|w_m\|_{H_0^1(\Omega)} = 1$. Therefore we can (by passing to a subsequence if necessary) suppose that

$$\begin{gathered} w_m \rightharpoonup w \quad \text{weakly in } H_0^1(\Omega) \text{ as } m \to \infty, \\ w_m \to w \quad \text{a.e. in } \Omega \text{ as } m \to \infty, \\ w_m \to w \quad \text{strongly in } L^{\nu}(\Omega) \text{ as } m \to \infty, 2 \le \nu < 2^*. \end{gathered} \tag{2.20}$$

Since $\widehat{\mathbb{X}}$ is finite dimensional, it follows that $w_m \to w$ strongly in $\widehat{\mathbb{X}}$ as $m \to \infty$, and $w \in \widehat{\mathbb{X}}$ with $\|w\|_{H_0^1(\Omega)} = 1$. Therefore, from (2.13) we obtain

$$0 = \lim_{m\to\infty} \frac{-M}{\|u_m\|^2_{H_0^1(\Omega)}} \le \lim_{m\to\infty} \frac{\overline{\Phi}(u_m)}{\|u_m\|^2_{H_0^1(\Omega)}} = -\infty.$$

Hence we arrive at a contradiction. So, there is $R = R(\widehat{\mathbb{X}}) > 0$ such that $\overline{\Phi}(u) \le 0$ for $u \in \widehat{\mathbb{X}}$ and $\|u\|_{H_0^1(\Omega)} \ge R$. $\square$

Now, we show that $\overline{\Phi}$ has an unbounded sequence of critical values. Let $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_k \le \cdots$ denote the eigenvalues of the problem

$$\begin{gathered} -\Delta u = \lambda u \quad \text{in } \ \Omega, \\ u = 0 \quad \text{on } \ \partial\Omega, \end{gathered} \tag{2.21}$$

and $e_1, e_2, \dots$ denote the corresponding eigenfunctions which normalized such that $\|e_j\|_{H_0^1(\Omega)} = 1$, for all $j = 1, 2, \dots$. For any $k > 0$, we put $\mathbb{V}_k = \operatorname{span}\{e_j; j \le k\}$ in $H_0^1(\Omega)$, and $\mathbb{V}_k^{\perp}$ its orthogonal complement. Choose an increasing sequence R_k such that $\overline{\Phi}(u) \le 0$ if $u \in \mathbb{V}_k, \|u\|_{H_0^1(\Omega)} \ge R_k$. Let B_{R_k} denote the closed ball of radius R_k in $H_0^1(\Omega), \mathbb{W}_k \equiv B_{R_k} \bigcap \mathbb{V}_k$, and

$$\begin{gathered} \Gamma_k = \Big\{h \in C(\mathbb{W}_k, H_0^1(\Omega)) : h \text{ is odd and } h(u) = u \text{ if } \|u\|_{H_0^1(\Omega)} = R_k\Big\}, \\ \mathbb{U}_k = \Big\{u = te_{k+1} + w :\ t \in [0, R_{k+1}], w \in B_{R_{k+1}} \cap \mathbb{V}_k, \|u\|_{H_0^1(\Omega)} \le R_{k+1}\Big\}, \\ \Lambda_k = \Big\{H \in C(\mathbb{U}_k, H_0^1(\Omega)) : H|_{\mathbb{W}_k} \in \Gamma_k \text{ and } H(u) = u \text{ if} \\ \|u\|_{H_0^1(\Omega)} = R_{k+1} \text{ or } u \in (B_{R_{k+1}} \backslash B_{R_k}) \cap \mathbb{V}_k\Big\}. \end{gathered} \tag{2.22}$$

Now we define

$$\gamma_k = \inf_{H \in \Lambda_k} \max_{u \in \mathbb{U}_k} \overline{\Phi}(H(u)), \quad k \in \mathbb{N}, \tag{2.23}$$

$$\beta_k = \inf_{h \in \Gamma_k} \max_{u \in \mathbb{W}_k} \overline{\Phi}(h(u)), \quad k \in \mathbb{N}. \tag{2.24}$$

It is obvious that $\gamma_k \geq \beta_k$. We will give the lower bounds for β_k in the next lemma.

Lemma 2.7. *Suppose that* (A2), (A6) *are satisfied. Then there are constants* $C_{17} > 0$ *and* $k_0 \in \mathbb{N}$ *such that for all* $k \geq k_0$,

$$\beta_k \geq C_{17} k^{\frac{2p}{N(p-2)}}. \tag{2.25}$$

Proof. By (A2) and (A6) we obtain

$$\overline{\Phi}(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - C_{18}\int_\Omega |u|^p \,\mathrm{d}x - C_{19}. \tag{2.26}$$

Set

$$K(u) = \frac{1}{2}\|u\|^2_{H^1_0(\Omega)} - C_{18}\|u\|^p_{L^p(\Omega)} \in C^2(H^1_0(\Omega), \mathbb{R}). \tag{2.27}$$

Then we can see that

$$\overline{\Phi}(u) \geq K(u) - C_{19}, \tag{2.28}$$

$$K''(u)(h,h) = \big((-\Delta - C_{18}p(p-1)|u|^{p-2})h, h\big) \quad \text{for all } u \in H^1_0(\Omega) \tag{2.29}$$

and that the functional $K(u)$ satisfies the following assumptions:

- (A7) $K(0) = 0$;
- (A8) $K(-u) = K(u)$ for all $u \in H^1_0(\Omega)$;
- (A9) for each finite dimensional subspace $E \subset H^1_0(\Omega)$, there is an $R = R(E) > 0$ such that
$$K(u) < 0, \quad \text{for all } u \in E \text{ with } \|u\|_{H^1_0(\Omega)} \geq R(E);$$
- (A10) $K'(u) = u + \kappa(u)$ for $u \in H^1_0(\Omega)$, where $\kappa : H^1_0(\Omega) \to H^1_0(\Omega)$ is a compact operator;
- (A11) If for some $M > 0$, $\{u_j\}_{j=1}^\infty \subset H^1_0(\Omega)$ satisfies $K(u_j) \leq M$ for all j, and $\|K'(u_j)\|_{(H^1_0(\Omega))^*} \to 0$ as $j \to \infty$, then there exists a subsequence $\{u_{j_k}\}_{k=1}^\infty$ which converges strongly in $H^1_0(\Omega)$;
- (A12) If for some $M > 0$, $\{u_j\}_{j=1}^\infty \subset \mathbb{V}_m$ satisfies $K(u_j) \leq M$ for all j and $\|(K|_{\mathbb{V}_m})'(u_j)\|_{(\mathbb{V}_m)^*} \to 0$ as $j \to \infty$, then there exists a subsequence $\{u_{j_k}\}_{k=1}^\infty$ which converges strongly in $\mathbb{V}_m$;
- (A13) If for some $M > 0, \{u_j\}_{j=1}^\infty \subset H^1_0(\Omega)$ satisfies $u_j \in \mathbb{V}_j$, $K(u_j) \leq M$ for all j and $\|(K|_{\mathbb{V}_j})'(u_j)\|_{(\mathbb{V}_j)^*} \to 0$ as $j \to \infty$, then there exists a subsequence $\{u_{j_k}\}_{k=1}^\infty$ which converges strongly in $H^1_0(\Omega)$.

Next, we define minimax values

$$\omega_k = \inf_{h \in \Gamma_k} \max_{u \in \mathbb{W}_k} K(h(u)), \quad k \in \mathbb{N}. \tag{2.30}$$

From (2.28), we obtain

$$\beta_k \geq \omega_k - C_{19}. \tag{2.31}$$

From [15, Theorem B], there is a $v_k \in H^1_0(\Omega)$ such that

$$K(v_k) \leq \omega_k, \tag{2.32}$$

$$K'(v_k) = 0, \tag{2.33}$$

$$\operatorname{index}_0 K''(v_k) \geq k, \tag{2.34}$$

where

$$\operatorname{index}_0 K''(v_k) := \max\big\{ \dim E : E \subset H^1_0(\Omega) \text{ is a subspace such that } K''(v_k)(h,h) \leq 0, \text{ for } h \in E \big\}.$$

Therefore, from (2.29) and (2.34), we obtain that

$$-\Delta - C_{18}p(p-1)|v_k|^{p-2} \text{ possesses at least } k \text{ non-positive eigenvalues.} \tag{2.35}$$

Let $\mathscr{N}(V)$ denote the number of non-positive eigenvalues (multiplicities counted) of the problem

$$\begin{gathered} -\Delta u - V(x)u = \lambda u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where $V(x) \in L^{N/2}(\Omega)$. Then from [15, Lemma 2.3] (or [5, 6]) there is a constant $C_N > 0$ such that

$$\mathscr{N}(V) \leq C_N \|V(x)\|^{\frac{N}{2}}_{L^{\frac{N}{2}}(\Omega)}. \tag{2.36}$$

From (2.35) and (2.36), we obtain

$$C_{20}k \leq \||v_k|^{p-2}\|^{\frac{N}{2}}_{L^{\frac{N}{2}}(\Omega)}. \tag{2.37}$$

On the other hand, from (2.33), we have

$$\|v_k\|^2_{H^1_0(\Omega)} = C_{18}p\|v_k\|^p_{L^p(\Omega)} \tag{2.38}$$

From (2.32), (2.37) and (2.38), we obtain

$$\omega_k \geq \frac{C_{18}}{2}(p-2)\|v_k\|^p_{L^p(\Omega)} \geq C_{21}k^{\frac{2p}{N(p-2)}} \text{ for all } k \in \mathbb{N}.$$

The proof is complete. □

Proof of Theorem 1.1. By Lemma 2.7, the proof is the same as that of [9, Theorem 1.1] (or, [11, Theorem 1.5]), so we omit it here. □

Proof of Theorem 1.3. The proof is a slightly modification of several of the lemmas above, we omit the details. □

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