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KIRCHHOFF-TYPE PROBLEMS WITH CRITICAL SOBOLEV EXPONENT IN A HYPERBOLIC SPACE

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Abstract. In this work we study a class of the critical Kirchhoff-type problems in a Hyperbolic space. Because of the Kirchhoff term, the nonlinearity u^q becomes "concave" for $2 < q < 4$, This brings difficulties when proving the boundedness of Palais Smale sequences. We overcome this difficulty by using a scaled functional related with a Pohozaev manifold. In addition, we need to overcome singularities on the unit sphere, so that we use variational methods to obtain our results.

1. Introduction

In this article we study the Kirchhoff-type problem

$$-\Big(a+b\int_{\mathbf{B}^3}|\nabla_{\mathbf{B}^3}u|^2dV_{\mathbf{B}^3}\Big)\Delta_{\mathbf{B}^3}u=\lambda|u|^{q-2}u+|u|^4u \quad \text{in } H^1(\mathbf{B}^3), \tag{1.1}$$

where a, b, λ are positive constants, $2 < q < 4$, $H^1(\mathbf{B}^3)$ is the usual Sobolev space on the disc of the Hyperbolic space $\mathbf{B}^3$, and $\Delta_{\mathbf{B}^3}$ denotes the Laplace Beltrami operator on $\mathbf{B}^3$. Problem (1.1) defined in whole space $\mathbb{R}^N$, with $N \geq 3$, and with the non-linearity behaving as a polynomial function of degree $2^* = \frac{2N}{N-2}$ was studied by Brezis and Nirenberg [7]. Posteriorly, several authors have studied this class of problems; see for instance Carrião, Costa, and Miyagaki [8].

In the Euclidean context, equation (1.1) is related to a stationary Kirchhoff equation (see [25])

$$u_{tt}-M\Big(\int_\Omega|\nabla_x u|^2dx\Big)\Delta_x u=f(x,t), \quad (x,t)\in\Omega\times(0,\infty),$$

where Ω is a bounded domain of $\mathbb{R}^N$, $M(s)=a+bs$ with $a, b > 0$, and f is a suitable function, which is an extension of the classical D'Alembert's wave equation. One characteristic of this model is that it considers the effects of the changes in the length of the strings during the vibrations. The main difficulty appears because the equation does not satisfy a pointwise identity any longer. It is generated by the presence of the term containing M in the equation, and it makes (1.1) a nonlocal problem.

Ma and Rivera [27] were the pioneers to study this problem by employing minimizing methods. In [1], the mountain pass theorem was used, while in [30] the

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Yang index and critical groups was used. In [21] the equation was studied using the minimization arguments and the Fountain theorem. Results can be seen in [12, 17, 36]. Results involving the Kirchhoff equation and critical exponents can be found in [2, 16, 19, 20, 26] and references therein. See also [11, 13, 14, 32] for some related results.

We also would like to cite the recent works by Xiang, Zhang and Rădulescu [38, 39]. In the first one, the authors studied the multiplicity of solutions for a class of quasilinear Kirchhoff system involving the fractional p-Laplacian. In the second paper, they proved the existence of local solution and a blow-up result for a class of nonlocal Kirchhoff diffusion problems.

Our main result reads as follows.

Theorem 1.1. *Under the assumptions that* $2 < q < 4$, *for* $\lambda > 0$ *sufficiently large, problem* (1.1) *has a nontrivial solution* $u \in H^1(\mathbf{B}^3)$.

This result extends the result in [20] with respect to the existence in a hyperbolic space. Also, in [9], when $a = 1$ and $b = 0$. It also extends [8], where the authors studied (1.1) with $4 < q < 6$ for $\lambda > 0$ arbitrary. We highlight that the case $2 < q < 4$ is more delicate and it is necessary additional tools.

Finally, we would like to emphasize that an extra difficulty of the present paper is to prove that the Palais Smale sequence is bounded. To overcome this difficulty, we use an appropriated modified functional (see $J_\theta(v)$ definition in next section). This functional gives us an additional property of the Palais Smale sequence which is fundamental to prove that the sequence is bounded (Lemmas 2.2 and 2.3). Precisely, the scaled functional J_θ works coupled with another appropriated functional, G, which has the property $G(v_k) \to 0$, where (v_k) is the Palais Smale sequence. Scaled functional was used by Jeanjean [23] and Jeanjean and Le Coz [24]. See also [19] and [22].

2. Proof of the main result

For the hyperbolic space $\mathcal{H}^n$, we use the stereographic projection, where each point $P' \in \mathcal{H}^n$ is projected to $P \in \mathbb{R}^n$, where P is the intersection of the straight line connecting P' and the point $(0, \dots, 0, -1)$. Explicitly the projection operator $G : \mathbb{R}^n \to \mathcal{H}^n$ and $G^{-1} : \mathcal{H}^n \to \mathbb{R}^n$ given by

$$G(x) = (x \cdot p(x), (1 + |x|^2)p/2) \quad \text{and} quad G^{-1}(y) = \frac{1}{y_{n+1}} y, \quad x, y \in \mathbb{R}^n,$$

where $p(x) = \frac{2}{1-|x|^2}$.

We consider the ball $B_1(0)$, and $\mathbf{B}^n$ endowed with the metric

$$ds = p(x)|dx|, \quad \text{where } p(x) = \frac{2}{1 - |x|^2}.$$

With this notation, the gradient, the Dirichlet integral and the Laplace-Beltrami operator corresponding to this metric are

$$\nabla_{\mathbf{B}^n} u = \frac{\nabla u}{p^2}, \quad Du = \int_{D'} |\nabla_{\mathbf{B}^n} u|^2 dV_{\mathbf{B}^n} = \int_D |\nabla u|^2 p^{n-2} dx,$$
$$\Delta_{\mathbf{B}^n} u = p^{-n} \operatorname{div}(p^{n-2} \nabla u).$$

We denote by $D \subset B_1(0)$ the stereographic projection of $D' \subset \mathcal{H}^n$. Details involving the hyperbolic space can be found in [3, 18, 31, 33, 34].

Defining $v := p^{1/2}u$, we have that u is solution of (1.1) if, and only if, v satisfies

$$
\begin{gathered}
(a+b\|v\|^2)(-\Delta v+(3/4)p^2 v)=\lambda p^\alpha |v|^{q-2}v+|v|^4 v, \quad \text{in } B_1(0)\\
v=0, \quad \text{on } \partial B_1(0),
\end{gathered}
\tag{2.1}
$$

where $\alpha=(6-q)/2$ and $\|v\|^2=\int_{B_1(0)}\left(|\nabla v|^2+(3/4)p^2v^2\right)$.

We denote by $H^1_{0,r}(\Omega)$, $\Omega := B_1(0)$ the subspace of $H^1_0(\Omega)$ of the radial functions which is endowed with the norm

$$
\|v\|^2=\int_\Omega \left(|\nabla v|^2+(3/4)p^2v^2\right).
$$

Since the Euclidean sphere with center at the origin $0\in\mathbb{R}^N$ is also a hyperbolic sphere with center at the origin $0\in\mathbf{B}^n$, $H^1_{0,r}(\Omega)$ can also be seen as the subspace of $H^1_0(\Omega)$ consisting of the hyperbolic radial functions. See this characterization as well as others remarks in [3, Appendix], for instance, $H^1_{0,r}(\Omega)$ is embedded compactly in $L^q(\Omega)$ for $2<q<2^*$, [3, Theorem 3.1]. Here, we use also [9, Lemma 3.1] and recall that $2^*=6$.

We consider the functional $J: H^1_{0,r}(\Omega)\to\mathbb{R}$ associated with problem (2.1),

$$
J(v)=\frac{a}{2}\|v\|^2+\frac{b}{4}\|v\|^4-\frac{\lambda}{q}\int_\Omega p^\alpha|v|^q-\frac{1}{6}\int_\Omega |v|^6, \tag{2.2}
$$

whose Gateaux derivative is

$$
J'(v)w=(a+b\|v\|^2)\int_\Omega\Big(\nabla v\cdot\nabla w+\frac{3}{4}p^2vw\Big)-\lambda\int_\Omega p^\alpha|v|^{q-2}vw-\int_\Omega |v|^4vw. \tag{2.3}
$$

The proof uses variational methods, more exactly, the mountain pass theorem. To this end, we have the following mountain pass geometry result.

Lemma 2.1 (Mountain pass geometry)**.**

(a) *There exist $\beta>0$ and $\rho>0$ such that $J(v)\ge\beta$ when $\|v\|=\rho$.*
(b) *There exists an element $e\in H^1_{0,r}(\Omega)$ with $\|e\|>\rho$ such that $J(e)<0$.*

Proof. (a) We observe that by [9, Lemma 2.1] (see also to [5, 6]) there exists a constant $C>0$, such that

$$
\int_\Omega p^\alpha v^q\le C\Big(\int_\Omega|\nabla v|^2\Big)^{q/2}\le C\Big[\int_\Omega\left(|\nabla v|^2+(3/4)p^2v^2\right)\Big]^{q/2}.
$$

Therefore,

$$
J(u)\ge\frac{a}{2}\|v\|^2+\frac{b}{4}\|v\|^4-\frac{C\lambda}{q}\Big[\int_\Omega\left(|\nabla v|^2+(3/4)p^2v^2\right)\Big]^{q/2}-\frac{1}{6}\int_\Omega|v|^6,
$$

and by the Sobolev continuous embedding, there exists a constant $\widetilde{C}>0$, satisfying

$$
J(u)\ge\frac{a}{2}\|v\|^2+\frac{b}{4}\|v\|^4-\frac{C\lambda}{q}\|v\|^q-\frac{\widetilde{C}}{6}\|v\|^6\ge\beta,
$$

where the conclusion follows by making $\|v\|=\rho$ sufficiently small.

Now, we prove the item (b). We take $0<v\in H^1_{0,r}(\Omega)$ and $0<t$. Therefore,

$$
J(tv)=\frac{at^2}{2}\|v\|^2+\frac{bt^4}{4}\|v\|^4-\frac{\lambda t^q}{q}\int_\Omega p^\alpha|v|^q-\frac{t^6}{6}\int_\Omega|v|^6.
$$

Therefore $J(tv)\to-\infty$, as $t\to+\infty$. Consequently, J satisfies the Mountain Pass Theorem geometry. $\square$

We recall that the pass mountain level is defined by

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], H^1_{0,r}(\Omega)) : \gamma(0) = 0, J(\gamma(1)) < 0\}$. For each $\theta > 0$, we define the functional

$$\begin{aligned} J_\theta(v) = \frac{a}{2}\int_\Omega \Big(|\nabla v|^2 + \frac{3}{4}\frac{1}{e^{5\theta}} p^2\Big(\frac{x}{e^{2\theta}}\Big)v^2\Big) + \frac{b}{4}\Big[\int_\Omega |\nabla v|^2 + \frac{3}{4}\frac{1}{e^{5\theta}} p^2\Big(\frac{x}{e^{2\theta}}\Big)v^2\Big]^2 \\ - \frac{\lambda}{q}\int_\Omega p^\alpha\Big(\frac{x}{e^{2\theta}}\Big)v^q - \frac{1}{6}\int_\Omega |v|^6. \end{aligned}$$

We also define $\Phi : \mathbb{R} \times H^1_{0,r}(\Omega) \to H^1_{0,r}(\Omega)$ by $\Phi(\theta, v) = e^\theta v\big(\frac{x}{e^{2\theta}}\big)$ and $I : \mathbb{R} \times H^1_{0,r}(\Omega) \to \mathbb{R}$ by $I(\theta, v) = J_\theta(\Phi(\theta, v))$.

Using Lemma 2.1, we have that the functional I satisfies the geometry of the Mountain Pass Theorem. Taking

$$\tilde{c} = \inf_{\tilde\gamma\in\tilde\Gamma} \sup_{t\in[0,1]} I(\tilde\gamma(t)),$$

where $\tilde\Gamma = \big\{\tilde\gamma \in C([0,1], \mathbb{R} \times H^1_{0,r}(\Omega)); \tilde\gamma(0) = (0,0), I(\tilde\gamma(1)) < 0\big\}$, we have $c = \tilde{c}$ because $\Gamma = \{\Phi \circ \tilde\gamma; \tilde\gamma \in \tilde\Gamma\}$.

Now, we define $G : H^1_{0,r}(\Omega) \to \mathbb{R}$ by

$$\begin{aligned} G(v) = 2a\int_\Omega |\nabla v|^2 + \frac{9a}{8}\int_\Omega p^2v^2 + 2b\Big(\int_\Omega |\nabla v|^2\Big)^2 + \frac{21b}{8}\int_\Omega |\nabla v|^2 \int_\Omega p^2v^2 \\ + \frac{27b}{32}\Big(\int_\Omega p^2v^2\Big)^2 - \frac{\lambda}{q}(q+6)\int_\Omega p^\alpha v^q - 2\int_\Omega |v|^6. \end{aligned}$$

As it was mentioned in the introduction, the functional G works coupled with the scaled functional J_θ. The functional G is a class of Pohozaev functional and it is defined to prove the boundedness of the Palais Smale sequence. The lemma below gives us the main property of G.

Lemma 2.2. *There exists a sequence $(v_k) \subset H^1_{0,r}(\Omega)$ such that*

$$J(v_k) \to c \quad J'(v_k) \to 0, \quad G(v_k) \to 0.$$

Proof. Applying [37, Theorem 2.8] as in [19] and [22, Proposition 4.2], we obtain a sequence (θ_k, v_k) such that

$$I(\theta_k, v_k) \to c, \quad I'(\theta_k, v_k) \to 0, \quad \theta_k \to 0.$$

We note that

$$\begin{aligned} I(\theta, v) &= \frac{a}{2}\Big(\int_\Omega \big|\nabla\Big(e^\theta v\big(\frac{x}{e^{2\theta}}\big)\Big)\big|^2 + \frac{3}{4}\frac{1}{e^{5\theta}}p^2\frac{x}{e^{2\theta}}v^2\big(\frac{x}{e^{2\theta}}\big)e^{2\theta}\Big) \\ &\quad + \frac{b}{4}\Big(\int_\Omega \big|\nabla\Big(e^\theta v\big(\frac{x}{e^{2\theta}}\big)\Big)\big|^2 + \frac{3}{4}\frac{1}{e^{5\theta}}p^2\frac{x}{e^{2\theta}}v^2\big(\frac{x}{e^{2\theta}}\big)e^{2\theta}\Big)^2 \\ &\quad - \frac{\lambda}{q}\int_\Omega p^\alpha\frac{x}{e^{2\theta}}\Big(e^\theta v\big(\frac{x}{e^{2\theta}}\big)\Big)^q - \frac{1}{6}\int_\Omega \big|e^\theta v\big(\frac{x}{e^{2\theta}}\big)\big|^6 \\ &= \frac{a}{2}\Big(e^{4\theta}\int_\Omega |\nabla v|^2 + \frac{3}{4}e^{3\theta}\int_\Omega p^2v^2\Big) \\ &\quad + \frac{b}{4}\Big[e^{8\theta}\Big(\int_\Omega |\nabla v|^2\Big)^2 + \frac{3}{4}e^{7\theta}\int_\Omega |\nabla v|^2\int_\Omega p^2v^2 \end{aligned}$$

$$+\frac{9}{16}e^{6\theta}\Big(\int_\Omega p^2v^2\Big)^2\Big]-\frac{\lambda}{q}e^{\theta(q+6)}\int_\Omega p^\alpha v^q-\frac{1}{6}2^{12\theta}\int_\Omega |v|^6.$$

Thus

$$\begin{aligned}\frac{\partial I}{\partial\theta}&=2ae^{4\theta}\int_\Omega|\nabla v|^2+\frac{9ae^{3\theta}}{8}\int_\Omega p^2v^2+\frac{21}{b}be^{7\theta}\int_\Omega|\nabla v|^2\int_\Omega p^2v^2\\&\quad+\frac{3}{2}\frac{9}{16}e^{6\theta}\Big(\int_\Omega p^2v^2\Big)^2-\frac{\lambda}{q}(q+6)e^{\theta(q+6)}\int_\Omega p^\alpha v^q-2e^{12\theta}\int_\Omega |v|^6.\end{aligned}\tag{2.4}$$

Considering $\theta_k\to 0$, by (2.4) and the definition of G for all $\epsilon>0$ there exists $k_0\in\mathbb{N}$ such that $k\ge k_0$

$$\Big|\frac{\partial I}{\partial\theta}(\theta_k,v_k)-G(v_k)\Big|<\epsilon.\tag{2.5}$$

Since $I'(\theta_k,v_k)\to 0$, by (2.5) we conclude that $G(v_k)\to 0$.

On the other hand, since $I(\theta_k,v_k)\to c$ and $I'(\theta_k,v_k)\to 0$ we obtain respectively

$$|I(\theta_k,v_k)-J(v_k)|<\epsilon,\tag{2.6}$$

$$|I'(\theta_k,v_k)(\xi,w)-J'(v_k)(w)|<\epsilon,\tag{2.7}$$

for all $k\ge k_0$. Using the facts that $I(\theta_k,v_k)\to c$ and $I'(\theta_k,v_k)\to 0$ by (2.6) and (2.7) we have $J(v_k)\to c$ and $J'(v_k)\to 0$ respectively. □

Next Lemma gives us the boundness for Palais Smale sequence.

Lemma 2.3. *The sequence $(v_k)\subset H^1_{0,r}(\Omega)$ obtained in Lemma 2.2 is bounded.*

Proof. We note that

$$\begin{aligned}J(v_k)-G(v_k)&=a\Big(\frac12-\frac{2}{q+6}\Big)\int_\Omega|\nabla v_k|^2+\frac{3a}{8}\Big(1-\frac{3}{q+6}\Big)\int_\Omega p^2v_k^2\\&\quad+b\Big(\frac14-\frac{2}{q+6}\Big)\int_\Omega|\nabla v_k|^2+\frac{3b}{8}\Big(1-\frac{7}{q+6}\Big)\int_\Omega|\nabla v_k|^2\int_\Omega p^2v_k^2\\&\quad+\frac{9b}{64}\Big(1-\frac{6}{q+6}\Big)\Big(\int_\Omega p^2v_k^2\Big)^2+\Big(\frac{2}{q+6}-\frac16\Big)\int_\Omega|v_k|^6.\end{aligned}$$

Since all the coefficients of the terms involving the integrals, on the right side of the equality are positive, $J(v_k)\to c$ and $G(v_k)\to 0$ by Lemma 2.2, we have (v_k) bounded. □

In next lemma, the number S is the best constant of Sobolev (see [35]). We follow the arguments of [7]. See also [9, 20, 19, 28]. We are going to omit some calculus, the reader can found the details in [8] where was studied the case $4<q<6$.

Lemma 2.4. *We have $c<\frac14 abS^3+\frac{1}{24}b^3S^6+\frac{1}{24}(b^2S^4+4aS)^{3/2}$, where*

$$S:=\inf_{u\in H^1_{0,r}(\Omega)}\frac{\int_\Omega|\nabla u|^2}{\big(\int_\Omega u^6\big)^{1/3}}.$$

Proof. First, we observe that it is sufficient to show that there exists a $v_0\in H^1_{0,r}(\Omega), v_0\ne 0$, such that

$$\sup_{t\ge 0}J(tv_0)<\frac14 abS^3+\frac{1}{24}b^3S^6+\frac{1}{24}(b^2S^4+4aS)^{3/2}.\tag{2.8}$$

Indeed, observing that $J(tv_0) \to -\infty$ as $t \to \infty$, there exists $R > 0$ such that $J(Rv_0) < 0$. Now, we write $u_1 := Rv_0$, and from Lemma 2.1, we have

$$0 < \beta \le c = \inf_{\gamma\in\Gamma} \max_{\tau\in[0,1]} J(\gamma(\tau)) \le \sup_{t\ge0} J(tv_0) < \frac{1}{4}abS^3+\frac{1}{24}b^3S^6+\frac{1}{24}(b^2S^4+4aS)^{3/2}.$$

Therefore, we are going to prove the existence of a function v_0 such that (2.8) holds.

We consider $0 < R < \frac{1}{2}$ a fixed number and let $\varphi \in C_0^\infty(\Omega)$ be a cut-off function with support at B_{2R}, such that φ is identically 1 on B_R and $0 \le \varphi \le 1$ on B_{2R}. Here, B_r denotes the ball in $\mathbb{R}^3$ with center at the origin and radius r.

Given $\varepsilon > 0$ we set $\psi_\varepsilon(x) := \varphi(x)\omega_\varepsilon(x)$, where

$$\omega_\varepsilon(x) = (3\varepsilon)^{1/4}\frac{1}{(\varepsilon+|x|^2)^{1/2}},$$

and ω_ε satisfies (see[35])

$$\int_{\mathbb{R}^3} |\nabla\omega_\varepsilon|^2 = \int_{\mathbb{R}^3} |\omega_\varepsilon|^6 = S^{3/2}. \tag{2.9}$$

From the definition of ω_ε, it can be shown that

$$\int_{B_R} |\nabla\omega_\varepsilon|^2 \le \int_{B_R} |\omega_\varepsilon|^6, \tag{2.10}$$

$$\int_{B_1-B_R} |\nabla\psi_\varepsilon|^2 = O(\varepsilon^{1/2}) \quad \text{as } \varepsilon \to 0. \tag{2.11}$$

Now, we define

$$v_\varepsilon := \frac{\psi_\varepsilon}{\left(\int_{B_{2R}} \psi_\varepsilon^6\right)^{1/6}}$$

and $X_\varepsilon := \int_{B_1} |\nabla v_\varepsilon|^2$. Then, we have

$$X_\varepsilon = \int_{B_R} \frac{|\nabla\psi_\varepsilon|}{B^2} + \int_{B_{2R}-B_R} \frac{|\nabla\psi_\varepsilon|}{B^2},$$

where $B := \left(\int_{B_{2R}} \psi_\varepsilon^6\right)^{1/6}$. Thus, since $\varphi \equiv 1$, and consequently $\nabla\varphi \equiv 0$ on B_R, we have

$$X_\varepsilon = \frac{1}{B^2}\int_{B_R} |\nabla\omega_\varepsilon|^2 + \int_{B_{2R}-B_R} |\nabla\psi_\varepsilon|^2.$$

By (2.10) and (2.11) we obtain

$$X_\varepsilon \le S + O(\varepsilon^{1/2}). \tag{2.12}$$

On the other hand, we have

$$\lim_{t\to+\infty} J(tv_\varepsilon) = -\infty, \quad \forall\varepsilon > 0.$$

This implies that there exists $t_\varepsilon > 0$ such that

$$\sup_{t\ge0} J(tv_\varepsilon) = J(t_\varepsilon v_\varepsilon). \tag{2.13}$$

Now, we are going to prove an estimate for t_ε. From (2.13), we have

$$\frac{d}{dt}J(tv_\varepsilon)_{|t=t_\varepsilon} = 0,$$

thus,

$$at_\varepsilon\|v_\varepsilon\|^2 + bt_\varepsilon^3\|v_\varepsilon\|^4 - \lambda t_\varepsilon^{q-1}\int_\Omega p^\alpha|v_\varepsilon|^q - t_\varepsilon^5\int_\Omega |v_\varepsilon|^6 = 0,$$

which implies

$$a\|v_\varepsilon\|^2 + bt_\varepsilon^2\|v_\varepsilon\|^4 - \lambda t_\varepsilon^{q-2}\int_\Omega p^\alpha |v_\varepsilon|^q - t_\varepsilon^4\int_\Omega |v_\varepsilon|^6 = 0.$$

Since $\int_\Omega |v_\varepsilon|^6 = 1$, we have

$$-a\|v_\varepsilon\|^2 - bt_\varepsilon^2\|v_\varepsilon\|^4 + t_\varepsilon^4 \le 0.$$

Hence

$$0 \le t_\varepsilon^2 \le \frac{b\|v_\varepsilon\|^4 + \big[(b\|v_\varepsilon\|^4)^2 + 4a\|v_\varepsilon\|^2\big]^{1/2}}{2} := t_0.$$

Since the function $t \mapsto \frac{a}{2}t^2\|v_\varepsilon\|^2 + \frac{b}{4}t^4\|v_\varepsilon\|^4 - \frac{t^6}{6}$ is increasing on $[0, t_0)$, denoting $C_1 = a\|v_\varepsilon\|^2$ and $C_2 = b\|v_\varepsilon\|^4$, we have

$$J(t_\varepsilon v_\varepsilon) \le \frac{C_1C_2}{4} + \frac{C_2^3}{24} + \frac{1}{24}(C_2^2 + 4C_1)^{3/2} - \frac{\lambda t_\varepsilon^q}{q}\int_\Omega p^\alpha v_\varepsilon^q.$$

Considering $A = 3/4\int_\Omega p^2 v_\varepsilon^2$, by definition of the norm, and the inequality (2.12), we obtain

$$\begin{aligned}
J(t_\varepsilon v_\varepsilon) &\le \frac{ab}{4}(X_\varepsilon + A)^3 + \frac{b^3}{24}(X_\varepsilon + A)^6 + \frac{1}{24}\left[b^2(X_\varepsilon + 4)^4 + 4a(X_\varepsilon + A)\right]^{3/2}\\
&\quad - \frac{\lambda t_\varepsilon^q}{q}\int_\Omega p^\alpha v_\varepsilon^q\\
&\le \frac{ab}{4}(S + O(\varepsilon^{1/2}) + A)^3 + \frac{b^3}{24}(S + O(\varepsilon^{1/2}) + A)^6\\
&\quad + \frac{1}{24}\left[b^2(S + O(\varepsilon^{1/2}) + A)^4 + 4a(S + O(\varepsilon^{1/2}) + A)\right]^{3/2}\\
&\quad - \frac{\lambda t_\varepsilon^q}{q}\int_\Omega p^\alpha v_\varepsilon^q.
\end{aligned}$$

Using several times the standard inequality (see e.g. [28, Page 778])

$$(a+b)^\beta \le a^\beta + \beta(a+b)^{\beta-1}b, \quad \forall\ \beta \ge 1,\ \forall a, b > 0,$$

we infer that

$$\begin{aligned}
J(t_\varepsilon v_\varepsilon) \le \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{1}{24}(b^2S^4 + 4aS)^{3/2} + O(\varepsilon^{1/2})\\
+ \int_{B_{2R}}\Big(\frac{3C}{4}p^2v_\varepsilon^2 - \lambda C_\varepsilon p^\alpha v_\varepsilon^q\Big),
\end{aligned} \tag{2.14}$$

for some constant $C > 0$, where $C_\varepsilon = t_\varepsilon^q/q$.

At this point, we can assume that there exists a positive constant C_0 such that $C_\varepsilon \ge C_0 > 0$ for all $\varepsilon > 0$. If it is not true, then we can find a sequence $\varepsilon_k \to 0$ as $k \to \infty$, such that $t_{\varepsilon_k} \to 0$ as $k \to \infty$, since $C_\varepsilon \ge 0$. Now, up to a subsequence, that we still denote by ε_k, we have $t_{\varepsilon_k}v_{\varepsilon_k} \to 0$, as $k \to \infty$. Therefore,

$$0 < c \le \sup_{t\ge 0} J(tv_{\varepsilon_k}) = J(t_{\varepsilon_k}v_{\varepsilon_k}) = J(0) = 0,$$

which is a contradiction.

Observing that $\int_{B_{2R}} p^2v_\varepsilon^2 < \infty$, we claim that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{1/2}}\int_{B_{2R}}\Big(\frac{3C}{4}p^2v_\varepsilon^2 - C_\varepsilon\lambda p^\alpha v_\varepsilon^q\Big) = -\infty.$$

Assuming the Claim is proved, from (2.14) we have

$$J(t_\varepsilon v_\varepsilon) < \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{1}{24}(b^2S^4 + 4aS)^{3/2},$$

for some $\varepsilon > 0$ sufficiently small, and the proof is complete.

Now, we prove the Claim. For this, it is sufficient to show that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{1/2}}\Big(\int_{B_R}\Big(\frac{3C}{4}p^2\omega_\varepsilon^2 - C_\varepsilon\lambda p^\alpha\omega_\varepsilon^q\Big)\Big) = -\infty \tag{2.15}$$

$$\int_{B_{2R}-B_R}\Big(\frac{3C}{4}p^2v_\varepsilon^2 - C_\varepsilon\lambda p^\alpha v_\varepsilon^q\Big) = O(\varepsilon^{1/2}). \tag{2.16}$$

First, we consider

$$\begin{aligned}
J_\varepsilon &= \frac{1}{\varepsilon^{1/2}}\int_{B_R}\Big(\frac{3C}{4}p^2\omega_\varepsilon^2 - C_\varepsilon\lambda p^\alpha\omega_\varepsilon^q\Big)\\
&= \frac{3C}{4\varepsilon^{1/2}}\int_{B_R}\Big(\frac{2}{1-|x|^2}\Big)^2\frac{(3\varepsilon)^{1/2}}{(\varepsilon+|x|^2)} - \frac{\lambda C_\varepsilon}{\varepsilon^{1/2}}\int_{B_R}\Big(\frac{2}{1-|x|^2}\Big)^\alpha\frac{(3\varepsilon)^{q/4}}{(\varepsilon+|x|^2)^{q/2}}\\
&= \tilde{C}\int_{B_R}\Big(\frac{2}{1-|x|^2}\Big)^2\frac{1}{(\varepsilon+|x|^2)} - \lambda\tilde{C}_\varepsilon\varepsilon^{\frac{(q-2)}{4}}\int_{B_R}\Big(\frac{2}{1-|x|^2}\Big)^\alpha\frac{1}{(\varepsilon+|x|^2)^{q/2}}\\
&= J_1 - J_2,
\end{aligned}\tag{2.17}$$

for some constant $\widetilde{C} > 0$. We observe that on B_R,

$$2 < \frac{2}{1-|x|^2} \le \frac{2}{1-R^2}. \tag{2.18}$$

Therefore, making the change of variables $x = \varepsilon^{1/2}y$ and using the polar coordinates, we obtain

$$J_1 \le \frac{4\tilde{C}}{(1-R^2)^2}\omega\varepsilon^{1/2}\int_0^{R\varepsilon^{-1/2}}\frac{r^2}{(1+r^2)}dr, \tag{2.19}$$

for some constant $\widetilde{C} > 0$. Similarly, for J_2, we have

$$J_2 \ge \lambda\tilde{C}_\varepsilon 2^\alpha w\varepsilon^{-\frac{q}{4}+1}\int_0^{R\varepsilon^{-1/2}}\frac{r^2}{(1+r^2)^{q/2}}dr, \tag{2.20}$$

where $\widetilde{C}_\varepsilon$ is a positive constant. Thus, combining (2.17), (2.19) and (2.20) we obtain

$$\begin{aligned}
J_\varepsilon &\le \frac{4\tilde{C}}{(1-R^2)^2}\omega\varepsilon^{1/2}\int_0^{R\varepsilon^{-1/2}}\frac{r^2}{(1+r^2)}dr\\
&\quad - \lambda\tilde{C}_\varepsilon 2^\alpha w\varepsilon^{-\frac{q}{4}+1}\int_0^{R\varepsilon^{-1/2}}\frac{r^2}{(1+r^2)^{q/2}}dr.
\end{aligned}\tag{2.21}$$

Observing that

$$\int_0^{R\varepsilon^{-1/2}}\frac{r^2}{1+r^2}dr = R\varepsilon^{-1/2} - \tan^{-1}(R\varepsilon^{-1/2})$$

we obtain

$$J_\varepsilon \le C - C\varepsilon^{1/2}\tan^{-1}(R\varepsilon^{-1/2}) - \lambda C\varepsilon^{-\frac{q}{4}+1}\int_0^{R\varepsilon^{-1/2}}\frac{r^2}{(1+r^2)^{q/2}}dr. \tag{2.22}$$

Now, as

$$\int_0^{R\varepsilon^{-1/2}} \frac{r^2}{(1+r^2)^{q/2}} dr \geq \int_0^{R\varepsilon^{-1/2}} \frac{1}{1+r^2} dr \geq C > 0,$$

for all $\varepsilon < \varepsilon_0$, with ε_0 small enough. At this moment, it is possible to see the main difference with the proof of [8, Lemma 2.3]. To control the sign of the expression of (2.15) it is necessary to use the assumption involving λ. Since, by assumption, λ is positive and sufficiently large, we can take $\lambda = \varepsilon^{-\frac{1}{2}}$ and we conclude that (2.15) holds.

The proof of (2.16) is the same of [8, (2.13)], This completes the proof. □

3. Proof of Theorem 1.1

Let $\{v_n\}$ be the sequence given by Lemma 2.2. Lemma 2.3 implies that $\{v_n\}$ is bounded in $H^1_{0,r}(\Omega)$. Thus, we can assume, passing to a subsequence, that $v_n \rightharpoonup v$, weakly in $H^1_{0,r}(\Omega)$ as $n \to \infty$. Arguing as in [9], we have

$$J'(v_n)w = o(1), \quad \forall w \in H^1_{0,r}(\Omega). \tag{3.1}$$

Now, we observe that

$$|J'(v_n)w - J'(v)w| \to 0, \tag{3.2}$$

as $n \to \infty$, for all $w \in C^\infty_{c,rad}(\Omega)$. From this, it follows that $J'(v)w = 0$, for all $w \in C^\infty_{c,rad}(\Omega)$. By denseness, we conclude that

$$J'(v)w = 0, \quad \forall w \in H^1_{0,r}(\Omega), \tag{3.3}$$

and v is a critical point of the functional J restricted to the space $H^1_{0,r}(\Omega)$.

Now, we follow the ideas in [4, 10, 15] (see also [29]). Since $H^1_{0,r}(\Omega)$ is a closed subspace of $H^1_0(\Omega)$, we can write

$$H^1_0(\Omega) = H^1_{0,r}(\Omega) \oplus H^1_{0,r}(\Omega)^\perp,$$

where $\cdot^\perp$ denotes the orthogonal complement of the space. Therefore, for each $w \in H^1_0(\Omega)$, there exist $\vartheta \in H^1_{0,r}(\Omega)$ and $\vartheta^\perp \in H^1_{0,r}(\Omega)^\perp$ such that

$$w = \vartheta + \vartheta^\perp. \tag{3.4}$$

As $H^1_{0,r}(\Omega)$ is a Hilbert space and $J'(v) \in H^1_{0,r}(\Omega)^*$, from the Riesz Representation Theorem there exists $z \in H^1_{0,r}(\Omega)$ such that

$$J'(v)w = \int_\Omega \nabla z \cdot \nabla w, \quad \forall w \in H^1_{0,r}(\Omega).$$

Thus, as $z \in H^1_{0,r}(\Omega)$ and $\vartheta^\perp \in H^1_{0,r}(\Omega)^\perp$, we have

$$J'(v)\vartheta^\perp = 0. \tag{3.5}$$

From (3.3), (3.4) and (3.5), for each $w \in H^1_0(\Omega)$, we obtain

$$J'(v)w = J'(v)\vartheta + I'(v)\vartheta^\perp = 0.$$

This allows us to conclude that v is a critical point of the functional J in $H^1_0(\Omega)$ and consequently v is a weak solution for problem (2.1).

If $v \neq 0$ we are done. Now, we suppose that $v \equiv 0$. Considering $v_n \rightharpoonup 0$, as $n \to \infty$, we have

$$J'(v_n)v_n = a\|v_n\|^2 + b\|v_n\|^4 - \lambda \int_\Omega p^\alpha |v_n|^q - \int_\Omega |v_n|^6 = o_n(1). \tag{3.6}$$

By [9, Lemma 3.1], we obtain

$$\lambda \int_\Omega p^\alpha |v_n|^q \to 0, \text{ as } n \to \infty, \tag{3.7}$$

Let $L_1 > 0$, $L_2 > 0$ be such that

$$a\|v_n\|^2 \to L_1 \quad \text{and} \quad b\|v_n\|^4 \to L_2, \quad \text{as } n \to \infty. \tag{3.8}$$

By (3.6), (3.7), and (3.8),

$$\int_\Omega |v_n|^6 \to L_1 + L_2, \quad \text{as } n \to \infty. \tag{3.9}$$

But

$$S\Big(\int_\Omega v_n^6\Big)^{1/3} \le \int_\Omega |\nabla v_n|^2, \tag{3.10}$$

which implies

$$aS\Big(\int_\Omega v_n^6\Big)^{1/3} \le a\int_\Omega |\nabla v_n|^2 \le a\int_\Omega \big(|\nabla v_n|^2 + (3/4)p^2 v_n^2\big) = a\|v_n\|^2, \tag{3.11}$$

$$bS^2\Big(\int_\Omega v_n^6\Big)^{2/3} \le b\Big[\int_\Omega |\nabla v_n|^2\Big]^2 \le b\Big[\int_\Omega \big(|\nabla v_n|^2 + (3/4)p^2 v_n^2\big)\Big]^2 = b\|v_n\|^4. \tag{3.12}$$

Thus, by (3.8), (3.9), (3.11) and (3.12),

$$L_1 \ge aS(L_1+L_2)^{1/3} \quad \text{and} \quad L_2 \ge bS^2(L_1+L_2)^{2/3}. \tag{3.13}$$

On the other hand, $J(v_n) = c + o(1)$. So

$$c = \frac{L_1}{2} + \frac{L_2}{4} - \frac{1}{6}(L_1+L_2) = \frac{L_1}{3} + \frac{L_2}{12}. \tag{3.14}$$

By (3.13) we have

$$(L_1+L_2)^{1/3} \ge \frac{bs^2 + (b^2s^4 + 4as)^{1/2}}{2}. \tag{3.15}$$

Hence by (3.13), (3.14) and (3.15),

$$\begin{aligned} c &\ge \frac{1}{3}L_1 + \frac{1}{12}L_2 \ge \frac{1}{3}aS(L_1+L_2)^{1/3} + \frac{1}{12}bS^2[(L_1+L_2)^{1/3}]^2 \\ &\ge \frac{1}{4}abS^3 + \frac{1}{24}b^3S^6 + \frac{1}{24}(b^2S^4 + 4aS)^{3/2}, \end{aligned}$$

which is a contradiction to Lemma 2.4. Therefore, we conclude that $v \ne 0$.

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