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OPTIMAL CONTROL FOR SOLUTIONS TO SOBOLEV STOCHASTIC EQUATIONS

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Abstract. This article concerns the optimal control problem for internal gravitational waves in a model with additive "white noise". This mathematical models based on the stochastic Sobolev equation, Dirichlet boundary conditions, and a Cauchy initial condition. The inhomogeneity describes random heterogeneities of the medium and fluctuations. By white noise we realize the Nelson-Gliklikh derivative of the Wiener process. The study was carried out within the framework of the theory of relatively bounded operators and the theory of Sobolev-type stochastic equations of higher order and the theory of (semi) groups of operators. We show the existence and uniqueness of a strong solutions, and obtain sufficient conditions for the existence of an optimal control of such solutions. The theorem about the existence and uniqueness of the optimal control is based on the works of J.-L. Lyons.

1. Introduction

Internal waves occur in the interface of two fluids with different densities. The maximum vertical displacement of particles in internal waves does not take place on the surface of the fluid, but inside it. This fact can be observed in the ocean at the location of desalinated water over heavier water with greater salinity. In such place a part of the power of the ship the engine is consumed on the excitation of internal waves, resulting in a decrease of sheep's speed. In the simplest case, the two-layer fluid model, internal waves are quite similar to the surface waves. They also concentrate near the interface. Assuming that the fluid fills entirely each half-space on both sides of the border, the dispersion relation for internal waves are identical to the dispersion relation $\omega^2 = gk$, where g is the acceleration due to gravity and k is the wave number for gravitational waves but with a different effective value gravity acceleration.

The mathematical model of waves in homogeneous incompressible fluid rotating with constant angular velocity Ω is described by the linear system of hydrodynamic

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equations

$$\begin{gathered}\nu_t + \frac{1}{\rho_0}\nabla p + 2[\Omega \times \nu] = 0,\\ \rho_t = 0,\\ \nabla \nu = 0,\end{gathered} \tag{1.1}$$

where $\nu = \{u, v, w\}$ is a vector of velocity, p is a vector of pressure which is directed perpendicularly to the free surface, ρ_0 is an equilibrium density (constant), and the buoyancy frequency is equal to zero. Let the Oz axis be directed collinear to the vector of Ω, than we obtain the equation for the vertical velocity component of the fluid particles (the Sobolev equation [25])

$$\Delta w_{tt} + F^2 w_{zz} = 0, \tag{1.2}$$

where F is the Coriolis parameter. The wave solutions that satisfy (1.2) are called inertial (or gyroscopic) waves propagating on the surface of a rotating fluid. A solution to equation (1.2) into an unbounded domain was obtained in [9] by the Green function method. Article [9] describes the behavior of solutions to two-dimensional Hamiltonian systems arising in the theory of small oscillations of rotating ideal fluid and constructs a mathematical model of the incipience of a vortex structure.

In this article, we study the inhomogeneous stochastic equation

$$\Delta \overset{\circ}{w}_{tt} + F^2 w_{zz} = \overset{\circ}{f}_K + u \tag{1.3}$$

with the initial-boundary conditions

$$w(x,t) = 0, \quad (x,t) \in \partial D \times \mathbb{R}, \tag{1.4}$$

$$w(x,0) = w_0(x), \quad \overset{\circ}{w}_t(x,0) = w_1(x), \tag{1.5}$$

where $\overset{\circ}{w}_t$ ($\overset{\circ}{w}_{tt}$) is the first (second) Nelson-Gliklikh derivative [10] of a random process w, $\overset{\circ}{f}_K(t)$ is the "white noise", which models heterogeneity of fluid and random fluctuations, u is a deterministic control function, $D \subset \mathbb{R}^3$ is a domain with a smooth boundary ∂D, $w_0(x)$ and $w_1(x)$ are random K-variables. By "white noise" we mean the Nelson-Gliklikh derivative of the Wiener process.

The concept of the Nelson-Gliklikh derivative was introduced in [10]. Also the first derivative of an arbitrary random process was found there, and further expanded in [11, 12]. Later, derivatives of higher-order random processes were calculated, and the first mathematical models with white noise, were investigated in [11]. The Nelson-Gliklikh derivative is based on the concept of the average derivative introduced by Nelson [19]. In addition to the approach to white noise, the Ito Stratonovich-Skorokhod approach is used by Kovac and Larson [14]. Also, in [3, 2], stochastic differential equations and their systems and partial differential equations were considered in the form of Ito differentials with uniform initial conditions. Melnikova developed the same approach. Article [18] introduces the spaces of generalized H-valued random variables, in which the H-valued white noise turns out to be smooth with respect to the variable t. It was shown in [21] that the Nelson-Gliklikh derivative of the Wiener process is in good agreement with the predictions of the Einstein-Smoluchowski theory of Brownian motion. Therefore, the Nelson-Gliklikh derivative of the Wiener process was called white noise. This approach is successfully applied to the study of the equation of the Sobolev type, a mathematical model based on one [5, 6, 7, 31], the dichotomies of the stochastic equation defined on the manifold [13], and to the study of mathematical models of measuring devices [23, 24]. After that, we study the optimal control problem on finding the pair $(\hat{w}, \hat{u})$, where $\hat{w}$ is the solution to problem (1.2), (1.4), and $\hat{u} \in \mathfrak{U}_{ad}$ is the control that satisfies the relation

$$J(\hat{w}, \hat{u}) = \min_{(w,u)\in\mathfrak{X}\times\mathfrak{U}_{ad}} J(w, u). \tag{1.6}$$

Here $J(w, u)$ is the quality functional constructed in a special way, and $\mathfrak{U}_{ad}$ is a set, which is closed and convex in the control space $\mathfrak{U}$.

Let us find a solution to the problem (1.2)–(1.5) in the framework of the theory of Sobolev type equations [1, 4, 8, 20, 26, 28]. First, consider the auxiliary Cauchy problem for the incomplete inhomogeneous Sobolev type equation of second order

$$A\ddot{w} = Bw + y, \tag{1.7}$$

$$\dot{w}(0) = w_1,\ w(0) = w_1, \tag{1.8}$$

where the operators $A, B \in \mathcal{L}(\mathfrak{X}; \mathfrak{Y})$, the function $y : [0, \tau) \subset \mathbf{R}_+ \to \mathfrak{Y}(\tau < \infty)$, and $\mathfrak{X}, \mathfrak{Y}$ are Hilbert spaces.

This article is based on the theory of relatively bounded operators [26], degenerate semigroups of operators[28], and the theory of incomplete Sobolev type equations of high order [30]. Firstly, the optimal control problem by solutions of Sobolev type equations appeared in [27]. Optimal control problems for models of mathematical physics represent a promising direction [16]. New statements of optimal control problems arise. For example, we note optimal control of solutions to stochastic equations [17, 29], optimal control of solutions to a multipoint initial-final value problem [22]. The applied research methods of the optimal control problem are based on the results obtained by J.-L. Lions [15].

This article is organized as follows. In Section 1, we introduce the basic definitions and concepts of the relatively bounded operators' theory. In Section 2, we define the space of random K-variables, the space of random K-process, and the space of K-noises. In Section 3, we consider a stochastic Sobolev type equation of second order and obtain a strong solution of a Cauchy problem, also a control problem for the stochastic Sobolev type equation of the second order is solved. In Section 4, the mathematical model is reduced to the Cauchy problem for an abstract operator-differential equation and the propagators for stochastic Sobolev equation (1.3) are constructed when D is a parallelepiped.

2. Relatively bounded operators

Necessary notation for the theory of relatively bounded operators can be found in [10, 11]. Proofs of all statements of this part can be found in [10, 11].

Definition 2.1. The set

$$\rho^A(B) = \{\mu \in \mathbb{C} : (\mu A - B)^{-1} \in \mathcal{L}(\mathfrak{Y}; \mathfrak{X})\}$$

is called the *resolvent set* of operator B with respect to the operator A (in short, A-resolvent set of the operator B). The set $\mathbb{C}\backslash\rho^A(B) = \sigma^A(B)$ is called a *spectrum* of operator B with respect to operator A (in short, A-spectrum of operator B).

Definition 2.2. The operator-functions

$$(\mu A - B)^{-1}, \quad R^A_\mu = (\mu A - B)^{-1}A, \quad L^A_\mu = A(\mu A - B)^{-1}$$

with the domain $\rho^A(B)$ are called the *resolvent*, *right resolvent*, *left resolvent* of operator B with respect to operator A (in short, A-resolvent, right A-resolvent, left A-resolvent of the operator B), respectively.

Definition 2.3. The operator B is called spectrally bounded with respect to operator A (in short, (A,σ)-bounded), if there exists $a > 0$ such that $\mu \in \mathbb{C}$ we have that

$$(|\mu| > a) \Rightarrow (\mu \in \rho^A(B)).$$

Lemma 2.4 ([4]). *Let the operator B be (A,σ)-bounded. Then the operators*

$$P = \frac{1}{2\pi i}\int_\Gamma R^A_\lambda(B)d\lambda, \quad Q = \frac{1}{2\pi i}\int_\Gamma L^A_\lambda(B)d\lambda$$

are projectors with $P : \mathfrak{X} \to \mathfrak{X}$ and $Q : \mathfrak{Y} \to \mathfrak{Y}$. Here $\Gamma = \{\lambda \in \mathbb{C} : |\lambda| = r > a\}$.

Let $\mathfrak{X}^0 = \ker P$, $\mathfrak{Y}^0 = \ker Q$, $\mathfrak{X}^1 = \operatorname{im} P$, $\mathfrak{Y}^1 = \operatorname{im} Q$. Denote by $A_k(B_k)$ the restriction of the operator A (B) to the subspace $\mathfrak{X}^k$, $k = 0, 1$.

Theorem 2.5 ([11]). *Let the operator B be (A,σ)-bounded. Then*

(i) *the operators $A_k, B_k : \mathfrak{X}^k \to \mathfrak{Y}^k$, $k = 0, 1$;*
(ii) *there exists the operator $B_0^{-1} \in \mathcal{L}(\mathfrak{Y}^0, \mathfrak{X}^0)$;*
(iii) *there exists the operator $A_1^{-1} \in \mathcal{L}(\mathfrak{Y}^1, \mathfrak{X}^1)$;*
(iv) *the operator B_1 belongs to $\mathcal{L}(\mathfrak{X}^1, \mathfrak{Y}^1)$.*

Under the conditions of Theorem 2.5 we construct the operators $H = B_0^{-1}A_0 \in \mathcal{L}(\mathfrak{X}^0)$ and $S = A_1^{-1}B_1 \in \mathcal{L}(\mathfrak{X}^1)$. Then

$$(\mu A - B)^{-1} = \Big(-\sum_{k=0}^{\infty}\mu^k H^k\Big)B_0^{-1}(\mathbb{I} - Q) + \sum_{k=1}^{\infty}\mu^{-k}S^{k-1}A_1^{-1}Q. \tag{2.1}$$

Definition 2.6. An infinitely distant point of the A-resolvent of the operator B is called

(i) *a removable singular point*, if $H \equiv \mathbb{O}$;
(ii) *a pole of the order p*, if $H^p \neq \mathbb{O}, H^{p+1} \equiv \mathbb{O}$; $p \in \mathbb{N}$,
(iii) *an essentially singular point*, if $H^q \neq \mathbb{O}$, for all $q \in \mathbb{N}$.

Definition 2.7. An (A,σ)-bounded operator B is called (A,p)-bounded, if the point ∞ is a pole of the order $p \in \{0\} \cup \mathbb{N}$ of its A-resolvent.

3. Space of differentiable noises

The Sobolev theory for equations has been applied to the spaces of K-noises [5, 6, 7, 21]. In this section, for completeness, we give only the necessary information about the spaces of differentiable K-noises, which are considered in [21, 5, 6, 7]. We denote by $\Omega = (\Omega, A, P)$ the total probability space. A measurable map $\xi : \Omega \to \mathbb{R}$ is called a *random variable*. The set of random variables whose expectations are equal to zero (i.e., $E\xi = 0$) and the variances are finite (i.e., $D\xi < +\infty$) form the Hilbert space $\mathbf{L}_2$ with the inner product $(\xi_1, \xi_2) = E\xi_1\xi_2$. We denote by A_0 a σ-subalgebra of the σ-algebra A and construct the space $\mathbf{L_2^0}$ of random variables measurable with respect to A_0, then $\mathbf{L_2^0}$ is a subspace of the space $\mathbf{L}_2$. Let $\xi \in \mathbf{L}_2$, then $\Pi : \mathbf{L}_2 \to \mathbf{L_2^0}$ is an orthoprojector, and $\Pi\xi$ is called the *conditional expectation* of a random variable ξ and denoted by $E(\xi|A_0)$.

Let $I = (0,T)$, $T \in \mathbb{R}_+$. Consider two mappings: $f : I \to \mathbf{L}_2$, which associates each $t \in I$ with a random variable $\xi \in \mathbf{L}_2$, and $g : \mathbf{L}_2 \times \Omega \to \mathbb{R}$, which associates each pair (ξ, ω) with a point $\xi(\omega) \in \mathbb{R}$. The map $\eta : I \times \Omega \to \mathbb{R}$ of the form $\eta = \eta(t,\omega) = g(f(t),\omega)$ is called (one-dimensional) *random process*. If all the trajectories of a random process are almost surely (a.s.) continuous, then this process is called *continuous*. The set of continuous random processes forms a Banach space, which we denote by $\mathbf{CL}_2$. An example of a continuous random process is the one-dimensional Wiener process $\beta = \beta(t)$, which can be represented as

$$\beta(t) = \sum_{k=0}^{\infty} \xi_k \sin\big(\frac{\pi}{2}(2k+1)t\big). \tag{3.1}$$

Here ξ_k are uncorrelated Gaussian random variables such that $E\xi_k = 0$, $D\xi_k = [\frac{\pi}{2}(2k+1)]^{-2}$.

Now we fix an arbitrary continuous random process $\eta \in \mathbf{CL}_2$ and $t \in I$. Let N_t^η be the σ-algebra generated by the random process $\eta(t)$, and $E_t^\eta = E(\cdot|N_t^\eta)$ be the conditional expectation.

Let $\eta \in \mathbf{CL}_2$, then by the *average derivative from the right* $D\eta(t,\cdot)$ (*from the left* $D_*\eta(t,\cdot)$) of the random process η at the point $t \in (\varepsilon,\tau)$ we mean a random variable

$$D\eta(t,\cdot) = \lim_{\Delta t \to 0+} E_t^\eta\Big(\frac{\eta(t+\Delta t,\cdot) - \eta(t,\cdot)}{\Delta t}\Big)$$
$$\Big(D_*\eta(t,\cdot) = \lim_{\Delta t \to 0+} E_t^\eta\Big(\frac{\eta(t,\cdot) - \eta(t-\Delta t,\cdot)}{\Delta t}\Big)\Big),$$

if the limit exists in the sense of the uniform metric on $\mathbb{R}$. A random process η is called *average differentiable from the right (left) on* I if there exists an average derivative from the right (left) at each point $t \in I$. Let $\eta \in \mathbf{CL}_2$ be a random process, which is average differentiable from the right and left on I. Then the *average symmetric derivative* is defined as $\overset{\circ}{\eta} = D_S\eta = \frac{1}{2}(D + D_*)\eta$. Further, we refer to the average symmetric derivative as the Nelson-Gliklikh derivative. By $\overset{\circ}{\eta}^{(l)}$, $l \in \mathbb{N}$, we denote the l-th Nelson-Glicklikh derivative of the random process η. Note the Nelson-Gliklikh derivative of a deterministic function coincides with the classical derivative. In the case of the one-dimensional Wiener process $\beta = \beta(t)$, the following statements are true:

(i) $\overset{\circ}{\beta}(t) = \frac{\beta(t)}{2t}$ for all $t \in \mathbb{R}_+$;
(ii) $\overset{\circ}{\beta}^{l)}(t) = (-1)^{l-1}\prod_{i=1}^{l-1}(2i-1)\frac{\beta(t)}{(2t)^l}$, $\quad l \in \mathbb{N}$, $l \geq 2$.

We construct the noise space $\mathbf{C}^l\mathbf{L}_2$, $l \in \mathbb{N}$, as the space of random processes from $\mathbf{CL}_2$, whose trajectories are almost sure (a.s.) differentiable in the sense of the Nelson-Gliklikh derivative on I up to the l-th order inclusively.

Let $\mathfrak{U}$ be a separable Hilbert space with the orthonormal basis $\{\varphi_k\}$. Each element $u \in \mathfrak{U}$ can be written as

$$u = \sum_{k=1}^{\infty} u_k\varphi_k.$$

Let $K = \{\nu_k\}$ be a monotonically non-increasing numerical sequence such that $\sum_{k=1}^{\infty}\nu_k^2 < +\infty$. Choose a sequence of random variables $\{\xi_k\} \subset \mathbf{L}_2$ such that $\sum_{k=1}^{\infty}\nu_k^2 D\xi_k < +\infty$. Then a Hilbert space $\mathbf{U}_K\mathbf{L}_2$ is called a *random K-variable space*, moreover $\xi = \sum_{k=1}^{\infty}\nu_k\xi_k\varphi_k$.

Choose a sequence $\{\eta_k\}$ from the space $\mathbf{CL}_2$ and define the $\mathfrak{U}$-valued continuous random K-process by the formula

$$\xi(t)=\sum_{k=1}^{\infty}\nu_k\xi_k(t)\varphi_k \tag{3.2}$$

provided that series converges uniformly on any compact set from I in the norm of $\mathbf{U}_K\mathbf{L}_2$. We introduce the Nelson-Gliklikh derivatives of the random K-process

$$\overset{\circ}{\xi}{}^{(l)}(t)=\sum_{k=1}^{\infty}\nu_k\overset{\circ}{\xi}{}_k^{(l)}(t)\varphi_k$$

provided that the Nelson-Gliklikh derivatives from the right exist up to the l-th order inclusive, and all the series converge uniformly on any compact set of I in the norm of $\mathbf{U}_K\mathbf{L}_2$. Therefore, we define the space $C^l(I;\mathbf{U}_K\mathbf{L}_2)$ of continuous random K–processes whose trajectories are a.s. continuously differentiable with respect to Nelson-Gliklikh up to the l-th order inclusive. For shortness, the space $C^l(I;\mathbf{U}_K\mathbf{L}_2)$ is called the space (of differentiable) K-noises.

4. Optimal control for stochastic Sobolev type equation

The following results are based on an obvious statement.

Lemma 4.1. *Let the operators $A,B\in\mathcal{L}(\mathfrak{U},\mathfrak{F})$. Then the operators A,B belong to $\mathcal{L}(\mathbf{U_K L_2},\mathbf{F}_K\mathbf{L}_2)$ additionally, if A is (B,p)-bounded in $\mathcal{L}(\mathfrak{U},\mathfrak{F})$ then A is (B,p)-bounded in the space $\mathcal{L}(\mathbf{U}_k\mathbf{L}_2;\mathbf{F}_k\mathbf{L}_2)$.*

Let $\mathfrak{X}=\mathbf{U}_K\mathbf{L}_2$, $\mathfrak{Y}=\mathbf{F}_K\mathbf{L}_2$ be Hilbert spaces of random K-variables, the operators $A,B\in\mathcal{L}(\mathfrak{X};\mathfrak{Y})$ be linear operators acting from $\mathfrak{X}$ to $\mathfrak{Y}$. Firstly, we solve the auxiliary problem (1.8), (1.7) within the framework of the theory of Sobolev type equations.

Definition 4.2. The operator-function $V^\bullet\in C^\infty(\mathbb{R};\mathcal{L}(\mathfrak{X}))$ is called the propagator of homogeneous equation (1.7), if the random K-process $w(t)=V^tv$ is a solution of (1.7) for all $v\in\mathfrak{X}$.

Theorem 4.3 ([14])**.** *Let the operator B be (A,σ)-bounded. Then the formula*

$$V_m^t=\frac{1}{2\pi i}\int_\Gamma \mu^{1-m}(\mu^2A-B)^{-1}Ae^{\mu t}d\mu,\ m=0,1, \tag{4.1}$$

where the contour $\Gamma=\{\mu\in\mathbb{C}:|\mu|=R>a\}$, defines the propagators of equation (1.7) for all $t\in\mathbb{R}$.

Lemma 4.4. *(i) $V_m^\bullet\in C^\infty(\mathbb{R};\mathcal{L}(\mathfrak{X};\mathfrak{X}^1))$, $(V_m^t)_t^{(l)}=V_{m-l}^t$, where $m=0,1$, $l=0,1$;*
(ii) $(V_m^t)_t^{(l)}\big|_{t=0}=\mathbb{O}$ for $m\neq l$, $(V_m^t)_t^{(m)}\big|_{t=0}=V_0^0=P$.

Consider Cauchy problem (1.8) for homogeneous equation (1.7).

Definition 4.5. The subspace $\mathcal{P}\subset\mathfrak{X}$ is called the phase space of homogeneous equation (1.7) if

(i) a.s. every trajectory of the solution $w=w(t)$ to equation (1.7) belongs to $\mathcal{P}$, i.e. $w(t)\in\mathcal{P},\forall t\in\mathbb{R}$;
(ii) there exists an unique solution to problem (1.8), (1.7) for any $w_0,w_1\in L_2(\Omega;\mathcal{P})$.

Theorem 4.6 ([25])**.** *Let the operator B be (A,p)-bounded. Let the vector function $y:(-\tau,\tau)\to\mathfrak{Y}$ be such that $y^0\in C^2((-\tau,\tau);\mathfrak{Y}^0)$, and $y^1\in C((-\tau,\tau);\mathfrak{Y}^1)$. Suppose that the initial values satisfy the relations*

$$(I-V_0^0)w_m=-\sum_{q=0}^{p}H^qB_0^{-1}\frac{d^{2q+m}}{dt^{2q+m}}y^0(0),\quad m=0,1.$$

Then there exists a unique solution to problem (1.7), (1.8)*, which can be represented in the form*

$$w(t)=-\sum_{q=0}^{p}H^qB_0^{-1}(\mathbb{I}-Q)y^{(2q)}(t)+\sum_{m=0}^{1}V_m^tw_m^1+\int_0^tV_1^{t-s}A_1^{-1}Qy(s)ds. \tag{4.2}$$

Definition 4.7. The random K-process $w\in H^2(\mathfrak{X})=\{w\in L_2(0,\tau;\mathfrak{X}):\ddot{w}\in L_2(0,\tau;\mathfrak{X})\}$ is said to be a strong solution of (1.7), if w converts (1.7) into identity almost surely on $(0,\tau)$. A strong solution $w=w(t)$ to equation (1.7) is called a strong solution to problem (1.7), (1.8) if w satisfies (1.8).

The concept of a "strong solution" used in Definition 4.7 is introduced to distinguish the solution to equation (1.7) in this sense and the solution to (1.7), which is usually called classical solution. The embedding $H^2(\mathfrak{X})\hookrightarrow C^1([0,\tau];\mathfrak{X})$ is continuous, therefore Definition 4.7 is correct. Note that the classical solution to (1.7), (1.8) is also a strong solution to the problem.

Let us define the space $H^2(\mathfrak{Y})=\{v\in L_2(0,\tau;\mathfrak{Y}):\ddot{v}\in L_2(0,\tau;\mathfrak{Y})$. Then $H^2(\mathfrak{Y})$ is a Hilbert space with the inner product

$$[v,w]=\sum_{q=0}^{2}\int_0^\tau\left\langle v^{(q)},w^{(q)}\right\rangle_{\mathfrak{Y}}dt.$$

Theorem 4.8 ([31])**.** *Let the operator B be (A,p)-bounded. Then there exists a unique strong solution to the problem* (1.8) *for equation* (1.7) *for any $w_0,w_1\in\mathfrak{X}$ and $y\in H^2(\mathfrak{Y})$.*

Let us consider the optimal control problem for the solutions to problem (1.7), (1.8) with the penalty functional

$$J(w,u)=\sum_{q=0}^{2}\int_0^\tau||w^{(q)}-\tilde{w}^{(q)}||^2dt+\sum_{q=0}^{2}\int_0^\tau\langle N_qu^{(q)},u^{(q)}\rangle_{\mathfrak{Y}}dt, \tag{4.3}$$

where $N_q\in\mathcal{L}(\mathfrak{Y})$, $q=0,1,2$, are positive definite and self-adjoint operators, w is a solution to problem (1.7), (1.8), $\tilde{w}(t)$ is the desired state of the system, and the heterogeneity function y is a unknown random K-process, which has sense a optimal control. The random K-process $\hat{u}\in H^2_\partial(\mathfrak{Y})$ minimizing functional (4.3) is called the optimal control to problem (1.7), (1.8).

We define the control space

$$H^2(\mathfrak{Y})=\{u\in L_2(0,\tau;\mathfrak{U}):\ddot{u}\in L_2(0,\tau;\mathfrak{U})\}.$$

The space $H^2(\mathfrak{Y})$ is Hilbert,because of the Hilbert property of $\mathfrak{Y}$, with the inner product

$$[v,w]=\sum_{q=0}^{2}\int_0^\tau\langle v^{(q)},w^{(q)}\rangle_{\mathfrak{U}}dt.$$

In the space $H^2(\mathfrak{Y})$, we consider the closed and convex subset $\mathfrak{Y}_{ad} = H^2_\partial(\mathfrak{Y})$, which is the *set of admissible controls.* In [31], the following theorem on the uniqueness of optimal control is proved.

Theorem 4.9. *Let the operator B be (A,p)-bounded. Then there exists the unique optimal control of solutions to problem* (1.8) *for equation* (1.7) *for any $w_0, w_1 \in \mathfrak{X}$ and $y \in H^2(\mathfrak{Y})$.*

5. Sobolev equation

Let us apply the abstract theory to the study of optimal control in the mathematical model (1.3)–(1.5). We consider the case when the domain D is a parallelepiped. There are domains with an analytic boundary for which the operator $\Delta^{-1}\frac{\partial^2}{\partial z^2}$ where it has continuous spectrum. Consider a space over the domain D such that the operator Δ^{-1} is a compact operator and the operator $\frac{\partial^2}{\partial z^2}$ is bounded, and therefore their composition is a compact operator. Therefore, the spectrum of $\Delta^{-1}\frac{\partial^2}{\partial z^2}$ is bounded. Later we will show that the A-spectrum of the operator B coincides with the spectrum of the operator $\Delta^{-1}\frac{\partial^2}{\partial z^2}$. Let the domain D be the parallelepiped $[0,a]\times[0,b]\times[0,c]$. Fix $p \in \mathbb{N}$ and define the spaces $\mathfrak{U} = \{u \in W_2^{p+2}(D) : u(x,y,z,t) = 0 \quad (x,y,z,t) \in \partial\Omega \times \mathbb{R}_+\}$ and $\mathfrak{F} = W_2^p(D)$. The space $\mathfrak{U}$ is a separable Hilbert space by construction. Denote by $-\lambda^2_{l,m,n} = -(\frac{\pi l}{a})^2 - (\frac{\pi m}{b})^2 - (\frac{\pi n}{c})^2$ the eigenvalues of the Dirichlet problem for the Laplace operator Δ. Denote by $\varphi_{l,m,n} = \sin(\frac{\pi l x}{a})\sin(\frac{\pi m y}{b})\sin(\frac{\pi n z}{c})$ the eigenfunctions corresponding to $-\lambda^2_{l,m,n}$.

We introduce the $\mathfrak{U}$-valued random K-processes. The sequence K is defined as follows: $K = \{\nu_{l,m,n} : \quad \nu_{l,m,n} = \lambda^{-2}_{l,m,n}\}$. By formula (3.2), we obtain the $\mathfrak{F}$-valued Wiener K-process in the form

$$w_K(t) = \sum_{l,m,n=1}^{\infty} \nu_{l,m,n}\beta_{l,m,n}(t)\varphi_{l,m,n},$$

where $\beta_{l,m,n}(t)$ is a product of three independent one-dimensional Wiener processes (2.1).

We define the operators

$$A = \Delta, \quad B = -F^2\frac{\partial^2}{\partial z^2}$$

as elements of the space $\mathcal{L}(\mathbf{U}_K\mathbf{L}^0_2; \mathbf{F}_K\mathbf{L}_2)$ by Lemma 4.1. Also, we define the inhomogeneity function as the Nelson-Gliklikh derivative of the Wiener process

$$g = \overset{\circ}{w}_K(t).$$

Therefore, we reduce mathematical model (1.3)–(1.5) to the Cauchy problem (1.8) for the abstract equation (1.7).

Since $\{\varphi_{l,m,n}\}$ is subset of $C^\infty(D)$, we have

$$\mu^2 A - B = \sum_{l,m,n=1}^{\infty}\Big[-\lambda^2_{l,m,n}\mu^2 - F^2\big(\frac{\pi n}{c}\big)^2\Big]\langle\varphi_{l,m,n},\cdot\rangle\varphi_{l,m,n},$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(D)$. The equation

$$\lambda^2_{l,m,n}\mu^2 + F^2(\frac{\pi n}{c})^2 = 0$$

determines the relative spectrum of the operator B:

$$\mu^{\pm}_{l,m,n} = \pm \frac{F\pi n}{c\sqrt{\lambda^2_{l,m,n}}} i.$$

Therefore, the A-spectrum of the operator $B\sigma^A(\overrightarrow{B}) = \{\mu^{\pm}_{l,m,n}\}$ is bounded.

We construct the propagators according to the Theorem 4.3. Since the relative spectrum of the operator B is discrete, we obtain

$$\begin{gathered} V^t_0 w_0 = \sum_{l,m,n=1}^{\infty} \cos\Big(\frac{F\pi n}{c\sqrt{\lambda^2_{l,m,n}}} t\Big) \langle \varphi_{l,m,n}, w_0\rangle \varphi_{l,m,n}, \\ V^t_1 w_1 = \sum_{l,m,n=1}^{\infty} \frac{F\pi n}{c\sqrt{\lambda^2_{l,m,n}}} \sin\Big(\frac{F\pi n}{c\sqrt{\lambda^2_{l,m,n}}} t\Big) \langle \varphi_{l,m,n}, w_1\rangle \varphi_{l,m,n}. \end{gathered} \tag{5.1}$$

Becasue the white noise $\overset{\circ}{w}_K(t)$ is not differentiable at $t = 0$, the integral in formula (4.2) does not make sense and conditions of Theorem 4.3 are not satisfied. To overcome this difficulty, we use the method proposed in [12]. We transform the second term from the right-hand side of the solution as follows

$$\begin{aligned} \int_{\varepsilon}^{t} V^{t-s}_1 \overset{\circ}{w}_K(t)(s)ds &= -V^{t-\varepsilon}_1 w_K(t) - \int_{\varepsilon}^{t} \frac{d}{ds}(V^{t-s}_0) w_K(s)ds \\ &= -V^{t-\varepsilon}_1 w_K(\varepsilon) + \int_{\varepsilon}^{t} V^{t-s}_0 w_K(s)ds. \end{aligned}$$

In this case, integration by parts makes sense for any $\varepsilon \in (0,t)$, $t \in \mathbb{R}_+$, by the definition of the Nelson-Gliklikh derivative. If $\varepsilon \to 0$, then we obtain

$$\int_0^t V^{t-s}_1 \overset{\circ}{w}_K(s)ds = \int_0^t V^{t-s}_0 w_K(s)ds.$$

Therefore, the conditions of Theorem 4.6 are fulfilled. Hence, there exists the unique solution to problem (1.2)–(1.5) given by

$$w(x,t) = V^t_0 w_0 + V^t_1 w_1 + \int_0^t V^{t-s}_0 w_K(s)ds. \tag{5.2}$$

By Theorem 4.8, the solution given by formula (5.2) is strong. Define the control space

$$H^2(\mathfrak{Y}) = \{u \in L_2(0,\tau;\mathfrak{Y}) : \ddot{u} \in L_2(0,\tau;\mathfrak{Y})\},$$

and consider the closed and convex subset $\mathfrak{Y}_{ad} = H^2_{\partial}(\mathfrak{Y})$, which is the set of admissible controls. The main result of the paper is the proof of the existence of the unique control $\hat{y} \in H^2_{\partial}(\mathfrak{Y})$ minimizing the functional $J(w,u)$. Fix $w_0, w_1 \in \mathfrak{X}$ and consider (5.2) as the map $D : y \to w(u)$. Then the map $D : H^2(\mathfrak{Y}) \to H^2(\mathfrak{X})$ is continuous. Therefore, the quality functional depends only on u, i.e. $J(w,u) = J(u)$.

We rewrite the quality functional (4.3) in the form

$$J(u) = \|w(t,u) - \tilde{w}\|^2_{H^2(\mathfrak{X})} + [v,u],$$

where $v^{(q)}(t) = N_q u^{(q)}(t),\ q = 0,1,2$. Hence

$$J(u) = \pi(u,u) - 2\lambda(u) + \|\tilde{w} - w(t,0)\|^2_{H^2(\mathfrak{X})},$$

where

$$\pi(u,u) = \|w(t,u) - w(t,0)\|^2_{H^2(\mathfrak{X})} + [v,u]$$

is a bilinear continuous coercive form on $H^2(\mathfrak{Y})$, and

$$\lambda(u) = \langle \tilde{w} - w(t,0), w(t,u) - w(t,0)\rangle_{H^2(\mathfrak{X})}$$

is a linear form, which is continuous on $H^2(\mathfrak{Y})$. Therefore, the conditions of [15, Theorem 1.1] are satisfied.

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References

[1] A. B. Al'shin, M. O. Korpusov, A. G. Sveshnikov; *Blow-up in Nonlinear Sobolev Type Equations.* de Gruyter, Berlin, 2011, DOI: 10.1515/9783110255294.

[2] M. Arato; *Linear Stochastic Systems with Constant Coefficients. A Statistical Approach.* Springer-Verlag, 1982, DOI: 10.1007/BFb0043631.

[3] G. Da Prato, J. Zabczyk; *Stochastic Equations in Infinite Dimensions.* Cambridge University Press, Cambridge, 1992, DOI: 10.1017/CBO9781107295513.

[4] G. V. Demidenko, S. V. Uspenskii; *Partial Differential Equations and Systems not Solvable with Respect to the Highest-order Derivative.* Marcel Dekker Inc, New York; Basel; Hong Kong, 2003.

[5] A. Favini, G. A. Sviridyuk, N. A. Manakova; Linear Sobolev Type Equations with Relatively p-Sectorial Operators in Space of "Noises". *Abstract and Applied Analysis,* article no. 697410, 2015, DOI: 10.1155/2015/697410.

[6] A. Favini, G. A. Sviridyuk, M. A. Sagadeeva; Linear Sobolev Type Equations with Relatively p-Radial Operators in Space of Noises. *Mediterranean Journal of Mathematics,* 13 (1.7), 4607–4621, 2016, DOI: 10.1007/s00009-016-0765-x.

[7] A. Favini, G. A. Sviridyuk, A. A. Zamyshlyaeva; One Class of Sobolev Type Equations of Higher Order with Additive "White Noise". *Communications on Pure and Applied Analysis,* 15 (1), 185–196, 2016, DOI: 10.3934/cpaa.2016.15.185.

[8] A. Favini, A. Yagi; *Degenerate Differential Equations in Banach Spaces.* Marcel Dekker Inc, New York; Basel; Hong Kong, 1999.

[9] M. V. Fokin; Hamiltonian Systems in the Theory of Small Oscillations of while Rotating Ideal Fluid I. *Mat. Tr.,* 2(4), pp. 155–206, 2001.

[10] Yu. E. Gliklikh; *Global and Stochastic Analysis with Applications to Mathematical Physics.* Springer-Verlag, 2011, DOI: 10.1007/978-0-85729-163-9.

[11] Yu.E. Gliklikh; Investigation of Leontieff Type Equations with White Noise Protect by the Methods of Mean Derivatives of Stochastic Processes. *Bulletin of the South Ural State University. Series: Mathematical Modelling, Programming and Computer Software,* 27 (286),issue 13, pp. 24–34, 2012.

[12] Yu. E. Gliklikh, E. Yu. Mashkov; Stochastic Leontieff Type Equations and Mean Derivatives of Stochastic Processes. *Bulletin of the South Ural State University. Series: Mathematical Modelling, Programming and Computer Software,* 6 (2), pp. 25–39, 2013.

[13] O. G. Kitaeva, D. E. Shafranov, G. A. Sviridyuk; Degenerate Holomorphic Semigroups of Operators in Spaces of K-Noises on Riemannian Manifolds. *Springer Proceedings in Mathematics and Statistics,* 325, pp. 279–292, 2020, DOI: 10.1007/978-3-030-46079-2_16.

[14] M. Kovacs, S. Larsson; Introduction to Stochastic Partial Differential Equations. In: *New Directions in the Mathematical and Computer Sciences*, ICMCS, Abuja, Nigeria, October 8–12, 2007.

[15] J.-L. Lyons; *Optimal Control of Systems Governed by Partial Differential Equations.* Springer-Verlag, 1972.

[16] N. A. Manakova; Mathematical Models and Optimal Control of the Filtration and Deformation Processes. *Bulletin of the South Ural State University, Series: Mathematical Modelling, Programming and Computer Software,* 8 (3), pp. 5–24, 2015, DOI: 10.14529/mmp150301.

[17] N. A. Manakova, G. A. Sviridyuk; An Optimal Control of the Solutions of the Initial-Final Problem for Linear Sobolev Type Equations with Strongly Relatively p-Radial Operator. *Springer Proceedings in Mathematics and Statistics,* 113, pp. 213–224, 2015, DOI: 10.1007/978-3-319-12145-1_13.

[18] I. V. Melnikova, A. I. Filinkov; Generalized Solutions to Abstract Stochastic Problems. *J. Integ. Transf. and Special Funct.,* 20 (3–4), pp. 199–206, 2009, DOI: 10.1080/10652460802567574.

[19] E. Nelson; *Dynamical Theories of Brownian Motion.* Princeton University Press, Princeton, 1967.

[20] S. G. Pyatkov; *Operator Theory. Nonclassical Problems.* VSP: Utrecht; Boston; Koln; Tokyo, 2002, DOI: 10.1515/9783110900163.

[21] M. A. Sagadeeva; Reconstruction of Observation from Distorted Data for the Optimal Dynamic Measurement Problem. *Bulletin of the South Ural State University, Series: Mathematical Modelling, Programming and Computer Software,* 12 (2), pp. 82–96, 2019, DOI: 10.14529/mmp190207.

[22] M. A. Sagadeeva, S. A. Zagrebina, N. A. Manakova; Optimal Control of Solutions of a Multipoint Initial-Final Problem for Non-autonomous Evolutionary Sobolev Type Equation. *Evolution Equations and Control Theory,* 8 (3), pp. 473–488, 2019, DOI: 10.3934/eect.2019023.

[23] A. L. Shestakov, G. A. Sviridyuk; Optimal Measurement of Dynamically Distorted Signals. *Bulletin of the South Ural State University, Series: Mathematical Modelling, Programming and Computer Software,* 17 (234), issue 8, pp. 70–75, 2011.

[24] A. L. Shestakov, G. A. Sviridyuk, Yu. V. Khudyakov; Dynamic Measurement in Spaces of "Noise". *Bulletin of the South Ural State University, Series "Computer Technologies, Automatic Control, Radio Electronics"*, 13 (2), pp. 4–11, 2013.

[25] S. L. Sobolev; On a New Problem of Mathematical Physics. *Izv. Akad. Nauk SSSR, Ser. Mat.,* 8(1), pp. 3–50, 1954.

[26] G. A. Sviridyuk; Sobolev-type Linear Equations and Strongly Continuous Semigroups of Resolving Operators with Kernels. *Russian Acad. Sci. Dokl. Math.,* 50 (1), pp. 137–142, 1995.

[27] G. A. Sviridyuk, A. A. Efremov; Optimal control of a class of linear degenerate equations. *Doklady Mathematics,* 59 (1), pp. 157–159, 1999.

[28] G. A. Sviridyuk, V. E. Fedorov; *Linear Sobolev Type Equations and Degenerate Semigroups of Operators.* VSP, Utrecht; Boston; Koln; Tokyo, 2003, DOI: 10.1515/9783110915501.

[29] G. A. Sviridyuk, N. A. Manakova; The Dynamical Models of Sobolev Type with Showalter-Sidorov Condition and Additive "Noise". *Bulletin of the South Ural State University, Series: Mathematical Modelling, Programming and Computer Software,* 7 (1), pp. 94–107, 2014, DOI: 10.14529/mmp140108.

[30] A. A. Zamyshlyaeva, E. V. Bychkov; The Cauchy Problem for the Sobolev Type Equation of Higher Order. *Bulletin of the South Ural State University, Series: Mathematical Modelling, Programming and Computer Software,* 11 (1), pp. 5–14, 2018, DOI: 10.14529/mmp180101.

[31] A. A. Zamyshlyaeva, O. N. Tsyplenkova; Optimal Control of Solutions to Showalter-Sidorov Problem for a High Order Sobolev Type Equation with Additive Noise. *Springer Proceedings in Mathematics and Statistics,* 325, pp. 425–446, 2020, DOI: 10.1007/978-3-030-46079-2_24.

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